Adamović D. et al. () "An application of collapsing levels to the representation theory of affine vertex algebras,"
International Mathematics Research Notices, Vol. , Article ID , 37 pages.
doi:10.1093/imrn/

An application of collapsing levels to the representation theory of affine vertex algebras

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We discover a large class of simple affine vertex algebras $V_k(\mathfrak{g})$, associated to basic Lie superalgebras $\mathfrak{g}$ at non–admissible collapsing levels k, having exactly one irreducible $\mathfrak{g}$–locally finite module in the category $\mathcal{O}$. In the case when $\mathfrak{g}$ is a Lie algebra, we prove a complete reducibility result for $V_k(\mathfrak{g})$–modules at an arbitrary collapsing level. We also determine the generators of the maximal ideal in the universal affine vertex algebra $V^k(\mathfrak{g})$ at certain negative integer levels. Considering some conformal embeddings in the simple affine vertex algebras $V_{-1/2}(C_n)$ and $V_{-4}(E_7)$, we surprisingly obtain the realization of non-simple affine vertex algebras of types B and D having exactly one non-trivial ideal.

1 Introduction

Affine vertex algebras are one of the most interesting and important classes of vertex algebras. Categories of modules for simple affine vertex algebra $V_k(\mathfrak{g})$, associated to a simple Lie algebra $\mathfrak{g}$, have mostly been studied in the case of positive integer levels $k \in \mathbb{Z}_{\geq 0}$. These categories enjoy many nice properties such as: finitely many irreducibles, semisimplicity, modular invariance of characters (cf. [26], [31], [34], [41]).

In recent years, affine vertex algebras have attracted a lot of attention because of their connection with affine $\mathcal{W}$–algebras $W_k(\mathfrak{g}, f)$, obtained by quantum Hamiltonian reduction (cf. [21], [23], [35], [36]). Since the quantum Hamiltonian reduction functor $H_f(\,\cdot\,)$ maps any integrable $\widehat{\mathfrak{g}}$–module to zero (cf. [12], [35]), in order to obtain interesting $\mathcal{W}$–algebras, one has to consider affine vertex algebras $V_k(\mathfrak{g})$, for $k \notin \mathbb{Z}_{\geq 0}$.

Received 1 Month 20XX; Revised 11 Month 20XX; Accepted 21 Month 20XX
Communicated by

It turns out that for certain non-admissible levels k (such as negative integer levels), the associated vertex algebras $V_k(\mathfrak{g})$ have finitely many irreducibles in category $\mathcal{O}$ (cf. [15], [17], [40]), and their characters satisfy certain modular-like properties (cf. [14]). These affine vertex algebras then give C_2–cofinite $\mathcal{W}$–algebras $W_k(\mathfrak{g}, f)$, for properly chosen nilpotent element f (cf. [37], [39]).

In this paper, we classify irreducible modules in the category KL_k (i.e. the category of $\mathfrak{g}$–locally finite $V_k(\mathfrak{g})$–modules in $\mathcal{O}^k$ (see Subsection 2.3) for a large family of collapsing levels k. Recall from [4] that a level k is called *collapsing* if the simple $\mathcal{W}$–algebra $W_k(\mathfrak{g}, \theta)$, associated to a minimal nilpotent element $e_{-\theta}$, is isomorphic to its affine vertex subalgebra $\mathcal{V}_k(\mathfrak{g}^\natural)$ (see Definition 2.2 and (7)). In the present paper we keep the notation of [4]. In particular, the highest root is normalized by the condition $(\theta, \theta) = 2$. We discover a large family of vertex algebras having one irreducible module in the category KL_k, which in a way extends the results on Deligne series from [15]. Part (1) is proven there in the Lie algebra case.

Theorem 1.1. Assume that the level k and the basic simple Lie superalgebra $\mathfrak{g}$ satisfy one of the following conditions:

(1) $k = -\frac{h^\vee}{6} - 1$ and $\mathfrak{g}$ is one of the Lie algebras of exceptional Deligne's series A_2, G_2, D_4, F_4, E_6, E_7, E_8, or $\mathfrak{g} = psl(m|m)$ $(m \geq 2)$, $osp(n+8|n)$ $(n \geq 2)$, $spo(2|1)$, $F(4)$, $G(3)$ (for both choices of θ);
(2) $k = -h^\vee/2 + 1$ and $\mathfrak{g} = osp(n + 4m + 8|n)$, $n \geq 2, m \geq 0$.
(3) $k = -h^\vee/2 + 1$ and $\mathfrak{g} = D_{2m}$, $m \geq 2$.
(4) $k = -10$ and $\mathfrak{g} = E_8$.

Then $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$–module in the category KL_k. □

We also prove a complete reducibility result in KL_k (cf. Theorem 5.9, Theorem 5.7):

Theorem 1.2. Assume that $\mathfrak{g}$ is a Lie algebra and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. Then KL_k is a semi-simple category in the following cases:

- k is a collapsing level.
- $W_k(\mathfrak{g}, \theta)$ is a rational vertex operator algebra.

□

It is interesting that in some cases we have that KL_k is a semi-simple category, but there can exist indecomposable but not irreducible $V_k(\mathfrak{g})$–modules in the category $\mathcal{O}$. In order to prove Theorem 1.2 we modified methods from [28] and [20] in a vertex algebraic setting. In particular we prove that the contravariant functor $M \mapsto M^\sigma$ from [20] acts on the category KL_k (cf. Lemma 3.6). Then for the proof of complete reducibility in KL_k it is enough to check that every highest weight $V_k(\mathfrak{g})$–module in KL_k is irreducible (cf. Theorem 5.5).

Representation theory of a simple affine vertex algebra $V_k(\mathfrak{g})$ is naturally connected with the structure of the maximal ideal in the universal affine vertex algebra $V^k(\mathfrak{g})$. In the second part of paper we present explicit formulas for singular vectors which generate the maximal ideal in $V^{2-2\ell}(D_{2\ell})$ (which is case (3) of Theorem 1.1) and $V^{-2}(D_\ell)$. In the second case, we show that the Hamiltonian reduction functor $H_\theta(\,\cdot\,)$ gives an equivalence of the category of $\mathfrak{g}$–locally finite $V_{-2}(D_\ell)$–modules KL_{-2} and the category of modules for a rational vertex algebra $V_{\ell-4}(A_1)$. Singular vectors in $V^k(\mathfrak{g})$ for certain negative integer levels k have also been constructed in [2].

We also apply our results to study the structure of conformally embedded subalgebras of some simple affine vertex algebras.

As in [6], for a subalgebra $\mathfrak{k}$ of a simple Lie algebra $\mathfrak{g}$, we denote by $\widetilde{V}(k,\mathfrak{k})$ the vertex subalgebra of $V_k(\mathfrak{g})$ generated by $x(-1)\mathbf{1}$, $x\in\mathfrak{k}$. If $\mathfrak{k}$ is a reductive quadratic subalgebra of $\mathfrak{g}$, then we say that $\widetilde{V}(k,\mathfrak{k})$ is conformally embedded in $V_k(\mathfrak{g})$ if the Sugawara-Virasoro vectors of both algebras coincide. We also say that $\mathfrak{k}$ is conformally embedded in $\mathfrak{g}$ at level k if $\widetilde{V}(k,\mathfrak{k})$ is conformally embedded in $V_k(\mathfrak{g})$.

We are able to prove that in the cases listed in Theorem 1.3 below, $\widetilde{V}(k,\mathfrak{k})$ is not simple. On the other hand, we show that $V_{-1/2}(C_5)$ contains a *simple* subalgebra $V_{-2}(B_2)\otimes V_{-5/2}(A_1)$ (see Corollary 7.4). For the conformal embedding of $D_6\times A_1$ into E_7 at level $k=-4$, we show that $\widetilde{V}(-4,D_6\times A_1)=\mathcal{V}_{-4}(D_6)\otimes V_{-4}(A_1)$ where $\mathcal{V}_{-4}(D_6)$ is a quotient of the universal affine vertex algebra $V^{-4}(D_6)$ by two singular vectors of conformal weights two and three (cf. (39)). Moreover, $\mathcal{V}_{-4}(D_6)$ has infinitely many irreducible modules in the category of $\mathfrak{g}$–locally finite modules, which we explicitly describe. All of them appear in $V_{-4}(E_7)$ as submodules or subquotients.

Theorem 1.3. Let $\mathcal{V}_k(D_\ell)$, $\mathcal{V}_k(B_\ell)$, be the vertex algebras defined in (25), (26), (39). Consider the following conformal embeddings:

(1) $D_\ell\times A_1$ into C_{2l} for $\ell\geq 4$ at level $k=-\frac{1}{2}$.

(2) $B_\ell\times A_1$ into C_{2l+1} for $\ell\geq 3$ at level $k=-\frac{1}{2}$.

(3) $D_6\times A_1$ into E_7 at level $k=-4$.

Then,

- $\widetilde{V}(-\frac{1}{2},D_\ell\times A_1)=\mathcal{V}_{-2}(D_\ell)\otimes V_{-\ell}(A_1)$ in case (1),
- $\widetilde{V}(-\frac{1}{2},B_\ell\times A_1)=\mathcal{V}_{-2}(B_\ell)\otimes V_{-\ell-1/2}(A_1)$ in case (2),
- $\widetilde{V}(-4,D_6\times A_1)=\mathcal{V}_{-4}(D_6)\otimes V_{-4}(A_1)$ in case (3).

Moreover, the algebras $\mathcal{V}_k(D_\ell)$, $\mathcal{V}_k(B_\ell)$, are non-simple, with a unique non-trivial ideal. $\square$

The decompositions of the embeddings above is still an open problem, and will be a subject of our forthcoming papers.

2 Preliminaries

We assume that the reader is familiar with the notion of vertex (super)algebra (cf. [18], [25], [32]) and of simple basic Lie superalgebras (see [30]) and their affinizations (see [31] for the Lie algebra case).

Let V be a conformal vertex algebra. Denote by $A(V)$ the associative algebra introduced in [41], called the Zhu algebra of V.

2.1 Basic Lie superalgebras and minimal gradings

For the reader's convenience we recall here the setting and notation of [4] regarding basic Lie superalgebras and their minimal gradings. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a simple finite dimensional basic Lie superalgebra. We choose a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}_{\bar{0}}$ and let Δ be the set of roots. Assume $\mathfrak{g}$ is not $osp(3|n)$. A root $-\theta$ is called *minimal* if it is even and there exists an additive function $\varphi : \Delta \to \mathbb{R}$ such that $\varphi_{|\Delta} \neq 0$ and $\varphi(\theta) > \varphi(\eta)$, $\forall\, \eta \in \Delta \setminus \{\theta\}$. Fix a minimal root $-\theta$ of $\mathfrak{g}$. We may choose root vectors e_θ and $e_{-\theta}$ such that

$$[e_\theta, e_{-\theta}] = x \in \mathfrak{h}, \qquad [x, e_{\pm\theta}] = \pm e_{\pm\theta}.$$

Due to the minimality of $-\theta$, the eigenspace decomposition of $ad\, x$ defines a *minimal* $\frac{1}{2}\mathbb{Z}$-grading ([36, (5.1)]):

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \tag{1}$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C} e_{\pm\theta}$. We thus have a bijective correspondence between minimal gradings (up to an automorphism of $\mathfrak{g}$) and minimal roots (up to the action of the Weyl group). Furthermore, one has

$$\mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x, \quad \mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}. \tag{2}$$

Note that $\mathfrak{g}^\natural$ is the centralizer of the triple $\{f_\theta, x, e_\theta\}$. We can choose $\mathfrak{h}^\natural = \{h \in \mathfrak{h} \mid (h|x) = 0\}$, as a Cartan subalgebra of the Lie superalgebra $\mathfrak{g}^\natural$, so that $\mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x$.

For a given choice of a minimal root $-\theta$, we normalize the invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ by the condition

$$(\theta|\theta) = 2. \tag{3}$$

The dual Coxeter number $h^\vee$ of the pair $(\mathfrak{g}, \theta)$ (equivalently, of the minimal gradation (1)) is defined to be half the eigenvalue of the Casimir operator of $\mathfrak{g}$ corresponding to $(\cdot|\cdot)$, normalized by (3). Since θ is the highest root, we have that $2h^\vee = (\theta|\theta + 2\rho)$ hence

$$(\rho|\theta) = h^\vee - 1. \tag{4}$$

The complete list of the Lie superalgebras $\mathfrak{g}^\natural$, the $\mathfrak{g}^\natural$–modules $\mathfrak{g}_{\pm 1/2}$ (they are isomorphic and self-dual), and $h^\vee$ for all possible choices of $\mathfrak{g}$ and of θ (up to isomorphism) is given in Tables 1,2,3 of [36]. We reproduce them below. Note that in these tables $\mathfrak{g} = osp(m|n)$ (resp. $\mathfrak{g} = spo(n|m)$) means that θ is the highest root of the simple component $so(m)$ (resp. $sp(n)$) of $\mathfrak{g}_{\bar{0}}$. Also, for $\mathfrak{g} = sl(m|n)$ or $psl(m|m)$ we always take θ to be the highest root of the simple component $sl(m)$ of $\mathfrak{g}_{\bar{0}}$ (for $m = 4$ we take one of the simple roots). Note that the exceptional Lie superalgebras $\mathfrak{g} = F(4)$ and $\mathfrak{g} = G(3)$ appear in both Tables 2 and 3, which corresponds to the two inequivalent choices of θ, the first one being a root of the simple component $sl(2)$ of $\mathfrak{g}_{\bar{0}}$.

Table 1

$\mathfrak{g}$ is a simple Lie algebra.

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$	$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$
$sl(n), n \geq 3$	$gl(n-2)$	$\mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$	n	F_4	$sp(6)$	$\bigwedge_0^3 \mathbb{C}^6$	9
$so(n), n \geq 5$	$sl(2) \oplus so(n-4)$	$\mathbb{C}^2 \otimes \mathbb{C}^{n-4}$	$n-2$	E_6	$sl(6)$	$\bigwedge^3 \mathbb{C}^6$	12
$sp(n), n \geq 2$	$sp(n-2)$	$\mathbb{C}^{n-2}$	$n/2+1$	E_7	$so(12)$	$spin_{12}$	18
G_2	$sl(2)$	$S^3\mathbb{C}^2$	4	E_8	E_7	$\dim = 56$	30

Table 2

$\mathfrak{g}$ is not a Lie algebra but $\mathfrak{g}^\natural$ is and $\mathfrak{g}_{\pm 1/2}$ is purely odd ($m \geq 1$).

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$	$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$
$sl(2\|m)$, $m \neq 2$	$gl(m)$	$\mathbb{C}^m \oplus (\mathbb{C}^m)^*$	$2-m$	$D(2,1;a)$	$sl(2) \oplus sl(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	0
$psl(2\|2)$	$sl(2)$	$\mathbb{C}^2 \oplus \mathbb{C}^2$	0	$F(4)$	$so(7)$	$spin_7$	-2
$spo(2\|m)$	$so(m)$	$\mathbb{C}^m$	$2-m/2$	$G(3)$	G_2	$\mathrm{Dim} = 0\|7$	$-3/2$
$osp(4\|m)$	$sl(2) \oplus sp(m)$	$\mathbb{C}^2 \otimes \mathbb{C}^m$	$2-m$				

Table 3

Both $\mathfrak{g}$ and $\mathfrak{g}^\natural$ are not Lie algebras ($m, n \geq 1$).

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$
$sl(m\|n)$, $m \neq n, m > 2$	$gl(m-2\|n)$	$\mathbb{C}^{m-2\|n} \oplus (\mathbb{C}^{m-2\|n})^*$	$m-n$
$psl(m\|m)$, $m > 2$	$sl(m-2\|m)$	$\mathbb{C}^{m-2\|m} \oplus (\mathbb{C}^{m-2\|m})^*$	0
$spo(n\|m)$, $n \geq 4$	$spo(n-2\|m)$	$\mathbb{C}^{n-2\|m}$	$1/2(n-m)+1$
$osp(m\|n)$, $m \geq 5$	$osp(m-4\|n) \oplus sl(2)$	$\mathbb{C}^{m-4\|n} \otimes \mathbb{C}^2$	$m-n-2$
$F(4)$	$D(2,1;2)$	$\mathrm{Dim} = 6\|4$	3
$G(3)$	$osp(3\|2)$	$\mathrm{Dim} = 4\|4$	2

In this paper we shall exclude the case of $\mathfrak{g} = sl(n+2|n)$, $n > 0$. In all other cases the Lie superalgebra $\mathfrak{g}^\natural$ decomposes in a direct sum of all its minimal ideals, called components of $\mathfrak{g}^\natural$:

$$\mathfrak{g}^\natural = \bigoplus_{i \in I} \mathfrak{g}_i^\natural,$$

where each summand is either the (at most 1-dimensional) center of $\mathfrak{g}^\natural$ or is a basic simple Lie superalgebra different from $psl(n|n)$. Let $C_{\mathfrak{g}_i^\natural}$ be the Casimir operator of $\mathfrak{g}_i^\natural$ corresponding to $(\cdot|\cdot)_{|\mathfrak{g}_i^\natural \times \mathfrak{g}_i^\natural}$. We define the dual Coxeter number $h_{0,i}^\vee$ of $\mathfrak{g}_i^\natural$ as half of the eigenvalue of $C_{\mathfrak{g}_i^\natural}$ acting on $\mathfrak{g}_i^\natural$ (which is 0 if $\mathfrak{g}_i^\natural$ is abelian).

Denote by $V_{\mathfrak{g}}(\mu)$ (or $V(\mu)$) the irreducible finite-dimensional highest weight $\mathfrak{g}$–module with highest weight μ. Denote by P_+ the set of highest weights of irreducible finite-dimensional representations of $\mathfrak{g}$.

Since $\mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x$, we have, in particular, that $\mu \in \mathfrak{h}^*$ can be uniquely written as

$$\mu = \mu_{|\mathfrak{h}^\natural} + \ell\theta, \tag{5}$$

with $\ell \in \mathbb{C}$. If $\mu \in P_+$, then, since $\theta(\mathfrak{h}^\natural) = 0$, $\mu(\theta^\vee) = 2\ell \in \mathbb{Z}$, so $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

2.2 Affine Lie algebras, vertex algebras, $\mathcal{W}$-algebras

Let $\widehat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$:

$$\widehat{\mathfrak{g}} = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with the usual commutation relations. We let δ be the fundamental imaginary root. Let $\alpha_0 = \delta - \theta$ the affine simple root. Since θ is even, hence non-isotropic, so that $\alpha_0^\vee = K - \theta^\vee$ makes sense.

Denote by $L(\lambda)$ (or $L_{\mathfrak{g}}(\lambda)$) the irreducible highest weight $\widehat{\mathfrak{g}}$–module with highest weight λ.

Denote by $V^k(\mathfrak{g})$ the universal affine vertex algebra associated to $\widehat{\mathfrak{g}}$ of level $k \in \mathbb{C}$. We shall assume that $k \neq -h^\vee$. Then (see e.g. [32]) $V^k(\mathfrak{g})$ is a conformal vertex algebra with Segal-Sugawara conformal vector $\omega_{\mathfrak{g}}$. Let $Y(\omega_{\mathfrak{g}}, z) = \sum L_{\mathfrak{g}}(n) z^{-n-2}$ be the corresponding Virasoro field. Denote by $V_k(\mathfrak{g})$ the (unique) simple quotient of $V^k(\mathfrak{g})$. Clearly, $V_k(\mathfrak{g}) \cong L_{\mathfrak{g}}(k\Lambda_0)$ as $\widehat{\mathfrak{g}}$–modules.

Denote by $W^k(\mathfrak{g},\theta)$ the affine $\mathcal{W}$–algebra obtained from $V^k(\mathfrak{g})$ by Hamiltonian reduction relative to a minimal nilpotent element $e_{-\theta}$. Denote by $W_k(\mathfrak{g},\theta)$ the simple quotient of $W^k(\mathfrak{g},\theta)$. Recall that the vertex algebra $W^k(\mathfrak{g},\theta)$ is strongly and freely generated by elements $J^{\{a\}}$, where a runs over a basis of $\mathfrak{g}^\natural$, $G^{\{v\}}$, where v runs over a basis of $\mathfrak{g}_{-1/2}$, and the Virasoro vector ω. The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight 1 and $3/2$, respectively, with respect to ω.

Let $\mathcal{V}^k(\mathfrak{g}^\natural)$ be the subalgebra of the vertex algebra $W^k(\mathfrak{g},\theta)$, generated by $\{J^{\{a\}} \mid a \in \mathfrak{g}^\natural\}$. The vertex algebra $\mathcal{V}^k(\mathfrak{g}^\natural)$ is isomorphic to a universal affine vertex algebra. More precisely, letting

$$k_i = k + \tfrac{1}{2}(h^\vee - h^\vee_{0,i}),\ i \in I, \tag{6}$$

the map $a \mapsto J^{\{a\}}$ extends to an isomorphism $\mathcal{V}^k(\mathfrak{g}^\natural) \simeq \bigotimes_{i\in I} V^{k_i}(\mathfrak{g}_i^\natural)$.

We also set $\mathcal{V}_k(\mathfrak{g}^\natural)$ to be the image of $\mathcal{V}^k(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g},\theta)$. Clearly we can write

$$\mathcal{V}_k(\mathfrak{g}^\natural) \simeq \bigotimes_{i\in I} \mathcal{V}_{k_i}(\mathfrak{g}_i^\natural), \tag{7}$$

where $\mathcal{V}_{k_i}(\mathfrak{g}_i^\natural)$ is some quotient (not necessarily simple) of $V^{k_i}(\mathfrak{g}_i^\natural)$.

2.3 Category $\mathcal{O}$ and Hamiltonian reduction functor

Recall that $\widehat{\mathfrak{g}}$-module M is in category $\mathcal{O}^k$ if it is $\widehat{\mathfrak{h}}$-diagonalizable with finite dimensional weight spaces, K acts as kId_M and M has a finite number of maximal weights.

There is a remarkable functor H_θ from $\mathcal{O}^k$ to the category of $W^k(\mathfrak{g},\theta)$-modules whose properties will be very important in the following. We recall them in a form suitable for our purposes (see [12] for details; there H_θ is denoted by H^0).

Theorem 2.1.

1. H_θ is exact.
2. If $L(\lambda)$ is a irreducible highest weight $\widehat{\mathfrak{g}}$-module, then $\lambda(\alpha_0^\vee) \in \mathbb{Z}_{\geq 0}$ implies $H_\theta(L(\lambda)) = \{0\}$. Otherwise $H_\theta(L(\lambda))$ is isomorphic to the irreducible $W^k(\mathfrak{g}, \theta)$-module with highest weight ϕ_λ defined by formula (67) in [12].

□

2.4 Collapsing levels

Definition 2.2. Assume $k \neq -h^\vee$. If $W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(\mathfrak{g}^\natural)$, we say that k is a *collapsing level.* □

Theorem 2.3. [4, Theorem 3.3] Let $p(k)$ be the polynomial listed in Table 4 below. Then k is a collapsing level if and only if $k \neq -h^\vee$ and $p(k) = 0$. In such cases,

$$W_k(\mathfrak{g}, \theta) = \bigotimes_{i \in I^*} V_{k_i}(\mathfrak{g}_i^\natural), \tag{8}$$

where $I^* = \{i \in I \mid k_i \neq 0\}$. If $I^* = \emptyset$, then $W_k(\mathfrak{g}, \theta) = \mathbb{C}$. □

Table 4

$\mathfrak{g}$	$p(k)$	$\mathfrak{g}$	$p(k)$
$sl(m\|n),\ n \neq m$	$(k+1)(k+(m-n)/2)$	E_6	$(k+3)(k+4)$
$psl(m\|m)$	$k(k+1)$	E_7	$(k+4)(k+6)$
$osp(m\|n)$	$(k+2)(k+(m-n-4)/2)$	E_8	$(k+6)(k+10)$
$spo(n\|m)$	$(k+1/2)(k+(n-m+4)/4)$	F_4	$(k+5/2)(k+3)$
$D(2,1;a)$	$(k-a)(k+1+a)$	G_2	$(k+4/3)(k+5/3)$
$F(4),\ \mathfrak{g}^\natural = so(7)$	$(k+2/3)(k-2/3)$	$G(3),\ \mathfrak{g}^\natural = G_2$	$(k-1/2)(k+3/4)$
$F(4),\ \mathfrak{g}^\natural = D(2,1;2)$	$(k+3/2)(k+1)$	$G(3),\ \mathfrak{g}^\natural = osp(3\|2)$	$(k+2/3)(k+4/3)$

2.5 Weyl vertex algebra

Let M_ℓ denote the Weyl vertex algebra (also called symplectic bosons) generated by even elements $a_i^\pm$, $i = 1, \ldots, \ell$ satisfying the following λ–brackets

$$[(a_i^\pm)_\lambda (a_j^\pm)] = 0, \quad [(a_i^+)_\lambda (a_j^-)] = \delta_{i,j}.$$

Recall also that the symplectic affine vertex algebra $V_{-1/2}(C_\ell)$ is realized as a $\mathbb{Z}_2$–orbifold of M_ℓ (see [22]).

3 The category KL_k

Let k be a noncritical level. Note that the Casimir element of $\widehat{\mathfrak{g}}$ can be expressed as $\Omega = d + L_{\mathfrak{g}}(0)$; it commutes with $\widehat{\mathfrak{g}}$–action.

Consider the category $\mathcal{C}^k$ of modules for the universal affine vertex algebra $V^k(\mathfrak{g})$, i.e. the category of restricted $\widehat{\mathfrak{g}}$–modules of level k. Regard $M \in \mathcal{C}^k$ as a $\widehat{\mathfrak{g}}$–module by letting d act as $-L_{\mathfrak{g}}(0)$. Let KL^k be the category of modules $M \in \mathcal{C}^k$ such that, as $\widehat{\mathfrak{g}}$-modules, are in $\mathcal{O}^k$ and which admit the following weight space decomposition with respect to $L_{\mathfrak{g}}(0)$:

$$M = \bigoplus_{\alpha \in \mathbb{C}} M(\alpha), \quad L_{\mathfrak{g}}(0)|M(\alpha) \equiv \alpha \, \mathrm{Id}, \; \dim M(\alpha) < \infty.$$

Our definition is related but different from the one introduced in [13]. Let KL_k be the category of all modules in KL^k which are $V_k(\mathfrak{g})$–modules.

Remark 3.1. If $V_k(\mathfrak{g})$ has finitely many irreducible modules in the category KL^k, one can show that every $V_k(\mathfrak{g})$–module M in KL_k is of finite length. This happens when k is admissible (cf. [12]) and when $V_k(\mathfrak{g})$ is quasi-lisse (cf. [14]). But when $V_k(\mathfrak{g})$ has infinitely many irreducible modules in KL^k (as in the cases considered in [33], [11]), then one can have modules in KL_k of infinite length. □

Recall that there is a one-to-one correspondence between irreducible $\mathbb{Z}_{\geq 0}$–graded modules for a conformal vertex algebra V (with a conformal vector ω, such that $Y(\omega, z) = \sum_{i \in \mathbb{Z}} L(i) z^{-i-2}$) and irreducible modules for the corresponding Zhu algebra $A(V)$ [41]. This implies, in particular, that there is a one-to-one correspondence between irreducible finite-dimensional $A(V)$–modules and irreducible $\mathbb{Z}_{\geq 0}$–graded V–modules whose graded components, which are eigenspaces for $L(0)$, are finite-dimensional. In the case of affine vertex algebras, we have the following simple interpretation.

Proposition 3.2. Let $\widetilde{V}_k(\mathfrak{g})$ be a quotient of $V^k(\mathfrak{g})$ (not necessary simple). Consider $\widetilde{V}_k(\mathfrak{g})$ as a conformal vertex algebra with conformal vector $\omega_{\mathfrak{g}}$. Then there is a one-to-one correspondence between irreducible $\widetilde{V}_k(\mathfrak{g})$ in the category KL^k and irreducible finite-dimensional $A(\widetilde{V}_k(\mathfrak{g}))$–modules. □

Corollary 3.3. Assume that $\mathfrak{g}$ is a simple *basic* Lie superalgebra and $\widetilde{V}_k(\mathfrak{g})$ is a quotient of $V^k(\mathfrak{g})$ such that the trivial module $\mathbb{C}$ is the unique finite-dimensional irreducible $A(\widetilde{V}_k(\mathfrak{g}))$–module. Then $\widetilde{V}_k(\mathfrak{g}) = V_k(\mathfrak{g})$. □

Proof. Assume that $\widetilde{V}_k(\mathfrak{g})$ is not simple. Then it contains a non-zero graded ideal $I \neq \widetilde{V}_k(\mathfrak{g})$ with respect to $L_{\mathfrak{g}}(0)$:

$$I = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} I(n + n_0), \quad L_{\mathfrak{g}}(0)|I(r) = r\mathrm{Id}, \quad I(n_0) \neq 0.$$

Since $I \neq \widetilde{V}_k(\mathfrak{g})$, we have that $n_0 > 0$, otherwise $\mathbf{1} \in I$.

We can consider $I(n_0)$ as a finite-dimensional module for $\mathfrak{g}$ and for the Zhu algebra $A(\widetilde{V}_k(\mathfrak{g}))$.

Since the Casimir element $C_{\mathfrak{g}}$ of $\mathfrak{g}$ acts on $I(n_0)$ as the non-zero constant $2(k + h^\vee)n_0$, we conclude that $C_{\mathfrak{g}}$ acts by the same constant on any irreducible $\mathfrak{g}$–subquotient of $I(n_0)$. But any irreducible subquotient of $I(n_0)$ is an irreducible finite–dimensional $A(\widetilde{V}_k(\mathfrak{g}))$–module, and therefore it is trivial. This implies that $C_{\mathfrak{g}}$ acts non-trivially on a trivial $\mathfrak{g}$–module. A contradiction. ■

Take the Chevalley generators e_i, f_i, h_i, $i = 0, \dots, \ell$, of the Kac–Moody Lie algebra $\widehat{\mathfrak{g}}$ such that e_i, f_i, h_i, $i = 1, \dots, \ell$, are the Chevalley generators of $\mathfrak{g}$. Let σ be the Chevalley antiautomorphism of $\widehat{\mathfrak{g}}$ defined by

$$e_i \mapsto f_i, \quad f_i \mapsto e_i, \;\; h_i \mapsto h_i, \quad d \mapsto d \quad (i = 0, \dots, \ell).$$

Assume that M is from the category $\mathcal{O}$ of non-critical level k. Then M admits the decomposition into weight spaces $M = \bigoplus_{\mu \in \Omega(M)} M_\mu$, where $\Omega(M)$ is the set of weights of M and $\dim M_\mu < \infty$ for every $\mu \in \Omega(M)$. For a finite-dimensional vector spaces U, let U^* denote its dual space. Then we have the contravariant functor $M \mapsto M^\sigma$ [20] acting on modules from the category $\mathcal{O}$. Here $M^\sigma = \bigoplus_{\mu \in \Omega(M)} M_\mu^*$ is the $\widehat{\mathfrak{g}}$–module uniquely determined by

$$\langle y w', w \rangle = \langle w', \sigma(y) w \rangle, \quad y \in \widehat{\mathfrak{g}}, \;\; w' \in M^\sigma, \;\; w \in M.$$

It is easy to see that M admits the decomposition

$$M = \bigoplus_{\alpha \in \mathbb{C}} M(\alpha), \quad L_{\mathfrak{g}}(0) | M(\alpha) \equiv \alpha \, \mathrm{Id} \tag{9}$$

such that :

- for any $\alpha \in \mathbb{C}$ we have $M(\alpha - n) = 0$ for $n \in \mathbb{Z}$ sufficiently large;
- for any $\mu \in \Omega(M)$ there exist $\alpha \in \mathbb{C}$ such that $M_\mu \subset M(\alpha)$.

Proposition 3.4. Assume that a module M is in the category $\mathcal{O}^k$. Then M is in the category KL^k if and only if M is $\mathfrak{g}$-locally finite. □

Proof. If M is in KL^k then it admits a decomposition as in (9). Since the spaces $M(\alpha)$ are $\mathfrak{g}$–stable and finite-dimensional, M is $\mathfrak{g}$–locally finite.

Let us prove the converse. If M is a highest weight module which is $\mathfrak{g}$–locally finite, then clearly all eigenspaces for $L_{\mathfrak{g}}(0)$ are finite-dimensional. Assume now that M is an arbitrary $\mathfrak{g}$–locally finite module in the category $\mathcal{O}^k$. Take $\alpha \in \mathbb{C}$ such that $M(\alpha) \neq \{0\}$. Then from [20, Proposition 3.1] we see that M has an increasing filtration (possibly infinite)

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M \tag{10}$$

such that for every $j \in \mathbb{Z}_{>0}$, $M_j / M_{j-1} \cong \widetilde{L}(\lambda_j)$ is a highest weight $V^k(\mathfrak{g})$–module with highest weight λ_j, which is $\mathfrak{g}$–locally finite. Let h_{λ_j} denotes the lowest conformal weight of $\widetilde{L}(\lambda_j)$. Since the factors M_i/M_{i-1} $(i \le j)$ of M_j are highest weight modules, their $L_{\mathfrak{g}}(0)$–eigenspaces are finite-dimensional. This implies that the $L_{\mathfrak{g}}(0)$–eigenspaces of M_j is finite-dimensional. By using the properties of the category $\mathcal{O}$ one sees the following:

- There exists a finite subset $\{d_1, \cdots, d_s\} \subset \mathbb{C}$ such that $\alpha \in \bigcup_{i=1}^{s}(d_i + \mathbb{Z}_{\geq 0})$.
- For $d \in \mathbb{C}$ there exist only finitely many subquotients $\widetilde{L}(\lambda_j)$ in (10) such that $h_{\lambda_j} = d$.

This implies that there is $j_0 \in \mathbb{Z}_{>0}$ such that $\alpha < h_{\lambda_j}$ for $j \geq j_0$. Therefore $M(\alpha) \subset M_{j_0}$. This proves that $M(\alpha)$ is finite-dimensional. ■

Remark 3.5. We will use several times the following fact, which is a consequence of the previous proposition: for any $k \notin \mathbb{Z}_{\geq 0}$ and any irreducible highest weight module $L(\lambda)$ in the category KL^k, one has $\lambda(\alpha_0^\vee) \notin \mathbb{Z}_{\geq 0}$. □

Since $\sigma(L_{\mathfrak{g}}(0)) = L_{\mathfrak{g}}(0)$, if M is in the category KL^k, then M^σ is also in the category KL^k. The next result shows that this functor acts on the category KL_k. In the proof we find an explicit relation of M^σ with the contragradient modules, defined for ordinary modules for vertex operator algebras [24].

Lemma 3.6.

(1) Assume that M is a $V_k(\mathfrak{g})$–module in the category $\mathcal{O}$. Then M^σ is also a $V_k(\mathfrak{g})$–module in the category $\mathcal{O}$.

(2) Assume that M is a $V_k(\mathfrak{g})$–module in the category KL_k. Then M^σ is also in KL_k. □

Proof. Assume that M is a $V_k(\mathfrak{g})$–module in the category $\mathcal{O}$. Take the weight decomposition $M = \bigoplus_{\mu\in\Omega(M)} M_\mu$, and set $M^c = \bigoplus_{\mu\in\Omega(M)} M_\mu^*$. By applying the same approach as in the construction of the contragredient module from [24, Section 5], we get a $V_k(\mathfrak{g})$–module $(M^c, Y_{M^c}(\cdot, z))$, with vertex operator map

$$\langle Y_{M^c}(v,z)w', w\rangle = \langle w', Y_M(e^{zL_{\mathfrak{g}}(1)}(-z^{-2})^{L_{\mathfrak{g}}(0)}v, z)w\rangle, \tag{11}$$

where $w' \in M^c$, $w \in M$. The $\widehat{\mathfrak{g}}$–action on M^c is uniquely determined by

$$\langle x(n)w', w\rangle = -\langle w', x(-n)w\rangle \quad (x \in \mathfrak{g}).$$

As a vector space $M^c = M^\sigma$, but we have different actions of $\widehat{\mathfrak{g}}$. (Note that, in general, M^c can be outside of the category $\mathcal{O}$.)

Take the Lie algebra automorphism $h \in Aut(\mathfrak{g})$ such that

$$e_i \mapsto -f_i, \quad f_i \mapsto -e_i, \ h_i \mapsto -h_i \quad (i = 1, \dots, \ell).$$

Then h can be lifted to an automorphism of $V^k(\mathfrak{g})$. Since the maximal ideal of $V^k(\mathfrak{g})$ is unique, then it is h–invariant, thus h is also an automorphism of $V_k(\mathfrak{g})$. Then we can define a $V_k(\mathfrak{g})$–module $(M_h^c, Y_{M_h^c}(\cdot, z))$ where

$$M_h^c := M^c, \quad Y_{M_h^c}(v,z) = Y_{M^c}(hv, z).$$

On $M^c_{\mathfrak{h}}$ we have

$$\langle e_i(n)w', w\rangle = \langle w', f_i(-n)w\rangle$$

$$\langle f_i(n)w', w\rangle = \langle w', e_i(-n)w\rangle$$

$$\langle h_i(n)w', w\rangle = \langle w', h_i(-n)w\rangle$$

where $i = 1, \ldots, \ell$. This implies that $M^c_{\mathfrak{h}} = M^\sigma$. This proves the assertion (1).

Assume now that M is in the category KL_k. Then all $L_{\mathfrak{g}}(0)$–eigenspaces are finite-dimensional, thus

$$M^c = \bigoplus_{\mu\in\Omega(M)} M^*_\mu = \bigoplus_{\alpha\in\mathbb{C}} M(\alpha)^*.$$

This implies the $V_k(\mathfrak{g})$–module $(M^c, Y_{M^c}(\cdot, z))$ coincides with the contragredient module [24], realized on the restricted dual space $\bigoplus_{\alpha\in\mathbb{C}} M(\alpha)^*$, with the vertex operator map (11). Since the $L_{\mathfrak{g}}(0)$–eigenspaces of M^c are finite-dimensional, we conclude that M^c and $M^\sigma = M^c_{\mathfrak{h}}$ are $V_k(\mathfrak{g})$–modules in KL_k. Claim (2) follows. ■

4 Constructions of vertex algebras with one irreducible module in KL_k via collapsing levels

By [4], if k is a collapsing level, then either $W_k(\mathfrak{g}, \theta) = \mathbb{C}$, $W_k(\mathfrak{g}, \theta) = M(1)$, or $W_k(\mathfrak{g}, \theta) = V_{k'}(\mathfrak{a})$ for a unique simple component $\mathfrak{a}$ of $\mathfrak{g}^\natural$. Here the level k' is computed with respect to the invariant bilinear form of $\mathfrak{a}$ normalized so that the minimal root has squared length 2. For $\mathfrak{a} = sl(m|n)$, $m \geq 2$, the minimal root is always chosen to be the lowest root of $sl(m)$. For $\mathfrak{a} = osp(m|n)$ we write $spo(n|m)$ vs. $osp(m|n)$ to specify the choice of the minimal root. In all other cases the minimal root of $\mathfrak{a}$ is unique.

To simplify notation define $V_{k'}(\mathfrak{g}^\natural)$ to be as follows:

$$V_{k'}(\mathfrak{g}^\natural) = \begin{cases} \mathbb{C} & \text{if } W_k(\mathfrak{g}, \theta) = \mathbb{C}; \text{ in this case we set } k' = 0; \\ M(1) & \text{if } W_k(\mathfrak{g}, \theta) = M(1); \text{ in this case we set } k' = 1; \\ V_{k'}(\mathfrak{a}) & \text{otherwise.} \end{cases}$$

In Table 5 we summarize all the relevant data.

Assume that $k \notin \mathbb{Z}_{\geq 0}$ and that:

(1) k is a collapsing level for $\mathfrak{g}$;

(2) $V_{k'}(\mathfrak{g}^\natural)$ is the unique irreducible $V_{k'}(\mathfrak{g}^\natural)$–module in the category $KL_{k'}$.

Assume that $L(\widehat{\Lambda})$ is an irreducible $V_k(\mathfrak{g})$-module in the category KL_k. Set $\mu = \widehat{\Lambda}_{|\mathfrak{h}}$. By Proposition 3.4 we have $\mu \in P_+$, hence, by (5), the weight μ has the form $\mu = \mu^\natural + \ell\theta$ with $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, where $\mu^\natural = \mu_{|\mathfrak{h}^\natural}$.

Table 5

$\mathfrak{g}$	$V_{k'}(\mathfrak{g}^\natural)$	k	k'
$sl(m\vert n)$, $m\neq n, m>3, m-2\neq n$	$V_{k'}(sl(m-2\vert n)))$	$\frac{n-m}{2}$	$\frac{n-m+2}{2}$
$sl(3\vert n)$, $n\neq 3, n\neq 1, n\neq 0$	$V_{k'}(sl(1\vert n)))$	$\frac{n-3}{2}$	$\frac{1-n}{2}$
$sl(3)$	$\mathbb{C}$	$-\frac{3}{2}$	0
$sl(2\vert n)$, $n\neq 2, n\neq 1, n\neq 0$	$V_{k'}(sl(n)))$	$\frac{n-2}{2}$	$-\frac{n}{2}$
$sl(2\vert 1)=spo(2\vert 2)$	$\mathbb{C}$	$-\frac{1}{2}$	0
$sl(m\vert n)$, $m\neq n, n+1, n+2, m\geq 2$	$M(1)$	-1	1
$psl(m\vert m)$, $m\geq 2$	$\mathbb{C}$	-1	0
$spo(n\vert m), m\neq n, n+2, n\geq 4$	$V_{k'}(spo(n-2\vert m))$	$\frac{m-n-4}{4}$	$\frac{m-n-2}{4}$
$spo(2\vert m), m\geq 5$	$V_{k'}(so(m))$	$\frac{m-6}{4}$	$\frac{4-m}{2}$
$spo(2\vert 3)$	$V_{k'}(sl(2))$	$-\frac{3}{4}$	1
$spo(2\vert 1)$	$\mathbb{C}$	$-\frac{5}{4}$	0
$spo(n\vert m), m\neq n+1, n\geq 2$	$\mathbb{C}$	$-1/2$	0
$osp(m\vert n), m\neq n, m\neq n+8, m\geq 7$	$V_{k'}(osp(m-4\vert n))$	$\frac{n-m+4}{2}$	$\frac{8-m+n}{2}$
$osp(m\vert n), n\neq m, 0; 4\leq m\leq 6$	$V_{k'}(osp(m-4\vert n))$	$\frac{n-m+4}{2}$	$\frac{m-n-8}{4}$
$osp(m\vert n), m\neq n+4, n+8; m\geq 4$	$V_{k'}(sl(2))$	-2	$\frac{m-n-8}{2}$
$osp(n+8\vert n), n\geq 0$	$\mathbb{C}$	-2	0
$D(2,1;a)$	$V_{k'}(sl(2))$	a	$-\frac{1+2a}{1+a}$
$D(2,1;a)$	$V_{k'}(sl(2))$	$-a-1$	$-\frac{1+2a}{a}$
$F(4)$	$V_{k'}(D(2,1;2))$	-1	$\frac{1}{2}$
$F(4)$	$\mathbb{C}$	$-3/2$	0
$F(4)$	$V_{k'}(so(7))$	$\frac{2}{3}$	-2
$F(4)$	$\mathbb{C}$	$-\frac{2}{3}$	0
E_6	$V_{k'}(sl(6))$	-4	-1
E_6	$\mathbb{C}$	-3	0
E_7	$V_{k'}(so(12))$	-6	-2
E_7	$\mathbb{C}$	-4	0
E_8	$V_{k'}(E_7)$	-10	-4
E_8	$\mathbb{C}$	-6	0
F_4	$V_{k'}(sp(6))$	-3	$-\frac{1}{2}$
F_4	$\mathbb{C}$	$-5/2$	0
G_2	$V_{k'}(sl(2))$	$-\frac{4}{3}$	1
G_2	$\mathbb{C}$	$-\frac{5}{3}$	0
$G(3)$	$V_{k'}(G_2)$	$\frac{1}{2}$	$-\frac{5}{3}$
$G(3)$	$\mathbb{C}$	$-\frac{3}{4}$	0
$G(3)$	$V_{k'}(osp(3\vert 2))$	$-\frac{2}{3}$	1
$G(3)$	$\mathbb{C}$	$-\frac{4}{3}$	0

Since $k \notin \mathbb{Z}_{\geq 0}$, by Theorem 2.1, $H_\theta(L(\widehat{\Lambda}))$ is a non-trivial irreducible module for $W_k(\mathfrak{g},\theta)$. Since $L(\widehat{\Lambda})$ is a quotient of the Verma module $M(\widehat{\Lambda})$, then, by exactness of H_θ, $H_\theta(L(\widehat{\Lambda}))$ is the quotient of a Verma module for $W_k(\mathfrak{g},\theta) = V_{k'}(\mathfrak{g}^\natural)$ hence it is an irreducible highest weight module. By [36, (6.14)] its highest weight as $\mathcal{V}_k(\mathfrak{g}^\natural)$-module is $\widehat{\Lambda}^\natural$ with $\widehat{\Lambda}^\natural(K) = k'$ and $\widehat{\Lambda}^\natural_{|\mathfrak{h}^\natural} = \mu^\natural$. Therefore

$$H_\theta(L(\widehat{\Lambda})) = L_{\mathfrak{g}^\natural}(\widehat{\Lambda}^\natural).$$

In particular $H_\theta(L(\widehat{\Lambda}))$ is in the category $KL_{k'}$.

Moreover, under the identification of the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ with $\mathfrak{g}_0 \oplus \mathfrak{g}_{1/2}$ via $ad(f)$ (see Example 6.2 of [36]), we get that x acts on $H_\theta(L(\widehat{\Lambda}))$ via $J_0^{\{f\}}$, and $J^{\{f\}}$ is the conformal vector of $W(k,\theta)$ (see the proof of Theorem 5.1 of [36]). Since the level is collapsing we know, by Proposition 4.1 of [4], that the conformal vector of $W_k(\mathfrak{g},\theta)$ coincides with the Segal-Sugawara vector conformal $\omega_{\mathfrak{g}^\natural}$ of $V_{k'}(\mathfrak{g}^\natural)$ hence, by (6.14) of [36] again, we obtain that the $(\omega_{\mathfrak{g}^\natural})_0$ acts on the lowest component of $H_\theta(L(\widehat{\Lambda}))$ by cI with

$$c = \frac{(\mu+2\rho,\mu)}{2(k+h^\vee)} - \mu(x). \tag{12}$$

Now condition (2) implies that $\mu^\natural = 0$, so $\mu = \ell\theta$ and

$$\frac{(\mu+2\rho,\mu)}{2(k+h^\vee)} - \mu(x) = \frac{(\ell\theta+2\rho,\ell\theta)}{2(k+h^\vee)} - \ell = 0.$$

By using formula (4), we get

$$\frac{2\ell^2+(2h^\vee-2)\ell}{2(k+h^\vee)} - \ell = \frac{\ell^2-(k+1)\ell}{k+h^\vee} = 0. \tag{13}$$

- Consider first the case $k = -h^\vee/2+1$ (this holds for $\mathfrak{g} = D_{2n}$, $n \geq 2$ and $\mathfrak{g} = osp(n+4m+8|n)$, $n \geq 0$). Then (13) gives that

$$\frac{2\ell^2+(h^\vee-4)\ell}{h^\vee+2} = 0. \tag{14}$$

  We get $\ell = 0$ or $2\ell + h^\vee - 4 = 0$.

- Next we consider the case $k = -h^\vee/6 - 1$. We get

$$\frac{6\ell^2+h^\vee\ell}{5h^\vee-6} = 0. \tag{15}$$

  We conclude that $\ell = 0$ or $\ell = -\frac{h^\vee}{6}$.

By using the above analysis and properties of Hamiltonian reduction, we get the following lemma, which extends a result of [15] for Lie algebras to the super case.

Lemma 4.1. Assume that $k = -\frac{h^\vee}{6} - 1$ and $\mathfrak{g}$ is one of the Lie algebras of exceptional Deligne's series A_2, G_2, D_4, F_4, E_6, E_7, E_8, or $\mathfrak{g} = psl(m|m)$ $(m \geq 2)$, $osp(n+8|n)$ $(n \geq 2)$, $spo(2|1)$, $F(4)$, $G(3)$ (for both choices of θ).

Assume that $L(\lambda)$ is a $V_k(\mathfrak{g})$–module in the category $\mathcal{O}$. Then one of the following condition holds:

(1) $\lambda(\alpha_0^\vee) \in \mathbb{Z}_{\geq 0}$;

(2) $\bar{\lambda}$ is either 0 or $\frac{-h^\vee}{6}\theta$, where $\bar{\lambda}$ is the restriction of λ to $\mathfrak{h}$.

□

Proof. By Theorem 2.1, if $L(\lambda)$ is a $V_k(\mathfrak{g})$–module for which $\lambda(\alpha_0^\vee) \notin \mathbb{Z}_{\geq 0}$, then $H_\theta(L(\lambda))$ is an irreducible $W_k(\mathfrak{g}, \theta) = H_\theta(V_k(\mathfrak{g}))$–module. The conditions on $\mathfrak{g}$ exactly correspond to the cases when $W_k(\mathfrak{g}, \theta)$ is one-dimensional (cf. [4], [15]), so the discussion that precedes the Lemma and relation (15) imply that $\bar{\lambda}$ is as in (2). ■

Lemma 4.1 implies:

Theorem 4.2. Assume that the level k and the Lie superalgebra $\mathfrak{g}$ satisfy one of the following conditions:

(1) $k = -\frac{h^\vee}{6} - 1$ and $\mathfrak{g}$ is one of the Lie algebras of exceptional Deligne's series A_2, G_2, D_4, F_4, E_6, E_7, E_8, or $\mathfrak{g} = psl(m|m)$ $(m \geq 2)$, $osp(n+8|n)$ $(n \geq 2)$, $spo(2|1)$, $F(4)$, $G(3)$ (for both choices of θ);

(2) $k = -h^\vee/2 + 1$ and $\mathfrak{g} = osp(n+4m+8|n)$, $n \geq 2, m \geq 0$.

(3) $k = -h^\vee/2 + 1$ and $\mathfrak{g} = D_{2m}$, $m \geq 2$.

(4) $k = -10$ and $\mathfrak{g} = E_8$.

Then $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$–module in the category KL_k. □

Proof. If the Lie superalgebra $\mathfrak{g}$ is as in (1), then Lemma 4.1 and Remark 3.5 imply that $\bar{\lambda}$ is either 0 or $\frac{-h^\vee}{6}\theta$. Since in all cases in (1) we have that $h^\vee \in \mathbb{Z}_{\geq 0}$, one obtains that the irreducible highest weight $\mathfrak{g}$–module with highest weight $\bar{\lambda} = \frac{-h^\vee}{6}\theta$ cannot be finite-dimensional. Therefore $L(\lambda)$ can not be a module in KL_k. This proves that $\bar{\lambda} = 0$ and therefore $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$–module in the category KL_k.

Let us consider the case $\mathfrak{g} = osp(n+4m+8|n)$. Then for every $m \in \mathbb{Z}_{\geq 0}$ we have:

$$h^\vee = 4m + 6, \tag{16}$$

$$k = -h^\vee/2 + 1 = -2(m+1), \tag{17}$$

$$2\ell + h^\vee - 4 \neq 0 \quad \forall \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}. \tag{18}$$

We prove the claim by induction. In the case $m = 0$, the claim was proved in (1). Assume now that the claim holds for $\mathfrak{g}' = osp(n+4(m-1)+8, n)$, and $k' = -2m$.

By Theorem 2.3, $k = -2(m+1)$ is a collapsing level and $W_k(\mathfrak{g},\theta) = V_{k'}(\mathfrak{g}')$.

By inductive assumption $V_{k'}(\mathfrak{g}')$ is the unique irreducible $V_{k'}(\mathfrak{g}')$ in the category $KL_{k'}$. By applying (14) and (18) we get that $\ell = 0$ and therefore $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$–module in the category KL_k. The assertion now follows by induction on m.

(3) is a special case of (2), by taking $n = 0$.

(4) follows from the fact that $H_\theta(V_{-10}(E_8)) = V_{-4}(E_7)$ and case (1) by applying formula (13). ■

Remark 4.3. Theorem 4.2 can be also proved by non-cohomological methods, using explicit formulas for singular vectors and Zhu algebra theory. As an illustration, we shall present in Theorem 8.6 a direct proof in the case of D_{2n} at level $k = -h^\vee/2 + 1$. □

In the following sections we shall study some other applications of collapsing levels. We shall restrict our analysis to the case of Lie algebras. In what follows we let $\omega_1, \dots, \omega_n$ be the fundamental weights for $\mathfrak{g}$ and $\Lambda_0, \dots, \Lambda_n$ the fundamental weights for $\widehat{\mathfrak{g}}$.

5 On complete reducibility in the category KL_k

In this Section we prove complete reducibility results in the category KL_k when $\mathfrak{g}$ is a Lie algebra. We start with a preliminary result, which also holds in the super setting.

Lemma 5.1. Assume that the Lie superalgebra $\mathfrak{g}$ and level k satisfy the conditions of Theorem 4.2. Assume that M is a highest weight $V_k(\mathfrak{g})$–module from the category KL_k. Then M is irreducible. □

Proof. By using the classification of irreducible modules from Theorem 4.2 we know that the highest weight of M is necessary $k\Lambda_0$, and therefore M is a $\mathbb{Z}_{\geq 0}$–graded with respect to $L_{\mathfrak{g}}(0)$. Denote a highest weight vector by $w_{k\Lambda_0}$. We have that

$$L_{\mathfrak{g}}(0)v = 0 \quad \Longleftrightarrow \quad v = \nu w_{k\Lambda_0} \quad (\nu \in \mathbb{C}).$$

Assume that M is not irreducible. Then it contains a non-zero graded submodule $N \neq M$ with respect to $L_{\mathfrak{g}}(0)$:

$$N = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} N(n + n_0), \quad L_{\mathfrak{g}}(0)_{|N(r)} = r\mathrm{Id}, \quad N(n_0) \neq 0.$$

Since $N \neq M$, we have that $n_0 > 0$, otherwise $w_{k\Lambda_0} \in M$.

We can consider $N(n_0)$ as a finite-dimensional module for $\mathfrak{g}$ and for the Zhu algebra $A(V_k(\mathfrak{g}))$. Note that Theorem 4.2 and Proposition 3.2 imply that any irreducible finite-dimensional $A(V_k(\mathfrak{g}))$–module is trivial. Since the Casimir element $C_{\mathfrak{g}}$ of $\mathfrak{g}$ acts on $N(n_0)$ as the non-zero constant $2(k + h^\vee)n_0$, we conclude that $C_{\mathfrak{g}}$ acts by the same constant on any irreducible $\mathfrak{g}$–subquotient of $N(n_0)$. But any irreducible subquotient of $N(n_0)$ is an irreducible finite–dimensional $A(V_k(\mathfrak{g}))$–module, and therefore it is trivial. This implies that $C_{\mathfrak{g}}$ acts non-trivially on a trivial $\mathfrak{g}$–module. A contradiction. ■

The following Lemma is a consequence of [28, Theorem 0.1].

Lemma 5.2. [28] Assume that $\mathfrak{g}$ is a simple Lie algebra and k is a rational number, $k > -h^\vee$. Then, in the category of $V_k(\mathfrak{g})$–modules, we have: $\mathrm{Ext}^1(V_k(\mathfrak{g}), V_k(\mathfrak{g})) = (0)$. □

Theorem 5.3. Assume that $\mathfrak{g}$ is a simple Lie algebra and that the level k satisfies the conditions of Theorem 4.2. Then any $V_k(\mathfrak{g})$–module M from the category KL_k is completely reducible. □

Proof. Since M is in KL_k we have that any irreducible subquotient of M is isomorphic to $V_k(\mathfrak{g})$. M has finite length. This implies that M is $\mathbb{Z}_{\geq 0}$–graded:

$$M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n), \quad L_{\mathfrak{g}}(0)_{|M(r)} = r\mathrm{Id}.$$

Assume that $M(0) = \mathrm{span}_{\mathbb{C}}\{w_1, \dots, w_s\}$. Then by Lemma 5.1 we have that $V_k(\mathfrak{g})w_i \cong V_k(\mathfrak{g})$ for every $i = 1, \dots, s$. Now using Lemma 5.2 we get $M \cong \oplus V_k(\mathfrak{g})w_i$ and therefore M is completely reducible. ■

Remark 5.4. We expect that the previous theorem holds in the case when $\mathfrak{g}$ is the Lie superalgebra from Theorem 4.2. We shall study this case in [7]. □

We shall now prove much more general result on complete reducibility in KL_k.

Theorem 5.5. Assume that level $k \in \mathbb{Q}$, $k > -h^\vee$, and the simple Lie algebra $\mathfrak{g}$ satisfy the following property:

$$\text{Every highest weight } V_k(\mathfrak{g})\text{–module in } KL_k \text{ is irreducible.} \tag{19}$$

Then the category KL_k is semi-simple. □

Proof. We shall present a sketch of the proof and omit some standard representation theoretic arguments which can be found in [20] and [28].

- Since every irreducible $V_k(\mathfrak{g})$-module in KL_k is isomorphic to $L(\lambda)$ for certain rational, non-critical weight λ, then [28, Theorem 0.1] implies that $\mathrm{Ext}^1(L(\lambda), L(\lambda)) = (0)$ in the category KL_k.
- We prove that in the category KL_k we have

$$Ext^1(L_1, L_2) = (0) \tag{20}$$

for any two irreducible modules L_1 and L_2 from KL_k.

It remains to consider the case $L_1 \neq L_2$. Take an exact sequence in KL_k:

$$0 \to L(\lambda_1) \to M \to L(\lambda_2) \to 0,$$

where $\lambda_1 \neq \lambda_2$. Then M contains a singular vector w_{λ_1} of highest weight λ_1 and a subsingular vector w_{λ_2} of weight λ_2 and w_{λ_1} generates a submodule isomorphic to $L(\lambda_1)$. Consider the case $\lambda_1 - \lambda_2 \notin Q_+$. Then λ_2 is a maximal element of the set $\Omega(M)$ of weights of M, and therefore the subsingular vector w_{λ_2} in M of weight λ_2 is a singular vector. By (19), it generates an irreducible module isomorphic to $L(\lambda_2)$ and we conclude that $M \cong L(\lambda_1) \oplus L(\lambda_2)$.

If $\lambda_1 - \lambda_2 \in Q_+$ we can use the contravariant functor $M \mapsto M^\sigma$ and get an exact sequence

$$0 \to L(\lambda_2) \to M^\sigma \to L(\lambda_1) \to 0.$$

Since M^σ is again a $V_k(\mathfrak{g})$–module in KL_k (cf. Lemma 3.6) by the first case we have that $M^\sigma = L(\lambda_1) \oplus L(\lambda_2)$. This implies that

$$M = L(\lambda_1)^\sigma \oplus L(\lambda_2)^\sigma = L(\lambda_1) \oplus L(\lambda_2).$$

- Assume now that M is a finitely generated module from KL_k. Then from [20, Proposition 3.1] we see that M has an increasing filtration

$$(0) = M_0 \subseteq M_1 \subseteq \cdots \tag{21}$$

such that

1. for every $j \in \mathbb{Z}_{>0}$, M_j/M_{j-1} is an highest weight module in category $\mathcal{O}$;
2. for any weight λ of M, there exists r such that $(M/M_r)_\lambda = 0$.

Since M is finitely generated as $\widehat{\mathfrak{g}}$–module, we can assume that its generators are weight vectors of weights say $\mu_1, ...\mu_p$. Since they are a finite number there certainly exists t such that $(M/M_t)_{\mu_i} = 0$ for all $i = 1, .., p$. Hence the filtration (21) is finite and stops at $M = M_t$. Since M is in category KL_k, we have that the factors of (21) are in category KL_k. Hence, by our assumption, they are irreducible. Therefore (21) is a composition series of finite length Using assumption (19), relation (20) and induction on t we get that

$$M \cong \bigoplus_{j=1}^{t} L(\lambda_j).$$

- Finally we shall consider the case when M is not finitely generated. Since M is in KL_k, it is countably generated. So $M = \cup_{n=1}^{\infty} M^{(n)}$ such that each $M^{(n)}$ is finitely generated $V_k(\mathfrak{g})$–module. By previous case

$M^{(n)}$ is completely reducible, so:

$$M^{(n)} = \bigoplus_{i=1}^{n_i} L(\lambda_{i,n}). \tag{22}$$

Therefore M is a sum of irreducible modules from KL_k and by using classical algebraic arguments one can see that M is a direct sum of countably many irreducible modules from KL_k appearing in decompositions (22).

The claim follows. ■

In order to apply Theorem 5.5, the basic step is to check relation (19). We have the following method.

Lemma 5.6. Let $k \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$. Assume that $H_\theta(U)$ is an irreducible, non-zero $W_k(\mathfrak{g}, \theta) = H_\theta(V_k(\mathfrak{g}))$–module for every non-zero highest weight $V_k(\mathfrak{g})$–module U from the category KL_k. Then every highest weight $V_k(\mathfrak{g})$–module in KL_k is irreducible. □

Proof. Assume that M is a highest weight $V_k(\mathfrak{g})$–module in KL_k. Then $H_\theta(M)$ is an irreducible $H_\theta(V_k(\mathfrak{g}))$–module. If M is not irreducible, then it contains a highest weight submodule U such that $\{0\} \subsetneqq U \subsetneqq M$. Modules U and M/U are again highest weight modules in KL_k. By the assumption of the Lemma we have that $H_\theta(U)$ is a non-trivial submodule of $H_\theta(M)$. Irreducibility of $H_\theta(M)$ implies that $H_\theta(U) = H_\theta(M)$, and therefore $H_\theta(M/U) = \{0\}$. A contradiction. ■

Theorem 5.7. Assume that $\mathfrak{g}$ is a simple Lie algebra and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ such that $W_k(\mathfrak{g}, \theta)$ is rational. Then KL_k is a semi-simple category. □

Proof. Assume that $\widetilde{L}(\lambda)$ is a highest weight $V_k(\mathfrak{g})$–module in KL_k. Clearly $\lambda(\alpha_0^\vee) \notin \mathbb{Z}_{\geq 0}$ and by Theorem 2.1 $H_\theta(\widetilde{L}(\lambda)) \neq (0)$. Since $H_\theta(\widetilde{L}(\lambda))$ is non-zero highest weight module for the rational vertex algebra $W_k(\mathfrak{g}, \theta)$, we conclude that $H_\theta(\widetilde{L}(\lambda))$ is irreducible. Now assertion follows from Theorem 5.5 and Lemma 5.6. ■

Remark 5.8. The previous theorem proves that the category KL_k is semisimple in the following (non-admissible) cases:

- $\mathfrak{g} = D_4, E_6, E_7, E_8$ and $k = -\frac{h^\vee}{6}$ using results from [39].

□

Moreover, using Theorem 5.5 and Lemma 5.6 we can prove the semi-simplicity of KL_k for all collapsing levels not accounted by Theorem 1.1. We list here only non-admissible levels, since in admissible case KL_k is semi-simple by [12].

Theorem 5.9. The category KL_k is semisimple in the following cases:

(1) $\mathfrak{g} = D_\ell$, $\ell \geq 3$ and $k = -2$;

(2) $\mathfrak{g} = B_\ell$, $\ell \geq 2$ and $k = -2$;

(3) $\mathfrak{g} = A_\ell$, $\ell \geq 2$ and $k = -1$;

(4) $\mathfrak{g} = A_{2\ell-1}$, $\ell \geq 2$, $k = -\ell$;

(5) $\mathfrak{g} = D_{2\ell-1}$, $\ell \geq 3$ and $k = -2\ell + 3$;

(6) $\mathfrak{g} = C_\ell$, $k = -1 - \ell/2$;

(7) $\mathfrak{g} = E_6$, $k = -4$;

(8) $\mathfrak{g} = E_7$, $k = -6$;

(9) $\mathfrak{g} = F_4$, $k = -3$.

□

Proof. We will give a proof of relations (1) and (2) in Corollaries 6.8 and 7.7, respectively. Case (1) for $\ell \neq 3$ will follow from Theorem 5.7. Note also that case (1) for $\ell = 3$ is a special case of case (4), and that case (2) for $\ell = 2$ is a special case of (6). The proof in cases (3) – (6) is similar, and it uses the classification of irreducible modules from [10], [11], [16] and the results on collapsing levels [4]. Cases (7) – (9) are reduced to cases we have already treated. Here are some details.

Case (3):

- [16], [4] $H_\theta(V_{-1}(A_\ell))$ is isomorphic to the Heisenberg vertex algebra $M(1)$ of central charge $c = 1$
- By using the fact that every highest weight $M(1)$–module is irreducible, we see that if U is a highest weight $V_{-1}(A_\ell)$–module in KL_{-1}, then $H_\theta(U)$ is a non-trivial irreducible $M(1)$–module.

Case (4):

- [16], [4] $H_\theta(V_{-\ell}(A_{2\ell-1})) = V_{-\ell+1}(A_{2\ell-3})$.
- For $\ell = 2$, we have that every highest weight $V_{-\ell+1}(A_{2\ell-3}) = V_{-1}(sl(2))$–module $\widetilde{L}(\lambda)$ in KL_{-1} with highest weight $\lambda = -(1 + j)\Lambda_0 + j\Lambda_1$, $j \in \mathbb{Z}_{\geq 0}$, is irreducible.
- By induction, we see that for every highest weight $V_{-\ell}(A_{2\ell-1})$–module U in $KL_{-\ell}$, $H_\theta(U)$ is a non-trivial irreducible $V_{-\ell+1}(A_{2\ell-3})$–module.

Case (5)

- $H_\theta(V_{-2\ell+3}(D_{2\ell-1})) \cong V_{-2\ell+5}(D_{2\ell-3})$.
- By induction we see that for or every highest weight $V_{-2\ell+3}(D_{2\ell-1})$–module U in $KL_{-2\ell+3}$, $H_\theta(U)$ is a non-trivial irreducible $V_{-2\ell+5}(D_{2\ell-3})$–module.

Case (6)

- $H_\theta(V_{-1-\ell/2}(C_\ell)) \cong V_{-1/2-\ell/2}(C_{\ell-1})$.
- For $\ell = 2$, we have that every highest weight $V_{-1/2-\ell/2}(C_{\ell-1}) = V_{-3/2}(sl(2))$–module in $KL_{-3/2}$ is irreducible.
- By induction, we see that for every highest weight $V_{-1-\ell/2}(C_\ell)$–module U in $KL_{-1-\ell/2}$, $H_\theta(U)$ is a non-trivial irreducible $V_{-1/2-\ell/2}(C_{\ell-1})$–module.

The proof follows by applying Theorem 5.5 and Lemma 5.6.

Cases (7) – (8)

We have

$$H_\theta(V_{-4}(E_6)) = V_{-1}(A_3), \quad H_\theta(V_{-6}(E_7)) = V_{-2}(D_6),$$

and these cases are settled in (3) and Theorem 1.1 (3) respectively. Case (9) follows from the fact that $H_\theta(V_{-3}(F_4))$ is isomorphic to the admissible affine vertex algebra $V_{-\frac{1}{2}}(C_3)$ which is semisimple in $KL_{-1/2}$ (cf. [1]). ■

Remark 5.10. The problem of complete-reducibility of modules in KL_k when $\mathfrak{g}$ is a Lie superalgebra will be also studied in [7]. An important tool in the description of the category KL_k will be the conformal embedding of $\widetilde{V}_k(\mathfrak{g}_0)$ to $V_k(\mathfrak{g})$ where $\mathfrak{g}_0$ is the even part of $\mathfrak{g}$. □

Note that in the category $\mathcal{O}$ we can have indecomposable $V_k(\mathfrak{g})$–modules in some cases listed in Theorem 5.9. See [10, Remark 5.8] for one example.

6 The vertex algebra $V^{-2}(D_\ell)$ and its quotients

In this section we exploit Hamiltonian reduction and the results on conformal embeddings from [4] to investigate the quotients of the vertex algebra $V^{-2}(D_\ell)$. In particular we are interested in a non-simple quotient $\mathcal{V}_{-2}(D_\ell)$ which appears in the analysis of certain dual pairs (see [6]) as well as in the simple quotient $V_{-2}(D_\ell)$. We will show that the vertex algebra $\mathcal{V}_{-2}(D_\ell)$ has infinitely many irreducible modules in the category KL^{-2}, while by [15], $V_{-2}(D_\ell)$ has finitely many irreducible modules in KL_{-2}. Recall that -2 is a collapsing level for D_ℓ [4].

Consider the vector

$$w_1 := (e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_4}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_4}(-1) + e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1))\mathbf{1}. \tag{23}$$

It is a singular vector in $V^{-2}(D_\ell)$ (cf. [15]). Note that this vector is contained in the subalgebra $V^{-2}(D_4)$ of $V^{-2}(D_\ell)$.

By using the explicit expression for singular vectors v_n in $V^{n-\ell+1}(D_\ell)$ (see (28)), we have that

$$w_2 := v_{\ell-3} = \Big(\sum_{i=2}^{\ell} e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1)\Big)^{\ell-3}\mathbf{1} \tag{24}$$

is a singular vector in $V^{-2}(D_\ell)$.

For $\ell = 4$ we also have a third singular vector (cf. [40])

$$w_3 := (e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3-\epsilon_4}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2-\epsilon_4}(-1) + e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1))\mathbf{1}.$$

6.1 The vertex algebra $\mathcal{V}_{-2}(D_\ell)$ for $\ell \geq 4$

Define the vertex algebra

$$\mathcal{V}_{-2}(D_\ell) = V^{-2}(D_\ell)\big/J_\ell, \tag{25}$$

where

$$J_\ell = \langle w_1, w_3 \rangle \quad (\ell = 4), \quad J_\ell = \langle w_1 \rangle \quad (\ell \geq 5).$$

The following proposition is essentially proven in [6].

Proposition 6.1.

(1) There is a non-trivial vertex algebra homomorphism $\overline{\Phi} : \mathcal{V}_{-2}(D_\ell) \to M_{2\ell}$ where $M_{2\ell}$ the Weyl vertex algebra of rank ℓ.

(2) $\mathcal{V}_{-2}(D_\ell)$ is not simple, and $L((-2-t)\Lambda_0 + t\Lambda_1)$, $t \in \mathbb{Z}_{\geq 0}$ are $\mathcal{V}_{-2}(D_\ell)$–modules. □

Proof. The homomorphism $\Phi : V^{-2}(D_\ell) \to M_{2\ell}$ was constructed in [6, Section 7]. By direct calculation one proves that $\Phi(w_1) = 0$ for $\ell \geq 4$ and $\Phi(w_3) = 0$ for $\ell = 4$. Finally [6, Lemma 7.1] implies that $L((-2-t)\Lambda_0 + t\Lambda_1)$, $t \in \mathbb{Z}_{\geq 0}$ are $\mathcal{V}_{-2}(D_\ell)$–modules. Since the simple vertex algebra $V_{-2}(D_\ell)$ has only finitely many irreducible modules in the category $\mathcal{O}$ [15], we have that $\mathcal{V}_{-2}(D_\ell)$ is not simple. ■

Next, we exploit the fact that in the case $\mathfrak{g} = D_\ell$, $k = -2$ is a collapsing level, i.e., in the affine W-algebra $W^k(\mathfrak{g}, \theta)$, all generators $G^{\{u\}}$ at conformal weight $3/2$, $u \in \mathfrak{g}_{-1/2}$, belong to the maximal ideal (see [4] for details). This implies that there exists a non-trivial ideal I in $V^{-2}(\mathfrak{g})$ such that $G^{\{u\}} \in H_\theta(I)$ for all $u \in \mathfrak{g}_{-1/2}$.

Note also that $\mathfrak{g}^\natural = A_1 \oplus D_{\ell-2}$, so we have that $V^{\ell-4}(A_1) \otimes V^0(D_{\ell-2})$ is a subalgebra of $W^{-2}(D_\ell, \theta)$. In the case $\ell = 4$ we identify D_2 with $A_1 \oplus A_1$.

Lemma 6.2. We have

- $x_{(-1)}\mathbf{1} \in H_\theta(J_\ell)$ for all $x \in D_{\ell-2} \subset \mathfrak{g}^\natural$,
- $G^{\{u\}} \in H_\theta(J_\ell)$ for all $u \in \mathfrak{g}_{-1/2}$.

□

Proof. Assume that $\ell \geq 5$. Since w_1 is a singular vector in $V^{-2}(D_\ell)$, the ideal J_ℓ is a highest weight module of highest weight $\lambda = -2\Lambda_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$. Now, the Main Theorem from [12] implies that $H_\theta(J_\ell)$ is a non-trivial highest weight module. By formula [36, (6.14)] the highest weight is $(0, \omega_2)$ and, by (12), the conformal weight of its highest weight vector is 1. Up to a non-zero constant, there is only one vector in $W^{-2}(D_\ell, \theta) = V^{\ell-4}(A_1) \otimes V^0(D_{\ell-2})$ that has these properties, namely $J_{(-1)}^{\{e_{\epsilon_3+\epsilon_4}\}}\mathbf{1}$, and therefore $H_\theta(J_\ell)$ contains all generators of $V^0(D_{\ell-2})$.

In the case $\ell = 4$, w_1 and w_3 generate submodules N_1 and N_3 of highest weights $\lambda_1 = -2\Lambda_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$, $\lambda_3 = -2\Lambda_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4$, respectively. Applying the same arguments as above we get that $J_{(-1)}^{\{e_{\epsilon_3\pm\epsilon_4}\}}\mathbf{1} \in H_\theta(I)$, which implies that $H_\theta(J_\ell)$ contains all generators of $V^0(D_2) = V^0(A_1) \otimes V^0(A_1)$.

Now, claim follows by applying the action of generators of $V^0(D_{\ell-2})$ to $G^{\{u\}}$ (see [4]). ■

Proposition 6.3. We have

(1) $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V^{\ell-4}(A_1)$.

(2) $H_\theta(L((-2-t)\Lambda_0 + t\Lambda_1)) \cong L_{A_1}((\ell-4-t)\Lambda_0 + t\Lambda_1),\ t \in \mathbb{Z}_{\geq 0}$.

(3) The set $\{L((-2-t)\Lambda_0 + t\Lambda_1) \mid t \in \mathbb{Z}_{\geq 0}\}$ provides a complete list of irreducible $\mathcal{V}_{-2}(D_\ell)$–modules from the category KL^{-2}. □

Proof. By Lemma 6.2 we see that the vertex algebra $H_\theta(\mathcal{V}_{-2}(D_\ell))$ is generated only by $x_{(-1)}\mathbf{1}$, $x \in A_1 \subset D_\ell^\natural$. So there are only two possibilities: either $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V^{\ell-4}(A_1)$ or $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V_{\ell-4}(A_1)$. Moreover, for every $t \in \mathbb{Z}_{\geq 0}$, $H_\theta(L((-2-t)\Lambda_0 + t\Lambda_1))$ must be the irreducible $H_\theta(\mathcal{V}_{-2}(D_\ell))$–module with highest weight $t\omega_1$ with respect to A_1. So $H_\theta(L((-2-t)\Lambda_0 + t\Lambda_1)) \cong L_{A_1}((\ell-4-t)\Lambda_0 + t\Lambda_1),\ t \in \mathbb{Z}_{\geq 0}$. Therefore, $H_\theta(\mathcal{V}_{-2}(D_\ell))$ contains infinitely many irreducible modules, which gives that $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V^{\ell-4}(A_1)$. In this way we have proved claims (1) and (2).

Let us now prove claim (3).

Assume that $L(k\Lambda_0 + \mu)$ $(\mu \in P_+,\ k = -2)$ is an irreducible $\mathcal{V}_k(D_\ell)$–module in the category KL^k. Then $H_\theta(L(k\Lambda_0 + \mu))$ is a non-trivial irreducible $V^{\ell-4}(A_1)$–module. The representation theory of $V^{\ell-4}(A_1)$ implies that:

$$H_\theta(L(k\Lambda_0 + \mu)) = L_{A_1}((\ell-4-j)\Lambda_0 + j\Lambda_1) \qquad \text{for } j \in \mathbb{Z}_{\geq 0}.$$

Since $D_\ell^\natural = A_1 \times D_{\ell-2}$, we conclude that $\mu^\natural = j\omega_1$ and therefore, by (5),

$$\mu = j\omega_1 + s\omega_2 = (s+j)\epsilon_1 + s\epsilon_2 \qquad (s \in \mathbb{Z}_{\geq 0}).$$

By using the action of $L(0) = \omega_0$ on the lowest component of $H_\theta(L(k\Lambda_0 + \mu))$ we get

$$\frac{(\mu + 2\rho, \mu)}{2(k + h^\vee)} - \mu(x) = \frac{j(j+2)}{4(\ell-2)} \qquad (x = \theta^\vee/2).$$

Since $2(k+h^\vee) = 2(-2+2\ell-2) = 4(\ell-2)$ and $\mu(x) = (2s+j)/2$ we get

$$(\mu+2\rho,\mu) - (h^\vee-2)(2s+j) = j(j+2).$$

By direct calculation we get

$$(\mu+2\rho,\mu) = (s+j)^2 + s^2 + h^\vee(s+j) + (h^\vee-2)s,$$

which gives an equation:

$$\begin{aligned}
& (s+j)^2 + s^2 + h^\vee(s+j) + (h^\vee-2)s - (h^\vee-2)(2s+j) = j(j+2). \\
\iff \quad & (s+j)^2 + s^2 + h^\vee(s+j) - (h^\vee-2)(s+j) = j(j+2). \\
\iff \quad & (s+j)(s+j+2) = j(j+2) \\
\iff \quad & s=0 \quad \text{or} \quad s=-2j-2.
\end{aligned}$$

Since $\mu \in P_+$ we conclude that $s=0$. Therefore $\mu = j\omega_1$ for certain $j \in \mathbb{Z}_{\geq 0}$. The proof of claim (3) is now complete. ■

6.2 The simple vertex algebra $V_{-2}(D_\ell)$

Next we use the fact that the simple affine W-algebra $W_{-2}(D_\ell,\theta)$ is isomorphic to the simple affine vertex algebra $V_{\ell-4}(A_1)$, for $\ell \geq 4$.

Proposition 6.4. The set $\{L((-2-j)\Lambda_0 + j\Lambda_1) \mid j \in \mathbb{Z}_{\geq 0}, j \leq \ell-4\}$ provides a complete list of irreducible $V_{-2}(D_\ell)$–modules from the category KL_{-2}. □

Proof. Assume that N is an irreducible $V_{-2}(D_\ell)$–module from the category KL_{-2}. Then N is also irreducible as $\mathcal{V}_{-2}(D_\ell)$–module, and therefore $N \cong L((-2-j)\Lambda_0 + j\Lambda_1)$ for certain $j \in \mathbb{Z}_{\geq 0}$. Since $H_\theta(N)$ must be an irreducible $H_\theta(V_{-2}(D_\ell)) = W_{-2}(D_\ell,\theta) = V_{\ell-4}(A_1)$–module, we get $j \leq \ell-4$, as desired. ■

Now we want to describe the maximal ideal in $V^{-2}(D_\ell)$. The next lemma states that any non-trivial ideal in $\mathcal{V}_{-2}(D_\ell)$ is automatically maximal.

Lemma 6.5. Let $\{0\} \neq I \subsetneq \mathcal{V}_{-2}(D_\ell)$ be any non-trivial ideal in $\mathcal{V}_{-2}(D_\ell)$. Then we have

(1) $H_\theta(I)$ is the maximal ideal in $V^{\ell-4}(A_1)$.

(2) I is a maximal ideal in $\mathcal{V}_{-2}(D_\ell)$ and $I = L(-2(\ell-2)\Lambda_0 + 2(\ell-3)\Lambda_1)$. □

Proof. Assume that I is a non-trivial ideal in $\mathcal{V}_{-2}(D_\ell)$. Then I can be regarded as a $\mathcal{V}_{-2}(D_\ell)$–module in the category KL^{-2} and therefore, by Proposition 6.3, (3), it contains a non-trivial subquotient isomorphic to $L((-2-j)\Lambda_0 + j\Lambda_1)$ for some $j \in \mathbb{Z}_{\geq 0}$. Since, by part (2) of the aforementioned Proposition, $H_\theta(L((-2-j)\Lambda_0 + j\Lambda_1)) \neq 0$ for every $j \in \mathbb{Z}_{\geq 0}$, we conclude that $H_\theta(I)$ is a non-trivial ideal in $H_\theta(\mathcal{V}_{-2}(D_\ell)) = V^{\ell-4}(A_1)$. But since $V^{\ell-4}(A_1)$, $\ell \geq 4$, contains a unique non-trivial ideal, which is automatically maximal, we have that $H_\theta(I)$ is a maximal ideal in $V^{\ell-4}(A_1)$. So

$$H_\theta(\mathcal{V}_{-2}(D_\ell)/I) \cong V_{\ell-4}(A_1).$$

Assume now that $\mathcal{V}_{-2}(D_\ell)/I$ is not simple. Then it contains a non-trivial singular vector v' of weight $-(2+j)\Lambda_0 + j\Lambda_1$ for $j \in \mathbb{Z}_{>0}$. By [12], we have that $H_\theta(V^{-2}(D_\ell).v')$ is a non-trivial ideal in $V_{\ell-4}(A_1)$ generated by a singular vector of A_1–weight $j\omega_1$. This is a contradiction. So I is the maximal ideal.

Since the maximal ideal in $V^{\ell-4}(A_1)$ is generated by a singular vector of A_1–weight $2(\ell-3)\omega_1$ and since the maximal ideal is simple, we conclude that $I = \mathcal{V}_{-2}(D_\ell).v_{sing}$ for a certain singular vector v_{sing} of weight $\lambda = -2(\ell-2)\Lambda_0 + 2(\ell-3)\Lambda_1$. It is also clear that this singular vector is unique, up to scalar factor. Therefore, $I = L(-2(\ell-2)\Lambda_0 + 2(\ell-3)\Lambda_1)$. ■

Note that in the previous lemma we proved the existence of a singular vector which generates the maximal ideal without presenting a formula for such a singular vector. Since the vector in (24) has the correct weight, we also have an explicit expression for this singular vector:

$$\Big(\sum_{i=2}^{\ell} e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1)\Big)^{\ell-3}\mathbf{1}$$

Corollary 6.6.

(1) The maximal ideal in $V^{-2}(D_\ell)$ is generated by the vectors w_1 and w_2 for $\ell \geq 5$ and by the vectors w_1, w_2, w_3 for $\ell = 4$.

(2) The homomorphism $\overline{\Phi} : \mathcal{V}_{-2}(D_\ell) \to M_{2\ell}$ is injective. In particular, the vertex algebra $\mathcal{V}_{-2}(D_\ell) \otimes V_{-\ell}(A_1)$ is conformally embedded into $V_{-1/2}(C_{2\ell})$.

(3) $\mathrm{ch}(\mathcal{V}_{-2}(D_\ell)) = \mathrm{ch}(V_{-2}(D_\ell)) + \mathrm{ch} L(-2(\ell-2))\Lambda_0 + 2(\ell-3)\Lambda_1)$. □

Remark 6.7. D. Gaiotto in [27] has started a study of the decomposition of $M_{2\ell}$ as a $V^{-2}(D_\ell) \otimes V_{-\ell}(A_1)$–module in the case $\ell = 4$. By combining results from [6, Section 8] and results from this Section we get that

$$\mathrm{Com}(V_{-\ell}(A_1), M_{2\ell}) \cong \mathcal{V}_{-2}(D_\ell).$$

So the vertex algebra responsible for the decomposition of $M_{2\ell}$ is exactly $\mathcal{V}_{-2}(D_\ell)$. Therefore in the decomposition of $M_{2\ell}$ only modules for $\mathcal{V}_{-2}(D_\ell)$ can appear. In our forthcoming papers we plan to apply the representation theory of $\mathcal{V}_{-2}(D_\ell)$ to the problem of finding branching rules. □

Corollary 6.8. For $\ell \geq 3$ the category KL_{-2} is semi-simple. □

Proof. The assertion in the case $\ell \geq 4$ follows from Theorem 5.7 since then $W_{-2}(D_\ell, \theta) = V_{\ell-4}(sl(2))$ is a rational vertex algebra.

In the case $\ell = 3$, we have that a highest weight $V_{-2}(D_3)$–module M is isomorphic to $\widetilde{L}((-2-j)\Lambda_0 + j\Lambda_1)$ where $j \in \mathbb{Z}_{\geq 0}$. The irreducibility of M follows easily from the fact that $H_\theta(M)$ is isomorphic to an irreducible $V_{-1}(sl(2))$–module $L_{A_1}(-1-j)\Lambda_0 + j\Lambda_1)$. Now claim follows from Theorem 5.5 and Lemma 5.6. ■

7 The vertex algebra $V^{-2}(B_\ell)$ and its quotients

In this section let $\ell \geq 2$. Note that $k = -2$ is a collapsing level for B_ℓ [4], and that the simple affine W-algebra $W_{-2}(B_\ell, \theta)$ is isomorphic to $V_{\ell-\frac{7}{2}}(A_1)$. This implies that $H_\theta(V_{-2}(B_\ell)) = V_{\ell-\frac{7}{2}}(A_1)$. But as in the case of the affine Lie algebra of type D, we can construct an intermediate vertex algebra $\mathcal{V}$ so that $H_\theta(\mathcal{V}) = V^{\ell-7/2}(A_1)$.

Remark 7.1. The formula for a singular vector of conformal weight two in $V^{-2}(B_\ell)$ was given in [15, Theorem 4.2] for $\ell \geq 3$, and in [15, Remark 4.3] for $\ell = 2$. Note that, for $\ell \geq 4$, the vector $\sigma(w_2)$ from [15] is equal to the vector w_1 from relation (23), i.e. it is contained in the subalgebra $V^{-2}(D_4)$. For $\ell = 3$, we have

$$w_1 = (e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2}(-1) + e_{\epsilon_1}(-1)e_{\epsilon_2+\epsilon_3}(-1))\mathbf{1}.$$

For $\ell = 2$, the singular vector of conformal weight two in $V^{-2}(B_2)$ is equal to

$$w_1 = (e_{\epsilon_1+\epsilon_2}(-1)e_{-\epsilon_2}(-1) + \frac{1}{2}h_{\epsilon_2}(-1)e_{\epsilon_1}(-1) - e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_2}(-1))\mathbf{1}.$$

□

Consider the singular vector in $V^{-2}(B_\ell)$ denoted by $\sigma(w_2)$ in [15, Theorem 4.2] and [17, Section 7]. Let us denote that singular vector by w_1 in this paper (see Remark 7.1 for explanation).

Then we have the quotient vertex algebra

$$\mathcal{V}_{-2}(B_\ell) = V^{-2}(B_\ell)\big/\langle w_1\rangle. \tag{26}$$

As in the case of the vertex algebra $\mathcal{V}_{-2}(D_\ell)$, we have the non-trivial homomorphism $\mathcal{V}_{-2}(B_\ell) \rightarrow M_{2\ell+1}$.

The proof of the following result is completely analogous to the proof of Proposition 6.3 and it is therefore omitted.

Proposition 7.2. We have

(1) There is a non-trivial homomorphism $\overline{\Phi} : \mathcal{V}_{-2}(B_\ell) \to M_{2\ell+1}$.

(2) $H_\theta(\mathcal{V}_{-2}(B_\ell)) = V^{\ell-7/2}(A_1)$.

(3) $H_\theta(L((-2-t)\Lambda_0 + t\Lambda_1)) \cong L_{A_1}((\ell - 7/2 - t)\Lambda_0 + t\Lambda_1),\ t \in \mathbb{Z}_{\geq 0}$.

(4) The set

$$\{L((-2-t)\Lambda_0 + t\Lambda_1) \mid t \in \mathbb{Z}_{\geq 0}\} \tag{27}$$

provides a complete list of irreducible $\mathcal{V}_{-2}(B_\ell)$–modules from the category KL^{-2}. □

We have the following result on classification of irreducible modules.

Proposition 7.3. Assume that $\ell \geq 3$. Then the set $\{L((-2-j)\Lambda_0 + j\Lambda_1) \mid j \in \mathbb{Z}_{\geq 0}, j \leq 2(\ell-3)+1\}$ provides a complete list of irreducible $V_{-2}(B_\ell)$–modules from the category KL_{-2}. □

Proof. The proof is analogous to the proof of Proposition 6.4: it uses the exactness of the functor H_θ and the representation theory of affine vertex algebras. In particular, we use the result from [8] which gives that the set

$$\{L(-(\ell-7/2)-j)\Lambda_0 + j\Lambda_1) \mid j \in \mathbb{Z}_{\geq 0}, j \leq 2(\ell-3)+1\}$$

provides a complete list of irreducible $V_{\ell-7/2}(A_1)$–modules from the category $KL_{\ell-7/2}$. ■

An important consequence is the simplicity of the vertex algebra $\mathcal{V}_{-2}(B_2)$.

Corollary 7.4. The vertex algebra $\mathcal{V}_{-2}(B_\ell)$ is simple if and only if $\ell = 2$. In particular, the set (27) provides a complete list of irreducible modules for $V_{-2}(B_2)$ in KL_{-2}. □

Proof. Since by Proposition 7.2, $\mathcal{V}_{-2}(B_\ell)$ has infinitely many irreducible modules in the category KL^{-2}, and, by Proposition 7.3, $V^{-2}(B_\ell)$ has finitely many irreducible modules in the category KL^{-2} (if $\ell \geq 3$), we conclude that $\mathcal{V}_{-2}(B_\ell)$ cannot be simple for $\ell \geq 3$.

Let us consider the case $\ell = 2$. Assume that $\mathcal{V}_{-2}(B_2)$ is not simple. Then it must contain an ideal I generated by a singular vector of weight $\lambda = (-2-j)\Lambda_0 + j\Lambda_1$ for certain $j > 0$. By applying the functor H_θ, we get a non-trivial ideal in $V^{-3/2}(A_1)$, against the simplicity of $V^{-3/2}(A_1)$. ■

Next we notice that $V^{\ell-7/2}(A_1)$ has a unique non-trivial ideal J which is generated by a singular vector of A_1–weight $2(\ell-2)\omega_1$. The ideal J is maximal and simple (cf. [5]). By combining this with properties of the functor H_θ from [12], one proves the existence of a unique maximal ideal I (which is also simple) in $\mathcal{V}_{-2}(B_\ell)$ such that $I \cong L(-2(\ell-1)\Lambda_0 + 2(\ell-2)\Lambda_1))$.

Remark 7.5. The explicit expression for a singular vector which generates I is more complicated that in the case D, and it won't be presented here. $\square$

In [6] we constructed a homomorphism $\mathcal{V}_{-2}(B_\ell) \otimes V_{-\ell-1/2}(A_1) \to M_{2\ell+1}$. The results of this section enable us to find the image of this homomorphism.

Corollary 7.6. We have:

(1) The vertex algebra $\mathcal{V}_{-2}(B_\ell) \otimes V_{-\ell-1/2}(A_1)$ is conformally embedded into $V_{-1/2}(C_{2\ell+1})$.

(2) The vertex algebra $\mathcal{V}_{-2}(B_\ell)$ for $\ell \geq 3$ contains a unique ideal $I \cong L(-2(\ell-1)\Lambda_0 + 2(\ell-2)\Lambda_1))$ and

$$\mathrm{ch}(\mathcal{V}_{-2}(B_\ell)) = \mathrm{ch}(V_{-2}(B_\ell)) + \mathrm{ch}(L(-2(\ell-1)\Lambda_0 + 2(\ell-2)\Lambda_1)).$$

$\square$

Finally, we apply Theorem 5.5 and prove that KL_{-2} is a semi-simple category.

Corollary 7.7. If $\ell \geq 2$, then every $V_{-2}(B_\ell)$–module in KL_{-2} is completely reducible. $\square$

Proof. It suffices to prove that every highest weight $V_{-2}(B_\ell)$–module in KL_{-2} is irreducible. Assume that $\ell \geq 3$. If $M \cong \widetilde{L}(\lambda)$ is a highest weight module in KL_{-2} then the highest weight is $\lambda = -(2+j)\Lambda_0 + j\Lambda_1$ where $0 \leq j \leq 2(\ell-3)j+1$. Since $H_\theta(L(\lambda))$ is a non-zero highest weight $V_{-\ell+7/2}(sl(2))$–module, then the complete reducibility result from [8] implies that $H_\theta(L(\lambda))$ is irreducible. The assertion now follows from Lemma 5.6. The proof in the case $\ell = 2$ is similar, and it uses the classification of irreducible $V_{-2}(B_2)$–modules from Corollary 7.4 and the fact that every highest weight $V_{-3/2}(sl(2)) = H_\theta(V_{-2}(B_2))$–module in $KL_{-3/2}$ is irreducible. $\blacksquare$

8 On the representation theory of $V_{2-\ell}(D_\ell)$

8.1 The vertex algebra $\overline{V}_{2-\ell}(D_\ell)$

Let $\mathfrak{g}$ be a simple Lie algebra of type D_ℓ. Recall that $2-\ell = -h^\vee/2+1$ is a collapsing level [4]. We have the singular vector

$$v_n = \Big(\sum_{i=2}^{\ell} e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1)\Big)^n \mathbf{1} \tag{28}$$

in $V^{n-\ell+1}(D_\ell)$, for any $n \in \mathbb{Z}_{>0}$. As in [40], we consider the vertex algebra

$$\overline{V}_{2-\ell}(D_\ell) = V^{2-\ell}(D_\ell)/\langle v_1 \rangle, \tag{29}$$

where $\langle v_1 \rangle$ denotes the ideal in $V^{2-\ell}(D_\ell)$ generated by the singular vector v_1. We recall the following result on the classification of irreducible $\overline{V}_{2-\ell}(D_\ell)$–modules in the category $KL^{2-\ell}$.

Proposition 8.1. [40]

(1) The set

$$\{V(t\omega_\ell), V(t\omega_{\ell-1}) \mid t \in \mathbb{Z}_{\geq 0}\}$$

provides a complete list of irreducible finite-dimensional modules for the Zhu algebra $A(\overline{V}_{2-\ell}(D_\ell))$.

(2) The set

$$\{L((2-t-\ell)\Lambda_0 + t\Lambda_\ell), L((2-t-\ell)\Lambda_0 + t\Lambda_{\ell-1}) \mid t \in \mathbb{Z}_{\geq 0}\}$$

provides a complete list of irreducible $\overline{V}_{2-\ell}(D_\ell)$–modules from the category $KL^{2-\ell}$. □

In the odd rank case $D_{2\ell-1}$, the modules from Proposition 8.1 (2) provide a complete list of irreducible $V_{3-2\ell}(D_{2\ell-1})$–modules from the category $KL_{3-2\ell}$ (cf. [11]). The paper [11] also contains a fusion rules result in the category $KL_{3-2\ell}$. Detailed fusion rules analysis will be presented elsewhere.

On the other hand, Theorem 4.2 implies that in the even rank case $D_{2\ell}$, $V_{2-2\ell}(D_{2\ell})$ is the unique irreducible $V_{2-2\ell}(D_{2\ell})$–module from the category $KL_{2-2\ell}$. In the next section we will give an explanation of this difference using singular vectors existing in the even rank case $D_{2\ell}$.

8.2 Singular vectors in $V^{n-2\ell+1}(D_{2\ell})$

In this section, we construct more singular vectors in $V^{n-2\ell+1}(D_{2\ell})$. In the case $n = 1$, we show that the maximal submodule of $V^{2-2\ell}(D_{2\ell})$ is generated by three singular vectors. We present explicit formulas for these singular vectors.

Let $\mathfrak{g}$ be a simple Lie algebra of type $D_{2\ell}$. Denote by $S_{2\ell}$ the group of permutations of 2ℓ elements. Let

$$\Pi_\ell = \left\{p \in S_{2\ell} \mid p^2 = 1,\ p(i) \neq i, \forall i \in \{1, \dots, 2\ell\}\right\}$$

be the set of fixed-points free involutions, which is well known to have $(2\ell-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2\ell-1)$ elements. For $i \neq j$, denote by $(i\,j) \in S_{2\ell}$ the transposition of i and j. Then, any $p \in \Pi_\ell$ admits a unique decomposition of the form:

$$p = (i_1\,j_1) \cdots (i_\ell\,j_\ell),$$

such that $i_h < j_h$ for $1 \leq h \leq \ell$, and $i_1 < \ldots < i_\ell$. Define a permutation $\bar{p} \in S_{2\ell}$ by:

$$\bar{p}(2h-1) = i_h,\ \bar{p}(2h) = j_h,\ 1 \leq h \leq \ell.$$

Thus, we have a well defined map $p \mapsto \bar{p}$ from Π_ℓ to $S_{2\ell}$. Define the function $s : \Pi_\ell \to \{\pm 1\}$ as follows:

$$s(p) = \operatorname{sign}(\bar{p}),$$

where $\mathrm{sign}(q)$ denotes the sign of the permutation $q \in S_{2\ell}$.

We have:

Theorem 8.2. The vector

$$w_n = \Big(\sum_{p\in\Pi_\ell} s(p) \prod_{\substack{i\in\{1,\dots,2\ell\}\\ i<p(i)}} e_{\epsilon_i+\epsilon_{p(i)}}(-1)\Big)^n \mathbf{1} \tag{30}$$

is a singular vector in $V^{n-2\ell+1}(D_{2\ell})$, for any $n \in \mathbb{Z}_{>0}$. □

Proof. Direct verification of relations $e_{\epsilon_k-\epsilon_{k+1}}(0)w_n = 0$, for $k = 1,\dots,2\ell-1$, $e_{\epsilon_{2\ell-1}+\epsilon_{2\ell}}(0)w_n = 0$ and $e_{-(\epsilon_1+\epsilon_2)}(1)w_n = 0$. ■

Remark 8.3. The vector w_n has conformal weight $n\ell$ and its $\mathfrak{g}$–highest weight equals $2n\omega_{2\ell} = n(\epsilon_1 + \ldots + \epsilon_{2\ell})$. In particular, for $n = 1$, the vector w_1 has conformal weight ℓ and highest weight $2\omega_{2\ell} = \epsilon_1 + \ldots + \epsilon_{2\ell}$. □

Example 8.4. Set $n = 1$ for simplicity. For $\ell = 2$ we recover the singular vector

$$w_1 = (e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_4}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_4}(-1) + e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1))\mathbf{1}$$

in $V^{-2}(D_4)$ of conformal weight 2 from [40]. For $\ell = 3$, the formula for the singular vector in $V^{-4}(D_6)$ of conformal weight 3 is more complicated. It is a sum of $5!! = 15$ monomials:

$$\begin{aligned}
w_1 = (&e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_4}(-1)e_{\epsilon_5+\epsilon_6}(-1) - e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_5}(-1)e_{\epsilon_4+\epsilon_6}(-1)\\
&+e_{\epsilon_1+\epsilon_2}(-1)e_{\epsilon_3+\epsilon_6}(-1)e_{\epsilon_4+\epsilon_5}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_5+\epsilon_6}(-1)\\
&+e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_5}(-1)e_{\epsilon_4+\epsilon_6}(-1) - e_{\epsilon_1+\epsilon_3}(-1)e_{\epsilon_2+\epsilon_6}(-1)e_{\epsilon_4+\epsilon_5}(-1)\\
&+e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_5+\epsilon_6}(-1) - e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_5}(-1)e_{\epsilon_3+\epsilon_6}(-1)\\
&+e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_6}(-1)e_{\epsilon_3+\epsilon_5}(-1) - e_{\epsilon_1+\epsilon_5}(-1)e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_4+\epsilon_6}(-1)\\
&+e_{\epsilon_1+\epsilon_5}(-1)e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_3+\epsilon_6}(-1) - e_{\epsilon_1+\epsilon_5}(-1)e_{\epsilon_2+\epsilon_6}(-1)e_{\epsilon_3+\epsilon_4}(-1)\\
&+e_{\epsilon_1+\epsilon_6}(-1)e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_4+\epsilon_5}(-1) - e_{\epsilon_1+\epsilon_6}(-1)e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_3+\epsilon_5}(-1)\\
&+e_{\epsilon_1+\epsilon_6}(-1)e_{\epsilon_2+\epsilon_5}(-1)e_{\epsilon_3+\epsilon_4}(-1))\mathbf{1}.
\end{aligned}$$

□

Denote by ϑ the automorphism of $V^{n-2\ell+1}(D_{2\ell})$ induced by the automorphism of the Dynkin diagram of $D_{2\ell}$ of order two such that

$$\vartheta(\epsilon_k - \epsilon_{k+1}) = \epsilon_k - \epsilon_{k+1},\ k = 1,\dots,2\ell-2, \tag{31}$$

$$\vartheta(\epsilon_{2\ell-1} - \epsilon_{2\ell}) = \epsilon_{2\ell-1} + \epsilon_{2\ell},\ \vartheta(\epsilon_{2\ell-1} + \epsilon_{2\ell}) = \epsilon_{2\ell-1} - \epsilon_{2\ell}. \tag{32}$$

Theorem 8.2 now implies that $\vartheta(w_n)$ is a singular vector in $V^{n-2\ell+1}(D_{2\ell})$, for any $n \in \mathbb{Z}_{>0}$, also. The vector $\vartheta(w_n)$ has conformal weight $n\ell$ and its highest weight for $\mathfrak{g}$ is $2n\omega_{2\ell-1} = n(\epsilon_1 + \ldots + \epsilon_{2\ell-1} - \epsilon_{2\ell})$.

We consider the associated quotient vertex algebra

$$\widetilde{V}_{n-2\ell+1}(D_{2\ell}) := V^{n-2\ell+1}(D_{2\ell})/\langle v_n, w_n, \vartheta(w_n)\rangle, \tag{33}$$

where v_n is given by relation (28) (for $D_{2\ell}$):

$$v_n = \Big(\sum_{i=2}^{2\ell} e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1)\Big)^n \mathbf{1}.$$

In particular, for $n = 1$ we have the vertex algebra

$$\widetilde{V}_{2-2\ell}(D_{2\ell}) = V^{2-2\ell}(D_{2\ell})/\langle v_1, w_1, \vartheta(w_1)\rangle.$$

Clearly, $\widetilde{V}_{2-2\ell}(D_{2\ell})$ is a quotient of vertex algebra $\overline{V}_{2-2\ell}(D_{2\ell})$ from Subsection 8.1. The associated Zhu algebra is

$$A(\widetilde{V}_{2-2\ell}(D_{2\ell})) = U(\mathfrak{g})/\langle \bar{v}, \bar{w}, \vartheta(\bar{w})\rangle,$$

where

$$\bar{v} = \sum_{i=2}^{2\ell} e_{\epsilon_1-\epsilon_i}e_{\epsilon_1+\epsilon_i}, \qquad \bar{w} = \sum_{p\in\Pi_\ell} s(p) \prod_{\substack{i\in\{1,\ldots,2\ell\}\\ i<p(i)}} e_{\epsilon_i+\epsilon_{p(i)}}.$$

Lemma 8.5. We have:

(1) $\bar{w}V(t\omega_{2\ell}) \neq 0$, for $t \in \mathbb{Z}_{>0}$.

(2) $\vartheta(\bar{w})V(t\omega_{2\ell-1}) \neq 0$, for $t \in \mathbb{Z}_{>0}$.

□

Proof. (1) Let $t = 1$. Denote by $v_{\omega_{2\ell}}$ the highest weight vector of $V(\omega_{2\ell})$, and by $v_{-\omega_{2\ell}}$ the lowest weight vector of $V(\omega_{2\ell})$. One can easily check, using the spinor realization of $V(\omega_{2\ell})$, that there exists a constant $C \neq 0$ such that

$$\bar{w}(v_{-\omega_{2\ell}}) = Cv_{\omega_{2\ell}}.$$

For general $t \in \mathbb{Z}_{>0}$, the claim follows using the embedding of $V(t\omega_{2\ell})$ into $V(\omega_{2\ell})^{\otimes t}$. Claim (2) follows similarly. ■

Theorem 8.6. We have:

(i) The trivial module $\mathbb{C}$ is the unique finite-dimensional irreducible module for $A(\widetilde{V}_{2-2\ell}(D_{2\ell}))$.

(ii) $V_{2-2\ell}(D_{2\ell})$ is the unique irreducible $\mathfrak{g}$–locally finite module for $\widetilde{V}_{2-2\ell}(D_{2\ell})$.

(iii) The vertex operator algebra $\widetilde{V}_{2-2\ell}(D_{2\ell})$ is simple, i.e.

$$V_{2-2\ell}(D_{2\ell}) = V^{2-2\ell}(D_{2\ell})/\langle v_1, w_1, \vartheta(w_1)\rangle.$$

□

Proof. (i) Proposition 8.1 implies that the set

$$\{V(t\omega_{2\ell}), V(t\omega_{2\ell-1}) \mid t \in \mathbb{Z}_{\geq 0}\}$$

provides a complete list of finite-dimensional irreducible modules for the algebra $U(\mathfrak{g})/\langle\bar{v}\rangle = A(\overline{V}_{2-2\ell}(D_{2\ell}))$. Lemma 8.5 shows that $V(t\omega_{2\ell})$ and $V(t\omega_{2\ell-1})$ are not modules for $A(\widetilde{V}_{2-2\ell}(D_{2\ell}))$, for $t \in \mathbb{Z}_{>0}$. Claim (i) follows. Claims (ii) and (iii) follow from (i) by applying Proposition 3.2 and Corollary 3.3. ■

Remark 8.7. A general character formula for certain simple affine vertex algebras at negative integer levels has been recently presented by V. G. Kac and M. Wakimoto in [38], (more precisely, $\mathfrak{g} = A_n, C_n$ for $k = -1$ and $\mathfrak{g} = D_4, E_6, E_7, E_8$ for $k = -2, -3, -4, 6$). Note that conditions (i)-(iii) of [38, Theorem 3.1] hold for vertex algebras $V_{-b}(D_n)$, $n > 4$, $b = 1, \ldots, n-2$, too. We conjecture that condition (iv) of this theorem holds as well; therefore formula (3.1) in [38] gives the character formula. □

9 Conformal embedding of $\widetilde{V}(-4, D_6 \times A_1)$ into $V_{-4}(E_7)$

In this section, we apply the results on representation theory of $V_{-4}(D_6)$ from previous sections to the conformal embedding of $\widetilde{V}(-4, D_6 \times A_1)$ into $V_{-4}(E_7)$. This gives us an interesting example of a maximal semisimple equal rank subalgebra such that the associated conformally embedded subalgebra is not simple.

We use the construction of the root system of type E_7 from [19], [29], and the notation for root vectors similar to the notation for root vectors for E_6 from [9].

For a subset $S = \{i_1, \ldots, i_k\} \subseteq \{1, 2, 3, 4, 5, 6\}$, $i_1 < \ldots < i_k$, with odd number of elements (so that $k = 1, 3$ or 5), denote by $e_{(i_1 \ldots i_k)}$ a suitably chosen root vector associated to the positive root

$$\frac{1}{2}\left(\epsilon_8 - \epsilon_7 + \sum_{i=1}^{6}(-1)^{p(i)}\epsilon_i\right),$$

such that $p(i) = 0$ for $i \in S$ and $p(i) = 1$ for $i \notin S$. We will use the symbol $f_{(i_1 \ldots i_k)}$ for the root vector associated to corresponding negative root.

Note now that the subalgebra of E_7 generated by positive root vectors

$$e_{\epsilon_6+\epsilon_5}, e_{\alpha_1} = e_{(1)}, e_{\alpha_3} = e_{\epsilon_2-\epsilon_1}, e_{\alpha_4} = e_{\epsilon_3-\epsilon_2}, e_{\alpha_2} = e_{\epsilon_1+\epsilon_2}, e_{\alpha_5} = e_{\epsilon_4-\epsilon_3} \tag{34}$$

and the associated negative root vectors is a simple Lie algebra of type D_6. There are 30 root vectors associated to positive roots for D_6:

$$\begin{aligned}
&e_{\epsilon_6+\epsilon_5},\ e_{\epsilon_8-\epsilon_7},\\
&e_{(i)},\ i \in \{1,2,3,4\},\\
&e_{(ijk)},\ i,j,k \in \{1,2,3,4\},\ i<j<k,\\
&e_{(i56)},\ i \in \{1,2,3,4\},\\
&e_{(ijk56)},\ i,j,k \in \{1,2,3,4\},\ i<j<k,\\
&e_{\pm\epsilon_i+\epsilon_j},\ i,j \in \{1,2,3,4\}, i<j.
\end{aligned} \tag{35}$$

Furthermore, the subalgebra of E_7 generated by $e_{\epsilon_6-\epsilon_5}$ and the associated negative root vector is a simple Lie algebra of type A_1. Thus, $D_6 \oplus A_1$ is a semisimple subalgebra of E_7.

It follows from [3], [9] that the affine vertex algebra $\widetilde{V}(-4, D_6 \times A_1)$ is conformally embedded in $V_{-4}(E_7)$. Remark that $\widetilde{V}(-4, A_1) = V_{-4}(A_1)$ (since $V^{-4}(A_1) = V_{-4}(A_1)$). This implies that $\widetilde{V}(-4, D_6 \times A_1) \cong \widetilde{V}(-4, D_6) \otimes V_{-4}(A_1)$.

It was shown in [15] that

$$\begin{aligned}
v_{E_7} = (&e_{\epsilon_8-\epsilon_7}(-1)e_{\epsilon_6+\epsilon_5}(-1) + e_{(156)}(-1)e_{(23456)}(-1)+\\
&+e_{(256)}(-1)e_{(13456)}(-1) + e_{(356)}(-1)e_{(12456)}(-1)+\\
&+e_{(456)}(-1)e_{(12356)}(-1))\mathbf{1}
\end{aligned} \tag{36}$$

is a singular vector in $V^{-4}(E_7)$. Moreover,

$$V_{-4}(E_7) \cong V^{-4}(E_7)/\langle v_{E_7}\rangle.$$

Vectors $(e_{(12346)}(-1))^s\mathbf{1}$, for $s \in \mathbb{Z}_{>0}$ are (non-trivial) singular vectors for the affinization of $D_6 \oplus A_1$ in $V_{-4}(E_7)$ of highest weights $-(s+4)\Lambda_0 + s\Lambda_6$ for $D_6^{(1)}$ and $-(s+4)\Lambda_0 + s\Lambda_1$ for $A_1^{(1)}$. Thus there exist highest weight modules $\widetilde{L}_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6)$ and $\widetilde{L}_{A_1}(-(s+4)\Lambda_0 + s\Lambda_1)$, for $D_6^{(1)}$ and $A_1^{(1)}$, respectively such that $(\widetilde{V}(-4, D_6) \otimes V_{-4}(A_1)).(e_{(12346)}(-1))^s\mathbf{1}$ is isomorphic to $\widetilde{L}_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6) \otimes \widetilde{L}_{A_1}(-(s+4)\Lambda_0 + s\Lambda_1)$. This implies that

$$L_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6) \otimes L_{A_1}(-(s+4)\Lambda_0 + s\Lambda_1)$$

are irreducible $\widetilde{V}(-4, D_6 \times A_1)$–modules, for $s \in \mathbb{Z}_{>0}$.

In particular, $L_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6)$ are irreducible (D_6–locally finite) $\widetilde{V}(-4, D_6)$–modules, for $s \in \mathbb{Z}_{>0}$. In the next proposition, we use the notation from (29), (30), (31), (32).

Proposition 9.1. We have:

(1) Assume that $\widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_6)$ and $\widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_5)$ are highest weight $\overline{V}_{-4}(D_6)$–modules from the category KL^{-4}, not necessarily irreducible. Then

$$\widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_6) \boxtimes \widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_5) = 0,$$

where $\boxtimes$ is the tensor functor for KL^{-4}–modules. In other words, we cannot have a non-zero $\overline{V}_{-4}(D_6)$–module M from KL^{-4} and a non-zero intertwining operator of type

$$\binom{M}{\widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_6) \quad \widetilde{L}_{D_6}(-6\Lambda_0 + 2\Lambda_5)}. \tag{37}$$

(2) Relations $w_1 \neq 0$ and $\vartheta(w_1) = 0$ hold in $V_{-4}(E_7)$. In particular, $\widetilde{V}(-4, D_6)$ is not simple. $\square$

Proof. For the proof of assertion (1) we first notice that the following decomposition of D_6–modules holds:

$$\begin{aligned} V_{D_6}(2\omega_6) \otimes V_{D_6}(2\omega_5) &= V_{D_6}(2\omega_5 + 2\omega_6) \oplus V_{D_6}(\omega_3 + \omega_5 + \omega_6) \oplus V_{D_6}(2\omega_3) \\ &\quad \oplus V_{D_6}(\omega_1 + \omega_5 + \omega_6) \oplus V_{D_6}(\omega_1 + \omega_3) \oplus V_{D_6}(2\omega_1). \end{aligned} \tag{38}$$

Assume that M is a non-zero $\overline{V}_{-4}(D_6)$–module in the category KL^{-4} such that there is a non-trivial intertwining operator of type (37). Then the Frenkel-Zhu formula for fusion rules implies that M must contain a non-trivial subquotient whose lowest graded component appears in the decomposition of $V_{D_6}(2\omega_6) \otimes V_{D_6}(2\omega_5)$. But by Proposition 8.1, the D_6–modules appearing in (38) cannot be lowest components of any $\overline{V}_{-4}(D_6)$–module. This proves assertion (1).

Assertion (1) implies that if $w_1 \neq 0$ and $\vartheta(w_1) \neq 0$ in $V_{-4}(E_7)$, then

$$Y(w_1, z)\vartheta(w_1) = 0.$$

A contradiction since $V_{-4}(E_7)$ is a simple vertex algebra. The same fusion rules argument shows that if $\vartheta(w_1) \neq 0$ in $V_{-4}(E_7)$, then

$$Y(\vartheta(w_1), z)e_{(12346)}(-1)^2\mathbf{1} = 0,$$

which again contradicts the simplicity of $V_{-4}(E_7)$. So, $\vartheta(w_1) = 0$.

But if $w_1 = 0$, then, by Theorem 8.6 (iii), we have that $\widetilde{V}(-4, D_6) = V_{-4}(D_6)$. Theorem 4.2 implies that $\widetilde{V}(-4, D_6)$ is not simple, since the simple vertex operator algebra $V_{-4}(D_6)$ has only one irreducible D_6–locally finite module. A contradiction. So $w_1 \neq 0$ and claim (2) follows. ∎

Set

$$\mathcal{V}_{-4}(D_6) = \frac{V^{-4}(D_6)}{< v_1, \vartheta(w_1) >}. \tag{39}$$

Theorem 9.2. We have:

(1) $\widetilde{V}(-4, D_6) \cong \mathcal{V}_{-4}(D_6)$.

(2) The set $\{L_{D_6}(-(s+4)\Lambda_0 + s\Lambda_6) \mid \quad s \in \mathbb{Z}_{\geq 0}\}$ provides a complete list of irreducible $\mathcal{V}_{-4}(D_6)$–modules. □

Proof. We first notice that $\widetilde{V}(-4, D_6)$ is a certain quotient of $\frac{V^{-4}(D_6)}{<v_1, \vartheta(w_1)>}$, and that

$$H_\theta\left(\frac{V^{-4}(D_6)}{< v_1, \vartheta(w_1) >}\right) = \mathcal{V}_{-2}(D_4).$$

Since $\mathcal{V}_{-2}(D_4)$ contains a unique non-trivial ideal which is maximal and simple, we conclude that $\frac{V^{-4}(D_6)}{<v_1, \vartheta(w_1)>}$ also contains a unique ideal, and it must be the ideal generated by w_1. Since in $\widetilde{V}(-4, D_6)$ we have that $w_1 \neq 0$, we conclude that

$$\widetilde{V}(-4, D_6) \cong \frac{V^{-4}(D_6)}{< v_1, \vartheta(w_1) >}.$$

The proof of assertion (2) follows from (1), the classification result of $\overline{V}_{-4}(D_6)$–modules from Proposition 8.1 and Lemma 8.5. ∎

10 Funding

This work was supported by the Croatian Science Foundation [grant number 2634 to D.A. and O.P.]; and the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme [KK.01.1.1.01 to D.A. and O.P.].

Acknowledgement

We would like to thank Maria Gorelik, Tomoyuki Arakawa and Anne Moreau for correspondence and discussions.

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