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A RELAXATION RESULT FOR STATE CONSTRAINED INCLUSIONS IN INFINITE DIMENSION

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Abstract. In this paper we consider a state constrained differential inclusion $\dot{x} \in \mathbb{A}x + F(t,x)$, with $\mathbb{A}$ generator of a strongly continuous semigroup in an infinite dimensional separable Banach space. Under an "inward pointing condition" we prove a relaxation result stating that the set of trajectories lying in the interior of the constraint is dense in the set of constrained trajectories of the convexified inclusion $\dot{x} \in \mathbb{A}x + \overline{\text{co}}F(t,x)$. Some applications to control problems involving PDEs are given.

1. **Introduction.** We study a class of infinite dimensional differential inclusions subject to state constraints. Interest in this kind of equations arises in several contexts. Differential inclusions find a natural application in a research area of great development, the control theory, and the infinite dimensional setting allows to apply our results to control problems involving PDEs. Hence, models describing many physical phenomena such as diffusion, vibration of strings, fluid dynamics, may be included in our analysis.

In this paper we are concerned with the differential inclusion

$$\dot{x}(t) \in \mathbb{A}x(t) + F(t,x(t)), \quad \text{a.e. } t \in [t_0, 1], \tag{1}$$

and the convexified differential inclusion

$$\dot{x}(t) \in \mathbb{A}x(t) + \overline{\text{co}}F(t,x(t)), \quad \text{a.e. } t \in [t_0, 1], \tag{2}$$

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with $\overline{\text{co}}F(t,x(t))$ the closed convex hull of $F(t,x(t))$. The operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$, X is an infinite dimensional separable Banach space, $F : I \times X \rightsquigarrow X$ is a set-valued map with closed non-empty images, $I = [0,1]$ and $t_0 \in I$. The trajectories of the differential inclusion (1) are understood in the mild sense (see [25]) and are subject to the state constraint. Namely given a set $K \subset X$, we restrict our attention to the trajectories satisfying

$$x(t) \in K, \qquad \text{for } t \in [t_0, 1]\,. \tag{3}$$

In this paper we shall always assume that K is the closure of an open subset of X. When satisfying the constraint, a trajectory x is called *feasible*.

Differential inclusions, and control systems, in presence of state constraints, are largely employed in applied sciences. One of the tools playing a key role in this context consists in approximating feasible trajectories by trajectories lying in the interior of the constraints. It is used for instance to establish regularity properties of value functions, to justify the use of the Maximum Principle in normal form, to prove existence and regularity results of optimal solutions. The classical technique employed to construct the approximating trajectories relies on the possibility of directing the velocity into the interior of the constraint K whenever approaching the boundary ∂K of K. To this aim, in the finite dimensional setting, an "inward pointing condition" was proposed by Soner, see [28], to get continuity of the value function associated to an optimal control problem with dynamics $\dot{x} \in F(x)$ independent of t. Since then, this subject has received considerable attention, a partial list of references includes [5, 6, 10, 16, 18, 19].

Defining the oriented distance from $x \in X$ to K by

$$d_K(x) = \begin{cases} \inf_{k \in K} \|x - k\|_X & \text{if } x \notin K \\ -\inf_{k \in (X \setminus K)} \|x - k\|_X & \text{otherwise,} \end{cases}$$

the inward pointing condition, in the case of time independent F and state constraints with a locally $C^{1,1}$ boundary, takes the following form:

$$\min_{v \in F(\bar{x})} \langle \nabla d_K(\bar{x}), v \rangle < -\rho, \qquad \forall\, \bar{x} \in \partial K\,, \tag{4}$$

for some $\rho > 0$, cf. [5, 18]. As in many applied models state constraints having nonsmooth boundary are present, a number of papers made extensions of (4) to the nonsmooth setting. However, contrary to the smooth case, here some regularity of the dynamics $F(t,x)$ is usually required both in t and in x. On the other hand, it may happen in some applications that the dynamics depends on t in a merely measurable way. In order to extend the theory to this situation, in the recent works [16, 17] a new inward pointing condition (equivalent to the classical one if K has smooth boundary) is proposed: for any "bad" velocity v pointing outside the constraint, there exists a "good" one $\bar{v}$ such that the difference $\bar{v} - v$ points inside in a uniform way. To be more precise, let $\partial d_K(x)$ denote the Clarke generalized gradient of d_K at $x \in X$. Its support function is defined by

$$\sigma(x; y) = \sup_{\xi \in \partial d_K(x)} \langle \xi,\, y \rangle, \quad \forall\, y \in X.$$

The new inward pointing condition is as follows

$$\begin{cases} \forall\, \bar{x} \in \partial K,\ \exists\, \rho > 0 \text{ such that if } \sigma(\bar{x};\, v) \geq 0 \text{ for some } t \in I,\ v \in F(t,\bar{x}), \\ \qquad\qquad \text{then} \quad \inf_{\bar{v} \in F(t,\bar{x})} \sigma(\bar{x}; \bar{v} - v) < -\rho\,. \end{cases} \tag{5}$$

Under this assumption, in [16, 17] some approximation results were proved in order to get uniqueness of solutions for a constrained Hamilton-Jacobi-Bellman equation.

The purpose of the present paper is to perform the analysis in the infinite dimensional setting, the natural framework for many phenomena described by PDEs. Also in this case we need results which permit to approximate feasible trajectories by trajectories staying in the interior of the state constraints. Assuming an inward pointing condition, Theorem 3.2 below guarantees the existence of the required approximation. Notice that, although the literature dealing with infinite dimensional control theory (and infinite dimensional differential inclusions) is quite rich, see e.g. the books [2, 3, 4, 14, 21, 22], the recent paper [11] and the bibliography therein, to our knowledge, no similar results are known in this setting. As an application, we obtain our main result, a relaxation theorem in infinite dimension (see Theorem 3.1).

We deal with great generality, allowing the state space X to be a separable Banach space. Hence, our analysis applies to some interesting and delicate frameworks as the space of essentially bounded functions and the space of continuous functions. For this reason, in this context, the relaxation theorem is obtained under a version of condition (5), requiring some uniformity on a neighborhood of ∂K and with respect to the semigroup. Nevertheless, as illustrated in Section 3, if some compactness assumptions are satisfied, a much more simple condition, analogue to the finite dimensional (5) is sufficient.

We consider the following inward pointing condition:

$$\forall\, \bar{x} \in \partial K,\ \exists\, \eta,\ \rho,\ M > 0 \text{ such that if } \max_{\tau \leq \eta} \sigma(z_0; S(\tau)\, v) \geq 0 \tag{6}$$
$$\text{for some } v \in \overline{\text{co}}F(t,x),\ z_0 \in B(x,\eta),\ t \in I,\ x \in K \cap B(\bar{x},\eta), \text{ then}$$
$$\Big\{ \bar{v} \in \overline{\text{co}}F(t,x) \,:\, \|\bar{v} - v\|_X \leq M\,, \sup_{z \in B(S(\tau)x,\eta),\, \tau \leq \eta} \sigma(z; S(\tau)\,(\bar{v} - v)) < -\rho \Big\} \neq \emptyset\,.$$

Notice that condition (6) deals with the set-valued map $\overline{\text{co}}F$, since, in order to prove the relaxation theorem, we need to approximate relaxed trajectories by relaxed trajectories lying in the interior of K. However, under additional compactness conditions, the first convex hull can be removed from (6).

Quite interesting for the applications is the case when X is a Hilbert space, see Section 5. In this framework we will provide an alternative version of condition (6), which drastically simplifies the analysis when the set of constraints K is convex. The inward pointing condition needed here involves projections on convex sets rather than generalized gradients of the oriented distance function which belong to the dual space X^*.

1.1. Outline of the paper.

Section 2 contains a list of notations, definitions, and assumptions in use. The main theorems are stated in Section 3. Some results which allow to simplify the inward pointing condition are also proposed. The Hilbert space setting is analyzed in Section 4, while Section 5 is devoted to some applications involving PDEs and integrodifferential equations. The final Section 6 and Appendix contain proofs and technical tools.

2. Preliminaries.

In this section we list the notation and the main assumptions in use throughout the paper.

2.1. **Notation.**

- $B(x,r)$ denotes the closed ball of center $x \in X$ and radius $r > 0$; B is the closed unit ball in X centered at 0;
- given a Banach space Y, $\mathbb{L}(X,Y)$ denotes the Banach space of bounded linear operators from X into Y, $\mathcal{C}(I,X)$ the space of continuous functions from I to X, $L^1(I,X)$ the space of Bochner integrable functions from I to X, and $L^\infty(I,X)$ the space of measurable essentially bounded functions from I to X;
- $\langle \cdot, \cdot \rangle$ stands for the duality pairing on $X^* \times X$;
- μ is the Lebesgue measure on the real line;
- $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, +1\}$ is the sign function: $t \mapsto \begin{cases} -1 & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases}$

We will use the following notion of solution.

Definition 2.1. Let $t_0 \in I$ and $x_0 \in X$. A function $x \in \mathcal{C}([t_0,1],X)$ is a (mild) solution of (1) with initial datum $x(t_0) = x_0$ if there exists a function $f \in L^1([t_0,1],X)$ such that

$$f(t) \in F(t, x(t)), \qquad \text{for a.e. } t \in (t_0, 1) \tag{7}$$

and

$$x(t) = S(t - t_0)\, x_0 + \int_{t_0}^{t} S(t-s)\, f(s)\, \mathrm{d}s, \qquad \text{for any } t \in [t_0, 1], \tag{8}$$

i.e. f is an integrable selection of the set valued map $t \rightsquigarrow F(t, x(t))$ and x is a mild solution (see [25]) of the initial value problem

$$\begin{cases} \dot{x}(t) = \mathbb{A}x(t) + f(t), & \text{for a.e. } t \in [t_0, 1] \\ x(t_0) = x_0. \end{cases} \tag{9}$$

In order to simplify the notation, for a mild solutions x of (1), we denote by f^x the corresponding measurable selection in (9).

Notice that, since $S(t)$ is a strongly continuous semigroup, there exists $M_S > 0$ such that

$$\|S(t)\|_{\mathbb{L}(X,X)} \le M_S, \qquad \text{for any } t \in I. \tag{10}$$

The differential inclusion (1) is a convenient tool to investigate for example the semilinear control system

$$\begin{cases} \dot{x}(t) = \mathbb{A}x(t) + f(t, x(t), u(t)), & \text{a.e. } t \in [t_0, 1] \\ u(t) \in U, \end{cases} \tag{11}$$

where U is an appropriate separable metric space of controls. Setting $F(t,x) = f(t,x,U)$, we can reduce (11) to (1) by applying a measurable selection theorem.

2.2. **Assumptions.** In our main theorems, we will assume the following conditions:

- positive invariance of K by the semigroup:

$$S(t)\, K \subset K, \qquad \forall\, t \in I; \tag{12}$$

- $\forall\, t \in I$ and any $x \in X$, $F(t,x)$ is closed, and, for any $x \in X$,

$$\text{the set-valued map } F(\cdot, x) \text{ is Lebesgue measurable;} \tag{13}$$

- $F(t, \cdot)$ is locally Lipschitz in the following sense: for any $R > 0$, there exists $k_R \in L^1(I, \mathbb{R}^+)$ such that, for a.e. $t \in I$ and any $x, y \in RB$,

$$F(t,x) \subset F(t,y) + k_R(t) \|x - y\|_X B; \tag{14}$$

- there exists $\phi \in L^1(I, \mathbb{R}^+)$ such that, for a.e. $t \in I$ and any $x \in X$,

$$F(t,x) \subset \phi(t)(1 + \|x\|_X)B. \tag{15}$$

3. The main results.

In this section we state the results of the paper whose proofs are postponed to Section 6. The first is a relaxation theorem.

Theorem 3.1. *Assume (6) and (12)–(15). Then, for any $\varepsilon > 0$ and any feasible trajectory $\hat{x}$ of (2), (3), there exists a trajectory x of (1) satisfying*

$$x(t_0) = \hat{x}(t_0), \qquad x(t) \in \operatorname{Int} K, \quad \text{for any } t \in (t_0, 1] \tag{16}$$

and

$$\|\hat{x} - x\|_{L^\infty([t_0,1],X)} \leq \varepsilon. \tag{17}$$

The key point in the proof of Theorem 3.1, is a result on approximation of feasible trajectories, by trajectories lying in the interior of the constraint K.

Theorem 3.2. *Assume (12)–(15) and that*

$$\forall\, \bar{x} \in \partial K,\ \exists\, \eta,\ \rho,\ M > 0 \text{ such that if } \max_{\tau \leq \eta} \sigma(z_0; S(\tau)\, v) \geq 0 \tag{18}$$

$$\text{for some } v \in F(t,x),\ z_0 \in B(x,\eta),\ t \in I,\ x \in K \cap B(\bar{x},\eta), \text{ then}$$

$$\Big\{ \bar{v} \in F(t,x) \,:\, \|\bar{v} - v\|_X \leq M\,, \sup_{z \in B(S(\tau)x,\eta),\, \tau \leq \eta} \sigma(z; S(\tau)\,(\bar{v} - v)) < -\rho \Big\} \neq \emptyset\,.$$

Then, for any $\varepsilon > 0$ and any feasible trajectory $\hat{x}$ of (1), (3), there exists a trajectory x of (1) satisfying (16) and (17).

In the following propositions pointwise versions of the inward pointing condition (18) are proposed, see the applications in Section 5.

Proposition 1. *Assume (14)–(15) with time independent $k_R, \phi \in \mathbb{R}^+$, that*

$$F(\cdot, x) \text{ is continuous for any } x \in X, \tag{19}$$

and

$$F(t, \bar{x}) \text{ is compact, for any } t \in I \text{ and any } \bar{x} \in \partial K. \tag{20}$$

Then, assumption (5) implies (18). Consequently, if (5) holds true with F replaced by $\overline{\text{co}}\, F$, then (6) is satisfied.

In the next proposition the convexity of values of F is needed on the boundary of K.

Proposition 2. *Let X be reflexive. Assume (19) and (14)–(15) with time independent $k_R, \phi \in \mathbb{R}^+$, that for any $\bar{x} \in \partial K$ and $t \in I$, $F(t, \bar{x})$ is convex, and*

$$\text{the map } \partial d_K(\cdot) \text{ is upper semicontinuous at } \bar{x}, \text{ and } \partial d_K(\bar{x}) \text{ is compact.} \tag{21}$$

Then, assumption (5) implies (18) and (6).

Notice that when d_K is C^1 on a neighborhood of ∂K, then condition (21) is satisfied. In the proof of Theorem 3.1, we need to approximate relaxed trajectories by relaxed trajectories lying in the interior of K. This is the reason why the inward pointing condition (6) required in this case involves the set-valued map $\overline{\text{co}} F$. By the way, the first convex hull in (6) can be removed in some special cases, as indicated in the next proposition.

Proposition 3. *Suppose that for every $t \in I$ and $\bar{x} \in \partial K$, $coF(t, \bar{x})$ is closed and the set-valued map*

$$[0,1] \ni t \rightsquigarrow co\left\{v \in F(t,\bar{x}) : \sigma(\bar{x}; v) \leq 0\right\} \tag{22}$$

is upper semicontinuous with closed values. Assume (19) and (14)–(15), with time independent $k_R, \phi \in \mathbb{R}^+$ and that either (20) is satisfied, or that X is a reflexive space and (21) is satisfied. Then (6) holds true whenever

$$\text{for any } \bar{x} \in \partial K, \text{ there exists } \rho > 0 \text{ such that} \tag{23}$$
$$\text{if } \sigma(\bar{x}; v) \geq 0 \text{ for some } v \in F(t,\bar{x}) \text{ and } t \in I, \text{ then } \inf_{\bar{v} \in co\, F(t,\bar{x})} \sigma(\bar{x}; \bar{v} - v) < -\rho\,.$$

In the following remark, the special case of affine forcing terms is analyzed, providing further simplification.

Remark 1. If d_K is C^1 on a neighborhood of ∂K and, for a subset $U \subset Y$,

$$F(t,x) = f_0(t,x) + g(t,x)U, \tag{24}$$

where Y is a Banach space,

$$f_0 : I \times X \to X \qquad \text{and} \qquad g : I \times X \to \mathbb{L}(Y, X),$$

then the classical inward pointing condition implies (5) with F and also with $\overline{co}\, F$ whenever either U is compact or Y is reflexive. Namely, assume (19), and (14), (15) for time independent $k_R, \phi \in \mathbb{R}^+$, with F replaced by f_0 and g. If either U is compact or Y is reflexive, then $\overline{co}\, F(t,x) = f_0(t,x) + g(t,x)\overline{co}\, U$. Then the inward pointing condition:

$$\forall\, \bar{x} \in \partial K,\ \forall\, t \in I,\ \exists\, \bar{u} \in U \text{ such that } \langle \nabla d_K(\bar{x}), f_0(t,\bar{x}) + g(t,\bar{x})\bar{u}\rangle < 0 \tag{25}$$

implies (5) both with F and with $\overline{co}\, F$. Indeed, by compactness of $[0,1]$ and continuity of $f_0(\cdot,\bar{x})$ and $g(\cdot,\bar{x})$, assumption (25) yields:

$$\forall\, \bar{x} \in \partial K,\ \exists\, \rho > 0, \forall\, t \in I,\ \exists\, \bar{u} \in U \text{ with } \langle \nabla d_K(\bar{x}), f_0(t,\bar{x}) + g(t,\bar{x})\bar{u}\rangle < -\rho. \tag{26}$$

Let $t \in I$ and $u \in \overline{co}\, U$ be so that $\langle \nabla d_K(\bar{x}), f_0(t,\bar{x}) + g(t,\bar{x})u\rangle \geq 0$. Thus

$$\langle \nabla d_K(\bar{x}), f_0(t,\bar{x})\rangle \geq -\langle \nabla d_K(\bar{x}), g(t,\bar{x})u\rangle.$$

Then, taking $\bar{u}$ as in (26), we obtain

$$\begin{aligned}\langle \nabla d_K(\bar{x}), &f_0(t,\bar{x}) + g(t,\bar{x})\bar{u} - \big(f_0(t,\bar{x}) + g(t,\bar{x})u\big)\rangle \\ &= \langle \nabla d_K(\bar{x}), g(t,\bar{x})\bar{u} - g(t,\bar{x})u\rangle \leq \langle \nabla d_K(\bar{x}), g(t,\bar{x})\bar{u} + f_0(t,\bar{x})\rangle < -\rho,\end{aligned}$$

yielding (5) with F and also with $\overline{co}\, F$.

Under the same assumptions and F given by (24), let us consider two examples, where condition (25) can be further simplified.

Case 1. $0 \in U$. If $\langle \nabla d_K(\bar{x}), f_0(t,\bar{x})\rangle < 0$, for any $\bar{x} \in \partial K$ and $t \in I$, then (25) holds for $\bar{u} = 0$.

Case 2. U is the unit sphere in Y. Here, if

$$\langle \nabla d_K(\bar{x}), f_0(t,\bar{x})\rangle < \|g(t,\bar{x})^* \nabla d_K(\bar{x})\|_Y \neq 0,$$

for any $\bar{x} \in \partial K$ and $t \in I$, then (25) holds for

$$\bar{u} = -\frac{g(t,\bar{x})^* \nabla d_K(\bar{x})}{\|g(t,\bar{x})^* \nabla d_K(\bar{x})\|_Y},$$

where $g(t,\bar{x})^*$ is the adjoint of $g(t,\bar{x})$.

Indeed, in this case, for any $\bar{x} \in \partial K$,

$$\langle \nabla d_K(\bar{x}), f_0(t,\bar{x}) + g(t,\bar{x})\bar{u}\rangle = \langle \nabla d_K(\bar{x}), f_0(t,\bar{x})\rangle - \|g(t,\bar{x})^* \nabla d_K(\bar{x})\|_Y < 0,$$

yielding (25).

4. **The case of Hilbert spaces.** Here we analyze the case when the state space X is Hilbert. In this setting, we show that if the state constraint is convex then the inward pointing condition can be drastically simplified by involving projections on convex sets instead of generalized gradients of the oriented distance function which do belong to the dual space X^*. This turns out to be very useful in the applications, as we will show in Section 5.

Let $\langle \cdot, \cdot \rangle_X$ be the scalar product in X and let K be a proper closed subset of X such that $K = \overline{\operatorname{Int} K}$. Denote by Z the set of points $z \in X \backslash \partial K$ admitting a unique projection $P_{\partial K}(z)$ on ∂K. This set is dense in X (see [26]). For every $z \in Z$, set

$$n_z = \frac{z - P_{\partial K}(z)}{\|z - P_{\partial K}(z)\|_X} \operatorname{sgn}(d_K(z))\,.$$

A new inward pointing condition involving n_z is proposed in this Hilbert framework in order to obtain results analogous to those from Section 3.

Theorem 4.1. *Assume* (12)–(15)*. Then,*

(i) the assertions of Theorem 3.1 are valid under the following inward pointing condition:

$$\forall\, \bar{x} \in \partial K,\ \exists\, \eta,\ \rho,\ M > 0 \text{ such that } \forall\, t \in I,\ \forall\, x \in K \cap B(\bar{x}, \eta), \tag{27}$$
$$\forall\, v \in \overline{co}\, F(t,x) \text{ satisfying } \sup_{\tau \le \eta,\, z \in Z \cap B(x,\eta)} \langle n_z,\, S(\tau)\, v\rangle_X \ge 0, \text{ we have}$$
$$\Big\{ \bar{v} \in \overline{co}\, F(t,x) \,:\, \|\bar{v} - v\|_X \le M\,, \sup_{\tau \le \eta,\, z \in Z \cap B(S(\tau)x, \eta)} \big\langle n_z, S(\tau)\,(\bar{v} - v)\big\rangle_X < -\rho \Big\} \neq \emptyset\,.$$

(ii) the assertions of Theorem 3.2 are valid under the following inward pointing condition:

$$\forall\, \bar{x} \in \partial K,\ \exists\, \eta,\ \rho,\ M > 0 \text{ such that } \forall\, t \in I,\ \forall\, x \in K \cap B(\bar{x}, \eta), \tag{28}$$
$$\forall\, v \in F(t,x) \text{ satisfying } \sup_{\tau \le \eta,\, z \in Z \cap B(x,\eta)} \langle n_z,\, S(\tau)\, v\rangle_X \ge 0, \text{ we have}$$
$$\Big\{ \bar{v} \in F(t,x) \,:\, \|\bar{v} - v\|_X \le M\,, \sup_{\tau \le \eta,\, z \in Z \cap B(S(\tau)x, \eta)} \big\langle n_z, S(\tau)\,(\bar{v} - v)\big\rangle_X < -\rho \Big\} \neq \emptyset\,.$$

Again, these conditions can be simplified when the data satisfy some compactness assumptions.

Proposition 4. *Assume* (14)–(15) *with time independent $k_R, \phi \in \mathbb{R}^+$ and* (19)*. Further suppose that either* (20) *is valid, or $F(t,\bar{x})$ is convex for any $t \in I$ and $\bar{x} \in \partial K$, and*

$$\forall\, \bar{x} \in \partial K,\ \exists\, r > 0 \text{ such that the set } \big\{ n_z : z \in Z \cap B(\bar{x}, r) \big\} \text{ is pre-compact.} \tag{29}$$

Then, the following assumption: for any $\bar{x} \in \partial K$, there exists $\rho > 0$ such that

$$\text{for any } t \in I \text{ and } v \in F(t,\bar{x}) \text{ satisfying } \inf_{\varepsilon > 0} \sup_{z \in Z \cap B(\bar{x},\varepsilon)} \langle n_z,\, v\rangle_X \ge 0, \tag{30}$$
$$\text{there exists } \bar{v} \in F(t,\bar{x}) \text{ such that } \inf_{\varepsilon > 0} \sup_{z \in Z \cap B(\bar{x},\varepsilon)} \big\langle n_z, \bar{v} - v\big\rangle_X < -\rho$$

implies (28)*.*

Remark 2. The proof of Proposition 4 provided in Section 6 implies that it is still valid if (29) is replaced by the following less restrictive assumption:

for $\bar{x} \in \partial K$ define $\mathcal{N}(\bar{x}) :=\text{Limsup}_{z\to\bar{x},\, z\in Z}\{n_z\}$ (the Kuratowski upper limit) and assume that for all $\bar{x} \in \partial K$ the set $\mathcal{N}(\bar{x})$ is compact and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$n_z \in \mathcal{N}(\bar{x}) + \varepsilon B \quad \forall\, z \in Z \cap B(\bar{x}, \delta).$$

In particular, if ∂K is of class C^1, then the above holds true.

4.1. **Convex state constraints.** If K is convex, then the inward pointing conditions (27), (28), and (30) can be weakened by replacing Z with $K^C := X \setminus K$. Indeed, any $z \in K^C$ admits a unique projection on ∂K and, as proved in the following proposition, for any $z \in \text{Int}\, K \cap Z$ we can find an element $w \in K^C$ such that $n_z = n_w$.

Proposition 5. *Let K be a closed convex set such that $K = \overline{\text{Int}\, K}$. Then, for any $z \in \text{Int}\, K \cap Z$, there exists $w \in K^C$ such that $z - P_{\partial K} z = P_{\partial K} w - w$. In particular, $n_z = n_w$.*

Proof. Let $z \in \text{Int}\, K \cap Z$ and $P_{\partial K}(z)$ be its unique projection on ∂K. By the Hahn-Banach theorem, there exists $p \in X$ such that $\|p\|_X = 1$ and

$$\langle p, P_{\partial K}(z)\rangle_X \le \langle p, k\rangle_X, \qquad \text{for any } k \in K.$$

Let

$$\mathcal{M}^+ = \big\{x \in X : \langle p, x - P_{\partial K}(z)\rangle_X \ge 0\big\} \supseteq K$$

and

$$\mathcal{M} = \partial\mathcal{M}^+ = \big\{x \in X : \langle p, x - P_{\partial K}(z)\rangle_X = 0\big\}.$$

Then $\mathcal{M}$ is a closed hyperplane in X and there exists a unique projection $P_{\mathcal{M}}(z)$ of z on $\mathcal{M}$. Actually, since $P_{\partial K}(z) \in \mathcal{M}$,

$$\|z - P_{\mathcal{M}}(z)\|_X \le \|z - P_{\partial K}(z)\|_X,$$

and, since $K \subset \mathcal{M}^+$ and z lies in the interior of K,

$$\|z - P_{\mathcal{M}}(z)\|_X \ge \|z - P_{\partial K}(z)\|_X,$$

we deduce that $P_{\mathcal{M}}(z) = P_{\partial K}(z)$. Take $w = z + 2(P_{\partial K}(z) - z)$. As $z \in \text{Int}\, K \subset \text{Int}\, \mathcal{M}^+$, we have

$$\langle p, w - P_{\partial K}(z)\rangle_X = \langle p, P_{\partial K}(z) - z\rangle_X < 0,$$

yielding $w \in X \setminus \mathcal{M}^+ \subset K^C$. Further, for any $x \in \mathcal{M}$,

$$0 = \langle z - P_{\mathcal{M}}(z), x - P_{\mathcal{M}}(z)\rangle = \langle z - P_{\partial K}(z), x - P_{\partial K}(z)\rangle = \langle P_{\partial K}(z) - w, x - P_{\partial K}(z)\rangle.$$

This implies that $P_{\partial K}(z) = P_{\mathcal{M}}(w)$. Finally, since $\mathcal{M}^+$ is a closed convex set, w admits a unique projection $P_{\mathcal{M}^+}(w) = P_{\mathcal{M}}(w) = P_{\partial K}(z)$. So, for any $k \in K \subset \mathcal{M}^+$,

$$\langle w - P_{\partial K}(z), k - P_{\partial K}(z)\rangle_X \le 0,$$

implying $P_{\partial K}(z) = P_{\partial K}(w)$. This ends the proof. □

5. **Examples.** The examples analyzed in this section describe some classical models involving partial differential equations and integrodifferential equations, to which we may apply our abstract results. In all the examples the state constraints satisfy the positive invariance property (12).

5.1. **A one-dimensional heat equation.** The first example is a one-dimensional parabolic equation describing the heat flux in a cylindrical bar, whose lateral surface is perfectly insulated and whose length is much larger than its cross-section. The Neumann boundary conditions are assumed, corresponding to the requirement that the heat flux at the two ends of the bar is zero. For $x = x(t,s) : [0,1] \times [0,1] \to \mathbb{R}$ we consider the following inclusion (we omit the variable s in the sequel)

$$\begin{cases} \dot{x}(t) \in \mathbb{A}x(t) + F(t, x(t)), & t \in [0,1] \\ x(0) = x_0. \end{cases}$$

The state space is $X = H^1(0,1)$ and the linear operator acting as $\mathbb{A}x = x'' - x$ with domain $D(\mathbb{A}) = \{x \in H^2([0,1],\mathbb{R}) : x'(0) = x'(1) = 0\}$ is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$, see e.g. [29, chapter II]. (The notation prime stands for the distributional derivative.) Classical results in PDEs ensure that, if the initial datum x_0 takes nonnegative values, then the solution x to $\dot{x}(t) = \mathbb{A}x(t)$, $x(0) = x_0$ takes nonnegative values. The reader is referred to [1] or [27], containing a number of examples of sets invariant under the action of the semigroup associated with $\mathbb{A}$. In particular, if the state constraint is the cone of nonnegative functions:

$$x(t) \in K = \{x \in X : x \geq 0\},$$

then the invariance property (12) is satisfied. Moreover, K is convex and $\operatorname{Int} K \neq \emptyset$. The state space is endowed with the scalar product

$$\langle x, y\rangle_X = x(0)y(0) + \langle x', y'\rangle_{L^2(0,1)}, \qquad \text{for any } x, y \in X,$$

whose associated norm

$$\|x\|_X^2 = |x(0)|^2 + \|x'\|^2_{L^2(0,1)} \qquad \text{for any } x \in X,$$

turns out to be equivalent to the usual one $\|\cdot\|_{H^1(0,1)}$. We show next that the set

$$\{n_z : z \in K^C\} \text{ is pre-compact.} \tag{31}$$

Since K is a closed convex cone, then any $z \in K^C$ can be uniquely represented as

$$z = P_{\partial K}(z) + b(z),$$

with $b(z) \in K^-$, here K^- is the negative polar cone to K. By [32],

$$K^- = \{p \in X : p' \text{ is nondecreasing and } p(0) \leq p'(s) \leq 0, \text{ for a.e. } s \in [0,1]\},$$

see also [24] where an explicit formula for b is provided. To prove (31), notice that

$$\left\{n_z = \frac{b(z)}{\|b(z)\|_X} : z \in K^C\right\} \subset Q := \left\{\frac{p}{\|p\|_X} : p \in K^-, p \neq 0\right\} \subset \partial B.$$

Any $y \in Q$ satisfies y' nondecreasing and

$$-1 \leq y(0) \leq y'(s) \leq 0, \qquad \text{for a.e. } s \in [0,1].$$

So, taking a sequence $\{y_n\}$ in Q,

$$\|y_n\|_{W^{1,\infty}(0,1)} := \|y_n\|_{L^\infty(0,1)} + \|y_n'\|_{L^\infty(0,1)} \leq 3,$$

implying that $y_n(0) \to y(0)$ (up to a subsequence). Since y_n' is nondecreasing, Helly's selection theorem, see [20], allows to deduce that, (again up to a subsequence),

$$y_n'(s) \to g(s), \qquad \text{for a.e. } s, \text{ with } g \in L^\infty(0,1),$$

and, applying Lebesgue dominated theorem, we deduce that $y_n \to g$ in $L^2(0,1)$. Further,

$$y_n(s) = y_n(0) + \int_0^s y_n'(\tau)\mathrm{d}\tau \to y(s) := y(0) + \int_0^s g(\tau)\mathrm{d}\tau, \qquad \text{for any } s \in [0,1].$$

Hence $g = y'$, $y \in W^{1,\infty}(0,1) \subset H^1(0,1)$, yielding the required pre-compactness.

Let F satisfy assumptions (14)–(15) with time independent $k_R, \phi \in \mathbb{R}^+$, (19), and let $F(t,\bar{x})$ be convex, for any $\bar{x} \in \partial K$ and $t \in I$. Taking into account Proposition 4 and the results in subsection 4.1, the inward pointing condition (28) is implied by the following assumption: for any $\bar{x} \in \partial K$ there exists $\rho > 0$ such that

$$\text{for any } t \in I \text{ and } v \in F(t,\bar{x}) \text{ satisfying } \inf_{\varepsilon>0} \sup_{z \in K^C \cap B(\bar{x},\varepsilon)} \langle n_z, v \rangle_X \geq 0,$$

$$\text{there exists } \bar{v} \in F(t,\bar{x}) \text{ such that } \inf_{\varepsilon>0} \sup_{z \in K^C \cap B(\bar{x},\varepsilon)} \big\langle n_z, \bar{v} - v \big\rangle_X < -\rho\,.$$

5.2. **Fourier's problem of the ring.** In the second example we consider the temperature distribution in a homogeneous isotropic circular ring with diameter small in comparison with its length and perfectly insulated lateral surfaces. This problem can be modeled by a one-dimensional equation with periodic boundary conditions

$$\begin{cases} \dot{x}(t) \in \mathbb{A}x(t) + F(t,x(t)), & t \in [0,1] \\ x(0) = x_0, \end{cases} \tag{32}$$

where $x = x(t,s) : [0,1] \times [0,1] \to \mathbb{R}$ (s is omitted as in the previous example), the state space is $X = H^1_{per}(0,1) := \big\{x \in H^1(0,1;\mathbb{R}) : x(0) = x(1)\big\}$, the linear operator acting as $\mathbb{A}x = x''$ with domain $D(\mathbb{A}) = H^2(0,1;\mathbb{R}) \cap H^1_{per}(0,1)$ is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$, see e.g. [9]. As before we supplement inclusion (32) with the state constraint

$$x(t) \in K = \big\{x \in X : x \geq 0\big\}.$$

Then, K satisfies condition (12), see for instance [23] dealing with invariant sets for semigroups. Again, K is a closed and convex set with non empty interior. Hence, by the results contained in subsection 4.1, the inward pointing conditions (27), (28), and (30) can be stated with Z replaced by K^C.

5.3. **A model for Boltzmann viscoelasticity.** The last example deals with the phenomena of isothermal viscoelasticity. An integrodifferential inclusion is involved, since, as outlined in the seminal works of Boltzmann and Volterra [7, 8, 30, 31], a correct description of the mechanical behavior of elastic bodies requires the notion of memory. The key assumption in this theory is that both the instantaneous stress and the past stresses influence the evolution of the displacement function $y = y(\boldsymbol{x},t) : \Omega \times \mathbb{R} \to \mathbb{R}$. Here $\Omega \subset \mathbb{R}^3$, a bounded domain with smooth boundary $\partial\Omega$, represents the region occupied by the elastic body. Omitting the variable $\boldsymbol{x}$ in the sequel, we study the following inclusion

$$\ddot{y}(t) + A\Big[y(t) - \int_0^\infty \mu(s)y(t-s)\,\mathrm{d}s\Big] \in \mathcal{F}(t,y(t)), \qquad t > 0, \tag{33}$$

where, $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, according to the assumption that the body is kept fixed at the boundary of Ω, the *memory kernel* μ, taking into account the viscoelastic behavior, is supposed to be a (nonnegative) nonincreasing and summable function on $\mathbb{R}^+$, with total mass

$$\kappa = \int_0^\infty \mu(s)\mathrm{d}s \in (0,1),$$

piecewise absolutely continuous, and thus differentiable almost everywhere with $\mu' \leq 0$. Equation (33) is supplemented with the following initial condition

$$y(0) = y_0, \qquad \dot{y}(0) = z_0, \qquad y(-s)_{|s>0} = \phi_0(s),$$

for some prescribed data y_0, z_0, ϕ_0, the latter taking into account the past history of y. Applying Dafermos' *history approach*, see [13], we can write (33) as a differential inclusion of type (1). To this aim, we first introduce an auxiliary variable which contains all the information about the unknown function up to the actual time

$$\eta^t(s) = y(t) - y(t-s), \qquad t \geq 0,\ s > 0$$

and we recast problem (33) as the system of two variables $y = y(t)$ and $\eta = \eta^t(s)$

$$\begin{cases} \ddot{y}(t) + A\Big[(1-\kappa)y(t) + \displaystyle\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big] \in \mathcal{F}(t, y(t)), \\ \dot{\eta}^t = T\eta^t + \dot{y}(t). \end{cases} \tag{34}$$

with initial conditions

$$y(0) = y_0, \qquad \dot{y}(0) = z_0, \qquad \eta^0 = \eta_0 = y_0 - \phi_0.$$

Here the operator T is the infinitesimal generator of the right-translation semigroup on the *memory space* $\mathcal{M} = L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$, namely,

$$T\eta = -\eta' \qquad \text{with domain} \qquad \mathrm{dom}(T) = \big\{\eta \in \mathcal{M} : \eta' \in \mathcal{M},\ \eta(0) = 0\big\}.$$

The notation prime standing for the distributional derivative, and $\eta(0) = \lim_{s\to 0} \eta(s)$ in $H^1_0(\Omega)$. In [11], details and related bibliography can be found, jointly with some applications of this model to optimal control problems. Now, defining the linear operator $\mathbb{A}$ on the state space $X = L^2(\Omega) \times H^1_0(\Omega) \times \mathcal{M}$, acting as

$$\mathbb{A}(y, z, \eta) = \Big(z, -A\big[(1-\kappa)y + \int_0^\infty \mu(s)\eta(s)\,\mathrm{d}s\big], T\eta + z\Big)$$

with domain

$$\mathrm{dom}(\mathbb{A}) = \left\{ (y,z,\eta) \in X \;\middle|\; \begin{array}{l} z \in H^1_0(\Omega)\,,\ \eta \in \mathrm{dom}(T), \\ (1-\kappa)y + \int_0^\infty \mu(s)\eta(s)\mathrm{d}s \in H^2(\Omega) \cap H^1_0(\Omega) \end{array} \right\}$$

and setting

$$x(t) = (y(t), z(t), \eta^t), \qquad x_0 = (y_0, z_0, \eta_0), \qquad F(t, x(t)) = \big(0, \mathcal{F}(t, y(t)), 0\big),$$

we view (34) as the following problem in X:

$$\begin{cases} \dot{x}(t) \in \mathbb{A}x(t) + F(t, x(t)), & \text{for } t \in I \\ x(0) = x_0. \end{cases} \tag{35}$$

The operator $\mathbb{A}$ generates a strongly continuous semigroup of contractions $S(t) : X \to X$ whose first component is a solution of (33) for $\mathcal{F} = 0$. Here, taking as a state constraint in (35)

$$K = B,$$

we deduce that (12) is satisfied. Further, let U, f_0 and g be as in Remark 1. Since $K = B$ in a Hilbert space, an appropriate adaptation on the basis of Section 4 of the inward pointing condition (25) reads: for any $\bar{x} \in X$ with $\|\bar{x}\|_X = 1$ and any $t \in I$, there exists $\bar{u} \in U$ such that

$$\langle \bar{x}, f_0(t,\bar{x}) + g(t,\bar{x})\bar{u}\rangle < 0.$$

In particular in the case when U is the unit sphere in $\mathbb{R}^N$, the condition

$$\langle \bar{x}, f_0(t,\bar{x})\rangle < \|g(t,\bar{x})^*\bar{x}\|_{\mathbb{R}^N} \neq 0,$$

implies the above inward pointing condition, for $\bar{u}$ defined by $\bar{u} = -\frac{g(t,\bar{x})^*\bar{x}}{\|g(t,\bar{x})^*\bar{x}\|_{\mathbb{R}^N}}$. As discussed in Remark 1, Proposition 3 holds in this case implying the validity of Theorems 3.1 and 3.2.

6. **Proofs.** We start by proving the main theorems contained in Section 3. To this aim we need some preliminary results.

Lemma 6.1. *Assume* (18). *Then, for any compact set $D \subset K$ with $D \cap \partial K \neq \emptyset$, there exist η', ρ, $M > 0$ such that*

$$\text{if } \sigma(z_0; S(\tau)v) \geq 0 \text{ for some } \tau \in [0,\eta'],\ v \in F(t,x),\ z_0 \in B(x,\eta'),\ t \in I \quad (36)$$
$$\text{and } x \in K \cap (D + \eta' B) \text{ satisfying } d_K(x) > -\eta', \text{ then}$$
$$\left\{\bar{v} \in F(t,x)\ :\ \|\bar{v} - v\|_X \leq M\,,\ \sup_{z \in B(S(\tau)x,\eta'),\, \tau \leq \eta'} \sigma(z; S(\tau)\,(\bar{v} - v)) < -\rho\right\} \neq \emptyset\,.$$

Proof. Fix a compact set $D \subset K$ with $D \cap \partial K \neq \emptyset$. We prove first that there exist η, ρ, $M > 0$ such that

$$\text{if } \sigma(z_0; S(\tau)v) \geq 0 \text{ for some } \tau \in [0,\eta],\ v \in F(t,x),\ z_0 \in B(x,\eta), \quad (37)$$
$$t \in I,\ x \in K \cap (D \cap \partial K + \eta B), \text{ then}$$
$$\left\{\bar{v} \in F(t,x)\ :\ \|\bar{v} - v\|_X \leq M\,,\ \sup_{z \in B(S(\tau)x,\eta),\, \tau \leq \eta} \sigma(z; S(\tau)\,(\bar{v} - v)) < -\rho\right\} \neq \emptyset\,.$$

From the compactness of $D \cap \partial K$ there exists a finite number of $x_k \in D \cap \partial K$, for $k = 1, \dots, N$, such that

$$D \cap \partial K \subset \bigcup_{k=1}^{N} \operatorname{Int} B(x_k, \eta_k)\,, \quad (38)$$

with $\eta_k > 0$ as in (18) corresponding to x_k. Now, define

$$\Lambda = \inf\left\{\|x - y\|_X : x \in D \cap \partial K, y \in \left(K \setminus \bigcup_{k=1}^{N} \operatorname{Int} B(x_k,\eta_k)\right)\right\}.$$

We have that $\Lambda > 0$. Indeed, if $\Lambda = 0$ we can find two sequences $\{x_i\} \subset D \cap \partial K$ and $\{y_i\} \subset K \setminus \bigcup_{k=1}^{N} \operatorname{Int} B(x_k,\eta_k)$ such that $\|x_i - y_i\|_X \to 0$. By the compactness of $D \cap \partial K$, taking a subsequence and keeping the same notation we deduce the existence of $\bar{x} \in D \cap \partial K$ such that $x_i \to \bar{x}$ in X, implying that also

$$y_i \to \bar{x} \in K \setminus \bigcup_{k=1}^{N} \operatorname{Int} B(x_k,\eta_k) \qquad \text{in X},$$

in contradiction with (38). Finally, let

$$0 < \eta < \min\left\{\frac{\Lambda}{2}\,, \eta_1\,, \dots\,, \eta_N\right\}$$

and define

$$\rho = \min\{\rho_1, \dots, \rho_N\}, \quad M = \max\{M_1, \dots, M_N\},$$

with ρ_k, M_k as in (18) associated to x_k, for $k = 1, \dots, N$. Then for any

$$x \in K \cap \big((D \cap \partial K) + \eta B\big) \subset \bigcup_{k=1}^{N} \operatorname{Int} B(x_k, \eta_k)$$

condition (37) holds true. Now, taking $0 < \eta' < \frac{\eta}{2}$ such that

$$\sup \left\{ d_K(x) \,:\, x \in D \backslash \big((D \cap \partial K) + \frac{\eta}{2} B\big) \right\} < -2\eta',$$

we obtain (36). □

Lemma 6.2. *Under the assumptions of Theorem 3.2, for every compact set $D \subset K$, there exist $\eta', \delta > 0$ such that, for any $\varepsilon' > 0$, $\bar t \in [t_0, 1]$, and any solution y to (1), (3) with $y(t) \in D + \eta' B$ for any $t \in [\bar t, 1]$, we can find a solution $x_{\varepsilon'}$ to (1) satisfying*

$$x_{\varepsilon'}(\bar t) = y(\bar t), \qquad x_{\varepsilon'}(t) \in \operatorname{Int} K, \quad \text{for any } t \in (\bar t, (\bar t + 2\delta) \wedge 1],$$
$$\|x_{\varepsilon'} - y\|_{L^\infty([\bar t, 1], X)} \le \varepsilon'.$$

Proof. Fix a compact set $D \subset K$. We may suppose that $D \cap \partial K \neq \emptyset$, because otherwise the Lemma is trivial. Hence, Lemma 6.1 implies (36). Let $\eta', \rho, M > 0$ be as in (36), $R > 0$ be such that $D + \eta' B \subset \frac{R}{4} B$. If k_R is as in (14), then it is not restrictive to assume that $\|k_R\|_{L^1} > 0$. Let

$$C = M_S M \Big(M_S \|k_R\|_{L^1} \mathrm{e}^{M_S \|k_R\|_{L^1}} + 1 \Big), \tag{39}$$

and $\delta' > 0$ be such that for any Lebesgue measurable $E \subset I$,

$$\int_E k_R(s) \mathrm{d}s < \frac{\rho \, \|k_R\|_{L^1}}{\rho + 4C}, \quad \int_E \phi(s) \mathrm{d}s < \frac{\eta'}{2 M_S (1 + R)} \qquad \text{whenever } \mu(E) \le \delta', \tag{40}$$

where M_S, ϕ and C are as in (10), (15), (39). Define

$$\delta = \frac{1}{2} \min \left\{ \eta', \delta' \right\} \tag{41}$$

and pick any

$$0 < \varepsilon' < \min \left\{ \frac{R}{2}, \frac{\eta'}{2} \right\}. \tag{42}$$

Let y be a solution to (1), (3) such that $y(t) \in D + \eta' B$ for any $t \in [\bar t, 1]$. Set

$$\Gamma = \Big\{ s \in [\bar t, 1] : d_K(y(s)) > -\eta' \ \text{ and } \ \exists\, z_0 \in B(y(s), \eta'),\ \max_{\tau \le \eta'} \sigma(z_0; S(\tau)\, f^y(s)) \ge 0 \Big\}$$

and

$$T = \begin{cases} (\bar t + 2\delta) \wedge 1, & \text{if } \mu\left(\Gamma \cap [\bar t, (\bar t + 2\delta) \wedge 1]\right) < \frac{\varepsilon'}{2C} \\ \min \Big\{ s > \bar t \,:\, \mu\left(\Gamma \cap [\bar t, s]\right) = \frac{\varepsilon'}{2C} \Big\}, & \text{otherwise.} \end{cases} \tag{43}$$

Lemma 6.1 and the measurable selection theorem ensure the existence of a measurable selection $\bar v(s) \in F(s, y(s))$ such that for any $s \in \Gamma$

$$\|\bar v(s) - f^y(s)\|_X \le M \tag{44}$$

and for any $\tau \le \eta'$, $z \in B(S(\tau)\, y(s), \eta')$, $\xi \in \partial d_K(z)$,

$$\big\langle \xi, S(\tau) \left(\bar v(s) - f^y(s) \right) \big\rangle < -\rho. \tag{45}$$

Then, we define

$$f_{\varepsilon'}(s) = \begin{cases} \bar{v}(s), & \text{if } s \in \Gamma \cap [\bar{t}, T] \\ f^y(s), & \text{otherwise} \end{cases}$$

and set

$$\begin{aligned} y_{\varepsilon'}(t) &= S(t-\bar{t})\, y(\bar{t}) + \int_{\bar{t}}^{t} S(t-s)\, f_{\varepsilon'}(s)\mathrm{d}s \\ &= S(t-\bar{t})\, y(\bar{t}) + \int_{\Gamma\cap[\bar{t}, t\wedge T]} S(t-s)\, \bar{v}(s)\mathrm{d}s + \int_{[\bar{t},t]\setminus(\Gamma\cap[\bar{t},T])} S(t-s)\, f^y(s)\mathrm{d}s. \end{aligned}$$

By the representation formula (8), estimates (10), (42), (43) and (44), we get for any $t \in [\bar{t}, 1]$

$$\begin{aligned} \|y_{\varepsilon'}(t) - y(t)\|_X &= \Big\| \int_{\bar{t}}^{t} S(t-s)\big[f_{\varepsilon'}(s) - f^y(s)\big]\,\mathrm{d}s \Big\|_X \\ &\le M_S \int_{\bar{t}}^{t} \|f_{\varepsilon'}(s) - f^y(s)\|_X\,\mathrm{d}s = M_S \int_{\Gamma\cap[\bar{t},t\wedge T]} \|\bar{v}(s) - f^y(s)\|_X\,\mathrm{d}s \\ &\le M_S M\,\mu\,(\Gamma \cap [\bar{t}, t\wedge T]) \le M_S M \frac{\varepsilon'}{2C} < \frac{R}{4}, \end{aligned} \tag{46}$$

implying that $y_{\varepsilon'}(t) \in \frac{R}{2}B$. Further, from (14), for a.e. $t \in [\bar{t}, 1]$,

$$f_{\varepsilon'}(t) \in F(t, y(t)) \subset F(t, y_{\varepsilon'}(t)) + k_R(t)\big\|y_{\varepsilon'}(t) - y(t)\big\|_X B.$$

Hence, setting

$$\gamma(t) = \operatorname{dist}\big(f_{\varepsilon'}(t), F(t, y_{\varepsilon'}(t))\big) \qquad \text{and} \qquad m(t) = M_S \mathrm{e}^{M_S \int_{\bar{t}}^{t} k_R(s)\mathrm{d}s},$$

as in Lemma A.1 in the Appendix, from (46) we get

$$\gamma(t) \le k_R(t)\|y_{\varepsilon'}(t) - y(t)\|_X \le k_R(t) M_S M\,\mu\,(\Gamma \cap [\bar{t}, t\wedge T]).$$

This implies in particular that

$$m(1)\int_{\bar{t}}^{1} \gamma(s)\mathrm{d}s \le M_S \mathrm{e}^{M_S\|k_R\|_{L^1}} M_S M\,\mu\,(\Gamma\cap[\bar{t},T])\|k_R\|_{L^1} \le C\frac{\varepsilon'}{2C} < \frac{R}{4}. \tag{47}$$

So, we can apply Lemma A.1 and deduce that, for any $\beta > 1$ there exists a solution $x_{\varepsilon'}$ on $[\bar{t}, 1]$ of the differential inclusion

$$\begin{cases} x'(t) \in \mathbb{A}x(t) + F(t, x(t)) \\ x(\bar{t}) = y(\bar{t}) \end{cases}$$

satisfying the estimates

$$\big\|x_{\varepsilon'}(t) - y_{\varepsilon'}(t)\big\|_X \le \beta M_S \mathrm{e}^{M_S \int_{\bar{t}}^{t} k_R(s)\mathrm{d}s} \int_{\bar{t}}^{t} \gamma(s)\mathrm{d}s$$

and

$$\big\|f^{x_{\varepsilon'}}(t) - f_{\varepsilon'}(t)\big\|_X \le k_R(t)\beta M_S \mathrm{e}^{M_S \int_{\bar{t}}^{t} k_R(s)\mathrm{d}s} \int_{\bar{t}}^{t} \gamma(s)\mathrm{d}s + \beta\gamma(t).$$

In particular, applying (46), arguing as in (47), and taking $\beta = 2$, we get, for any $t \in [\bar{t}, 1]$,

$$\begin{aligned} \|x_{\varepsilon'}(t) - y(t)\|_X &\le \|x_{\varepsilon'}(t) - y_{\varepsilon'}(t)\|_X + \|y_{\varepsilon'}(t) - y(t)\|_X \\ &\le 2M_S \mathrm{e}^{M_S\|k_R\|_{L^1}} M_S M \|k_R\|_{L^1} \frac{\varepsilon'}{2C} + M_S M \frac{\varepsilon'}{2C} \\ &\le 2C\frac{\varepsilon'}{2C} = \varepsilon'. \end{aligned} \tag{48}$$

Finally, by the definition of δ and (40), we recover for $t \in [\bar{t}, (\bar{t}+2\delta)\wedge 1]$,

$$\|x_{\varepsilon'}(t) - y_{\varepsilon'}(t)\|_X \le 2M_S e^{M_S\|k_R\|_{L^1}} M_S M\, \mu\,(\Gamma\cap[\bar{t}, t\wedge T]) \int_{\bar{t}}^{t} k_R(s)\mathrm{d}s \qquad (49)$$
$$\le \frac{2C}{\|k_R\|_{L^1}} \mu\,(\Gamma\cap[\bar{t}, t\wedge T]) \int_{\bar{t}}^{t} k_R(s)\,\mathrm{d}s \le \frac{\rho}{2}\,\mu\,(\Gamma\cap[\bar{t}, t\wedge T]),$$

and

$$\left\|f^{x_{\varepsilon'}}(t) - f_{\varepsilon'}(t)\right\|_X \le k_R(t)\frac{\rho}{2}\,\mu\,(\Gamma\cap[\bar{t}, t\wedge T]) + 2k_R(t)M_S M\,\mu\,(\Gamma\cap[\bar{t}, t\wedge T])$$
$$= k_R(t)\,\mu\,(\Gamma\cap[\bar{t}, t\wedge T])\Big(\frac{\rho}{2} + 2M_S M\Big).$$

Now, fix $t \in (\bar{t}, (\bar{t}+2\delta)\wedge 1]$. We claim that $x_{\varepsilon'}(t) \in \operatorname{Int} K$.

Let us first assume that $\mu\,(\Gamma\cap[\bar{t}, t\wedge T]) = 0$. Observe that in this case we cannot have $t > T$, otherwise $\mu\,(\Gamma\cap[\bar{t}, T]) = 0$ and, by the definition of T, $T = (\bar{t}+2\delta)\wedge 1$. The last equality is impossible, since $t \le (\bar{t}+2\delta)\wedge 1$. Therefore $\mu\,(\Gamma\cap[\bar{t}, t]) = 0$. Thus, by the definition of Γ, for almost every $s \in [\bar{t}, t]$ we have that either $d_K(y(s)) \le -\eta'$ or $d_K(y(s)) > -\eta'$ and

$$\sup_{z\in B(y(s),\eta'),\tau\le\eta'} \sigma(z; S(\tau)\, f^y(s)) < 0\,. \qquad (50)$$

Suppose that $d_K(y(t)) \ge -\frac{\eta'}{2}$, otherwise from (42) and (48), $x_{\varepsilon'}(t) \in \operatorname{Int} K$. By continuity, there exists $\bar{s} \in [\bar{t}, t)$ such that

$$\|y(t) - S(t-\bar{s})\, y(\bar{s})\|_X < \frac{\eta'}{2}\,, \qquad (51)$$

and, for any $s \in [\bar{s}, t]$,

$$\|y(t) - y(s)\|_X < \frac{\eta'}{2} \qquad \text{and} \qquad d_K(y(s)) > -\eta'\,. \qquad (52)$$

Then we have (50). By the mean value theorem (see [12]) there exist $z \in [y(t), S(t-\bar{s})\, y(\bar{s})]$ and $\xi \in \partial d_K(z)$ such that

$$d_K(y(t)) = d_K(S(t-\bar{s})\, y(\bar{s})) + \big\langle \xi, y(t) - S(t-\bar{s})\, y(\bar{s})\big\rangle\,.$$

Since, by (51)–(52), $z \in B(y(s), \eta')$, for any $s \in [\bar{s}, t]$, applying the representation formula (8), the invariance assumption (12), and (50) we obtain

$$d_K(y(t)) \le \int_{\bar{s}}^{t} \big\langle \xi, S(t-s)\, f^y(s)\big\rangle\,\mathrm{d}s < 0\,.$$

As $\mu\,(\Gamma\cap[\bar{t}, t]) = 0$ implies $x_{\varepsilon'} \equiv y$ on $[\bar{t}, t]$, we have proved that $x_{\varepsilon'}(t) \in \operatorname{Int} K$.

Now, we consider the case $\mu\,(\Gamma\cap[\bar{t}, t\wedge T]) > 0$. Applying again the mean value theorem, we get for some $z \in [y(t), x_{\varepsilon'}(t)]$ and $\xi \in \partial d_K(z)$,

$$d_K(x_{\varepsilon'}(t)) = d_K(y(t)) + \langle \xi, x_{\varepsilon'}(t) - y(t)\rangle\,. \qquad (53)$$

It follows that $z \in S(t-s)\, y(s) + \eta' B$, for any $s \in [\bar{t}, t]$. Indeed, recalling (8), (40), (42), and (48), we obtain

$$\|z - S(t-s)y(s)\|_X \le \|z - y(t)\|_X + \left\| \int_s^t S(t-r)\, f^y(r)\,\mathrm{d}r \right\|_X$$
$$\le \|x_{\varepsilon'}(t) - y(t)\|_X + M_S(1+R)\int_s^t \phi(r)\,\mathrm{d}r < \varepsilon' + \frac{\eta'}{2} < \eta'\,.$$

Hence, from (41), (45), (49) and (53) we obtain

$$\begin{aligned} d_K(x_{\varepsilon'}(t)) &\leq \langle \xi, x_{\varepsilon'}(t) - y(t)\rangle = \langle \xi, x_{\varepsilon'}(t) - y_{\varepsilon'}(t)\rangle + \langle \xi, y_{\varepsilon'}(t) - y(t)\rangle \\ &\leq \|x_{\varepsilon'}(t) - y_{\varepsilon'}(t)\|_X + \int_{\Gamma\cap[\bar{t}, t\wedge T]} \big\langle \xi, S(t-s)\left(\bar{v}(s) - f^y(s)\right)\big\rangle \,\mathrm{d}s \\ &< \frac{\rho}{2}\,\mu\left(\Gamma\cap[\bar{t}, t\wedge T]\right) - \rho\,\mu\left(\Gamma\cap[\bar{t}, t\wedge T]\right) = -\frac{\rho}{2}\,\mu\left(\Gamma\cap[\bar{t}, t\wedge T]\right) < 0\,. \end{aligned}$$

This completes the proof. □

Proof of Theorem 3.2. Fix $t_0 \in I$, $\varepsilon > 0$ and a solution $\hat{x}$ of (1), (3). It is enough to consider the case when $\hat{x}([t_0, 1]) \cap \partial K \neq \emptyset$. By the Gronwall Lemma and assumption (15), there exists $R > 0$ such that every trajectory x of (1), with $x(t_0) = \hat{x}(t_0)$, satisfies $\|x(t)\|_X \leq R$ for all $t \geq t_0$. Then, let $D = \big\{\hat{x}(t) : t \in [t_0, 1]\big\}$, η' be as in Lemma 6.1, and take

$$\tilde{\varepsilon} = \min\big\{\varepsilon, \eta', 1\big\}.$$

Observe that the same η' can be used in the claim of Lemma 6.2 (see the proof of Lemma 6.2). One of the key point of the proof is a Filippov type theorem, see Lemma A.1 in the Appendix. Set

$$C = \max\Big\{1, M_S \mathrm{e}^{M_S\|k_{2R}\|_{L^1}}\Big\}\,.$$

The claimed trajectory x is obtained by a backward iteration. Let δ be as in Lemma 6.2, and set

$$N = \max\big\{n \in \mathbb{N} : t_0 + n\delta < 1\big\}, \qquad s_n = t_0 + n\delta, \quad \text{ if } 0 \leq n \leq N, \qquad s_{N+1} = 1.$$

We show that, for any $0 \leq n \leq N$, there exists a solution y_n to (1) satisfying

$$y_n(s_n) = \hat{x}(s_n), \qquad y_n(t) \in \operatorname{Int} K, \quad \text{ for any } t \in (s_n, 1], \tag{54}$$

$$\|y_n - \hat{x}\|_{L^\infty([s_n,1],X)} \leq \frac{\tilde{\varepsilon}}{2C^N}\sum_{k=n}^{N}\frac{C^k}{2^k}. \tag{55}$$

In particular, $y_0(t_0) = \hat{x}(t_0)$,

$$\|y_0 - \hat{x}\|_{L^\infty([t_0,1],X)} \leq \frac{\tilde{\varepsilon}}{2C^N}\sum_{k=0}^{N}\frac{C^k}{2^k} \leq \frac{\tilde{\varepsilon}}{2}\sum_{k=0}^{\infty}\frac{1}{2^k} = \tilde{\varepsilon}$$

implying that the function $x = y_0$ satisfies the requirements of our theorem.

Let $n = N$. Lemma 6.2 ensures the existence of a trajectory y_N solving (1) and satisfying (54), (55) for $n = N$. Now, we assume that $n < N$ and that for any $n + 1 \leq i \leq N$, there exists a solution y_i to (1) on $[s_i, 1]$ such that (54) and (55) hold. Using y_{n+1} and applying Lemmas 6.2 and A.1, we construct the desired y_n. We begin by defining y_n on the interval $[s_{n+1}, 1]$. Let $t_N = s_N + \frac{1-s_N}{2}$ and, for $n < N - 1$,

$$t_{n+1} = s_{n+1} + \frac{\delta}{2}, \qquad \text{and} \qquad \varrho = \frac{1}{2}\min\big\{-d_K(y_{n+1}(t)) : t \in [t_{n+1}, 1]\big\} > 0\,.$$

Define

$$\varepsilon_n = \min\left\{\frac{\varrho}{C}, \frac{\tilde{\varepsilon}}{2C^{N+1}}\frac{C^n}{2^n}, \frac{R}{2C}\right\}.$$

By Lemma A.1, for any $z_0 \in y_{n+1}(t_{n+1}) + \varepsilon_n B$, there exists a solution x_n to (1) on $[t_{n+1}, 1]$ such that $x_n(t_{n+1}) = z_0$ and

$$\|x_n(t) - y_{n+1}(t)\|_X \leq C\varepsilon_n, \qquad \text{for any } t \in [t_{n+1}, 1],$$

implying that $x_n(t) \in \operatorname{Int} K$, for any $t \in [t_{n+1}, 1]$, and

$$\|x_n - y_{n+1}\|_{L^\infty([t_{n+1},1],X)} \leq \frac{\tilde{\varepsilon}}{2C^N} \frac{C^n}{2^n}. \tag{56}$$

Define

$$z_n(t) = \begin{cases} \hat{x}(t) & \text{in } [s_n, s_{n+1}) \\ y_{n+1}(t) & \text{in } [s_{n+1}, t_{n+1}]. \end{cases}$$

Since z_n solves (1), (3) and, by the inductive assumption (55), $z_n(t) \in D + \eta' B$ for any $t \in [s_n, t_{n+1}]$, we can apply Lemma 6.2 and obtain that there exists a solution $\tilde{z}_n$ to (1) such that $\tilde{z}_n(s_n) = \hat{x}(s_n)$, $\tilde{z}_n(t) \in \operatorname{Int} K$, for any $t \in (s_n, t_{n+1}]$, and

$$\|z_n(t) - \tilde{z}_n(t)\|_X \leq \varepsilon_n, \qquad \text{for any } t \in [s_n, t_{n+1}]. \tag{57}$$

Taking the solution x_n obtained above from Lemma A.1 in the case $z_0 = \tilde{z}_n(t_{n+1})$, the function

$$y_n(t) = \begin{cases} \tilde{z}_n(t) & \text{in } [s_n, t_{n+1}) \\ x_n(t) & \text{in } [t_{n+1}, 1], \end{cases}$$

satisfies (54). Further, by the very definition of y_n, the induction hypothesis and the estimates (56), (57), we get

$$\|y_n(t) - \hat{x}(t)\|_X = \|\tilde{z}_n(t) - z_n(t)\|_X \leq \varepsilon_n \leq \frac{\tilde{\varepsilon}}{2C^N} \sum_{k=n}^{N} \frac{C^k}{2^k}, \qquad \text{in } [s_n, s_{n+1}],$$

$$\begin{aligned} \|y_n(t) - \hat{x}(t)\|_X &\leq \|y_n(t) - y_{n+1}(t)\|_X + \|y_{n+1}(t) - \hat{x}(t)\|_X \\ &= \|\tilde{z}_n(t) - z_n(t)\|_X + \|y_{n+1}(t) - \hat{x}(t)\|_X \\ &\leq \varepsilon_n + \frac{\tilde{\varepsilon}}{2C^N} \sum_{k=n+1}^{N} \frac{C^k}{2^k} \leq \frac{\tilde{\varepsilon}}{2C^N} \sum_{k=n}^{N} \frac{C^k}{2^k}, \qquad \text{in } [s_{n+1}, t_{n+1}], \end{aligned}$$

and

$$\begin{aligned} \|y_n(t) - \hat{x}(t)\|_X &\leq \|y_n(t) - y_{n+1}(t)\|_X + \|y_{n+1}(t) - \hat{x}(t)\|_X \\ &= \|x_n(t) - y_{n+1}(t)\|_X + \|y_{n+1}(t) - \hat{x}(t)\|_X \\ &\leq C\varepsilon_n + \frac{\tilde{\varepsilon}}{2C^N} \sum_{k=n+1}^{N} \frac{C^k}{2^k} \leq \frac{\tilde{\varepsilon}}{2C^N} \sum_{k=n}^{N} \frac{C^k}{2^k}, \qquad \text{in } [t_{n+1}, 1], \end{aligned}$$

ending the proof. □

Proof of Theorem 3.1. Let $\varepsilon, t_0, \hat{x}$ as in the statement of the theorem. Our proof follows some ideas from [6, Lemma 5.2], for a finite dimensional problem. We distinguish two cases:

$$\hat{x}(t_0) \in \operatorname{Int} K \qquad \text{and} \qquad \hat{x}(t_0) \in \partial K.$$

In the first case, Theorem 3.2 implies the existence of a solution y to (2) such that

$$y(t_0) = \hat{x}(t_0), \qquad y(t) \in \operatorname{Int} K, \text{ for any } t \geq t_0, \qquad \text{and} \qquad \|\hat{x} - y\|_{L^\infty([t_0,1],X)} \leq \frac{\varepsilon}{2}.$$

Further, taking $\varrho = \frac{1}{2} \min \{ -d_K(y(t)) : t \in [t_0, 1] \}$, we have

$$y(t) + \varrho B \subset \operatorname{Int} K, \qquad \text{for any } t \in [t_0, 1].$$

Applying the relaxation theorem, see [15, Theorem 2.1], we deduce that there exists a solution x to system (1) such that

$$x(t_0) = \hat{x}(t_0) \quad \text{and} \quad \|x - y\|_{L^\infty([t_0,1],X)} \leq \min \left\{ \varrho, \frac{\varepsilon}{2} \right\},$$

implying that

$$x(t) \in y(t) + \varrho B \subset \mathrm{Int}K, \qquad \text{for any } t \geq t_0,$$

and

$$\|\hat{x} - x\|_{L^\infty([t_0,1],X)} \leq \|\hat{x} - y\|_{L^\infty([t_0,1],X)} + \|y - x\|_{L^\infty([t_0,1],X)} \leq \varepsilon.$$

It remains to consider the case $\hat{x}(t_0) \in \partial K$. We construct a sequence of solutions y_i to (1) converging in suitable space to a function x satisfying the claim of the theorem. To perform our project we first apply again Theorem 3.2, in order to find a solution y to (2) such that

$$y(t_0) = \hat{x}(t_0), \qquad y(t) \in \mathrm{Int}K, \text{ for any } t > t_0, \qquad \text{and} \qquad \|\hat{x}-y\|_{L^\infty([t_0,1],X)} \leq \frac{\varepsilon}{4}.$$

Then, we take a monotone sequence $s_i \downarrow t_0$, $i = 1, 2, \dots$ and applying a relaxation theorem from [15], we obtain trajectories x_i solving (1) on $[s_i, 1]$ and satisfying

$$x_i(s_i) = y(s_i) \qquad \text{and} \qquad \|x_i - y\|_{L^\infty([s_i,1],X)} < \alpha_i, \tag{58}$$

for some $\alpha_i > 0$ to be determined below. Notice that, by the Gronwall Lemma, there exists $R > 0$ such that every trajectory x of (1), with $x(t_1) = y(t_1)$ for some $t_1 \geq t_0$, satisfies $\|x\|_{L^\infty([t_1,1],X)} \leq \frac{R}{2}$. Let, for any $i \geq 1$,

$$\varepsilon_i = \frac{1}{4} \min\Big\{ \min_{t\in[s_i,1]} \big(- d_K(y(t))\big) \,,\, \frac{\varepsilon}{2} \,,\, R\Big\} \text{ and } \alpha_i = \frac{1}{M_S \mathrm{e}^{M_S\|k_R\|_{L^1}} + 1} \frac{\varepsilon_i}{2^i} \leq \varepsilon_i,$$

so that

$$x_i(t) \in y(t) + \varepsilon_i B \subset \mathrm{Int}K, \qquad \text{for all } t \in [s_i, 1]. \tag{59}$$

Now, set $s_0 = 1$, $y_1 \equiv x_1$ on $[s_1, s_0]$, and, for any $i \geq 2$,

$$y_i(t) = \begin{cases} x_i(t) & \text{on } [s_i, s_{i-1}] \\ \tilde{y}_i(t) & \text{on } [s_{i-1}, 1]. \end{cases}$$

Here, $\tilde{y}_i$ is a solution to (1) in $[s_{i-1}, 1]$, with the initial datum $\tilde{y}_i(s_{i-1}) = x_i(s_{i-1})$, obtained by applying Lemma A.1 to y_{i-1}. Let $g_i(t) \in F(t, y_i(t))$ be such that

$$y_i(t) = S(t - s_i)y(s_i) + \int_{s_i}^t S(t-s)g_i(s)\mathrm{d}s.$$

Since

$$\|\tilde{y}_i(s_{i-1}) - y_{i-1}(s_{i-1})\|_X = \|x_i(s_{i-1}) - x_{i-1}(s_{i-1})\|_X = \|x_i(s_{i-1}) - y(s_{i-1})\|_X < \alpha_i,$$

we get the following estimates, for any $t \in [s_{i-1}, 1]$,

$$\|y_i(t) - y_{i-1}(t)\|_X = \|\tilde{y}_i(t) - y_{i-1}(t)\|_X \leq M_S \mathrm{e}^{M_S\|k_R\|_{L^1}} \alpha_i < \frac{\varepsilon_i}{2^i},$$

and, for a.e. $t \in [s_{i-1}, 1]$,

$$\|g_i(t) - g_{i-1}(t)\|_X \leq k_R(t) M_S \mathrm{e}^{M_S\|k_R\|_{L^1}} \alpha_i < k_R(t)\frac{\varepsilon_i}{2^i}.$$

Then, recalling that $\varepsilon_i \leq \varepsilon_k$ and $[s_k, 1] \subset [s_i, 1]$, for any $i \geq k$, we obtain, for any $j > i \geq k \geq 1$, and any $t \in [s_k, 1]$,

$$\|y_j(t) - y_i(t)\|_X \leq \sum_{m=i+1}^{j} \|y_m(t) - y_{m-1}(t)\|_X < \frac{\varepsilon_{i+1}}{2^{i+1}} \sum_{m=0}^{\infty} \frac{1}{2^m} = \frac{\varepsilon_{i+1}}{2^i} \leq \frac{\varepsilon_i}{2^i}, \tag{60}$$

and, for a.e. $t \in [s_k, 1]$,

$$\|g_j(t) - g_i(t)\|_X \leq \sum_{m=i+1}^{j} \|g_m(t) - g_{m-1}(t)\|_X < k_R(t)\frac{\varepsilon_{i+1}}{2^i} \leq k_R(t)\frac{\varepsilon_i}{2^i}. \tag{61}$$

Thus $\{y_j\}_{j>k}$ is a Cauchy sequence in $C([s_k,1],X)$ converging uniformly to some continuous function x, and, for a.e. $t\in[s_k,1]$, $\{g_j(t)\}_{j>k}$ is a Cauchy sequence in X converging to some limit $g(t)$. We claim that, for any $t\in[t_0,1]$,

$$x(t)=S(t-t_0)\hat{x}(t_0)+\int_{t_0}^{t}S(t-s)g(s)\mathrm{d}s \tag{62}$$

and, for a.e. $t\in[t_0,1]$,

$$g(t)\in F(t,x(t)). \tag{63}$$

Indeed, from assumption (15), for any $1\le k<j$ and a.e. $t\in[s_k,1]$,

$$\|g_j(t)\|_X\le\phi(t)\big(1+\|y_j(t)\|_X\big)\le\phi(t)\big(1+R\big),$$

and the dominated convergence theorem jointly with the strong continuity of the semigroup allow to prove that, for any $k\ge1$ and any $t\in[s_k,1]$,

$$\lim_{j\to\infty}\Big[S(t-s_j)y(s_j)+\int_{t_0}^{t}S(t-s)g_j(s)\chi_{(s_j,1)}(s)\mathrm{d}s\Big]=S(t-t_0)\hat{x}(t_0)+\int_{t_0}^{t}S(t-s)g(s)\mathrm{d}s\,.$$

On the other hand, for any $1\le k<j$ and any $t\in[s_k,1]$,

$$y_j(t)=S(t-s_j)y(s_j)+\int_{t_0}^{t}S(t-r)g_j(r)\chi_{(s_j,1)}(r)\mathrm{d}r$$

and $\lim_{j\to\infty}y_j(t)=x(t)$. Now, as the limit above holds for any k, setting $x(t_0)=\hat{x}(t_0)$ we obtain (62). To prove (63), notice that (60)–(61), and assumption (14) imply that, for any $1\le k<i$ and a.e. $t\in[s_k,1]$,

$$g(t)\in F(t,y_i(t))+k_R(t)\frac{\varepsilon_i}{2^i}B\subset F(t,x(t))+2k_R(t)\frac{\varepsilon_i}{2^i}B\,.$$

So, since $F(t,x(t))$ is closed, (63) holds true for a.e. t.

Finally, we need to prove that, for any $t>t_0$, $\|\hat{x}(t)-x(t)\|_X\le\varepsilon$ and $x(t)\in\mathrm{Int}K$. For this aim, let us fix $i\ge1$ and $t\in[s_i,s_{i-1}]$. We take $j>i$ such that

$$\|y_j(t)-x(t)\|_X\le\frac{\varepsilon_i}{2^i}\le\frac{\varepsilon}{4}\,,$$

by the definition of ε_i. Applying (58) and (60), we obtain for any $t\in[s_i,s_{i-1}]$

$$\begin{aligned}\|y(t)-y_j(t)\|_X&\le\|y(t)-x_i(t)\|_X+\|x_i(t)-y_j(t)\|_X\\&=\|y(t)-x_i(t)\|_X+\|y_i(t)-y_j(t)\|_X\le\varepsilon_i+\frac{\varepsilon_i}{2^i}\le\frac{\varepsilon}{2}\,.\end{aligned}$$

Hence,

$$\begin{aligned}\|\hat{x}(t)-x(t)\|_X&\le\|\hat{x}(t)-y(t)\|_X+\|y(t)-x(t)\|_X\\&\le\frac{\varepsilon}{4}+\|y(t)-y_j(t)\|_X+\|y_j(t)-x(t)\|_X\le\varepsilon\,.\end{aligned}$$

Moreover, the previous estimates, the choice of ε_i and (59), (60) yield for every $j\ge i$ as above

$$x(t)\in y_j(t)+\frac{\varepsilon_i}{2^i}B\subset y_i(t)+\varepsilon_iB=x_i(t)+\varepsilon_iB\subset y(t)+2\varepsilon_iB\subset\mathrm{Int}K\,.$$

This completes the proof. □

Proof of Theorem 4.1. In order to prove the theorem, it is sufficient to show that the conclusions of Lemma 6.2 remain valid if we replace assumption (18) by (28). Afterwards, the proof follows exactly as in the case of Theorems 3.2 and 3.1.

By the same arguments as those of Lemma 6.1, assumption (28) implies that for any compact set $D \subset K$ with $D \cap \partial K \neq \emptyset$, there exist $\eta', \rho, M > 0$ such that

$$\begin{aligned} &\forall\, t \in I,\ \forall\, x \in K \cap (D + 2\eta' B), \text{ for any } v \in F(t,x) \text{ satisfying} \\ &d_K(x) > -2\eta' \quad \text{and} \quad \sup_{\tau \leq \eta'} \sup_{z \in Z \cap B(x, 2\eta')} \langle n_z,\, S(\tau)\, v\rangle_X \geq 0 \\ &\exists\, \bar v \in F(t,x) \cap B(v, M) \quad \text{with} \quad \sup_{\tau \leq \eta'} \sup_{z \in Z \cap B(S(\tau)x, 2\eta')} \big\langle n_z, S(\tau)\,(\bar v - v)\big\rangle_X < -\rho\,. \end{aligned} \tag{64}$$

Now fix a compact set $D \subset K$ such that $D \cap \partial K \neq \emptyset$. Let $\eta', \rho, M > 0$ satisfy (64) and $R > 0$ be such that $D + \eta' B \subset \frac{R}{4} B$. Define C, δ', δ and ε' as in (39), (40), (41) and (42). Let y be a solution to (1), (3) such that $y(t) \in D + \eta' B$ for any $t \in [\bar t, 1]$ and set

$$\Gamma = \Big\{ s \in [\bar t, 1] : d_K(y(s)) > -2\eta'\,, \sup_{\tau \leq \eta'} \sup_{z \in Z \cap B(y(s), 2\eta')} \langle n_z, S(\tau)\, f^y(s)\rangle_X \geq 0 \Big\}. \tag{65}$$

Finally, define $T > 0$ and trajectories $y_{\varepsilon'}$ and $x_{\varepsilon'}$ as in the proof of Lemma 6.2. Again, we obtain

$$\|x_{\varepsilon'} - y\|_{L^\infty([\bar t, 1], X)} \leq \varepsilon' \tag{66}$$

and

$$\|x_{\varepsilon'} - y_{\varepsilon'}\|_{L^\infty([\bar t, (\bar t + 2\delta) \wedge 1], X)} \leq \frac{\rho}{2}\, \mu\,(\Gamma \cap [\bar t, t \wedge T])\,. \tag{67}$$

Fix $t \in (\bar t, (\bar t + 2\delta) \wedge 1]$. We shall verify that $x_{\varepsilon'}(t) \in \operatorname{Int} K$.

Consider first the case where $\mu\,(\Gamma \cap [\bar t, t]) = 0$. By (65) for almost every $s \in [\bar t, t]$ we have that either $d_K(y(s)) \leq -2\eta'$ or $d_K(y(s)) > -2\eta'$ and

$$\sup_{\tau \leq \eta'} \sup_{z \in Z \cap B(y(s), 2\eta')} \langle n_z, S(\tau)\, f^y(s)\rangle_X < 0\,.$$

Suppose that $d_K(y(t)) \geq -\frac{\eta'}{2}$, otherwise, from (42) and (66), it follows that $x_{\varepsilon'}(t) \in \operatorname{Int} K$. Again, by continuity, there exists $\bar s \in [\bar t, t)$ such that

$$\|y(t) - S(t - \bar s)\, y(\bar s)\|_X < \frac{\eta'}{2}\,, \tag{68}$$

and, for any $s \in [\bar s, t]$,

$$\|y(t) - y(s)\|_X < \frac{\eta'}{2} \qquad \text{and} \qquad d_K(y(s)) > -\eta'\,. \tag{69}$$

Then we have

$$\sup_{\tau \leq \eta'} \sup_{z \in Z \cap B(y(s), 2\eta')} \langle n_z, S(\tau)\, f^y(s)\rangle_X < 0\,, \qquad \text{a.e. } s \in [\bar s, t]\,. \tag{70}$$

By Lemma A.2,

$$d_K(y(t)) \leq d_K(S(t - \bar s)\, y(\bar s)) + \sup_{z \in Z \cap ([S(t - \bar s) y(\bar s), y(t)] + \eta' B)} \big\langle n_z, y(t) - S(t - \bar s)\, y(\bar s)\big\rangle_X\,.$$

Since $\mu\,(\Gamma \cap [\bar t, t]) = 0$ implies $x_{\varepsilon'} \equiv y$ on $[\bar t, t]$, and (68)–(69) yield

$$[S(t - \bar s) y(\bar s), y(t)] \subset B(y(s), \eta') \qquad \forall\, s \in [\bar s, t]\,,$$

applying the representation formula (8), the invariance assumption (12), and (70) we obtain

$$d_K(x_{\varepsilon'}(t)) = d_K(y(t)) \le \int_{\bar{s}}^{t} \sup_{z\in Z\cap B(y(s),2\eta')} \big\langle n_z, S(t-s)\, f^y(s)\big\rangle_X \,\mathrm{d}s < 0\,.$$

Now, we consider the case $\mu\,(\Gamma\cap[\bar{t},t\wedge T])>0$. Applying again Lemma A.2 we obtain

$$d_K(x_{\varepsilon'}(t)) \le d_K(y(t)) + \sup_{z\in Z\cap([y(t),x_{\varepsilon'}(t)]+\eta' B)} \langle n_z, x_{\varepsilon'}(t)-y(t)\rangle_X\,. \tag{71}$$

By (8), (40), (42), and (66), we have

$$[y(t),x_{\varepsilon'}(t)] \subset S(t-s)\,y(s)+\eta' B \qquad \forall\, s\in[\bar{t},t]\,.$$

Then, from (28), (41), (67), (71) and the definition of $y_{\varepsilon'}$ we obtain

$$\begin{aligned}
&d_K(x_{\varepsilon'}(t)) \le \sup\nolimits_{z\in Z\cap B(S(t-s)\,y(s),2\eta')}\langle n_z, x_{\varepsilon'}(t)-y_{\varepsilon'}(t)\rangle_X +\\
&\sup\nolimits_{z\in Z\cap B(S(t-s)\,y(s),2\eta')}\langle n_z, y_{\varepsilon'}(t)-y(t)\rangle_X \le \|x_{\varepsilon'}(t)-y_{\varepsilon'}(t)\|_X +\\
&\textstyle\int_{\Gamma\cap[\bar{t},t\wedge T]} \sup_{z\in Z\cap B(S(t-s)\,y(s),2\eta')} \big\langle n_z, S(t-s)\,\big[\bar{v}(s)-f^y(s)\big]\big\rangle_X \,\mathrm{d}s\\
&< \tfrac{\rho}{2}\,\mu\,(\Gamma\cap[\bar{t},t\wedge T]) - \rho\,\mu\,(\Gamma\cap[\bar{t},t\wedge T]) = -\tfrac{\rho}{2}\,\mu\,(\Gamma\cap[\bar{t},t\wedge T])<0\,,
\end{aligned}$$

proving the theorem. □

The proofs of Propositions 1, 2, 3 and 4 conclude the section.

Proof of Proposition 2. Let $\bar{x}\in\partial K$ and take $R=\|\bar{x}\|_X+1$. Notice that, by (15), for any $t\in I$ and any $x\in B(\bar{x},1)$,

$$F(t,x)\subset \phi(1+R)B \tag{72}$$

implying, in particular, that

$$\|v-w\|_X \le M := 2\phi(1+R),$$

for any $v,w\in F(t,x)$.

We prove our proposition using a contradiction argument. Assume that (18) does not hold. Then, we can find sequences

$$t_i\in I,\ x_i\in B\Big(\bar{x},\frac{1}{i}\Big)\cap K,\ z_i\in B\Big(x_i,\frac{1}{i}\Big),\ \xi_i\in\partial d_K(z_i),\ 0\le\tau_i\le\frac{1}{i},\ v_i\in F(t_i,x_i)$$

such that

$$\langle \xi_i, S(\tau_i)v_i\rangle \ge -\frac{1}{i}, \tag{73}$$

and, for any $w\in F(t_i,x_i)$, there exist

$$0\le\sigma_i\le\tau_i,\quad y_i\in B\Big(S(\sigma_i)x_i,\frac{1}{i}\Big),\quad \zeta_i\in\partial d_K(y_i)$$

satisfying

$$\langle \zeta_i, S(\sigma_i)(w-v_i)\rangle \ge -\frac{1}{i}. \tag{74}$$

Passing to the limit $i\to\infty$ we reach a contradiction. Indeed, by (72),

$$v_i\in F(t_i,x_i)\subset\phi(1+R)B.$$

Further, $\|z_i-x_i\|_X\le\frac{1}{i}$, hence, assumption (21) implies that for any $k\in\mathbb{N}$ and some i_k,

$$\xi_i\in\partial d_K(z_i)\subset\partial d_K(\bar{x})+\frac{1}{k}B, \qquad \text{for any } i\ge i_k,$$

yielding $\|\xi_i - \bar{\xi}_i\|_{X^*} \leq \frac{1}{k}$, for some $\bar{\xi}_i \in \partial d_K(\bar{x})$ and, as $\partial d_K(\bar{x})$ is a compact set, up to a subsequence, $\bar{\xi}_i \to \xi_0 \in \partial d_K(\bar{x})$ implying that also $\xi_i \to \xi_0$. So, since X is a reflexive space, up to subsequences,

$$t_i \to t, \qquad z_i \to \bar{x} \quad \text{in } X, \qquad \xi_i \to \xi_0 \quad \text{in } X^*, \qquad v_i \rightharpoonup v \quad \text{weakly in } X, \tag{75}$$

for some $t \in I$, $\xi_0 \in \partial d_K(\bar{x})$ and $v \in X$. Finally, as $\tau_i \to 0$, from (73) we deduce that

$$0 \leq \lim_{i\to\infty} \langle \xi_i, S(\tau_i)v_i\rangle = \lim_{i\to\infty} \langle S(\tau_i)^*\xi_i, v_i\rangle = \langle \xi_0, v\rangle\,. \tag{76}$$

Here $S(t)^*$ stands for the adjoint of $S(t)$. Further, by (14) and (19), for any $\varepsilon > 0$ there exists i_ε such that,

$$F(t_i, x_i) \subset F(t,\bar{x}) + \varepsilon B \quad \text{and} \quad F(t,\bar{x}) \subset F(t_i,x_i) + \varepsilon B, \qquad \text{for any } i \geq i_\varepsilon\,. \tag{77}$$

The set $F(t,\bar{x}) + \varepsilon B$ is convex and closed. By the Mazur lemma we get $v \in F(t,\bar{x}) + \varepsilon B$. Since $\varepsilon > 0$ is arbitrary, we deduce that $v \in F(t,\bar{x})$.

From (5) and (76), there exists $\bar{v} \in F(t,\bar{x})$ such that

$$\sigma(\bar{x}; \bar{v} - v) < -\rho. \tag{78}$$

By (77), there exists a sequence $w_i \in F(t_i, x_i)$ such that $w_i \to \bar{v}$ in X. Now, taking σ_i, y_i, ζ_i as in (74), and passing to the limit, we finally get, up to subsequences,

$$y_i \to \bar{x} \qquad \text{in } X, \quad \zeta_i \to \zeta \qquad \text{in } X^*, \tag{79}$$

and

$$0 \leq \lim_{i\to\infty} \langle \zeta_i, S(\sigma_i)(w_i - v_i)\rangle = \lim_{i\to\infty} \langle S(\sigma_i)^*\zeta_i, w_i - v_i\rangle = \langle \zeta, \bar{v} - v\rangle\,. \tag{80}$$

As $\zeta \in \partial d_K(\bar{x})$, (80) contradicts (78) and proves (18). Since $F(t,\bar{x})$ is convex and closed for any $\bar{x} \in \partial K$ (6) follows. □

Proof of Proposition 1. The proof is similar to the one of Proposition 2. The only difference concerns the limits in (75), (76), (79), (80) where we obtain instead:

$$\begin{gathered} \xi_i \overset{*}{\rightharpoonup} \xi_0 \quad \text{weakly-star in } X^*, \quad v_i \to v \in F(t,\bar{x}) \quad \text{in } X, \\ \langle \xi_i, S(\tau_i)v_i\rangle \to \langle \xi_0, v\rangle \geq 0\,, \quad \zeta_i \overset{*}{\rightharpoonup} \zeta \quad \text{weakly-star in } X^*, \\ 0 \leq \langle \zeta_i, S(t_i)(w_i - v_i)\rangle \to \langle \zeta, \bar{v} - v\rangle\,. \end{gathered}$$

The properties of the Clarke generalized gradient [12, Proposition 2.1.5] imply that ξ_0 and ζ belong to $\partial d_K(\bar{x})$. The assumption (20) is used here to prove that, up to a subsequence, $v_i \to v \in F(t,\bar{x})$. Indeed, thanks to (14) and (19), for any $k \in \mathbb{N}$ there exists i_k such that

$$v_i \in F(t_i, x_i) \subset F(t,\bar{x}) + \frac{1}{k}B, \qquad \text{for any } i \geq i_k$$

implying that $\|v_i - \bar{v}_i\|_X \leq \frac{1}{k}$, for some $\bar{v}_i \in F(t,\bar{x})$ and, as $F(t,\bar{x})$ is a compact set, up to a subsequence, $\bar{v}_i \to v \in F(t,\bar{x})$ yielding that also $v_i \to v$. The contradiction is reached also in this case, ending the proof. □

Proof of Proposition 3. The proof relies on the proof of [17, Lemma 3.7]. Fix $\bar{x} \in \partial K$. By Propositions 1 and 2, it is sufficient to prove that there exists $\rho > 0$ such that

$$\begin{gathered} \text{for any } t \in I \text{ and } v \in \mathrm{co}F(t,\bar{x}) \text{ satisfying } \sigma(\bar{x}; v) \geq 0, \\ \text{there exists } \bar{v} \in \mathrm{co}F(t,\bar{x}) \text{ satisfying } \sigma(\bar{x}; \bar{v} - v) \leq -\rho\,. \end{gathered} \tag{81}$$

Below we set

$$V_t^+ = \Big\{ v \in F(t,\bar{x}) \ : \ \sigma(\bar{x}; v) \geq 0 \Big\}$$

and

$$V_t^- = \Big\{ v \in F(t,\bar{x}) \ : \ \sigma(\bar{x}; v) < 0 \Big\} .$$

We claim that for every $t \in I$ and $v \in \mathrm{co}F(t,\bar{x})$ satisfying $\sigma(\bar{x}; v) \geq 0$, there exist $\lambda^+ > 0$, $\lambda^- \geq 0$, $v^+ \in \mathrm{co}\, V_t^+$, $v^- \in X$ and $\bar{w} \in \mathrm{co}F(t,\bar{x})$ such that $\lambda^+ + \lambda^- = 1$, $v = \lambda^+ v^+ + \lambda^- v^-$, where $v^- = 0$ if $\lambda^- = 0$ and $v^- \in \mathrm{co}\, V_t^-$ otherwise, and that

$$\sigma(\bar{x}; \bar{w} - v^+) \leq -\rho .$$

Indeed, for t and v as above, let $\lambda_\alpha > 0$, $v_\alpha \in F(t,\bar{x})$, $\alpha = 1, \dots, n$, be such that $\sum_{\alpha=1}^n \lambda_\alpha = 1$ and $\sum_{\alpha=1}^n \lambda_\alpha v_\alpha = v$. By reordering we may assume that $v_1, \dots, v_m \in V_t^+$ and $v_{m+1}, \dots, v_n \in V_t^-$. Observe that $m \geq 1$, since otherwise $\sigma(\bar{x}; v) < 0$. Define $\lambda^+ = \sum_{\alpha=1}^m \lambda_\alpha$, $\lambda^- = 1 - \lambda^+$, $v^+ = \frac{1}{\lambda^+} \sum_{\alpha=1}^m \lambda_\alpha v_\alpha$, $v^- = \frac{1}{\lambda^-} \sum_{\alpha=m+1}^n \lambda_\alpha v_\alpha$ if $m < n$ and $v^- = 0$ if $m = n$. Then $v = \lambda^+ v^+ + \lambda^- v^-$ and $v^+ \in \mathrm{co}\, V_t^+$. By (23) for each $\alpha = 1, \dots, m$, there exists $w_\alpha \in \mathrm{co}F(t,\bar{x})$ such that

$$\sigma(\bar{x}; w_\alpha - v_\alpha) \leq -\frac{\rho}{4} .$$

Then the vector $\bar{w} = \frac{1}{\lambda^+} \sum_{\alpha=1}^m \lambda_\alpha w_\alpha$ is as in our claim.

We shall prove (81) by contradiction. Suppose that for some sequences $t_i \in I$, $v_i \in \mathrm{co}F(t_i,\bar{x})$ satisfying

$$\sigma(\bar{x}; v_i) \geq 0 \tag{82}$$

and for any choice of $w \in \mathrm{co}F(t_i,\bar{x})$ we have

$$\sigma(\bar{x}; w - v_i) > -\frac{1}{i}. \tag{83}$$

For each i let $\lambda_i^+ > 0$, $\lambda_i^- \geq 0$, $\lambda_i^+ + \lambda_i^- = 1$, $v_i^+ \in \mathrm{co}\, V_{t_i}^+$, $v_i^- \in X$ be such that $v_i^- \in \mathrm{co}\, V_{t_i}^-$ if $\lambda_i^- > 0$ and $v_i^- = 0$ otherwise, $v_i = \lambda_i^+ v_i^+ + \lambda_i^- v_i^-$. Then for every $j \geq 1$ there exists $i(j)$ such that for every $i \geq i(j)$ we can find $\bar{w}_i \in \mathrm{co}F(t_i,\bar{x})$ satisfying

$$\sigma(\bar{x}; \bar{w}_i - v_i^+) \leq -\rho + \frac{1}{j} .$$

Taking subsequences and keeping the same notations, we may assume that for some t, λ^+, λ^-, v^+, v^-, $\bar{w}$ the following is satisfied: $t_i \to t$, $\lambda_i^+ \to \lambda^+$, $\lambda_i^- \to \lambda^-$, $\lambda^+ + \lambda^- = 1$, $v_i^+ \to v^+ \in \mathrm{co}F(t,\bar{x})$, $v_i^- \to v^- \in X$, $v_i \to v$, $\bar{w}_i \to \bar{w} \in \mathrm{co}F(t,\bar{x})$ and

$$\sigma(\bar{x}; \bar{w} - v^+) \leq -\rho .$$

The last four convergences are meant to be either weak or strong, depending on the set of assumptions under consideration. Then $v \in \mathrm{co}F(t,\bar{x})$. By (82), one can prove as in the above propositions that

$$\sigma(\bar{x}; v) \geq 0 . \tag{84}$$

Moreover, since the set-valued map (22) is upper semicontinuous with closed images, we get

$$v^- \in \mathrm{co}\Big\{ v \in F(t,\bar{x}) \ : \ \sigma(\bar{x}; v) \leq 0 \Big\} \subset \mathrm{co}F(t,\bar{x}) \cap \Big\{ v \in X \ : \sigma(\bar{x}; v) \leq 0 \Big\}$$

whenever $\lambda^- > 0$. Clearly $v = \lambda^+ v^+ + \lambda^- v^-$. Furthermore, by (83), for all $w \in \mathrm{co}F(t,\bar{x})$,

$$\sigma(\bar{x}; w - v) \geq 0 . \tag{85}$$

If $\lambda^- = 0$, then we get a contradiction. Hence $\lambda^- > 0$ and, consequently, $v^- \in \mathrm{co}\{v \in F(t,\bar{x}) \ : \ \sigma(\bar{x}; v) \le 0\}$. In particular, $v^- \in \mathrm{co}F(t,\bar{x})$ and $\sigma(\bar{x}; v^-) \le 0$.

Suppose first that $\sigma(\bar{x}; v^-) = 0$. Then for some $\mu_j > 0$ and for some $\tilde{v}_j \in F(t,\bar{x})$ such that $\sigma(\bar{x}; \tilde{v}_j) = 0$, we have $\sum_{j=0}^{l} \mu_j = 1$ and $v^- = \sum_{j=0}^{l} \mu_j \tilde{v}_j$. By the assumption,

$$\sigma(\bar{x}; w_j - \tilde{v}_j) \le -\frac{\rho}{2}$$

for some $w_j \in \mathrm{co}F(t,\bar{x})$ and each j. Set $w^- = \sum_{j=0}^{l} \mu_j w_j$. Then $w^- \in \mathrm{co}F(t,\bar{x})$ and

$$\sigma(\bar{x}; w^- - v^-) \le -\frac{\rho}{2}\,.$$

Setting $w = \lambda^+ v^+ + \lambda^- w^-$, we obtain a contradiction with (85).

On the other hand, if $\sigma(\bar{x}; v^-) < 0$, then (84) yields $\lambda^+ > 0$. Setting $w = \lambda^+ \bar{w} + \lambda^- v^- \in \mathrm{co}F(t,\bar{x})$, we obtain

$$\sigma(\bar{x}; w - v) = \lambda^+ \sigma(\bar{x}; \bar{w} - v^+) \le -\lambda^+ \rho < 0\,.$$

This contradicts (85) and proves our claim. □

Proof of Proposition 4. Consider first the case where $F(t,\bar{x})$ is convex for any $t \in I$ and $\bar{x} \in \partial K$, and (29) holds. Let $\bar{x} \in \partial K$ and take $R = \|\bar{x}\| + 1$. The proof is similar to the one of Proposition 2. Observe that for any $t \in I$ and any $x \in B(\bar{x}, 1)$, $\|v - w\|_X \le M := 2\phi(1 + R)$, for any $v, w \in F(t, x)$.

Assume by contradiction that (28) does not hold. Then, we can find sequences

$$t_i \in I, \quad x_i \in B\Big(\bar{x}, \frac{1}{i}\Big) \cap K, \quad v_i \in F(t_i, x_i), \quad 0 \le \tau_i \le \frac{1}{i} \quad \text{and} \quad z_i \in B\Big(x_i, \frac{1}{i}\Big) \cap Z$$

such that

$$\langle n_{z_i}, S(\tau_i) v_i \rangle_X \ge -\frac{1}{i}\,, \tag{86}$$

and, for any $w \in F(t_i, x_i)$, there exist

$$0 \le \sigma_i \le \tau_i \quad \text{and} \quad y_i \in Z \cap B\Big(S(\sigma_i) x_i, \frac{1}{i}\Big) \tag{87}$$

satisfying

$$\langle n_{y_i}, S(\sigma_i)(w - v_i) \rangle_X \ge -\frac{1}{i}\,. \tag{88}$$

By our assumptions, up to subsequences, $t_i \to t$, $v_i \rightharpoonup v$ weakly in X, $n_{z_i} \to n$ in X, as $i \to +\infty$, for some $t \in I$, $v \in X$ and $n \in X$ of norm equal to one. Since $z_i \to \bar{x}$, by (86), for any $\varepsilon > 0$,

$$0 \le \lim_{i \to \infty} \langle n_{z_i}, S(\tau_i) v_i \rangle = \lim_{i \to \infty} \langle S(\tau_i)^* n_{z_i}, v_i \rangle = \langle n, v \rangle \le \sup_{z \in Z \cap B(\bar{x}, \varepsilon)} \langle n_z, v \rangle\,. \tag{89}$$

Therefore,

$$\inf_{\varepsilon > 0} \sup_{z \in Z \cap B(\bar{x}, \varepsilon)} \langle n_z, v \rangle_X \ge 0\,,$$

where $v \in F(t, \bar{x})$, because of the closedness and convexity of $F(t,\bar{x})$ and (14), (19). Hence, assumption (30) yields the existence of $\bar{v} \in F(t,\bar{x})$ satisfying

$$\inf_{\varepsilon > 0} \sup_{z \in Z \cap B(\bar{x}, \varepsilon)} \langle n_z, \bar{v} - v \rangle_X < -\rho\,. \tag{90}$$

Let $w_i \in F(t_i, x_i)$ be a sequence converging in X to $\bar{v}$, and let σ_i and y_i be as in (87), (88). Since $y_i \to \bar{x}$, applying (88) and arguing as in (89), for every $\varepsilon > 0$ we have

$$0 \le \lim_{i\to\infty} \langle n_{y_i}, S(\sigma_i)(w_i - v_i)\rangle_X = \lim_{i\to\infty} \langle S(\sigma_i)^* n_{y_i}, w_i - v_i\rangle_X \le \sup_{z\in Z\cap B(\bar{x},\varepsilon)} \langle n_z, \bar{v} - v\rangle_X \,.$$

This contradicts (90).

In the case of compact $F(t,\bar{x})$, the proof follows the same lines, using now the convergence $n_{z_i} \rightharpoonup n \in B$ weakly in X and $v_i \to v \in F(t,\bar{x})$, up to a subsequence, see the proof of Proposition 1 for the details. □

Appendix. This section contains two technical results needed in the course of the investigation. The first is an infinite dimensional version of the Filippov Theorem, see [15, Theorem 1.2], modified here in a suitable form for our scopes.

Lemma A.1. *Let $\delta_0 \ge 0$ and $t_0 \in I$. Assume (13)–(14), let y be the mild solution to (9) in $[t_0, 1]$, for some $f \in L^1([t_0,1], X)$. Set $R = \frac{1}{2}\max_{t\in[t_0,1]} \|y(t)\|_X$,*

$$\gamma(t) = \operatorname{dist}\big(f(t), F(t, y(t))\big) \qquad \text{and} \qquad m(t) = M_S e^{M_S \int_{t_0}^t k_R(s)\mathrm{d}s}.$$

If $m(1)\big(\delta_0 + \int_{t_0}^1 \gamma(s)\mathrm{d}s\big) < \frac{R}{2}$, then, for any $y_0 \in y(t_0) + \delta_0 B$ and any $\beta > 1$, there exists a solution x to (1) in $[t_0, 1]$, satisfying $x(t_0) = y_0$,

$$\|x(t) - y(t)\|_X \le m(t)\Big(\delta_0 + \beta \int_{t_0}^t \gamma(s)\mathrm{d}s\Big), \qquad \text{for any } t \in [t_0, 1],$$

and

$$\|f^x(t) - f(t)\|_X \le k_R(t) m(t)\Big(\delta_0 + \beta \int_{t_0}^t \gamma(s)\mathrm{d}s\Big) + \beta\gamma(t), \qquad \text{for a.e. } t \in [t_0, 1],$$

where $f^x \in L^1([t_0,1], X)$ is so that (7) and (8) hold true for $f = f^x$.

Proof. The proof proceeds exactly as in [15, Theorem 1.2]. The only difference is in the first line of page 109, while applying Lemma 1.3 of [15]. The assumption $\beta > 1$ is needed to ensure that, for a.e. t, the set

$$F(t, y(t)) \cap \{f(t) + \beta\gamma(t)B\} \neq \emptyset.$$

Indeed, if $\gamma(t) = 0$, the definition of γ ensures that $f(t) \in F(t, y(t))$, while, if $\gamma(t) > 0$, since $\beta > 1$, from the very definition of distance and the measurable selection theorem we get that there exists a measurable selection $w(t) \in F(t, y(t))$ such that $\|f(t) - w(t)\|_X \le \beta\gamma(t)$. Thus

$$w(t) \in F(t, y(t)) \cap \{f(t) + \beta\gamma(t)B\}.$$

Notice that in finite dimension we can take $\beta = 1$, recovering the original Filippov Theorem. □

The second result is a version of the mean value theorem for the oriented distance d_K in Hilbert spaces. Here we make use of the notations introduced in Section 4.

Lemma A.2. *Let $(X, \langle\, ,\, \rangle_X)$ be a Hilbert space. For every $x, y \in X$ we have*

$$d_K(y) - d_K(x) \le \inf_{\varepsilon > 0} \sup_{z\in Z\cap([x,y]+\varepsilon B)} \langle n_z, y - x\rangle_X \,.$$

Proof. Let $x,\, y \in X$ and fix $\varepsilon > 0$. It is enough to consider the case $x \neq y$. For every $s \in [0,1]$ set $\gamma(s) = x + s(y-x)$. Then $d_K(\gamma(\cdot))$ is absolutely continuous. It is sufficient to prove that for almost every $s \in [0,1]$

$$\frac{\mathrm{d}}{\mathrm{d}s} d_K(\gamma(s)) \leq \sup_{z \in Z \cap B(\gamma(s),\varepsilon)} \langle n_z, y-x\rangle_X \,,$$

implying that

$$d_K(y) - d_K(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} d_K(\gamma(s))\,\mathrm{d}s \leq \sup_{z \in Z \cap ([x,y]+\varepsilon B)} \langle n_z, y-x\rangle_X \,.$$

The maps $d_K(\gamma(\cdot))$ and $d_K^2(\gamma(\cdot))$ are almost everywhere differentiable. Denote by $D \subset (0,1)$ the set on which both functions are differentiable and fix $s \in D$. We distinguish three cases.

Case 1. $\gamma(s) \notin K$. Let $L > 0$ be a Lipschitz constant for d_K^2 on $\gamma([0,1]) + B$. Recall that $|d_K(x)| = \mathrm{dist}(x, \partial K)$, for all $x \in X$. Then for each $0 < h < \min\{\varepsilon, 1-s\}$ and every $z \in Z \cap (B(\gamma(s), h^2) \backslash K)$ we have

$$d_K^2(\gamma(s)) \geq \|z - P_{\partial K}(z)\|_X^2 - L\,h^2 \tag{A.1}$$

and

$$\begin{aligned} &d_K^2(\gamma(s+h)) \leq \|\gamma(s+h) - P_{\partial K}(z)\|_X^2 = \|\gamma(s) - z\|_X^2 \\ &+2\langle \gamma(s) - z\,,\, z - P_{\partial K}(z) + h(y-x)\rangle_X + \|z - P_{\partial K}(z) + h(y-x)\|_X^2 \\ &= \|z - P_{\partial K}(z) + h(y-x)\|_X^2 + o(h)\,. \end{aligned}$$

Consequently, for all $h > 0$ small enough,

$$\begin{aligned} &\tfrac{d_K^2(\gamma(s+h)) - d_K^2(\gamma(s))}{h} \leq \inf_{z \in Z \cap (B(\gamma(s),h^2)\backslash K)} \tfrac{\|z - P_{\partial K}(z) + h(y-x)\|^2 - \|z - P_{\partial K}(z)\|^2}{h} + o(1) \\ &= \inf_{z \in Z \cap (B(\gamma(s),h^2)\backslash K)} 2\,\langle z - P_{\partial K}(z)\,,\, y-x\rangle_X + o(1) \\ &= \inf_{z \in Z \cap (B(\gamma(s),h^2)\backslash K)} 2\,d_K(\gamma(s))\,\langle n_z\,,\, y-x\rangle_X + o(1) \\ &\leq \sup_{z \in Z \cap B(\gamma(s),\varepsilon)} 2\,d_K(\gamma(s))\,\langle n_z\,,\, y-x\rangle_X + o(1)\,. \end{aligned}$$

Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} d_K^2(\gamma(s)) \leq \sup_{z \in Z \cap B(\gamma(s),\varepsilon)} 2\,d_K(\gamma(s))\,\langle n_z\,,\, y-x\rangle_X$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s} d_K(\gamma(s)) = \frac{1}{2\,d_K(\gamma(s))} \cdot \frac{\mathrm{d}}{\mathrm{d}s} d_K^2(\gamma(s)) \leq \sup_{z \in Z \cap B(\gamma(s),\varepsilon)} \langle n_z\,,\, y-x\rangle_X \,.$$

Case 2. $\gamma(s) \in \mathrm{Int}\, K$. Similarly to Case 1, for every $0 < h < \min\{\varepsilon, s\}$ and every $z \in Z \cap (B(\gamma(s), h^2) \backslash K^C)$ we have (A.1) and

$$\begin{aligned} &d_K^2(\gamma(s-h)) \leq \|\gamma(s-h) - P_{\partial K}(z)\|_X^2 \qquad\qquad (A.2) \\ &= \|\gamma(s) - z\|^2 + 2\langle \gamma(s) - z\,,\, z - P_{\partial K}(z) - h(y-x)\rangle + \|z - P_{\partial K}(z) - h(y-x)\|^2 \\ &= \|z - P_{\partial K}(z) - h(y-x)\|_X^2 + o(h)\,. \end{aligned}$$

Inequalities (A.1) and (A.2) yield

$$\frac{d_K^2(\gamma(s-h))-d_K^2(\gamma(s))}{h} \leq \inf_{z\in Z\cap(B(\gamma(s),h^2)\setminus K^C)} 2\,\langle P_{\partial K}(z)-z\,,\,y-x\rangle_X + o(1)$$
$$= \inf_{z\in Z\cap(B(\gamma(s),h^2)\setminus K^C)} 2\,(-d_K(\gamma(s)))\,\langle n_z\,,\,y-x\rangle_X + o(1)$$
$$\leq 2\,(-d_K(\gamma(s)))\sup_{z\in Z\cap B(\gamma(s),\varepsilon)}\langle n_z\,,\,y-x\rangle_X + o(1)\,.$$

Then we obtain

$$\frac{\mathrm{d}}{ds}d_K^2(\gamma(s)) \geq 2\,d_K(\gamma(s)) \sup_{z\in Z\cap B(\gamma(s),\varepsilon)} \langle n_z\,,\,y-x\rangle_X$$

and again

$$\frac{\mathrm{d}}{ds}d_K(\gamma(s)) = \frac{1}{2\,d_K(\gamma(s))}\cdot\frac{\mathrm{d}}{ds}d_K^2(\gamma(s)) \leq \sup_{z\in Z\cap B(\gamma(s),\varepsilon)} \langle n_z\,,\,y-x\rangle_X\,.$$

Case 3. $\gamma(s)\in\partial K$. Let us first suppose that $|\frac{\mathrm{d}}{\mathrm{d}s}d_K(\gamma(s))| = 2C > 0$. Then for all $h>0$ small enough we have $|d_K(\gamma(s+h))| \geq Ch$, so that the point s is isolated in the set $\{s\in D\,:\,\gamma(s)\in\partial K\}$. Consequently,

$$\mu\left(\left\{s\in D\,:\,\gamma(s)\in\partial K\,,\;\frac{\mathrm{d}}{\mathrm{d}s}d_K(\gamma(s))\neq 0\right\}\right) = 0\,.$$

It remains to verify that in the case where $\frac{\mathrm{d}}{\mathrm{d}s}d_K(\gamma(s)) = 0$, we have

$$\sup_{z\in Z\cap B(\gamma(s),\varepsilon)} \langle n_z\,,\,y-x\rangle_X \geq 0\,.$$

Assume by contradiction that there exists $\alpha>0$ such that

$$\sup_{z\in Z\cap B(\gamma(s),\varepsilon)} \langle n_z\,,\,y-x\rangle_X \leq -\alpha\,.$$

Let $0<h<\varepsilon/4\|y-x\|_X$, $\{\xi_i\}_{i\in\mathbb{N}}\subset B(\gamma(s),\frac{\varepsilon}{2})\cap\mathrm{Int}\,K$ converge to $\gamma(s)$ as $i\to+\infty$ and set, for every $0\leq r\leq h$, $\gamma_i(r)=\xi_i+r(y-x)$. Fix $i\in\mathbb{N}$ and define

$$h_i = \sup\{0<\tilde{h}<h\;:\;\gamma_i([0,\tilde{h}])\subset\mathrm{Int}\,K\}$$

Proceeding as in Case 2, we can prove that

$$\frac{\mathrm{d}}{\mathrm{d}r}d_K(\gamma_i(r)) \leq \sup_{z\in Z\cap B(\gamma_i(r),\frac{\varepsilon}{4})} \langle n_z\,,\,y-x\rangle_X \leq -\alpha\,, \qquad \text{for a.e. } r\in[0,h_i]\,.$$

Then, $\gamma_i(h_i)\in\mathrm{Int}\,K$ and, consequently, $h_i=h$. Summarizing, we have obtained that for every $i\in\mathbb{N}$

$$\frac{\mathrm{d}}{\mathrm{d}r}d_K(\gamma_i(r)) \leq \sup_{z\in Z\cap B(\gamma_i(r),\frac{\varepsilon}{4})} \langle n_z\,,\,y-x\rangle_X \leq -\alpha\,, \qquad \text{for a.e. } r\in[0,h]\,.$$

Therefore, for every $i\in\mathbb{N}$ and $h\in(0,\frac{\varepsilon}{4\|y-x\|_X})$ we have $d_K(\gamma_i(h))-d_K(\gamma_i(0))\leq -\alpha\,h$. Taking the limit as $i\to+\infty$, we obtain

$$d_K(\gamma(s+h))-d_K(\gamma(s))\leq -\alpha\,h\,,$$

that is impossible, since $\frac{\mathrm{d}}{\mathrm{d}s}d_K(\gamma(s))=0$. This completes the proof. □

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