Homogenization of the discrete diffusive coagulation-fragmentation equations in perforated domains.

Laurent Desvillettes[1] and Silvia Lorenzani[2]

[1] *IMJ-PRG, Université Paris Diderot, France*

[2] *Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy*

Abstract

The asymptotic behavior of the solution of an infinite set of Smoluchowski's discrete coagulation-fragmentation-diffusion equations with non-homogeneous Neumann boundary conditions, defined in a periodically perforated domain, is analyzed. Our homogenization result, based on Nguetseng-Allaire two-scale convergence, is meant to pass from a microscopic model (where the physical processes are properly described) to a macroscopic one (which takes into account only the effective or averaged properties of the system). When the characteristic size of the perforations vanishes, the information given on the microscale by the non-homogeneous Neumann boundary condition is transferred into a global source term appearing in the limiting (homogenized) equations. Furthermore, on the macroscale, the geometric structure of the perforated domain induces a correction in the diffusion coefficients.

1 Introduction

This paper is devoted to the homogenization of an infinite set of Smoluchowski's discrete coagulation-fragmentation-diffusion equations in a periodically perforated domain. The system of evolution equations considered describes the dynamics of cluster growth, that is the mechanisms allowing clusters to coalesce to form larger clusters or break apart into smaller ones. These clusters can diffuse in space with a diffusion constant which depends on their size. Since the size of clusters is not limited a priori, the system of reaction-diffusion equations that we consider consists of an infinite number of equations. The structure of the chosen equations, defined in a perforated medium with a non-homogeneous Neumann condition on the boundary of the perforations, is useful in investigating several phenomena arising in porous media [14], [8], [13] or in the field of biomedical research [11].

Typically, in a porous medium, the domain consists of two parts: a fluid phase where colloidal species or chemical substances, transported by diffusion, are dissolved and a solid skeleton (formed by grains or pores) on the boundary of which deposition processes or chemical reactions take place. In recent years, the Smoluchowski equation has been also considered in biomedical research to model the aggregation and diffusion of β-amyloid peptide (Aβ) in the cerebral tissue, a process associated with the development of Alzheimer's disease. One can define a perforated geometry, obtained by removing from a fixed domain (which represents the cerebral tissue) infinitely many small holes (the neurons). The production of Aβ in monomeric form from the neuron membranes can be modeled by coupling the Smoluchowski equation for the concentration of monomers with a non-homogeneous Neumann condition on the boundaries of the holes.

The mathematical complexity underlying the models that can be proposed to describe such processes has been fully addressed in our work. Furthermore, the results of this paper constitute a generalization of some of the results contained in [14], [11], by considering an infinite system of equations where both the coagulation and fragmentation processes are taken into account. Unlike previous theoretical works, where existence and uniqueness of solutions for an infinite system of coagulation-fragmentation equations (with homogeneous Neumann boundary conditions) have been studied [19], [15], in this paper our focus lies on a distinct aspect, that is, the averaging of the system of Smoluchowski's equations over arrays of periodically-distributed microstructures.

Our homogenization result, based on Nguetseng-Allaire two-scale convergence [17], [1], is meant to pass from a microscopic model (where the physical processes are properly described) to a macroscopic one (which takes into account only the effective or averaged properties of the system).

1.1 Setting of the problem

Let Ω be a bounded open set in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Let Y be the unit periodicity cell $[0,1[^3$ having the paving property. We perforate Ω by removing from it a set T_ϵ of periodically distributed holes defined as follows. Let us denote by T an open subset of Y with a smooth boundary Γ, such that $\overline{T} \subset \operatorname{Int} Y$. Set $Y^* = Y \setminus T$ which is called in the literature the solid or material part. We define $\tau(\epsilon\overline{T})$ to be the set of all translated images of $\epsilon\overline{T}$ of the form $\epsilon(k+\overline{T})$, $k \in \mathbb{Z}^3$. Then,

$$T_\epsilon := \Omega \cap \tau(\epsilon\overline{T}).$$

Introduce now the periodically perforated domain Ω_ϵ defined by

$$\Omega_\epsilon = \Omega \setminus \overline{T}_\epsilon.$$

For the sake of simplicity, we make the following standard assumption on the holes [6], [9]:
there exists a 'security' zone around $\partial\Omega$ without holes, i.e.

$$\exists\, \delta > 0 \text{ such that } dist\,(\partial\Omega, T_\epsilon) \geq \delta. \tag{1}$$

Therefore, Ω_ϵ is a connected set. The boundary $\partial\Omega_\epsilon$ of Ω_ϵ is then composed of two parts. The first one is the union of the boundaries of the holes strictly contained in Ω. It is denoted by Γ_ϵ and is defined by

$$\Gamma_\epsilon := \cup\Big\{\partial(\epsilon(k+\overline{T})) \mid \epsilon(k+\overline{T}) \subset \Omega\Big\}.$$

The second part of $\partial\Omega_\epsilon$ is its fixed exterior boundary denoted by $\partial\Omega$. It is easily seen that (see [2], Eq. (3))

$$\lim_{\epsilon\to 0} \epsilon \mid \Gamma_\epsilon|_{N-1} =\mid \Gamma|_{N-1} \frac{\mid \Omega \mid_N}{\mid Y \mid_N}, \tag{2}$$

where $\mid \cdot \mid_N$ is the N-dimensional Hausdorff measure.

Throughout this paper, ϵ will denote the general term of a sequence of positive real numbers which converges to zero. We will consider in the following a discrete coagulation-fragmentation-diffusion model for the evolution of clusters [3], [4]. Denoting by $u_i^\epsilon := u_i^\epsilon(t,x) \geq 0$ the density of clusters with integer size $i \geq 1$ at position $x \in \Omega_\epsilon$ and time $t \geq 0$ and by d_i the diffusion constant for clusters of size i, the corresponding system can be written as a family of equations in Ω_ϵ:

$$
\begin{cases}
\dfrac{\partial u_1^\epsilon}{\partial t} - \nabla_x \cdot (d_1 \nabla_x u_1^\epsilon) + u_1^\epsilon \sum_{j=1}^\infty a_{1,j} u_j^\epsilon = \sum_{j=1}^\infty B_{1+j}\, \beta_{1+j,1}\, u_{1+j}^\epsilon & \text{in } [0,T] \times \Omega_\epsilon, \\[2ex]
\dfrac{\partial u_1^\epsilon}{\partial \nu} \equiv \nabla_x u_1^\epsilon \cdot n = 0 & \text{on } [0,T] \times \partial\Omega, \\[2ex]
\dfrac{\partial u_1^\epsilon}{\partial \nu} \equiv \nabla_x u_1^\epsilon \cdot n = \epsilon\, \psi(t, x, \frac{x}{\epsilon}) & \text{on } [0,T] \times \Gamma_\epsilon, \\[2ex]
u_1^\epsilon(0,x) = U_1 & \text{in } \Omega_\epsilon,
\end{cases}
\tag{3}
$$

where ψ is a given bounded function satisfying the following conditions:

(i) $\psi(t, x, \frac{x}{\epsilon}) \in C^1(0,T;B)$ with $B = C^1[\overline{\Omega}; C^1_\#(Y)]$

(ii) $\psi(t=0, x, \frac{x}{\epsilon}) = 0$

and U_1 is a positive constant such that $U_1 \leq \|\psi\|_{L^\infty([0,T];B)}$.

In addition, if $i \geq 2$,

$$
\begin{cases}
\dfrac{\partial u_i^\epsilon}{\partial t} - \nabla_x \cdot (d_i \nabla_x u_i^\epsilon) = Q_i^\epsilon + F_i^\epsilon & \text{in } [0,T] \times \Omega_\epsilon, \\[2ex]
\dfrac{\partial u_i^\epsilon}{\partial \nu} \equiv \nabla_x u_i^\epsilon \cdot n = 0 & \text{on } [0,T] \times \partial\Omega, \\[2ex]
\dfrac{\partial u_i^\epsilon}{\partial \nu} \equiv \nabla_x u_i^\epsilon \cdot n = 0 & \text{on } [0,T] \times \Gamma_\epsilon, \\[2ex]
u_i^\epsilon(0,x) = 0 & \text{in } \Omega_\epsilon,
\end{cases}
\tag{4}
$$

where the terms Q_i^ϵ, F_i^ϵ due to coagulation and fragmentation, respectively, are given by

$$Q_i^\epsilon := \frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j}\, u_{i-j}^\epsilon\, u_j^\epsilon - \sum_{j=1}^{\infty} a_{i,j}\, u_i^\epsilon\, u_j^\epsilon \tag{5}$$

$$F_i^\epsilon := \sum_{j=1}^{\infty} B_{i+j}\, \beta_{i+j,i}\, u_{i+j}^\epsilon - B_i\, u_i^\epsilon. \tag{6}$$

The parameters B_i, $\beta_{i,j}$ and $a_{i,j}$, for integers $i, j \geq 1$, represent the total rate B_i of fragmentation of clusters of size i, the average number $\beta_{i,j}$ of clusters of size j produced due to fragmentation of a cluster of size i, and the coagulation rate $a_{i,j}$ of clusters of size i with clusters of size j. These parameters represent rates, so they are always nonnegative; single particles do not fragment further, and mass should be conserved when a cluster fragments into smaller pieces, so one always imposes

$$a_{i,j} = a_{j,i} \geq 0, \quad \beta_{i,j} \geq 0, \quad (i, j \geq 1) \tag{7}$$

$$B_1 = 0, \quad B_i \geq 0 \quad (i \geq 2) \tag{8}$$

$$i = \sum_{j=1}^{i-1} j\, \beta_{i,j} \quad (i \geq 2) \tag{9}$$

In order to prove the bounds presented in the sequel, we need to impose an additional restriction on the fragmentation coefficients: For each $m \geq 1$ there exists $\gamma_m > 0$ such that

$$B_j\, \beta_{j,m} \leq \gamma_m\, a_{m,j} \qquad \text{for } j \geq m+1. \tag{10}$$

1.2 Main statement and comments

Our aim is to study the homogenization of the set of equations (3)-(4) as $\epsilon \to 0$, i.e., to study the behaviour of $u_i^\epsilon (i \geq 1)$ as $\epsilon \to 0$ and obtain the equations satisfied by the limit. There is no clear notion of convergence for the sequence $u_i^\epsilon (i \geq 1))$ which is defined on a varying set Ω_ϵ. This difficulty is specific to the case of perforated domains. A natural way to get rid of this difficulty is given by Nguetseng-Allaire two-scale convergence [17], [1].

Theorem 1.1. *If $\epsilon > 0$, there exists a strong solution*

$$u_i^\epsilon \in L^2([0,T]; H^2(\Omega_\epsilon)) \cap H^1([0,T]; L^2(\Omega_\epsilon)) \quad (i \geq 1)$$

to system (3) - (4)*, which is moreover nonnegative, that is*

$$u_i^\epsilon(t,x) \geq 0 \quad \textit{for } (t,x) \in (0,T) \times \Omega_\epsilon \ .$$

Let $u_i^\epsilon(t,x)$ $(i \geq 1)$ *be a family of such strong solutions to problems (3)-(4). The sequences* $\widetilde{u_i^\epsilon}$ *and* $\widetilde{\nabla_x u_i^\epsilon}$ $(i \geq 1)$ *two-scale converge (up to a subsequence) to:* $[\chi(y)\, u_i(t,x)]$ *and* $[\chi(y)(\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))]$ $(i \geq 1)$*, respectively, where tilde denotes the extension by zero outside* Ω_ϵ *and* $\chi(y)$ *represents the characteristic function of* Y^**. The limiting functions* $(u_i(t,x), u_i^1(t,x,y))$ $(i \geq i)$ *are the unique solutions in* $L^2(0,T;H^1(\Omega)) \times L^2([0,T] \times \Omega; H^1_\#(Y)/\mathbb{R})$ *of the following two-scale homogenized systems.*

If $i = 1$*:*

$$\begin{cases} \theta \dfrac{\partial u_1}{\partial t}(t,x) - div_x \Big[d_1\, A\, \nabla_x u_1(t,x) \Big] + \theta\, u_1(t,x) \sum_{j=1}^{\infty} a_{1,j}\, u_j(t,x) \\ = \theta \sum_{j=1}^{\infty} B_{1+j}\, \beta_{1+j,1}\, u_{1+j}(t,x) + d_1 \displaystyle\int_\Gamma \psi(t,x,y)\, d\sigma(y) & \textit{in } [0,T] \times \Omega, \\ [A\, \nabla_x u_1(t,x)] \cdot n = 0 & \textit{on } [0,T] \times \partial\Omega, \\ u_1(0,x) = U_1 & \textit{in } \Omega. \end{cases} \tag{11}$$

If $i \geq 2$

$$\begin{cases} \theta \dfrac{\partial u_i}{\partial t}(t,x) - div_x \Big[d_i\, A\, \nabla_x u_i(t,x) \Big] + \theta\, u_i(t,x) \sum_{j=1}^{\infty} a_{i,j}\, u_j(t,x) \\ +\theta\, B_i\, u_i(t,x) = \dfrac{\theta}{2} \sum_{j=1}^{i-1} a_{j,i-j} u_j(t,x)\, u_{i-j}(t,x) \\ +\theta \sum_{j=1}^{\infty} B_{i+j}\, \beta_{i+j,i}\, u_{i+j}(t,x) & \textit{in } [0,T] \times \Omega, \\ [A\, \nabla_x u_i(t,x)] \cdot n = 0 & \textit{on } [0,T] \times \partial\Omega, \\ u_i(0,x) = 0 & \textit{in } \Omega. \end{cases} \tag{12}$$

where

$$u_i^1(t,x,y) = \sum_{j=1}^{N} w_j(y)\, \frac{\partial u_i}{\partial x_j}(t,x) \quad (i \geq 1),$$

$$\theta = \int_Y \chi(y) dy = |Y^*|$$

is the volume fraction of material, and A is a matrix with constant coefficients defined by

$$A_{jk} = \int_{Y^*} (\nabla_y w_j + \hat{e}_j) \cdot (\nabla_y w_k + \hat{e}_k)\, dy,$$

with $\hat{e}_j$ being the j-th unit vector in $\mathbb{R}^N$, and $(w_j)_{1\leq j\leq N}$ the family of solutions of the cell problem

$$\begin{cases} -div_y[\nabla_y w_j + \hat{e}_j] = 0 & \text{in } Y^* \\ (\nabla_y w_j + \hat{e}_j) \cdot n = 0 & \text{on } \Gamma \\ y \to w_j(y) \quad Y - periodic \end{cases} \tag{13}$$

1.3 Structure of the rest of the paper

The paper is organized as follows. In Section 2 we derive all the a priori estimates needed for two-scale homogenization. In particular, in order to prove the uniform L^2-bound on the infinite sums appearing in our set of Eqs. (3)-(4), we extend to the case of non-homogeneous Neumann boundary conditions a duality method first devised by M. Pierre and D. Schmitt [18] and largely exploited afterwards [3], [4]. Then, Section 3 is devoted to the proof of our main results on the homogenization of the infinite Smoluchowski discrete coagulation-fragmentation-diffusion equations in a periodically perforated domain. Finally, Appendix A and Appendix B are introduced to summarize, respectively, some fundamental inequalities in Sobolev spaces tailored for perforated media and some basic results on the two-scale convergence method (used to perform the homogenization procedure).

2 Estimates

We first obtain the *a priori* estimates for the sequences u_i^ϵ, $\nabla_x u_i^\epsilon$, $\partial_t u_i^\epsilon$ in $[0,T]\times\Omega_\epsilon$, that are independent of ϵ.

Lemma 2.1. *Assume that*

$$\sup_i d_i < +\infty.$$

Then, for all $T > 0$, the weak solutions to system (3)-(4) satisfy the following bound:

$$\int_0^T \int_{\Omega_\epsilon} \left[\sum_{i=1}^{\infty} i\, d_i\, u_i^\epsilon(t,x) \right] \left[\sum_{i=1}^{\infty} i\, u_i^\epsilon(t,x) \right] dt\, dx \le C, \tag{14}$$

where C is a positive constant independent of ϵ.

Proof. Let us consider the following fundamental identity or weak formulation of the coagulation and fragmentation operators [3], [4]:

$$\sum_{i=1}^{\infty} \varphi_i\, Q_i^\epsilon = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}\, u_i^\epsilon\, u_j^\epsilon\, (\varphi_{i+j} - \varphi_i - \varphi_j), \tag{15}$$

$$\sum_{i=1}^{\infty} \varphi_i\, F_i^\epsilon = - \sum_{i=2}^{\infty} B_i\, u_i^\epsilon \Big(\varphi_i - \sum_{j=1}^{i-1} \beta_{i,j}\, \varphi_j \Big), \tag{16}$$

which holds for any sequence of numbers φ_i. By choosing $\varphi_i := i$ above and thanks to (9), we have

$$\sum_{i=1}^{\infty} i\, Q_i^\epsilon = \sum_{i=1}^{\infty} i\, F_i^\epsilon = 0. \tag{17}$$

Therefore, summing together Eq. (3) and Eq. (4) multiplied by i, taking into account the result (17), we get

$$\frac{\partial}{\partial t} \left[\sum_{i=1}^{\infty} i\, u_i^\epsilon \right] - \Delta_x \left[\sum_{i=1}^{\infty} i\, d_i\, u_i^\epsilon \right] = 0. \tag{18}$$

Denoting

$$\rho^\epsilon(t,x) = \sum_{i=1}^{\infty} i\, u_i^\epsilon(t,x), \tag{19}$$

and

$$A^\epsilon(t,x) = [\rho^\epsilon(t,x)]^{-1} \sum_{i=1}^{\infty} i\, d_i\, u_i^\epsilon(t,x), \tag{20}$$

the following system can be derived from Eqs. (3), (4) and (18)

$$\begin{cases} \dfrac{\partial \rho^\epsilon}{\partial t} - \Delta_x (A^\epsilon \, \rho^\epsilon) = 0 & \text{in } [0,T] \times \Omega_\epsilon, \\ \\ \nabla_x (A^\epsilon \, \rho^\epsilon) \cdot n = 0 & \text{on } [0,T] \times \partial\Omega, \\ \\ \nabla_x (A^\epsilon \, \rho^\epsilon) \cdot n = d_1 \, \epsilon \, \psi(t, x, \frac{x}{\epsilon}) & \text{on } [0,T] \times \Gamma_\epsilon, \\ \\ \rho^\epsilon(0,x) = U_1 & \text{in } \Omega_\epsilon. \end{cases} \tag{21}$$

We observe that (for all $t \in [0,T]$)

$$\|A^\epsilon(t,\cdot)\|_{L^\infty(\Omega_\epsilon)} \leq \sup_i \, d_i. \tag{22}$$

Multiplying the first equation in (21) by the function w^ϵ defined by the following dual problem:

$$\begin{cases} -\left(\dfrac{\partial w^\epsilon}{\partial t} + A^\epsilon \, \Delta_x w^\epsilon\right) = A^\epsilon \, \rho^\epsilon & \text{in } [0,T] \times \Omega_\epsilon, \\ \\ \nabla_x w^\epsilon \cdot n = 0 & \text{on } [0,T] \times \partial\Omega, \\ \\ \nabla_x w^\epsilon \cdot n = 0 & \text{on } [0,T] \times \Gamma_\epsilon, \\ \\ w^\epsilon(T,x) = 0 & \text{in } \Omega_\epsilon, \end{cases} \tag{23}$$

and integrating by parts on $[0,T] \times \Omega_\epsilon$, we end up with the identity

$$\begin{aligned} &\int_0^T \int_{\Omega_\epsilon} A^\epsilon(t,x) \, (\rho^\epsilon(t,x))^2 \, dt \, dx = \int_{\Omega_\epsilon} w^\epsilon(0,x) \, \rho^\epsilon(0,x) \, dx \\ &+ \epsilon \, d_1 \int_0^T \int_{\Gamma_\epsilon} \psi(t,x,\frac{x}{\epsilon}) \, w^\epsilon(t,x) \, dt \, d\sigma_\epsilon(x) := I_1 + I_2, \end{aligned} \tag{24}$$

where $d\sigma_\epsilon$ is the measure on Γ_ϵ.

Let us now estimate the terms I_1 and I_2. From Hölder's inequality we obtain

$$I_1 = \int_{\Omega_\epsilon} w^\epsilon(0,x)\,\rho^\epsilon(0,x)\,dx \le U_1\,|\Omega_\epsilon|^{1/2}\,\|w^\epsilon(0,\cdot)\|_{L^2(\Omega_\epsilon)}. \tag{25}$$

Applying once more the Hölder inequality and using the estimate (22), it holds

$$\begin{aligned}
&\int_{\Omega_\epsilon} |w^\epsilon(0,x)|^2\,dx = \int_{\Omega_\epsilon} \left| \int_0^T \sqrt{A^\epsilon}\,\frac{\partial_t\,w^\epsilon}{\sqrt{A^\epsilon}}\,dt \right|^2 dx \\
&\le T\,\|A^\epsilon\|_{L^\infty(\Omega_\epsilon)} \int_0^T \int_{\Omega_\epsilon} (A^\epsilon)^{-1} \left| \frac{\partial}{\partial t} w^\epsilon(t,x) \right|^2 dt\,dx \\
&\le T\,(\sup_i\, d_i) \int_0^T \int_{\Omega_\epsilon} (A^\epsilon)^{-1} \left| \frac{\partial}{\partial t} w^\epsilon(t,x) \right|^2 dt\,dx.
\end{aligned} \tag{26}$$

By exploiting the dual problem (23), Eq. (26) becomes

$$\begin{aligned}
&\int_{\Omega_\epsilon} |w^\epsilon(0,x)|^2\,dx \le T\,(\sup_i\, d_i) \int_0^T \int_{\Omega_\epsilon} (A^\epsilon)^{-1}\,|A^\epsilon\,\Delta_x w^\epsilon + A^\epsilon\,\rho^\epsilon|^2\,dt\,dx \\
&\le T\,(\sup_i\, d_i) \int_0^T \int_{\Omega_\epsilon} (A^\epsilon)^{-1} \left[2\,(A^\epsilon)^2\,(\Delta_x w^\epsilon)^2 + 2\,(A^\epsilon)^2\,(\rho^\epsilon)^2 \right] dt\,dx.
\end{aligned} \tag{27}$$

Let us now estimate the first term on the right-hand side of (27). Multiplying the first equation in (23) by $(\Delta_x w^\epsilon)$,

$$\int_{\Omega_\epsilon} (\Delta_x w^\epsilon) \left(\frac{\partial w^\epsilon}{\partial t} \right) dx + \int_{\Omega_\epsilon} A^\epsilon\,(\Delta_x w^\epsilon)^2\,dx = -\int_{\Omega_\epsilon} A^\epsilon\,\rho^\epsilon\,(\Delta_x w^\epsilon)\,dx \tag{28}$$

and integrating by parts on Ω_ϵ, we get

$$-\frac{\partial}{\partial t} \int_{\Omega_\epsilon} \frac{|\nabla_x w^\epsilon|^2}{2}\,dx + \int_{\Omega_\epsilon} A^\epsilon\,(\Delta_x w^\epsilon)^2\,dx = -\int_{\Omega_\epsilon} A^\epsilon\,\rho^\epsilon (\Delta_x w^\epsilon)\,dx. \tag{29}$$

Then, integrating once more over the time interval $[0,T]$ and using Young's inequality on the right-hand side of (29), one finds that

$$\int_{\Omega_\epsilon} |\nabla_x w^\epsilon(0,x)|^2\,dx + \int_0^T \int_{\Omega_\epsilon} A^\epsilon\,(\Delta_x w^\epsilon)^2\,dt\,dx \le \int_0^T \int_{\Omega_\epsilon} (\rho^\epsilon)^2\,A^\epsilon\,dt\,dx. \tag{30}$$

Since the first term on the left-hand side of (30) is nonnegative, we conclude that

$$\int_0^T \int_{\Omega_\epsilon} A^\epsilon\,(\Delta_x w^\epsilon)^2\,dt\,dx \le \int_0^T \int_{\Omega_\epsilon} (\rho^\epsilon)^2\,A^\epsilon\,dt\,dx. \tag{31}$$

Inserting Eq. (31) into Eq. (27), one has

$$\int_{\Omega_\epsilon} |w^\epsilon(0,x)|^2\,dx \le 4\,T\,(\sup_i\, d_i)\int_0^T \int_{\Omega_\epsilon} A^\epsilon\,(\rho^\epsilon)^2\,dt\,dx. \tag{32}$$

Therefore, we end up with the estimate

$$I_1 \le 2\,U_1\left[|\Omega_\epsilon|\,T\,\sup_i\, d_i\right]^{1/2}\left[\int_0^T \int_{\Omega_\epsilon} A^\epsilon\,(\rho^\epsilon)^2\,dt\,dx\right]^{1/2}. \tag{33}$$

By using Lemma A.1 and Hölder's inequality, the term I_2 in (24) can be rewritten as

$$\begin{aligned} I_2 &= \epsilon\,d_1\int_0^T \int_{\Gamma_\epsilon} \psi(t,x,\frac{x}{\epsilon})\,w^\epsilon(t,x)\,dt\,d\sigma_\epsilon(x) \\ &\le C\,d_1\int_0^T \|\psi(t)\|_B\left\{\left[\int_{\Omega_\epsilon}|w^\epsilon|^2\,dx\right]^{1/2}+\epsilon\left[\int_{\Omega_\epsilon}|\nabla_x w^\epsilon|^2\,dx\right]^{1/2}\right\}, \end{aligned} \tag{34}$$

where we have taken into account the following estimate (see Lemma B.1):

$$\epsilon\int_{\Gamma_\epsilon}|\psi(t,x,\frac{x}{\epsilon})|^2\,d\sigma_\epsilon(x) \le \tilde{C}\,\|\psi(t)\|_B^2 \tag{35}$$

with $\tilde{C}$ being a positive constant independent of ϵ and $B = C^1[\overline{\Omega};C^1_\#(Y)]$. Note that we do not really need the C^1 in the estimate above, continuity would indeed be sufficient.

Since $\psi \in L^\infty([0,T];B)$, using the Cauchy-Schwarz inequality, Eq. (34) reads

$$I_2 \le C_1\,d_1\,\|w^\epsilon\|_{L^2(0,T;L^2(\Omega_\epsilon))}+C_2\,d_1\,\epsilon\,\|\nabla_x w^\epsilon\|_{L^2(0,T;L^2(\Omega_\epsilon))} := J_1+J_2, \tag{36}$$

where C_1 and C_2 are positive constants independent of ϵ. Let us now estimate the terms J_1 and J_2. Using Hölder's inequality and estimate (22), by following the same strategy as the one leading to (32), we have

$$\begin{aligned} &\int_0^T \int_{\Omega_\epsilon}|w^\epsilon(t,x)|^2\,dt\,dx = \int_0^T \int_{\Omega_\epsilon}\left|\int_t^T \sqrt{A^\epsilon}\,\frac{\partial_s w^\epsilon(s,x)}{\sqrt{A^\epsilon}}\,ds\right|^2 dt\,dx \\ &\le T^2\,(\sup_i\, d_i)\int_0^T \int_{\Omega_\epsilon}(A^\epsilon)^{-1}\left|\frac{\partial w^\epsilon(t,x)}{\partial t}\right|^2 dt\,dx \\ &\le 4\,T^2\,(\sup_i\, d_i)\int_0^T \int_{\Omega_\epsilon} A^\epsilon\,(\rho^\epsilon)^2\,dt\,dx. \end{aligned} \tag{37}$$

Therefore,

$$
\begin{aligned}
J_1 &= C_1\, d_1 \left[\int_0^T \int_{\Omega_\epsilon} |w^\epsilon(t,x)|^2 \, dt\, dx \right]^{1/2} \\
&\leq 2\, C_1\, d_1\, T\, (\sup_i \; d_i)^{1/2} \left[\int_0^T \int_{\Omega_\epsilon} A^\epsilon \, (\rho^\epsilon)^2 \, dt\, dx \right]^{1/2} .
\end{aligned} \tag{38}
$$

In order to estimate J_2, we go back to Eq. (29), then integrating over $[t, T]$, one has

$$
\begin{aligned}
&\frac{1}{2} \int_t^T \int_{\Omega_\epsilon} \frac{\partial}{\partial s} |\nabla_x w^\epsilon(s,x)|^2 \, ds\, dx - \int_t^T \int_{\Omega_\epsilon} A^\epsilon \, (\Delta_x w^\epsilon)^2 \, ds\, dx \\
&= \int_t^T \int_{\Omega_\epsilon} A^\epsilon \, \rho^\epsilon \, (\Delta_x w^\epsilon) \, ds\, dx.
\end{aligned} \tag{39}
$$

Young's inequality applied to the right-hand side of Eq. (39) leads to

$$
\int_{\Omega_\epsilon} |\nabla_x w^\epsilon(t,x)|^2 \, dx + \int_t^T \int_{\Omega_\epsilon} A^\epsilon \, (\Delta_x w^\epsilon)^2 \, ds\, dx \leq \int_t^T \int_{\Omega_\epsilon} A^\epsilon \, (\rho^\epsilon)^2 \, ds\, dx. \tag{40}
$$

Taking into account that the second term on the left-hand side of (40) is nonnegative and integrating once more over time, we get

$$
\int_0^T \int_{\Omega_\epsilon} |\nabla_x w^\epsilon(t,x)|^2 \, dt\, dx \leq T \int_0^T \int_{\Omega_\epsilon} A^\epsilon \, (\rho^\epsilon)^2 \, dt\, dx. \tag{41}
$$

Therefore, we conclude that

$$
\begin{aligned}
J_2 =& C_2\, d_1\, \epsilon \left[\int_0^T \int_{\Omega_\epsilon} |\nabla_x w^\epsilon(t,x)|^2 \, dt\, dx \right]^{1/2} \\
&\leq C_2\, d_1\, \epsilon\, (T)^{1/2} \left[\int_0^T \int_{\Omega_\epsilon} A^\epsilon \, (\rho^\epsilon)^2 \, dt\, dx \right]^{1/2} .
\end{aligned} \tag{42}
$$

By combining (38) and (42), we end up with the estimate

$$
I_2 \leq d_1 \left[2\, C_1\, T \sqrt{\sup_i \; d_i} + C_2\, \epsilon \sqrt{T} \right] \left[\int_0^T \int_{\Omega_\epsilon} A^\epsilon \, (\rho^\epsilon)^2 \, dt\, dx \right]^{1/2} . \tag{43}
$$

Hence, inserting the estimates (33) and (43) in Eq. (24), one obtains

$$
\int_0^T \int_{\Omega_\epsilon} A^\epsilon(t,x) \, (\rho^\epsilon(t,x))^2 \, dt\, dx \leq C_3^2 \tag{44}
$$

where

$$C_3 = \max\left(2\,U_1\,\sqrt{|\Omega_\epsilon|\,T\,\sup_i d_i},\, d_1\,[2\,C_1\,T\,\sqrt{\sup_i d_i} + C_2\,\sqrt{T}]\right). \tag{45}$$

Thus, recalling the definitions of A^ϵ and ρ^ϵ, the assertion of the Lemma follows immediately.

□

Corollary: If we assume that $\inf_i d_i > 0$ and

$$a_{i,j} \leq \mathrm{Cst}\,(i^{1-\zeta} + j^{1-\zeta}) \tag{46}$$

for some $\zeta > 0$, then, the estimate (14) leads to the following bound:

$$\int_0^T \int_{\Omega_\epsilon} \left|\sum_{j=1}^{\infty} a_{i,j}\, u_j^\epsilon(t,x)\right|^2 dt\,dx \leq C \tag{47}$$

with C independent of ϵ.

Lemma 2.2. *Let $T > 0$ be arbitrary and u_1^ϵ be a classical solution of (3). Then,*

$$\|u_1^\epsilon\|_{L^\infty(0,T;L^\infty(\Omega_\epsilon))} \leq |U_1| + \|u_1^\epsilon\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))} + \gamma_1. \tag{48}$$

Proof. Let us test the first equation of (3) with the function

$$\phi_1 \equiv p\,(u_1^\epsilon)^{(p-1)} \qquad p \geq 2.$$

We stress that the function ϕ_1 is strictly positive and continuously differentiable in $[0,t] \times \overline{\Omega}$, for all $t > 0$. Integrating, the divergence theorem yields

$$\begin{aligned}
&\int_0^t ds \int_{\Omega_\epsilon} \frac{\partial}{\partial s}(u_1^\epsilon)^p(s)\,dx + d_1\,p\,(p-1)\int_0^t ds \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon|^2\,(u_1^\epsilon)^{(p-2)}\,dx \\
&= -p\int_0^t ds \int_{\Omega_\epsilon} a_{1,1}\,(u_1^\epsilon)^{(p+1)}\,dx - p\int_0^t ds \int_{\Omega_\epsilon} (u_1^\epsilon)^p \sum_{j=2}^{\infty} a_{1,j}\,u_j^\epsilon\,dx \\
&+ p\int_0^t ds \int_{\Omega_\epsilon} (u_1^\epsilon)^{(p-1)} \sum_{j=2}^{\infty} B_j\,\beta_{j,1}\,u_j^\epsilon\,dx + \epsilon\,d_1\,p\int_0^t ds \int_{\Gamma_\epsilon} \psi(s,x,\frac{x}{\epsilon})\,(u_1^\epsilon)^{(p-1)}\,d\sigma_\epsilon(x) \\
&\leq -p\int_0^t ds \int_{\Omega_\epsilon} \sum_{j=2}^{\infty}[a_{1,j}\,u_1^\epsilon - B_j\,\beta_{j,1}]\,u_j^\epsilon\,(u_1^\epsilon)^{(p-1)}\,dx \\
&+ \epsilon\,d_1\,p\int_0^t ds \int_{\Gamma_\epsilon} \psi(s,x,\frac{x}{\epsilon})\,(u_1^\epsilon)^{(p-1)}\,d\sigma_\epsilon(x).
\end{aligned} \tag{49}$$

By exploiting the assumption (10), we end up with the estimate

$$
\begin{aligned}
&\int_0^t ds \int_{\Omega_\epsilon} \frac{\partial}{\partial s}(u_1^\epsilon)^p(s)\,dx + d_1\,p\,(p-1)\int_0^t ds \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon|^2\,(u_1^\epsilon)^{(p-2)}\,dx \\
&\le \epsilon\, d_1\, p \int_0^t ds \int_{\Gamma_\epsilon} \psi(s,x,\frac{x}{\epsilon})\,(u_1^\epsilon)^{(p-1)}\,d\sigma_\epsilon(x) \\
&+ p\,\gamma_1^p \int_0^t ds \int_{\Omega_\epsilon} \sum_{j=2}^{\infty} a_{1,j}\,u_j^\epsilon\,dx
\end{aligned}
\tag{50}
$$

The Hölder inequality applied to the right-hand side of (50) and the duality estimate lead to

$$
\begin{aligned}
&\int_{\Omega_\epsilon} (u_1^\epsilon(t,x))^p\,dx + d_1\,p\,(p-1)\int_0^t ds \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon|^2\,(u_1^\epsilon)^{(p-2)}\,dx \\
&\le \int_{\Omega_\epsilon} U_1^p\,dx + \epsilon\,d_1\,p\,\|\psi\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))}\int_0^t ds\int_{\Gamma_\epsilon}(u_1^\epsilon)^{(p-1)}\,d\sigma_\epsilon(x) + C_T\,p\,\gamma_1^p.
\end{aligned}
\tag{51}
$$

Since the second term on the left-hand side of (51) is nonnegative, one has

$$
\begin{aligned}
&\int_{\Omega_\epsilon} (u_1^\epsilon(t,x))^p\,dx \le \int_{\Omega_\epsilon} U_1^p\,dx \\
&+ \epsilon\,d_1\,p\,\|\psi\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))}\int_0^t ds\int_{\Gamma_\epsilon}[1+(u_1^\epsilon)^p]\,d\sigma_\epsilon(x) + C_T\,p\,\gamma_1^p \\
&\le \int_{\Omega_\epsilon} U_1^p\,dx + \epsilon\,d_1\,p\,\|\psi\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))}\,T\,|\Gamma_\epsilon| \\
&+ \epsilon\,d_1\,p\,\|\psi\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))}\int_0^t ds\int_{\Gamma_\epsilon}(u_1^\epsilon)^p\,d\sigma_\epsilon(x) + C_T\,p\,\gamma_1^p.
\end{aligned}
\tag{52}
$$

Hence, we conclude that

$$
\sup_{t\in[0,T]}\lim_{p\to\infty}\left[\int_{\Omega_\epsilon}(u_1^\epsilon(t,x))^p\,dx\right]^{1/p} \le |U_1| + \|u_1^\epsilon\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))} + \gamma_1. \tag{53}
$$

□

The boundedness of $u_1^\epsilon(t,x)$ in $L^\infty([0,T]\times\Gamma_\epsilon)$, uniformly in ϵ, can be immediately deduced from Lemma 2.3 below.

Lemma 2.3. *Let $T>0$ be arbitrary and u_1^ϵ be a classical solution of (3). Then,*

$$
\|u_1^\epsilon\|_{L^\infty(0,T;L^\infty(\Gamma_\epsilon))} \le C\,\max(\|\psi\|_{L^\infty(0,T;B)},\gamma_1) \tag{54}
$$

where C is independent of ϵ.

In order to establish Lemma 2.3, we will first need the following preliminary result [11].

Theorem 2.1 ([11], Theorem 5.2)**.** *Assume that there exist positive constants* T*,* $\hat{k} = \|\psi\|_{L^\infty(0,T;B)}$*,* γ*, such that for all* $k \geq \hat{k}$ *we have*

$$\|u_\epsilon^{(k)}\|_{Q_\epsilon(T)}^2 := \sup_{0\leq t\leq T} \int_{\Omega_\epsilon} |u_\epsilon^{(k)}|^2\,dx + \int_0^T dt \int_{\Omega_\epsilon} |\nabla u_\epsilon^{(k)}|^2\,dx \leq \epsilon\,\gamma\,k^2 \int_0^T dt\,|B_k^\epsilon(t)| \tag{55}$$

where $u_\epsilon^{(k)}(t) := (u_1^\epsilon(t) - k)_+$ *and* $B_k^\epsilon(t)$ *is the set of points on* Γ_ϵ *at which* $u_1^\epsilon(t,x) > k$*.*

Then

$$ess\ sup_{(t,x)\in[0,T]\times\Gamma_\epsilon} u_1^\epsilon(t,x) \leq 2\,m\,\hat{k} \tag{56}$$

where the positive constant m *is independent of* ϵ*.*

Proof. *of Lemma 2.3*

Since this proof is close to the proof of Lemma 5.2 in [11], we only sketch it. Let $T > 0$ and $k \geq 0$ be fixed. Define: $u_\epsilon^{(k)}(t) := (u_1^\epsilon(t) - k)_+$ for $t \geq 0$, with derivatives:

$$\frac{\partial u_\epsilon^{(k)}}{\partial t} = \frac{\partial u_1^\epsilon}{\partial t}\,\mathbb{1}_{\{u_1^\epsilon > k\}} \tag{57}$$

$$\nabla_x u_\epsilon^{(k)} = \nabla_x u_1^\epsilon\,\mathbb{1}_{\{u_1^\epsilon > k\}}. \tag{58}$$

Moreover,

$$u_\epsilon^{(k)}\ |_{\partial\Omega} = (u_1^\epsilon\ |_{\partial\Omega} - k)_+ \tag{59}$$

$$u_\epsilon^{(k)}\ |_{\Gamma_\epsilon} = (u_1^\epsilon\ |_{\Gamma_\epsilon} - k)_+ \tag{60}$$

Let us assume $k \geq \hat{k}$, where $\hat{k} := \|\psi\|_{L^\infty(0,T;B)}$. Then,

$$u_1^\epsilon(0,x) = U_1 \leq \hat{k} \leq k. \tag{61}$$

For $t \in [0, T_1]$ with $T_1 \leq T$, we get

$$\frac{1}{2}\int_{\Omega_\epsilon} |u_\epsilon^{(k)}(t)|^2\,dx = \int_0^t \frac{d}{ds}\left[\frac{1}{2}\int_{\Omega_\epsilon} |u_\epsilon^{(k)}(s)|^2\,dx\right] ds$$
$$= \int_0^t ds \int_{\Omega_\epsilon} \frac{\partial u_\epsilon^{(k)}(s)}{\partial s}\, u_\epsilon^{(k)}(s)\,dx. \tag{62}$$

Taking into account Eq. (57) and Eq. (3), we obtain that for all $s \in [0, T_1]$

$$\int_{\Omega_\epsilon} \frac{\partial u_\epsilon^{(k)}(s)}{\partial s}\, u_\epsilon^{(k)}(s)\,dx = \int_{\Omega_\epsilon} \frac{\partial u_1^\epsilon(s)}{\partial s}\, u_\epsilon^{(k)}(s)\,dx$$
$$= \int_{\Omega_\epsilon} \left[d_1\,\Delta_x u_1^\epsilon - u_1^\epsilon \sum_{j=1}^{\infty} a_{1,j} u_j^\epsilon + \sum_{j=1}^{\infty} B_{1+j}\,\beta_{1+j,1}\,u_{1+j}^\epsilon\right] u_\epsilon^{(k)}(s)\,dx$$
$$= \epsilon\,d_1 \int_{\Gamma_\epsilon} \psi\left(s, x, \frac{x}{\epsilon}\right) u_\epsilon^{(k)}(s)\,d\sigma_\epsilon(x) - d_1 \int_{\Omega_\epsilon} \nabla_x u_1^\epsilon(s)\cdot\nabla_x u_\epsilon^{(k)}(s)\,dx$$
$$- \int_{\Omega_\epsilon} (u_1^\epsilon(s))^2\,a_{1,1}\,u_\epsilon^{(k)}(s)\,dx - \int_{\Omega_\epsilon} u_1^\epsilon(s) \sum_{j=2}^{\infty}\left[a_{1,j}\,u_j^\epsilon(s)\right] u_\epsilon^{(k)}(s)\,dx \tag{63}$$
$$+ \int_{\Omega_\epsilon}\left[\sum_{j=2}^{\infty} B_j\,\beta_{j,1}\,u_j^\epsilon(s)\right] u_\epsilon^{(k)}(s)\,dx$$
$$\le \epsilon\,d_1 \int_{\Gamma_\epsilon} \psi\left(s, x, \frac{x}{\epsilon}\right) u_\epsilon^{(k)}(s)\,d\sigma_\epsilon(x) - d_1 \int_{\Omega_\epsilon} \nabla_x u_1^\epsilon(s)\cdot\nabla_x u_\epsilon^{(k)}(s)\,dx$$
$$- \int_{\Omega_\epsilon} \sum_{j=2}^{\infty}\left[a_{1,j}\,u_1^\epsilon(s) - B_j\,\beta_{j,1}\right] u_j^\epsilon(s)\,u_\epsilon^{(k)}(s)\,dx.$$

By using the assumption (10), Lemma A.1 and Young's inequality, one has, taking $k \ge \gamma_1$,

$$\int_{\Omega_\epsilon} \frac{\partial u_\epsilon^{(k)}(s)}{\partial s}\, u_\epsilon^{(k)}(s)\,dx \le \frac{\epsilon\,d_1}{2} \int_{B_k^\epsilon(s)} \left|\psi\left(s, x, \frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x)$$
$$+ \frac{C_1\,d_1}{2} \int_{A_k^\epsilon(s)} |u_\epsilon^{(k)}(s)|^2\,dx - d_1\left(1 - \frac{C_1\epsilon^2}{2}\right) \int_{\Omega_\epsilon} |\nabla_x u_\epsilon^{(k)}(s)|^2\,dx \tag{64}$$

where we denote by $A_k^\epsilon(t)$ and $B_k^\epsilon(t)$ the set of points in Ω_ϵ and on Γ_ϵ, respectively, at which $u_1^\epsilon(t,x) > k$. It holds:

$$|A_k^\epsilon(t)| \le |\Omega_\epsilon|,$$

$$|B_k^\epsilon(t)| \le |\Gamma_\epsilon|,$$

where $|\cdot|$ is the Hausdorff measure. Inserting Eq. (64) into Eq. (62) and varying over t, we end up with the estimate:

$$
\begin{aligned}
&\sup_{0\le t\le T_1}\left[\frac{1}{2}\int_{\Omega_\epsilon}|u_\epsilon^{(k)}(t)|^2\,dx\right]+d_1\left(1-\frac{C_1\,\epsilon^2}{2}\right)\int_0^{T_1}dt\int_{\Omega_\epsilon}|\nabla_x u_\epsilon^{(k)}(t)|^2\,dx\\
&\le\frac{C_1\,d_1}{2}\int_0^{T_1}dt\int_{A_k^\epsilon(t)}|u_\epsilon^{(k)}(t)|^2\,dx+\frac{\epsilon\,d_1}{2}\int_0^{T_1}dt\int_{B_k^\epsilon(t)}\left|\psi\left(t,x,\frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x)
\end{aligned}
\tag{65}
$$

Introducing the following norm

$$
\|u\|^2_{Q_\epsilon(T)}:=\sup_{0\le t\le T}\int_{\Omega_\epsilon}|u(t)|^2\,dx+\int_0^T dt\int_{\Omega_\epsilon}|\nabla u(t)|^2\,dx, \tag{66}
$$

the inequality (65) can be rewritten as follows:

$$
\begin{aligned}
\min\left\{\frac{1}{2},d_1\left(1-\frac{C_1\,\epsilon^2}{2}\right)\right\}\|u_\epsilon^{(k)}\|^2_{Q_\epsilon(T_1)}\le{}&\frac{C_1\,d_1}{2}\int_0^{T_1}dt\int_{A_k^\epsilon(t)}|u_\epsilon^{(k)}(t)|^2\,dx\\
&+\frac{\epsilon\,d_1}{2}\int_0^{T_1}dt\int_{B_k^\epsilon(t)}\left|\psi\left(t,x,\frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x).
\end{aligned}
\tag{67}
$$

Let us estimate the right-hand side of (67). From Hölder's inequality, we obtain

$$
\int_0^{T_1}dt\int_{A_k^\epsilon(t)}|u_\epsilon^{(k)}(t)|^2\,dx\le\|u_\epsilon^{(k)}\|^2_{L^{\overline{r}_1}(0,T_1;L^{\overline{q}_1}(\Omega_\epsilon))}\,\|\mathbb{1}_{A_k^\epsilon}\|_{L^{r_1'}(0,T_1;L^{q_1'}(\Omega_\epsilon))}, \tag{68}
$$

with $r_1'=\dfrac{r_1}{r_1-1}$, $q_1'=\dfrac{q_1}{q_1-1}$, $\overline{r}_1=2\,r_1$, $\overline{q}_1=2\,q_1$, where, for $N>2$, $\overline{r}_1\in(2,\infty)$ and $\overline{q}_1\in(2,\frac{2\,N}{(N-2)})$ have been chosen such that

$$
\frac{1}{\overline{r}_1}+\frac{N}{2\,\overline{q}_1}=\frac{N}{4}.
$$

In particular, $r_1',q_1'<\infty$, so that (68) yields

$$
\int_0^{T_1}dt\int_{A_k^\epsilon(t)}|u_\epsilon^{(k)}(t)|^2\,dx\le\|u_\epsilon^{(k)}\|^2_{L^{\overline{r}_1}(0,T_1;L^{\overline{q}_1}(\Omega_\epsilon))}\,|\Omega|^{1/q_1'}\,T_1^{1/r_1'}. \tag{69}
$$

If we choose (for ϵ small enough)

$$
T_1^{1/r_1'}<\frac{\min\{1,d_1\}}{2C_1d_1c^2}\,|\Omega|^{-1/q_1'}\le\frac{\min\left\{\frac{1}{2},d_1\left(1-\frac{C_1\,\epsilon^2}{2}\right)\right\}}{C_1d_1c^2}\,|\Omega|^{-1/q_1'},
$$

then from (113) (and c being the constant appearing in this formula) it follows that

$$
\frac{C_1\,d_1}{2}\int_0^{T_1}dt\int_{A_k^\epsilon(t)}|u_\epsilon^{(k)}(t)|^2\,dx\le\frac{1}{2}\min\left\{\frac{1}{2},d_1\left(1-\frac{C_1\,\epsilon^2}{2}\right)\right\}\|u_\epsilon^{(k)}\|^2_{Q_\epsilon(T_1)}. \tag{70}
$$

Analogously, from Hölder's inequality we have, for $k \geq \hat{k}$

$$\begin{aligned} \frac{\epsilon\, d_1}{2} \int_0^{T_1} dt \int_{B_k^\epsilon(t)} \left| \psi\left(t, x, \frac{x}{\epsilon}\right) \right|^2 d\sigma_\epsilon(x) &\leq \frac{\epsilon\, d_1\, k^2}{2} \left(\frac{\hat{k}^2}{k^2}\right) \| \mathbb{1}_{B_k^\epsilon} \|_{L^1(0,T_1;L^1(\Gamma_\epsilon))} \\ &\leq \frac{\epsilon\, d_1\, k^2}{2} \int_0^{T_1} dt\, |B_k^\epsilon(t)|. \end{aligned} \tag{71}$$

Thus (67) yields

$$\| u_\epsilon^{(k)} \|^2_{Q_\epsilon(T_1)} \leq \epsilon\, \gamma\, k^2 \int_0^{T_1} dt\, |B_k^\epsilon(t)|. \tag{72}$$

Hence, by Theorem 2.1 we obtain

$$\| u_1^\epsilon \|_{L^\infty(0,T_1;L^\infty(\Gamma_\epsilon))} \leq 2\, m\, \max(\hat{k}, \gamma_1)$$

where the positive constant m is independent of ϵ. Analogous arguments are valid for the cylinder $[T_s, T_{s+1}] \times \Omega_\epsilon$, $s = 1, 2, \ldots, p-1$ with

$$\Big[T_{s+1} - T_s \Big]^{1/r_1'} < \frac{\min\{1, d_1\}}{2 C_1 d_1 c^2}\, |\Omega|^{-1/q_1'}$$

and $T_p \equiv T$. Thus, after a finite number of steps, we obtain the estimate (54).

□

Lemma 2.4. *The sequence $\nabla_x u_1^\epsilon$ is bounded in $L^2([0,T] \times \Omega_\epsilon)$, uniformly in ϵ.*

This Lemma can be easily proved by following the same arguments presented in [11] (Lemma 5.4), provided that the assumption (10) is taken into account.

Lemma 2.5. *Let $u_i^\epsilon(t,x)$ $(i \geq 2)$ be a classical solution of (4). Then*

$$\| u_i^\epsilon \|_{L^\infty(0,T;L^\infty(\Omega_\epsilon))} \leq K_i \tag{73}$$

uniformly with respect to ϵ, where

$$K_i = 1 + \frac{\left[\displaystyle\sum_{j=1}^{i-1} a_{j,i-j} K_j K_{i-j} \right]}{(B_i + a_{i,i})} + \gamma_i. \tag{74}$$

Proof. The Lemma can be proved directly by induction following the proof reported in [19] (Lemma 2.2, p. 284). Since we have a zero initial condition for the system

(4), we have chosen a function slightly different than what was done in [19] to test the ith equation of (4):

$$\phi_i \equiv p\,(u_i^\epsilon)^{(p-1)} \qquad p \geq 2.$$

We stress that the functions ϕ_i are strictly positive and continuously differentiable in $[0,t] \times \overline{\Omega}$, for all $t > 0$.

Therefore, multiplying the ith equation in system (4) by ϕ_i and reorganizing the terms appearing in the sums, we can write the estimate

$$||u_i^\epsilon||^p_{L^p(\Omega_\epsilon)} + d_i\, p\,(p-1) \int_0^t \int_{\Omega_\epsilon} |\nabla_x u_i^\epsilon|^2\, (u_i^\epsilon)^{p-2} dxds$$
$$\leq \int_0^t \int_{\Omega_\epsilon} \left[\frac{1}{2} \sum_{j=1}^{i-1} a_{i-j,j}\, u_j^\epsilon\, u_{i-j}^\epsilon - a_{i,i}\, |u_i^\epsilon|^2 - B_i\, u_i^\epsilon \right] p\,(u_i^\epsilon)^{p-1} dxds$$
$$- \int_0^t \int_{\Omega_\epsilon} \left[\sum_{j=1}^{i-1} a_{i,j}\, u_i^\epsilon\, u_j^\epsilon + \sum_{j=i+1}^{\infty} (a_{i,j}\, u_i^\epsilon - B_j\, \beta_{j,i})\; u_j^\epsilon \right] p\,(u_i^\epsilon)^{p-1} dxds.$$

We now work using an induction on i. Supposing that we already know that $\|u_j^\epsilon\|_{L^\infty(0,T;L^\infty(\Omega_\epsilon))} \leq K_j$ for all $j < i$, and using assumption (10), the previous estimate leads to

$$||u_i^\epsilon||^p_{L^p(\Omega_\epsilon)} \leq \int_0^t \int_{\Omega_\epsilon} \left[\frac{1}{2} \sum_{j=1}^{i-1} a_{i-j,j}\, K_j\, K_{i-j} - a_{i,i}\, |u_i^\epsilon|^2 - B_i\, u_i^\epsilon \right] p\,(u_i^\epsilon)^{p-1} dxds$$
$$+ \int_0^t \int_{\Omega_\epsilon} \sum_{j=i+1}^{\infty} a_{i,j}\,(-u_i^\epsilon + \gamma_i)\; u_j^\epsilon\, p\,(u_i^\epsilon)^{p-1} dxds =: I_1 + I_2.$$

Then, thanks for example to Young's inequality,

$$I_1 \leq \left[\left(\sum_{j=1}^{i-1} a_{i-j,j}\, K_j\, K_{i-j} \right)^p (B_i + a_{i,i})^{1-p} \right] |\Omega_\epsilon|\, T + p\, a_{i,i}\, |\Omega_\epsilon|\, T,$$

and

$$I_2 \leq \int_0^t \int_{\Omega_\epsilon} \sum_{j=i+1}^{\infty} a_{i,j}\,(\gamma_i - u_i^\epsilon)\, u_j^\epsilon\, 1_{\{u_i^\epsilon \leq \gamma_i\}}\, p\,(u_i^\epsilon)^{p-1} dxds$$
$$\leq p\, \gamma_i^p \int_0^t \int_{\Omega_\epsilon} \left(\sum_{j=i+1}^{\infty} a_{i,j}\, u_j^\epsilon \right) dxds$$
$$\leq C\, p\, \gamma_i^p (|\Omega_\epsilon|\, T)^{1/2},$$

where Cauchy-Schwarz inequality and the duality Lemma 2.1 (more precisely Eq. (47)) have been exploited.

Using these estimates for bounding $||u_i^\epsilon||_{L^p(\Omega_\epsilon)}$ and letting $p \to \infty$, we end up with the desired estimate.

□

Lemma 2.6. *The sequence $\nabla_x u_i^\epsilon$ $(i \geq 2)$ is bounded in $L^2([0,T] \times \Omega_\epsilon)$, uniformly in ϵ.*

This Lemma can be easily proved by following the same arguments presented in [11] (Lemma 5.6), provided that the assumption (10) is taken into account.

Lemma 2.7. *The sequence $\partial_t u_i^\epsilon$ $(i \geq 1)$ is bounded in $L^2([0,T] \times \Omega_\epsilon)$, uniformly in ϵ.*

Proof. Since this proof is close to the proof of Lemma 5.9 in [11], we only sketch it.

Case $i = 1$: Let us multiply the first equation in (3) by the function $\partial_t u_1^\epsilon(t,x)$. Integrating, the divergence theorem yields

$$\begin{aligned}
&\int_{\Omega_\epsilon} \left| \frac{\partial u_1^\epsilon(t,x)}{\partial t} \right|^2 dx + \frac{d_1}{2} \int_{\Omega_\epsilon} \frac{\partial}{\partial t} (|\nabla_x u_1^\epsilon(t,x)|^2)\, dx \\
&= \epsilon\, d_1 \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) \frac{\partial u_1^\epsilon}{\partial t}\, d\sigma_\epsilon(x) - \int_{\Omega_\epsilon} u_1^\epsilon \left(\sum_{j=1}^{\infty} a_{1,j}\, u_j^\epsilon \right) \frac{\partial u_1^\epsilon}{\partial t}\, dx \\
&+ \int_{\Omega_\epsilon} \left(\sum_{j=1}^{\infty} B_{1+j}\, \beta_{1+j,1}\, u_{1+j}^\epsilon \right) \frac{\partial u_1^\epsilon}{\partial t}\, dx.
\end{aligned} \tag{75}$$

Using Young's inequality and exploiting the boundedness of $u_1^\epsilon(t,x)$ in $L^\infty(0,T; L^\infty(\Omega_\epsilon))$, one gets

$$\begin{aligned}
&C_1 \int_{\Omega_\epsilon} \left| \frac{\partial u_1^\epsilon(t,x)}{\partial t} \right|^2 dx + \frac{d_1}{2} \int_{\Omega_\epsilon} \frac{\partial}{\partial t} (|\nabla_x u_1^\epsilon(t,x)|^2)\, dx \\
&\leq \epsilon\, d_1 \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) \frac{\partial u_1^\epsilon}{\partial t}\, d\sigma_\epsilon(x) + C_2 \int_{\Omega_\epsilon} \left| \sum_{j=1}^{\infty} a_{1,j}\, u_j^\epsilon \right|^2 dx \\
&+ C_3 \int_{\Omega_\epsilon} \left| \sum_{j=2}^{\infty} B_j\, \beta_{j,1}\, u_j^\epsilon \right|^2 dx,
\end{aligned} \tag{76}$$

where C_1, C_2 and C_3 are positive constants independent of ϵ. Integrating over $[0,t]$ with $t \in [0,T]$, thanks to (47) and (10), we end up with the estimate

$$
\begin{aligned}
C_1 \int_0^t ds \int_{\Omega_\epsilon} \left| \frac{\partial u_1^\epsilon}{\partial s} \right|^2 dx + \frac{d_1}{2} \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon(t,x)|^2 \, dx \le C_4 \\
+ \epsilon \, d_1 \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) u_1^\epsilon(t,x) \, d\sigma_\epsilon(x) \\
- \epsilon \, d_1 \int_0^t ds \int_{\Gamma_\epsilon} \frac{\partial}{\partial s} \psi\left(s, x, \frac{x}{\epsilon}\right) u_1^\epsilon(s,x) \, d\sigma_\epsilon(x),
\end{aligned} \tag{77}
$$

since $\psi\left(t=0, x, \frac{x}{\epsilon}\right) \equiv 0$.

Applying once more Young's inequality and taking into account the estimate (35) and Lemma A.1, Eq. (77) can be rewritten as follows

$$
C_1 \int_0^t ds \int_{\Omega_\epsilon} \left| \frac{\partial u_1^\epsilon}{\partial s} \right|^2 dx + \frac{d_1}{2}(1 - \epsilon^2 \, C_5) \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon(t,x)|^2 \, dx \le C_6, \tag{78}
$$

where the positive constants C_1, C_5, C_6 are independent of ϵ, since $\psi \in L^\infty(0,T;B)$, u_1^ϵ is bounded in $L^\infty(0,T;L^\infty(\Omega_\epsilon))$, $\nabla_x u_1^\epsilon$ is bounded in $L^2(0,T;L^2(\Omega_\epsilon))$ and the following inequality holds:

$$
\epsilon \int_{\Gamma_\epsilon} \left| \partial_t \psi\left(t, x, \frac{x}{\epsilon}\right) \right|^2 d\sigma_\epsilon(x) \le C_7 \, \|\partial_t \psi(t)\|_B^2 \le C_8, \tag{79}
$$

with C_7 and C_8 independent of ϵ. For a sequence ϵ of positive numbers going to zero: $(1 - \epsilon^2 C_5) \ge 0$. Then, the second term on the left-hand side of (78) is nonnegative, and one has:

$$
\|\partial_t u_1^\epsilon\|^2_{L^2(0,T;L^2(\Omega_\epsilon))} \le C, \tag{80}
$$

where $C \ge 0$ is a constant independent of ϵ.

Case $i \ge 2$: Let us multiply the first equation in (4) by the function $\partial_t u_i^\epsilon(t,x)$. Integrating, the divergence theorem yields

$$
\begin{aligned}
&\int_{\Omega_\epsilon} \left| \frac{\partial u_i^\epsilon(t,x)}{\partial t} \right|^2 dx + \frac{d_i}{2} \int_{\Omega_\epsilon} \frac{\partial}{\partial t}(|\nabla_x u_i^\epsilon(t,x)|^2) \, dx \\
&= \frac{1}{2} \int_{\Omega_\epsilon} \left(\sum_{j=1}^{i-1} a_{i-j,j} \, u_{i-j}^\epsilon \, u_j^\epsilon \right) \frac{\partial u_i^\epsilon}{\partial t} \, dx - \int_{\Omega_\epsilon} u_i^\epsilon \left(\sum_{j=1}^{\infty} a_{i,j} \, u_j^\epsilon \right) \frac{\partial u_i^\epsilon}{\partial t} \, dx \\
&+ \int_{\Omega_\epsilon} \left(\sum_{j=1}^{\infty} B_{i+j} \, \beta_{i+j,i} \, u_{i+j}^\epsilon \right) \frac{\partial u_i^\epsilon}{\partial t} \, dx - \int_{\Omega_\epsilon} B_i \, u_i^\epsilon \, \frac{\partial u_i^\epsilon}{\partial t} \, dx.
\end{aligned} \tag{81}
$$

Using Young's inequality and exploiting the boundedness of $u_i^\epsilon(t,x)$ in $L^\infty(0,T;L^\infty(\Omega_\epsilon))$, one gets

$$\begin{aligned} & C_1 \int_{\Omega_\epsilon} \left| \frac{\partial u_i^\epsilon(t,x)}{\partial t} \right|^2 dx + \frac{d_i}{2} \int_{\Omega_\epsilon} \frac{\partial}{\partial t} (|\nabla_x u_i^\epsilon(t,x)|^2)\, dx \\ & \le C_2 + C_3 \int_{\Omega_\epsilon} \left| \sum_{j=1}^{\infty} a_{i,j}\, u_j^\epsilon \right|^2 dx + C_4 \int_{\Omega_\epsilon} \left| \sum_{j=i+1}^{\infty} B_j\, \beta_{j,i}\, u_j^\epsilon \right|^2 dx, \end{aligned} \tag{82}$$

where C_1, C_2, C_3 and C_4 are positive constants independent of ϵ.

Integrating over $[0,t]$ with $t \in [0,T]$, thanks to (47) and (10), we end up with the estimate

$$C_1 \int_0^t ds \int_{\Omega_\epsilon} \left| \frac{\partial u_i^\epsilon}{\partial s} \right|^2 dx + \frac{d_i}{2} \int_{\Omega_\epsilon} |\nabla_x u_i^\epsilon(t,x)|^2\, dx \le C_5, \tag{83}$$

with $C_5 \ge 0$ independent of ϵ. Since the second term on the left-hand side of (83) is nonnegative, we conclude that

$$\|\partial_t u_i^\epsilon\|^2_{L^2(0,T;L^2(\Omega_\epsilon))} \le C, \tag{84}$$

where $C \ge 0$ is a constant independent of ϵ.

□

3 Proof of the main result

We start here the proof of our main Theorem 1.1.

3.1 Existence of solutions for a given $\epsilon > 0$

We first explain how the estimates of the previous section can be used in the proof of existence, for a given ε, of a solution to system (3) - (4). In order to do so, we introduce a finite size truncation of this system, which writes, once the dependence w.r.t. ε has been eliminated for readability,

$$\begin{cases} \dfrac{\partial u_1^n}{\partial t} - \nabla_x \cdot (d_1 \, \nabla_x u_1^n) + u_1^n \, \sum_{j=1}^n a_{1,j} u_j^n = \sum_{j=1}^{n-1} B_{1+j} \, \beta_{1+j,1} \, u_{1+j}^n & \text{in } [0,T] \times \Omega_\epsilon, \\ \dfrac{\partial u_1^n}{\partial \nu} \equiv \nabla_x u_1^n \cdot n = 0 & \text{on } [0,T] \times \partial\Omega, \\ \dfrac{\partial u_1^n}{\partial \nu} \equiv \nabla_x u_1^n \cdot n = \epsilon \, \psi(t, x, \frac{x}{\epsilon}) & \text{on } [0,T] \times \Gamma_\epsilon, \\ u_1^n(0,x) = U_1 & \text{in } \Omega_\epsilon, \end{cases} \tag{85}$$

and, if $i = 2, .., n$,

$$\begin{cases} \dfrac{\partial u_i^n}{\partial t} - \nabla_x \cdot (d_i \, \nabla_x u_i^n) = Q_i^n + F_i^n & \text{in } [0,T] \times \Omega_\epsilon, \\ \dfrac{\partial u_i^n}{\partial \nu} \equiv \nabla_x u_i^n \cdot n = 0 & \text{on } [0,T] \times \partial\Omega, \\ \dfrac{\partial u_i^n}{\partial \nu} \equiv \nabla_x u_i^n \cdot n = 0 & \text{on } [0,T] \times \Gamma_\epsilon, \\ u_i^n(0,x) = 0 & \text{in } \Omega_\epsilon, \end{cases} \tag{86}$$

where the truncated coagulation and breakup kernels Q_i^n, F_i^n write

$$Q_i^n := \frac{1}{2} \sum_{j=1}^{i-1} a_{i-j,j} \, u_{i-j}^n \, u_j^n - \sum_{j=1}^{n} a_{i,j} \, u_i^n \, u_j^n \tag{87}$$

$$F_i^n := \sum_{j=1}^{n-i} B_{i+j} \, \beta_{i+j,i} \, u_{i+j}^n - B_i \, u_i^n. \tag{88}$$

We observe, then, that the duality lemma (that is, Lemma 2.1) is still valid in this setting (with a proof that exactly follows the proof written above), so that we end up with the a priori estimate

$$\int_0^T \int_{\Omega_\varepsilon} \left| \sum_{i=1}^n i \, u_i^n(t,x) \right|^2 dt dx \leq C, \tag{89}$$

where C is a constant which does not depend on n.

Using now a proof analogous to that of Lemma 2.5, we can obtain the a priori estimate

$$||u_i^n||_{L^\infty([0,T]\times\Omega_\varepsilon)} \le C,$$

where C is a constant which also does not depend on n (in fact we will not use the uniformity w.r.t. n of this bound in the sequel).

At this point, we use standard theorems for systems of reaction-diffusion equations in order to get the existence and uniqueness of a smooth solution to system (85) - (86) (for a given $n \in I\!N - \{0\}$). We refer to [10], Prop. 3.2 p. 97 and Thm. 3.3 p. 105 for a complete description of a case with a slightly different boundary condition (homogeneous Neumann instead of Neumann) and a different right-hand side (but having the same crucial property, that is leading to an L^∞ a priori bound on the components of the unknown).

We now briefly explain how to pass to the limit when $n \to \infty$ in such a way that the limit of u_i^n satisfies the system (3) - (4). First, we notice that thanks to the duality estimate (89), each component sequence $(u_i^n)_{n\ge i}$ is bounded in $L^2([0,T]\times\Omega_\varepsilon)$. As a consequence, we can extract a subsequence from $(u_i^n)_{n\ge i}$ still denoted by $(u_i^n)_{n\ge i}$ (the extraction is done diagonally in such a way that it gives a subsequence which is common for all i) which converges in $L^2([0,T]\times\Omega_\varepsilon)$ weakly towards some function $u_i \in L^2([0,T]\times\Omega_\varepsilon)$. Using then the a priori estimate (consequence of the duality lemma, the assumptions on the coagulation and fragmentation coefficients, and natural bound on u_i^n in L^1 coming from a direct integration of the equations)

$$||\frac{\partial u_i^n}{\partial t} - \nabla_x \cdot (d_i\, \nabla_x u_i^n)||_{L^1([0,T]\times\Omega_\varepsilon)} \le C_i,$$

where C_i may depend on i but not on n, we see that the convergence in fact holds in $L^2([0,T]\times\Omega_\varepsilon)$ strong. This is sufficient to pass to the limit in system (85) - (86) and get system (3) - (4).

3.2 Homogenization

Now that existence for a given ϵ is obtained, we provide the proof of the homogenization part of Theorem 1.1.

In view of Lemmas 2.2-2.6 the sequences $\widetilde{u_i^\epsilon}$ and $\widetilde{\nabla_x u_i^\epsilon}$ $(i \geq 1)$ are bounded in $L^2([0,T] \times \Omega)$, and by application of Theorem B.1 and Theorem B.3, they two-scale converge, up to a subsequence, to: $[\chi(y)\, u_i(t,x)]$ and $[\chi(y)(\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))]$ $(i \geq 1)$ (cf. [1]). Similarly, in view of Lemma 2.7, it is possible to prove that the sequence $\left(\widetilde{\frac{\partial u_i^\epsilon}{\partial t}}\right)$ $(i \geq 1)$ two-scale converges to: $\left[\chi(y)\, \frac{\partial u_i}{\partial t}(t,x)\right]$ $(i \geq 1)$. We can now find the homogenized equations satisfied by $u_i(t,x)$ and $u_i^1(t,x,y)$.

In the case $i = 1$, let us multiply the first equation of (3) by the test function

$$\phi_\epsilon \equiv \phi(t,x) + \epsilon\, \phi_1\left(t, x, \frac{x}{\epsilon}\right),$$

where $\phi \in C^1([0,T]\times\overline{\Omega})$ and $\phi_1 \in C^1([0,T]\times\overline{\Omega}; C^\infty_\#(Y))$. Integrating, the divergence theorem yields

$$\begin{aligned}
&\int_0^T \int_{\Omega_\epsilon} \frac{\partial u_1^\epsilon}{\partial t}\, \phi_\epsilon(t,x,\frac{x}{\epsilon})\, dt\, dx + d_1 \int_0^T \int_{\Omega_\epsilon} \nabla_x u_1^\epsilon \cdot \nabla \phi_\epsilon\, dt\, dx \\
&+ \int_0^T \int_{\Omega_\epsilon} u_1^\epsilon \sum_{j=1}^\infty a_{1,j}\, u_j^\epsilon\, \phi_\epsilon\, dt\, dx = \epsilon\, d_1 \int_0^T \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) \phi_\epsilon\, dt\, d\sigma_\epsilon(x) \\
&+ \int_0^T \int_{\Omega_\epsilon} \sum_{j=1}^\infty B_{1+j}\, \beta_{1+j,1}\, u_{1+j}^\epsilon\, \phi_\epsilon\, dt\, dx.
\end{aligned} \tag{90}$$

Passing to the two-scale limit, thanks to Theorem B.2 and Theorem B.5, we get

$$\begin{aligned}
&\int_0^T \int_\Omega \int_{Y^*} \frac{\partial u_1}{\partial t}(t,x)\, \phi(t,x)\, dt\, dx\, dy \\
&+ d_1 \int_0^T \int_\Omega \int_{Y^*} [\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y)] \cdot [\nabla_x \phi(t,x) + \nabla_y \phi_1(t,x,y)]\, dt\, dx\, dy \\
&+ \int_0^T \int_\Omega \int_{Y^*} u_1(t,x) \sum_{j=1}^\infty a_{1,j}\, u_j(t,x)\, \phi(t,x)\, dt\, dx\, dy \\
&= d_1 \int_0^T \int_\Omega \int_\Gamma \psi(t,x,y)\, \phi(t,x)\, dt\, dx\, d\sigma(y) \\
&+ \int_0^T \int_\Omega \int_{Y^*} \sum_{j=1}^\infty B_{1+j}\, \beta_{1+j,1}\, u_{1+j}(t,x)\, \phi(t,x)\, dt\, dx\, dy.
\end{aligned} \tag{91}$$

The passage to the limit in the infinite sums can be performed since thanks to the assumptions on $a_{i,j}$, B_j and $\beta_{i,j}$, and to the duality lemma (using Cauchy-Schwarz inequality),

$$\int_0^T \int_{\Omega_\epsilon} \sum_{j=K}^{\infty} B_{1+j}\, \beta_{1+j,1}\, u_{1+j}^{\epsilon}\, \phi_\epsilon\, dt\, dx$$

$$\leq \int_0^T \int_{\Omega_\epsilon} \sum_{j=K}^{\infty} \gamma_1\, a_{1,1+j}\ u_{1+j}^{\epsilon}\ dt\, dx\, ||\phi_\epsilon||_\infty$$

$$\leq C_T\, K^{-\zeta}.$$

An integration by parts shows that (91) is a variational formulation associated to the following homogenized system:

$$-div_y[d_1(\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y))] = 0 \qquad \text{in } [0,T]\times\Omega\times Y^*, \tag{92}$$

$$[\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y)]\cdot n = 0 \qquad \text{on } [0,T]\times\Omega\times\Gamma, \tag{93}$$

$$\begin{aligned} &\theta\,\frac{\partial u_1}{\partial t}(t,x) - div_x\left[d_1 \int_{Y^*} (\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y))dy\right] \\ &+ \theta\, u_1(t,x) \sum_{j=1}^{\infty} a_{1,j}\, u_j(t,x) = d_1 \int_\Gamma \psi(t,x,y)\, d\sigma(y) \\ &+ \theta \sum_{j=1}^{\infty} B_{1+j}\, \beta_{1+j,1}\, u_{1+j}(t,x) \qquad \text{in } [0,T]\times\Omega, \end{aligned} \tag{94}$$

$$\left[\int_{Y^*} (\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y))\, dy\right]\cdot n = 0 \qquad \text{on } [0,T]\times\partial\Omega, \tag{95}$$

where

$$\theta = \int_Y \chi(y)dy = |Y^*|$$

is the volume fraction of material. Furthermore, by continuity, we have that

$$u_1(0,x) = U_1 \qquad \text{in } \Omega.$$

Taking advantage of the constancy of the diffusion coefficient d_1, Eqs. (92) and (93) can be reexpressed as follows:

$$\triangle_y u_1^1(t,x,y) = 0 \qquad \text{in } [0,T]\times\Omega\times Y^*, \tag{96}$$

$$\nabla_y u_1^1(t,x,y)\cdot n = -\nabla_x u_1(t,x)\cdot n \qquad \text{on } [0,T]\times\Omega\times\Gamma. \tag{97}$$

Then, $u_1^1(t,x,y)$ satisfying (96)-(97) can be written as

$$u_1^1(t,x,y) = \sum_{j=1}^{N} w_j(y)\,\frac{\partial u_1}{\partial x_j}(t,x), \tag{98}$$

where $(w_j)_{1\le j\le N}$ is the family of solutions of the cell problem:

$$\begin{cases} -\nabla_y\cdot[\nabla_y w_j + \hat{e}_j] = 0 & \text{in } Y^*,\\ (\nabla_y w_j + \hat{e}_j)\cdot n = 0 & \text{on } \Gamma,\\ y \to w_j(y) \quad Y-\text{periodic}. \end{cases} \tag{99}$$

By using the relation (98) in Eqs. (94) and (95), the system (11) can be immediately derived (cf. [1]).

In the case $i\ge 2$, let us multiply the first equation of (4) by the test function

$$\phi_\epsilon \equiv \phi(t,x) + \epsilon\,\phi_1\left(t,x,\frac{x}{\epsilon}\right),$$

where $\phi\in C^1([0,T]\times\overline{\Omega})$ and $\phi_1\in C^1([0,T]\times\overline{\Omega};C^\infty_\#(Y))$. Integrating, the divergence theorem yields

$$\begin{aligned}
&\int_0^T\int_{\Omega_\epsilon}\frac{\partial u_i^\epsilon}{\partial t}\,\phi_\epsilon(t,x,\frac{x}{\epsilon})\,dt\,dx + d_i\int_0^T\int_{\Omega_\epsilon}\nabla_x u_i^\epsilon\cdot\nabla\phi_\epsilon\,dt\,dx\\
&= -\int_0^T\int_{\Omega_\epsilon} u_i^\epsilon\sum_{j=1}^{\infty} a_{i,j}\,u_j^\epsilon\,\phi_\epsilon\,dt\,dx + \frac{1}{2}\int_0^T\int_{\Omega_\epsilon}\sum_{j=1}^{i-1} a_{j,i-j}\,u_j^\epsilon\,u_{i-j}^\epsilon\,\phi_\epsilon\,dt\,dx\\
&+\int_0^T\int_{\Omega_\epsilon}\sum_{j=1}^{\infty} B_{i+j}\,\beta_{i+j,i}\,u_{i+j}^\epsilon\,\phi_\epsilon\,dt\,dx - \int_0^T\int_{\Omega_\epsilon} B_i\,u_i^\epsilon\,\phi_\epsilon\,dt\,dx.
\end{aligned} \tag{100}$$

Passing to the two-scale limit, thanks to Theorem B.2, we get

$$\int_0^T \int_\Omega \int_{Y^*} \frac{\partial u_i}{\partial t}(t,x)\,\phi(t,x)\,dt\,dx\,dy$$
$$+ d_i \int_0^T \int_\Omega \int_{Y^*} [\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y)] \cdot [\nabla_x \phi(t,x) + \nabla_y \phi_1(t,x,y)]\,dt\,dx\,dy$$
$$= - \int_0^T \int_\Omega \int_{Y^*} u_i(t,x) \sum_{j=1}^{\infty} a_{i,j}\, u_j(t,x)\,\phi(t,x)\,dt\,dx\,dy$$
$$+ \frac{1}{2} \int_0^T \int_\Omega \int_{Y^*} \sum_{j=1}^{i-1} a_{j,i-j}\, u_j(t,x)\, u_{i-j}(t,x)\,\phi(t,x)\,dt\,dx\,dy$$
$$+ \int_0^T \int_\Omega \int_{Y^*} \sum_{j=1}^{\infty} B_{i+j}\,\beta_{i+j,i}\, u_{i+j}(t,x)\,\phi(t,x)\,dt\,dx\,dy$$
$$- \int_0^T \int_\Omega \int_{Y^*} B_i\, u_i(t,x)\,\phi(t,x)\,dt\,dx\,dy. \tag{101}$$

The passage to the limit in the infinite sums can be performed since thanks to the assumptions on $a_{i,j}$, B_j and $\beta_{i,j}$, and to the duality lemma,

$$\int_0^T \int_{\Omega_\epsilon} \sum_{j=K}^{\infty} B_{i+j}\,\beta_{i+j,i}\, u_{i+j}^\epsilon\,\phi_\epsilon\,dt\,dx$$
$$\leq \int_0^T \int_{\Omega_\epsilon} \sum_{j=K}^{\infty} \gamma_i\, a_{i,i+j}\;\; u_{i+j}^\epsilon \;\; dt\,dx\,||\phi_\epsilon||_\infty$$
$$\leq C_{T,i}\, K^{-\zeta}.$$

An integration by parts shows that (101) is a variational formulation associated to the following homogenized system:

$$-div_y[d_i(\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))] = 0 \qquad \text{in } [0,T]\times\Omega\times Y^*, \tag{102}$$

$$[\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y)] \cdot n = 0 \qquad \text{on } [0,T]\times\Omega\times\Gamma, \tag{103}$$

$$\begin{aligned} &\theta\,\frac{\partial u_i}{\partial t}(t,x) - div_x\left[d_i \int_{Y^*} (\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))dy\right] \\ &= -\theta\, u_i(t,x) \sum_{j=1}^{\infty} a_{i,j}\, u_j(t,x) + \frac{\theta}{2} \sum_{j=1}^{i-1} a_{j,i-j} u_j(t,x)\, u_{i-j}(t,x) \\ &+ \theta \sum_{j=1}^{\infty} B_{i+j}\,\beta_{i+j,i}\, u_{i+j}(t,x) - \theta\, B_i\, u_i(t,x) \qquad \text{in } [0,T]\times\Omega, \end{aligned} \tag{104}$$

$$\left[\int_{Y^*} (\nabla_x u_i(t,x) + \nabla_y u_i^1(t,x,y))\, dy\right] \cdot n = 0 \qquad \text{on } [0,T] \times \partial\Omega, \tag{105}$$

where

$$\theta = \int_Y \chi(y) dy = |Y^*|$$

is the volume fraction of material. Moreover, by continuity

$$u_i(0,x) = 0 \qquad \text{in } \Omega.$$

Taking advantage of the constancy of the diffusion coefficient d_i, Eqs. (102) and (103) can be reexpressed as follows

$$\triangle_y u_i^1(t,x,y) = 0 \qquad \text{in } [0,T] \times \Omega \times Y^*, \tag{106}$$

$$\nabla_y u_i^1(t,x,y) \cdot n = -\nabla_x u_i(t,x) \cdot n \qquad \text{on } [0,T] \times \Omega \times \Gamma. \tag{107}$$

Then, $u_i^1(t,x,y)$ satisfying (106)-(107) can be written as

$$u_i^1(t,x,y) = \sum_{j=1}^N w_j(y) \frac{\partial u_i}{\partial x_j}(t,x) \tag{108}$$

where $(w_j)_{1 \le j \le N}$ is the family of solutions of the cell problem (99). By using the relation (108) in Eqs. (104) and (105), the system (12) can be immediately derived (cf. [1]).

A Appendix A

Lemma A.1. *The following estimate holds: If $v \in Lip(\Omega_\epsilon)$, then*

$$\|v\|^2_{L^2(\Gamma_\epsilon)} \le C_1 \left[\epsilon^{-1} \int_{\Omega_\epsilon} |v|^2\, dx + \epsilon \int_{\Omega_\epsilon} |\nabla_x v|^2\, dx\right], \tag{109}$$

where C_1 is a constant which does not depend on ϵ.

The inequality (109) can be easily obtained from the standard trace theorem by means of a scaling argument [2].

Lemma A.2. *Suppose that the domain Ω_ϵ is such that assumption (1) is satisfied. Then there exists a family of linear continuous extension operators*

$$P_\epsilon : W^{1,p}(\Omega_\epsilon) \to W^{1,p}(\Omega)$$

and a constant $C > 0$ independent of ϵ such that

$$P_\epsilon v = v \quad in\ \Omega_\epsilon,$$

and

$$\int_\Omega |P_\epsilon v|^p dx \leq C \int_{\Omega_\epsilon} |v|^p dx \ , \tag{110}$$

$$\int_\Omega |\nabla(P_\epsilon v)|^p dx \leq C \int_{\Omega_\epsilon} |\nabla v|^p dx \tag{111}$$

for each $v \in W^{1,p}(\Omega_\epsilon)$ and for any $p \in (1, +\infty)$.

For the proof of this Lemma see for instance [5].

As a consequence of the existence of extension operators one can derive the Sobolev inequalities in $W^{1,p}(\Omega_\epsilon)$ with a constant independent of ϵ.

Lemma A.3 (Anisotropic Sobolev inequalities in perforated domains)**.**
(i) For arbitrary $v \in H^1(0,T; L^2(\Omega_\epsilon)) \cap L^2(0,T; H^1(\Omega_\epsilon))$ and q_1 and r_1 satisfying the conditions

$$\begin{cases} \dfrac{1}{r_1} + \dfrac{N}{2q_1} = \dfrac{N}{4}, \\ r_1 \in [2, \infty], \ q_1 \in [2, \frac{2N}{N-2}] \quad for\ N > 2, \end{cases} \tag{112}$$

the following estimate holds:

$$\|v\|_{L^{r_1}(0,T;L^{q_1}(\Omega_\epsilon))} \leq c\, \|v\|_{Q_\epsilon(T)}, \tag{113}$$

where c is a positive constant independent of ϵ, and

$$\|v\|^2_{Q_\epsilon(T)} := \sup_{0 \leq t \leq T} \int_{\Omega_\epsilon} |v(t)|^2\, dx + \int_0^T dt \int_{\Omega_\epsilon} |\nabla v(t)|^2\, dx; \tag{114}$$

(ii) For arbitrary $v \in H^1(0,T;L^2(\Omega_\epsilon)) \cap L^2(0,T;H^1(\Omega_\epsilon))$ and q_2 and r_2 satisfying the conditions

$$\begin{cases} \frac{1}{r_2} + \frac{(N-1)}{2q_2} = \frac{N}{4}, \\ r_2 \in [2,\infty],\ q_2 \in [2, \frac{2(N-1)}{(N-2)}] \quad \text{for } N \geq 3, \end{cases} \tag{115}$$

the following estimate holds:

$$\|v\|_{L^{r_2}(0,T;L^{q_2}(\Gamma_\epsilon))} \leq c\, \epsilon^{-\frac{N}{2} - \frac{(1-N)}{q_2}} \|v\|_{Q_\epsilon(T)}, \tag{116}$$

where c is a positive constant independent of ϵ and the norm $\|v\|_{Q_\epsilon(T)}$ is defined as in (114).

For the proof of this Lemma, see [11].

B Appendix B

Let us summarize some definitions and results on two-scale convergence [1], [2], [17], [7], [12], [16].

Definition B.1. *A sequence of functions v^ϵ in $L^2([0,T] \times \Omega)$ two-scale converges to $v_0 \in L^2([0,T] \times \Omega \times Y)$ if*

$$\lim_{\epsilon \to 0} \int_0^T \int_\Omega v^\epsilon(t,x)\, \phi\Big(t,x,\frac{x}{\epsilon}\Big)\, dt\, dx = \int_0^T \int_\Omega \int_Y v_0(t,x,y)\, \phi(t,x,y)\, dt\, dx\, dy, \tag{117}$$

for all $\phi \in C^1([0,T] \times \overline{\Omega}; C^\infty_\#(Y))$.

Theorem B.1. *If v^ϵ is a bounded sequence in $L^2([0,T] \times \Omega)$, then there exists a function $v_0(t,x,y)$ in $L^2([0,T] \times \Omega \times Y)$ such that, up to a subsequence, v^ϵ two-scale converges to v_0.*

The following theorem is useful in obtaining the limit of the product of two two-scale convergent sequences.

Theorem B.2. *Let v^ϵ be a sequence of functions in $L^2([0,T] \times \Omega)$ which two-scale converges to a limit $v_0 \in L^2([0,T] \times \Omega \times Y)$. Suppose furthermore that*

$$\lim_{\epsilon \to 0} \int_0^T \int_\Omega |v^\epsilon(t,x)|^2\, dt\, dx = \int_0^T \int_\Omega \int_Y |v_0(t,x,y)|^2\, dt\, dx\, dy. \tag{118}$$

Then, for any sequence w^ϵ in $L^2([0,T] \times \Omega)$ that two-scale converges to a limit $w_0 \in L^2([0,T] \times \Omega \times Y)$, we have

$$\lim_{\epsilon \to 0} \int_0^T \int_\Omega v^\epsilon(t,x)\, w^\epsilon(t,x)\, \phi\left(t, x, \frac{x}{\epsilon}\right) dt\, dx \\ = \int_0^T \int_\Omega \int_Y v_0(t,x,y)\, w_0(t,x,y)\, \phi(t,x,y)\, dt\, dx\, dy \tag{119}$$

for all $\phi \in C^1([0,T] \times \overline{\Omega}; C^\infty_\#(Y))$.

Remark: Note that, in the setting of this paper, identity (118) can be obtained by standard computations, used in problems with perforated domains. One uses the properties of the extension operators P_ϵ stated in Lemma A.2. For instance, using $v^\epsilon(t,x) := u_1^\epsilon$, we see that $P_\epsilon v^\epsilon$ converges strongly in L^2 towards $v_0 := u_1$. As a consequence, $|P_\epsilon v^\epsilon|^2$ converges towards v_0^2 strongly in L^1, and therefore it also 2-scales converges towards the same quantity. Finally, the properties of P_ϵ enable us to obtain identity (118).

The next theorems yield a characterization of the two-scale limit of the gradients of bounded sequences v^ϵ. This result is crucial for applications to homogenization problems.

We identify $H^1(\Omega) = W^{1,2}(\Omega)$, where the Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ v | v \in L^p(\Omega), \frac{\partial v}{\partial x_i} \in L^p(\Omega), i = 1, \ldots, N \right\},$$

and we denote by $H^1_\#(Y)$ the closure of $C^\infty_\#(Y)$ for the H^1-norm.

Theorem B.3. *Let v^ϵ be a bounded sequence in $L^2(0,T;H^1(\Omega))$ that converges weakly to a limit $v(t,x)$ in $L^2(0,T;H^1(\Omega))$. Then, v^ϵ two-scale converges to $v(t,x)$, and there exists a function $v_1(t,x,y)$ in $L^2([0,T] \times \Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence, ∇v^ϵ two-scale converges to $\nabla_x v(t,x) + \nabla_y v_1(t,x,y)$.*

Theorem B.4. *Let v^ϵ and $\epsilon \nabla v^\epsilon$ be two bounded sequences in $L^2([0,T] \times \Omega)$. Then, there exists a function $v_1(t,x,y)$ in $L^2([0,T] \times \Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence, v^ϵ and $\epsilon \nabla v^\epsilon$ two-scale converge to $v_1(t,x,y)$ and $\nabla_y v_1(t,x,y)$, respectively.*

The main result of two-scale convergence can be generalized to the case of sequences defined in $L^2([0,T] \times \Gamma_\epsilon)$.

Theorem B.5. *Let v^ϵ be a sequence in $L^2([0,T]\times\Gamma_\epsilon)$ such that*

$$\epsilon \int_0^T \int_{\Gamma_\epsilon} |v^\epsilon(t,x)|^2 \, dt \, d\sigma_\epsilon(x) \le C, \tag{120}$$

where C is a positive constant, independent of ϵ. There exist a subsequence (still denoted by ϵ) and a two-scale limit $v_0(t,x,y) \in L^2([0,T]\times\Omega; L^2(\Gamma))$ such that $v^\epsilon(t,x)$ two-scale converges to $v_0(t,x,y)$ in the sense that

$$\lim_{\epsilon\to 0} \epsilon \int_0^T \int_{\Gamma_\epsilon} v^\epsilon(t,x)\, \phi\left(t,x,\frac{x}{\epsilon}\right) dt\, d\sigma_\epsilon(x) = \int_0^T \int_\Omega \int_\Gamma v_0(t,x,y)\, \phi(t,x,y)\, dt\, dx\, d\sigma(y) \tag{121}$$

for any function $\phi \in C^1([0,T]\times\overline{\Omega}; C^\infty_\#(Y))$.

The proof of Theorem B.5 is very similar to the usual two-scale convergence theorem [1]. It relies on the following lemma [2]:

Lemma B.1. *Let $B = C[\overline{\Omega}; C_\#(Y)]$ be the space of continuous functions $\phi(x,y)$ on $\overline{\Omega}\times Y$ which are Y-periodic in y. Then, B is a separable Banach space which is dense in $L^2(\Omega; L^2(\Gamma))$, and such that any function $\phi(x,y) \in B$ satisfies*

$$\epsilon \int_{\Gamma_\epsilon} \left|\phi(x,\frac{x}{\epsilon})\right|^2 d\sigma_\epsilon(x) \le C\, \|\phi\|_B^2, \tag{122}$$

and

$$\lim_{\epsilon\to 0} \epsilon \int_{\Gamma_\epsilon} \left|\phi\left(x,\frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x) = \int_\Omega \int_\Gamma |\phi(x,y)|^2\, dx\, d\sigma(y). \tag{123}$$

Acknowledgements

S.L. is supported by GNFM of INdAM, Italy.

References

[1] G. Allaire, *Homogenization and two-scale convergence.* Siam J. Math. Anal., **23(6)**: 1482-1518, 1992.

[2] G. Allaire, A. Damlamian and U. Hornung, *Two-scale convergence on periodic surfaces and applications.* In: Proceedings of the International Conference on Mathematical Modelling of Flow through Porous Media. A. Bourgeat et al. eds., pp. 15-25, World Scientific pub., Singapore, 1996.

[3] J. A. Canizo, L. Desvillettes and K. Fellner, *Absence of gelation for models of coagulation-fragmentation with degenerate diffusion.* Il Nuovo Cimento, **33**: 79-86, 2010.

[4] J. A. Canizo, L. Desvillettes and K. Fellner, *Regularity and mass conservation for discrete coagulation-fragmentation equations with diffusion.* Ann. I. H. Poncare', **27**: 639-654, 2010.

[5] V. Chiadò Piat and A. Piatnitski, Γ-*convergence approach to variational problems in perforated domains with Fourier boundary conditions.* ESAIM: COCV, **16**: 148-175, 2010.

[6] D. Cioranescu and J. Saint Jean Paulin, *Homogenization of reticulated structures.* Springer-Verlag, New York, 1999.

[7] G. W. Clark and R. E. Showalter, *Two-scale convergence of a model for flow in a partially fissured medium.* Electronic Journal of Differential Equations, **1999(2)**: 1-20, 1999.

[8] C. Conca, J. I. Diaz and C. Timofte, *Effective chemical processes in porous media.* Mathematical Models and Methods in Applied Sciences, **13**: 1437-1462, 2003.

[9] A. Damlamian and P. Donato, *Which sequences of holes are admissible for periodic homogenization with Neumann boundary condition?.* ESAIM: COCV, **8**: 555-585, 2002.

[10] L. Desvillettes, *About entropy methods for reaction-diffusion equations.* Riv. Mat. Univ. Parma, **7**, (7): 81-123, 2007.

[11] B. Franchi and S. Lorenzani, *From a microscopic to a macroscopic model for Alzheimer disease: Two-scale homogenization of the Smoluchowski equation in perforated domains*. Journal of Nonlinear Science, **26** (3): 717-753, 2016.

[12] U. Hornung, *Applications of the homogenization method to flow and transport in porous media*. Summer school on flow and transport in porous media. Beijing, China, 8-26 August 1988, pages 167-222. World Scientific, Singapore, 1992.

[13] U. Hornung and W. Jäger, *Diffusion, Convection, Adsorption, and Reaction of Chemicals in Porous Media*. Journal of Differential Equations, **92**: 199-225, 1991.

[14] O. Krehel, T. Aiki and A. Muntean, *Homogenization of a thermo-diffusion system with Smoluchowski interactions*. Networks and Heterogeneous Media, **9**: 739-762, 2014.

[15] P. Laurençot and S. Mischler, *Global existence for the discrete diffusive coagulation-fragmentation equations in* L^1. Rev. Mat. Iberoamericana, **18**: 731-745, 2002.

[16] A. K. Nandakumaran and M. Rajesh, *Homogenization of a parabolic equation in perforated domain with Neumann boundary condition*. Proc. Indian Acad. Sci., **112(1)**: 195-207, 2002.

[17] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*. Siam J. Math. Anal., **20**: 608-623, 1989.

[18] M. Pierre and D. Schmitt, *Blowup in reaction-diffusion systems with dissipation of mass*. SIAM Review, **42**: 93-106 , 2000.

[19] D. Wrzosek, *Existence of solutions for the discrete coagulation-fragmentation model with diffusion*. Topol. Methods Nonlinear Anal., **9(2)**: 279-296, 1997.