EXISTENCE AND ORBITAL STABILITY OF THE GROUND STATES WITH PRESCRIBED MASS FOR THE L^2-CRITICAL AND SUPERCRITICAL NLS ON BOUNDED DOMAINS

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Given $\rho > 0$, we study the elliptic problem

$$\text{find } (U, \lambda) \in H_0^1(B_1) \times \mathbb{R} \text{ such that } \begin{cases} -\Delta U + \lambda U = U^p, \\ \int_{B_1} U^2\, dx = \rho, \end{cases} \quad U > 0,$$

where $B_1 \subset \mathbb{R}^N$ is the unitary ball and p is Sobolev-subcritical. Such a problem arises in the search for solitary wave solutions for nonlinear Schrödinger equations (NLS) with power nonlinearity on bounded domains. Necessary and sufficient conditions (about ρ, N and p) are provided for the existence of solutions. Moreover, we show that standing waves associated to least energy solutions are orbitally stable for every ρ (in the existence range) when p is L^2-critical and subcritical, i.e., $1 < p \le 1 + 4/N$, while they are stable for almost every ρ in the L^2-supercritical regime $1 + 4/N < p < 2^* - 1$. The proofs are obtained in connection with the study of a variational problem with two constraints of independent interest: to maximize the L^{p+1}-norm among functions having prescribed L^2- and H_0^1-norms.

1. Introduction

In this paper, we study standing wave solutions of the nonlinear Schrödinger equation (NLS)

$$\begin{cases} i\dfrac{\partial \Phi}{\partial t} + \Delta\Phi + |\Phi|^{p-1}\Phi = 0, & (t, x) \in \mathbb{R} \times B_1, \\ \Phi(t, x) = 0, & (t, x) \in \mathbb{R} \times \partial B_1 \end{cases} \tag{1-1}$$

with B_1 the unitary ball of $\mathbb{R}^N$, $N \ge 1$, and $1 < p < 2^* - 1$, where $2^* = \infty$ if $N = 1, 2$ and $2^* = 2N/(N-2)$ otherwise. In what follows, p is always subcritical for the Sobolev immersion while criticality will be understood in the L^2-sense; see below. The main tool in our investigation will be the analysis of the variational problem

$$\max\left\{ \int_\Omega |u|^{p+1}\, dx : u \in H_0^1(\Omega),\ \int_\Omega u^2\, dx = 1,\ \int_\Omega |\nabla u|^2\, dx = \alpha \right\}$$

and in particular of its asymptotic properties in dependence of the parameter α. As we will show, when the bounded domain $\Omega \subset \mathbb{R}^N$ is chosen to be B_1, the two problems are strongly related.

NLS on bounded domains appear in different physical contexts. For instance, in nonlinear optics, with $N = 2$ and $p = 3$, they describe the propagation of laser beams in hollow-core fibers [Agrawal

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2013; Fibich and Merle 2001]. In Bose–Einstein condensation, when $N \le 3$ and $p = 3$, they model the presence of an infinite well-trapping potential [Bartsch and Parnet 2014]. When considered in the whole space $\mathbb{R}^N$, this equation admits the L^2-critical exponent $p = 1 + 4/N$; indeed, in the subcritical case $1 < p < 1 + 4/N$, ground state solutions are orbitally stable while in the critical and supercritical one they are always unstable [Cazenave and Lions 1982; Cazenave 2003]. Notice that the exponent $p = 3$ is subcritical when $N = 1$, critical when $N = 2$ and supercritical when $N = 3$. In the case of a bounded domain, only a few papers analyze the effect of boundary conditions on stability, namely [Fibich and Merle 2001] and the more recent [Fukuizumi et al. 2012] by Fukuizumi, Selem and Kikuchi. In these papers, it is proved that also in the critical and supercritical cases there exist standing waves that are orbitally stable (even though a full classification is not provided, even in the subcritical range). This shows that the presence of the boundary has a stabilizing effect.

As is well known, two quantities are conserved along trajectories of (1-1): the energy

$$\mathscr{E}(\Phi) = \int_{B_1} \Big(\frac{1}{2} |\nabla \Phi|^2 - \frac{1}{p+1} |\Phi|^{p+1} \Big)\, dx$$

and the mass

$$\mathscr{Q}(\Phi) = \int_{B_1} |\Phi|^2 \, dx.$$

A standing wave is a solution of the form $\Phi(t, x) = e^{i\lambda t} U(x)$, where the real-valued function U solves the elliptic problem

$$\begin{cases} -\Delta U + \lambda U = |U|^{p-1} U & \text{in } B_1, \\ U = 0 & \text{on } \partial B_1. \end{cases} \tag{1-2}$$

In (1-2), one can either consider the chemical potential $\lambda \in \mathbb{R}$ to be given or to be an unknown of the problem. In the latter case, it is natural to prescribe the value of the mass so that λ can be interpreted as a Lagrange multiplier.

Among all possible standing waves, typically the most relevant are ground state solutions. In the literature, the two points of view mentioned above lead to different definitions of ground state; see for instance [Adami et al. 2013]. When λ is prescribed, ground states can be defined as minimizers of the action functional

$$\mathscr{A}_\lambda(\Phi) = \mathscr{E}(\Phi) + \tfrac{1}{2} \lambda \mathscr{Q}(\Phi)$$

among its nontrivial critical points (recall that $\mathscr{A}_\lambda$ is not bounded from below); see for instance [Berestycki and Lions 1983, p. 316]. Equivalently, they can be defined as minimizers of $\mathscr{A}_\lambda$ on the associated Nehari manifold. Even though these solutions of (1-2) are sometimes called least energy solutions, we will refer to them as *least action solutions*. In case λ is not given, one may define the ground states as the minimizers of $\mathscr{E}$ under the mass constraint $\mathscr{Q}(U) = \rho$ for some prescribed $\rho > 0$ [Cazenave and Lions 1982, p. 555]. It is worth noticing that this second definition is fully consistent only in the subcritical case

$$p < 1 + \frac{4}{N}$$

since in the supercritical case $\mathscr{E}|_{\{\mathscr{Q}=\rho\}}$ is unbounded from below [Cazenave 2003]; see also Appendix A.

Remark 1.1. When working on the whole space $\mathbb{R}^N$, the two points of view above are in some sense equivalent. Indeed, in such a situation, it is well known [Kwong 1989] that the problem

$$-\Delta Z + Z = Z^p, \qquad Z \in H^1(\mathbb{R}^N), \quad Z > 0$$

admits a solution $Z_{N,p}$ that is unique (up to translations), radial and decreasing in r. Therefore, both the problem with fixed mass and the one with given chemical potential can be uniquely solved in terms of a suitable scaling of $Z_{N,p}$. On the other hand, NLS on $\mathbb{R}^N$ with a nonhomogeneous nonlinearity cannot be treated in this way, and the fixed mass problem becomes hard to tackle [Bellazzini et al. 2013; Bartsch and de Valeriola 2013; Jeanjean 1997; Jeanjean et al. 2014].

When working on bounded domains, the two papers [Fibich and Merle 2001; Fukuizumi et al. 2012] mentioned above deal with least action solutions. In this paper, we make a first attempt to study the case of prescribed mass. Since we consider p also in the critical and supercritical ranges, we have to restrict the minimization process to constrained critical points of $\mathscr{E}$.

Definition 1.2. Let $\rho > 0$. A positive solution of (1-2) with prescribed L^2-mass ρ is a positive critical point of $\mathscr{E}|_{\{\mathscr{Q}=\rho\}}$, that is, an element of the set

$$\mathscr{P}_\rho = \{U \in H_0^1(B_1) : \mathscr{Q}(U) = \rho,\ U > 0,\ \text{there exists } \lambda \text{ such that } -\Delta U + \lambda U = U^p\}.$$

A positive *least energy solution* is a minimizer of the problem

$$e_\rho = \inf_{\mathscr{P}_\rho} \mathscr{E}.$$

Remark 1.3. When p is subcritical, as we mentioned, the above procedure is equivalent to the minimization of $\mathscr{E}|_{\{\mathscr{Q}=\rho\}}$ with no further constraint. On the other hand, when p is supercritical, the set $\mathscr{P}_\rho$ on which the minimization is settled may be strongly irregular. Contrary to what happens for least action solutions, no natural Nehari manifold seems to be associated to least energy solutions. Furthermore, since we work on a bounded domain, the dependence of $\mathscr{P}_\rho$ on ρ cannot be understood in terms of dilations. As a consequence, no regularized version of the minimization problem defined above seems available.

Remark 1.4. Since $\mathscr{A}_\lambda$ and the corresponding Nehari manifold are even, one can immediately see that least action solutions do not change sign so that they can be chosen to be positive. On the other hand, since $U \in \mathscr{P}_\rho$ does not necessarily imply $|U| \in \mathscr{P}_\rho$, in the previous definition, we require the positivity of U. Nonetheless, this condition can be removed in some cases, for instance when p is subcritical or when it is critical and ρ is small (see also Remark 5.10).

Our main results deal with the existence and orbital stability of the least energy solutions of (1-2) (the definition of orbital stability is recalled at the beginning of Section 6 below).

Theorem 1.5. *Under the above notations, the following hold*:

(1) *If* $1 < p < 1 + 4/N$, *then for every* $\rho > 0$, *the set* $\mathscr{P}_\rho$ *has a unique element, which achieves* e_ρ.

(2) *If* $p = 1 + 4/N$, *for* $0 < \rho < \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$, *the set* $\mathscr{P}_\rho$ *has a unique element, which achieves* e_ρ; *for* $\rho \geq \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$, *we have* $\mathscr{P}_\rho = \varnothing$.

(3) *If* $1+4/N < p < 2^*-1$, *there exists* $\rho^* > 0$ *such that* e_ρ *is achieved if and only if* $0 < \rho \le \rho^*$. *Moreover,* $\mathcal{P}_\rho = \varnothing$ *for* $\rho > \rho^*$ *whereas*

$$\#\mathcal{P}_\rho \ge 2 \quad \text{for } 0 < \rho < \rho^*.$$

In this latter case, $\mathcal{P}_\rho$ *contains positive solutions of* (1-2) *that are not least energy solutions.*

Remark 1.6. As a consequence, we have that, for p and ρ as in case (3) of the previous theorem, the problem

$$\text{find } (U,\lambda) \in H_0^1(B_1) \times \mathbb{R} : \begin{cases} -\Delta U + \lambda U = U^p, \\ \int_{B_1} U^2\,dx = \rho \end{cases}$$

admits multiple positive radial solutions.

Concerning the stability, following [Fukuizumi et al. 2012], we apply the abstract results in [Grillakis et al. 1987], which require the local existence for the Cauchy problem associated to (1-1). Since this is not known to hold for all the cases we consider, we take it as an assumption and refer to [Fukuizumi et al. 2012, Remark 1] for further details.

Theorem 1.7. *Suppose that for each* $\Phi_0 \in H_0^1(B_1,\mathbb{C})$ *there exist* $t_0 > 0$, *only depending on* $\|\Phi_0\|$, *and a unique solution* $\Phi(t,x)$ *of* (1-1) *with initial datum* Φ_0 *in the interval* $I = [0, t_0)$.

Let U *denote a least energy solution of* (1-2) *as in Theorem 1.5, and let* $\Phi(t,x) = e^{i\lambda t}U(x)$.

(1) *If* $1 < p \le 1+4/N$, *then* Φ *is orbitally stable.*

(2) *If* $1+4/N < p < 2^*-1$, *then* Φ *is orbitally stable for a.e.* $\rho \in (0,\rho^*]$.

In case (2) of the previous theorem, we expect orbital stability for every $\rho \in (0,\rho^*)$ and instability for $\rho = \rho^*$; see Remark 6.4 ahead.

As we mentioned, [Fibich and Merle 2001; Fukuizumi et al. 2012] consider least action solutions, that is, minimizers associated to

$$a_\lambda = \inf\{\mathcal{A}_\lambda(U) : U \in H_0^1(B_1),\ U \not\equiv 0,\ \mathcal{A}'_\lambda(U) = 0\}.$$

In this situation, the existence and positivity of the least energy solution is not an issue. Indeed, it is well known that problem (1-2) admits a unique positive solution R_λ if and only if $\lambda \in (-\lambda_1(B_1), +\infty)$, where $\lambda_1(B_1)$ is the first eigenvalue of the Dirichlet Laplacian. Such a solution achieves a_λ. Concerning the stability, in the critical case [Fibich and Merle 2001] and in the subcritical one [Fukuizumi et al. 2012], it is proved that $e^{i\lambda t}R_\lambda$ is orbitally stable whenever $\lambda \sim -\lambda_1(B_1)$ and $\lambda \sim +\infty$. Furthermore, stability for all $\lambda \in (-\lambda_1(B_1), +\infty)$ is proved in the second paper in dimension $N = 1$ for $1 < p \le 5$ whereas in the first paper numerical evidence of it is provided in the critical case. In this context, our contribution is the following:

Theorem 1.8. *Let us assume local existence as in Theorem 1.7, and let* R_λ *be the unique positive solution of* (1-2). *If* $1 < p \le 1+4/N$, *then* $e^{i\lambda t}R_\lambda$ *is orbitally stable for every* $\lambda \in (-\lambda_1(B_1), +\infty)$.

Remark 1.9. In [Fukuizumi et al. 2012], it is also shown that, in the supercritical case $p > 1+4/N$, the standing wave associated to R_λ is orbitally *unstable* for $\lambda \sim +\infty$. In view of Theorem 1.7(2), this marks a substantial qualitative difference between the two notions of ground state.

Remark 1.10. Working in B_1 allows one to obtain radial symmetry, uniqueness properties and nondegeneracy of solutions (which in turn implies smooth dependence of the solutions on suitable parameters). These properties are not necessary for the existence results of Theorem 1.5, most of which hold also in general bounded domains, but they are crucial in our proof of stability.

As we mentioned, we will prove the above results as a byproduct of the analysis of a different variational problem that we think is of independent interest. The main feature of such a problem is due to the fact that it involves an optimization with two constraints. Let $\Omega \subset \mathbb{R}^N$ be a general bounded domain. For any fixed $\alpha > \lambda_1(\Omega)$, we consider the maximization problem

$$M_\alpha = \sup\left\{ \int_\Omega |u|^{p+1}\,dx : u \in H_0^1(\Omega),\ \int_\Omega u^2\,dx = 1,\ \int_\Omega |\nabla u|^2\,dx = \alpha \right\}, \tag{1-3}$$

which is related to the validity of Gagliardo–Nirenberg type inequalities (Appendix A).

Theorem 1.11. *Given $\alpha > \lambda_1(\Omega)$, M_α is achieved by a positive function $u_\alpha \in H_0^1(\Omega)$, and there exist $\mu_\alpha > 0$ and $\lambda_\alpha > -\lambda_1(\Omega)$ such that*

$$-\Delta u_\alpha + \lambda_\alpha u_\alpha = \mu_\alpha u_\alpha^p, \qquad \int_\Omega u_\alpha^2\,dx = 1, \qquad \int_\Omega |\nabla u_\alpha|^2\,dx = \alpha. \tag{1-4}$$

Moreover, as $\alpha \to \lambda_1(\Omega)^+$,

$$u_\alpha \to \varphi_1, \qquad \mu_\alpha \to 0^+, \qquad \lambda_\alpha \to -\lambda_1(\Omega)$$

(φ_1 denotes the first positive eigenfunction, normalized in L^2).

As $\alpha \to +\infty$,

$$\frac{\alpha}{\lambda_\alpha} \to \frac{N(p-1)}{N+2-p(N-2)},$$

and

(1) *if $1 < p < 1+4/N$, then $\mu_\alpha \to +\infty$,*
(2) *if $p = 1+4/N$, then $\mu_\alpha \to \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^{p-1}$ and*
(3) *if $1+4/N < p < 2^*-1$, then $\mu_\alpha \to 0$.*

Furthermore, as $\alpha \to +\infty$, u_α is a one-spike solution, and a suitable scaling of u_α approaches the function $Z_{N,p}$ defined in Remark 1.1.

More detailed asymptotics are provided in Sections 3 and 4. This problem is related to the previous one in the following way. Taking $u > 0$ and $\mu > 0$ as in (1-4), the function $U = \mu^{1/(p-1)}u$ belongs to $\mathcal{P}_\rho$ for $\rho = \mu^{2/(p-1)}$. Incidentally, if one considers the minimization problem

$$m_\alpha = \inf\left\{ \int_\Omega |u|^{p+1}\,dx : u \in H_0^1(\Omega),\ \int_\Omega u^2\,dx = 1,\ \int_\Omega |\nabla u|^2\,dx = \alpha \right\},$$

then one obtains a solution of (1-4) with $\mu < 0$ and $\lambda < -\lambda_1(\Omega)$. This allows one to recover the well-known theory of ground states for the defocusing Schrödinger equation $i\partial\Phi/\partial t + \Delta\Phi - |\Phi|^{p-1}\Phi = 0$; see Appendix B. Moreover, when $\alpha \sim \lambda_1(\Omega)$, there exist exactly two solutions (u, μ, λ) of (1-4) that achieve M_α and m_α, respectively. More precisely, in the context of Ambrosetti–Prodi theory [1972; 1993], we prove that $(u, \mu, \lambda) = (\varphi_1, 0, -\lambda_1(\Omega))$ is an ordinary singular point for a suitable map, which yields sharp asymptotic estimates as $\alpha \to \lambda_1(\Omega)^+$. On the other hand, the estimates on M_α as $\alpha \to +\infty$ lean on suitable pointwise a priori controls [Esposito and Petralla 2011]: controls of this kind were initiated and performed for the first time for critical nonlinear elliptic problems by Druet, Hebey and Robert [Druet et al. 2004] (see also [Druet et al. 2012]).

We stress that these results about the two-constraints problem hold for a general bounded domain Ω. Going back to the case $\Omega = B_1$, positive solutions for (1-2) have been the object of an intensive study by a number of authors, in particular regarding uniqueness issues; among others, we refer to [Gidas et al. 1979; Kwong 1989; Kwong and Li 1992; Zhang 1992; Kabeya and Tanaka 1999; Korman 2002; Tang 2003; Felmer et al. 2008]. In our framework, we can exploit the synergy with such uniqueness results in order to fully characterize the positive solutions of (1-4). We do this in the following statement, which collects the results of Proposition 5.4 and of Appendix B below:

Theorem 1.12. *Let* $\Omega = B_1$ *and*

$$\mathcal{S} = \{(u, \mu, \lambda, \alpha) \in H_0^1(\Omega) \times \mathbb{R}^3 : u > 0 \text{ and } (1\text{-}4) \text{ holds}\}.$$

Then

$$\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^- \cup \{(\varphi_1, 0, -\lambda_1(B_1), \lambda_1(B_1))\},$$

where both $\mathcal{S}^+$ *and* $\mathcal{S}^-$ *are smooth curves parametrized by* $\alpha \in (\lambda_1(B_1), +\infty)$, *corresponding to* $\mathcal{S} \cap \{\mu > 0\}$ *and* $\mathcal{S} \cap \{\mu < 0\}$, *respectively. In addition,* $(u, \mu, \lambda, \alpha) \in \mathcal{S}^+$ $(\mathcal{S}^-)$ *if and only if* u *achieves* M_α (m_α).

Remark 1.13. As a consequence of the previous theorem, we have that the smooth set $\mathcal{S}^+$ defined through the maximization problem M_α can be used as a surrogate of the Nehari manifold in order to "regularize" the minimization procedure introduced in Definition 1.2.

To conclude, we mention that in [Noris et al. 2014], by exploiting part of the strategy we have described, we were able to find stable solutions with small mass for the cubic Schrödinger system with trapping potential on $\mathbb{R}^N$.

This paper is structured as follows. In Section 2, we address the preliminary study of the two-constraint problems associated to M_α and m_α. Afterwards, in Section 3, we focus on the case where $\alpha \sim \lambda_1(\Omega)$, seen as an Ambrosetti–Prodi-type problem. Section 4 is devoted to the asymptotics as $\alpha \to +\infty$ for M_α, which concludes the proof of Theorem 1.11. In Section 5, we restrict our attention to the case $\Omega = B_1$, proving all the existence results (in particular Theorem 1.5), qualitative properties and more precise asymptotics for the map $\alpha \mapsto (u, \mu, \lambda)$ that parametrizes $\mathcal{S}^+$. In particular, we show that $\mu'(\alpha) > 0$ whenever $p \le 1 + 4/N$ whereas it changes sign in the supercritical case. Relying on such monotonicity properties, the stability issues are addressed in Section 6, which contains the proofs of Theorems 1.7 and 1.8. Finally, in Appendix A, we collect some known results for the reader's convenience, whereas Appendix B is devoted to the study of $\mathcal{S}^-$, which concludes the proof of Theorem 1.12.

2. A variational problem with two constraints

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$. For every $\alpha \geq \lambda_1(\Omega)$ fixed, we consider the variational problems

$$m_\alpha = \inf_{u \in \mathcal{U}_\alpha} \int_\Omega |u|^{p+1}\,dx, \qquad M_\alpha = \sup_{u \in \mathcal{U}_\alpha} \int_\Omega |u|^{p+1}\,dx,$$

where

$$\mathcal{U}_\alpha = \left\{ u \in H_0^1(\Omega) : \int_\Omega u^2\,dx = 1,\ \int_\Omega |\nabla u|^2\,dx \leq \alpha \right\}.$$

As we will see, these definitions of M_α and m_α are equivalent to the ones given in the introduction. To start with, we state the following straightforward properties:

Lemma 2.1. *For every fixed* $\alpha \geq \lambda_1(\Omega)$,

(i) $\mathcal{U}_\alpha \neq \varnothing$,

(ii) $\mathcal{U}_\alpha$ *is weakly compact in* $H_0^1(\Omega)$,

(iii) *the functional* $u \mapsto \int_\Omega |u|^{p+1}\,dx$ *is weakly continuous and bounded in* $\mathcal{U}_\alpha$ *and*

(iv) $\|u\|_{L^{p+1}(\Omega)} \geq |\Omega|^{-(p-1)/2(p+1)}$ *for every* $u \in \mathcal{U}_\alpha$.

Lemma 2.2. *For every fixed* $\alpha > \lambda_1(\Omega)$, *the set*

$$\widetilde{\mathcal{U}}_\alpha = \left\{ u \in H_0^1(\Omega) : \int_\Omega u^2\,dx = 1,\ \int_\Omega |\nabla u|^2\,dx = \alpha,\ \int_\Omega u\varphi_1\,dx \neq 0 \right\}$$

is a submanifold of $H_0^1(\Omega)$ *of codimension* 2.

Proof. Setting $F(u) = \left(\int_\Omega u^2\,dx - 1, \int_\Omega |\nabla u|^2\,dx\right)$, it suffices to prove that, for every $u \in \widetilde{\mathcal{U}}_\alpha$, the range of $F'(u)$ is $\mathbb{R}^2$. We have

$$\tfrac{1}{2}F'(u)[u] = (1, \alpha), \qquad \tfrac{1}{2}F'(u)[\varphi_1] = \int_\Omega u\varphi_1\,dx \cdot (1, \lambda_1(\Omega)),$$

which are linearly independent as $\alpha > \lambda_1(\Omega)$.

Lemma 2.3. *For every fixed* $\alpha > \lambda_1(\Omega)$, *there exists* $u \in \widetilde{\mathcal{U}}_\alpha$, *with* $u \geq 0$, *such that* $m_\alpha = \int_\Omega u^{p+1}\,dx$. *Moreover, there exist* $\lambda, \mu \in \mathbb{R}$, *with* $\mu \neq 0$, *such that*

$$-\Delta u + \lambda u = \mu u^p \quad \text{in } \Omega. \tag{2-1}$$

A similar result holds for M_α.

Proof. Let us prove the result for m_α. First, the infimum is attained by a function $u \in \mathcal{U}_\alpha$ by Lemma 2.1; by possibly taking $|u|$, we can suppose that $u \geq 0$. Let us show that $u \in \widetilde{\mathcal{U}}_\alpha$. Notice that, with $u \geq 0$ and $u \not\equiv 0$, it holds that $\int_\Omega u\varphi_1\,dx \neq 0$. Assume by contradiction that $\int_\Omega |\nabla u|^2\,dx < \alpha$; then we have

$$\int_\Omega u^{p+1}\,dx = \inf\left\{ \int_\Omega |v|^{p+1}\,dx : v \in H_0^1(\Omega),\ \int_\Omega v^2\,dx = 1,\ \int_\Omega |\nabla v|^2\,dx < \alpha \right\},$$

and there exists a Lagrange multiplier $\mu \in \mathbb{R}$ so that

$$\int_\Omega u^p z\,dx = \mu \int_\Omega uz\,dx \quad \text{for all } z \in H_0^1(\Omega).$$

Hence, $\mu \equiv u^{p-1} \in H_0^1(\Omega)$, which contradicts the fact that $\int_\Omega u^2\,dx = 1$. Therefore $u \in \widetilde{\mathcal{U}}_\alpha$ so that, by Lemma 2.2, the Lagrange multiplier theorem applies, thus providing the existence of $k_1, k_2 \in \mathbb{R}$ such that

$$\int_\Omega u^p z\,dx = k_1 \int_\Omega \nabla u \cdot \nabla z\,dx + k_2 \int_\Omega uz\,dx \quad \text{for all } z \in H_0^1(\Omega).$$

By the previous argument, we see that $k_1 \neq 0$; hence, setting $\mu = 1/k_1$ and $\lambda = k_2/k_1$, the proposition is proved. □

Proposition 2.4. *Given $\alpha > \lambda_1(\Omega)$, the Lagrange multipliers μ and λ associated to m_α as in Lemma 2.3 satisfy $\mu < 0$ and $\lambda < -\lambda_1(\Omega)$. Similarly, in the case of M_α, it holds that $\mu > 0$ and $\lambda > -\lambda_1(\Omega)$.*

Proof. Let (u, λ, μ) be any triplet associated to m_α as in Lemma 2.3. We will prove that $\mu < 0$. Set

$$w(t) = tu + s(t)\varphi_1,$$

where $t \in \mathbb{R}$ is close to 1, $s(1) = 0$ and $s(t)$ is such that

$$1 = \int_\Omega w(t)^2\,dx = t^2 + 2ts(t) \int_\Omega u\varphi_1\,dx + s(t)^2. \tag{2-2}$$

Since

$$\partial_s \left(t^2 + 2ts \int_\Omega u\varphi_1\,dx + s^2 \right) \bigg|_{(t,s)=(1,0)} = 2 \int_\Omega u\varphi_1\,dx \neq 0,$$

then the implicit function theorem applies, and the map $t \mapsto w(t)$ is of class C^1 in a neighborhood of $t = 1$. Differentiating (2-2) with respect to t at $t = 1$, we obtain

$$0 = \int_\Omega w'(1)w(1)\,dx = \int_\Omega w'(1)u\,dx = 1 + s'(1) \int_\Omega u\varphi_1\,dx,$$

which implies $s'(1) = -1/\int_\Omega u\varphi_1\,dx$ and $w'(1) = u - \varphi_1/\int_\Omega u\varphi_1\,dx$. Thus,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla w(t)|^2\,dx \bigg|_{t=1} &= \int_\Omega \nabla u \cdot \nabla w'(1)\,dx \\ &= \int_\Omega |\nabla u|^2\,dx - \frac{\int_\Omega \nabla u \cdot \nabla \varphi_1\,dx}{\int_\Omega u\varphi_1\,dx} = \alpha - \lambda_1(\Omega) > 0. \end{aligned} \tag{2-3}$$

In particular, this implies the existence of $\varepsilon > 0$ such that $w(t) \in \mathcal{U}_\alpha$ for $t \in (1-\varepsilon, 1]$. Therefore, by the definition of m_α, $\|w(1)\|_{p+1} \leq \|w(t)\|_{p+1}$ for every $t \in (1-\varepsilon, 1]$, and

$$\frac{d}{dt} \int_\Omega |w(t)|^{p+1}\,dx \bigg|_{t=1} \leq 0. \tag{2-4}$$

On the other hand, using (2-1) and the fact that $\int_\Omega u w'(1)\,dx = 0$, we have

$$\frac{\mu}{p+1}\frac{d}{dt}\int_\Omega |w(t)|^{p+1}\,dx\bigg|_{t=1} = \mu\int_\Omega u^p w'(1)\,dx = \int_\Omega (-\Delta u + \lambda u)w'(1)\,dx$$
$$= \int_\Omega \nabla u\cdot\nabla w'(1)\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w(t)|^2\,dx\bigg|_{t=1} > 0$$

by (2-3). By comparing with (2-4), we obtain that $\mu < 0$.

The case of M_α can be handled in the same way, obtaining that in such situation $\mu > 0$. Finally, by multiplying (2-1) by φ_1, we obtain

$$(\lambda_1(\Omega)+\lambda)\int_\Omega u\varphi_1\,dx = \mu\int_\Omega u^p\varphi_1\,dx.$$

As $u, \varphi_1 \geq 0$, we deduce that $\lambda_1(\Omega)+\lambda$ has the same sign as μ.

We conclude this section with the following boundedness result, which we will need later on:

Lemma 2.5. *Take a sequence* $\{(u_n, \mu_n, \lambda_n)\}_n$ *such that*

$$\int_\Omega u_n^2\,dx = 1, \qquad \int_\Omega |\nabla u_n|^2\,dx =: \alpha_n \text{ is bounded}$$

and

$$-\Delta u_n + \lambda_n u_n = \mu_n u_n^p. \tag{2-5}$$

Then the sequences $\{\lambda_n\}_n$ *and* $\{\mu_n\}_n$ *are bounded.*

Proof. By multiplying (2-5) by u_n, we see that

$$\alpha_n + \lambda_n = \mu_n\int_\Omega u_n^{p+1}\,dx;$$

thus, if one of the sequences $\{\lambda_n\}_n$ or $\{\mu_n\}_n$ is bounded, the other is also bounded. Recall that, by assumption, u_n is bounded in $H_0^1(\Omega)$; hence, it converges in the L^{p+1}-norm to some $u \in H_0^1(\Omega)$ up to a subsequence. Moreover, $u \not\equiv 0$ as $\int_\Omega u^2\,dx = 1$.

For concreteness, suppose without loss of generality that $\mu_n \to +\infty$ and that $\lambda_n \to +\infty$. From the previous identity, we also have that

$$\frac{\lambda_n}{\mu_n} = \int_\Omega u_n^{p+1}\,dx - \frac{\alpha_n}{\mu_n} \to \int_\Omega u^{p+1}\,dx =: \gamma \neq 0$$

up to a subsequence. Now take any $\varphi \in H_0^1(\Omega)$ and use it as test function in (2-5). We obtain

$$\int_\Omega \nabla u_n\cdot\nabla\varphi\,dx = \mu_n\int_\Omega u_n^p\varphi\,dx - \lambda_n\int_\Omega u_n\varphi\,dx$$
$$= \mu_n\biggl(\int_\Omega u_n^p\varphi\,dx - \frac{\lambda_n}{\mu_n}\int_\Omega u_n\varphi\,dx\biggr).$$

As $\mu_n \to +\infty$, we must have

$$\int_\Omega u_n^p\varphi\,dx - \frac{\lambda_n}{\mu_n}\int_\Omega u_n\varphi\,dx \to 0.$$

On the other hand,

$$\int_\Omega u_n^p \varphi\,dx - \frac{\lambda_n}{\mu_n}\int_\Omega u_n\varphi\,dx \to \int_\Omega u^p\varphi\,dx - \gamma\int_\Omega u\varphi\,dx.$$

Thus, we have $u^p \equiv \gamma u$, which is a contradiction.

3. Asymptotics as $\alpha \to \lambda_1(\Omega)^+$

In this section, we will completely describe the solutions of the problem

$$-\Delta u + \lambda u = \mu u^p, \quad u \in H_0^1(\Omega),\ u > 0, \qquad \int_\Omega u^2\,dx = 1 \tag{3-1}$$

for $\alpha := \int_\Omega |\nabla u|^2\,dx$ in a (right) neighborhood of $\lambda_1(\Omega)$. For that, we will follow the theory presented in [Ambrosetti and Prodi 1993, §3.2], which we now briefly recall.

Definition 3.1. Let X and Y be Banach spaces, $U \subseteq X$ an open set and $\Phi \in C^2(U, Y)$. A point $x \in U$ is said to be *ordinary singular* for Φ if

(a) $\mathrm{Ker}(\Phi'(x))$ is one-dimensional, spanned by a certain $\phi \in X$,

(b) $R(\Phi'(x))$ is closed and has codimension 1 and

(c) $\Phi''(x)[\phi,\phi] \notin R(\Phi'(x))$,

where $\mathrm{Ker}(\Phi'(x))$ and $R(\Phi'(x))$ denote respectively the kernel and the range of the map $\Phi'(x) : X \to Y$.

We will need the following result:

Theorem 3.2 [Ambrosetti and Prodi 1993, §3.2, Lemma 2.5]. *Under the previous notations, let $x^* \in U$ be an ordinary singular point for Φ. Take $y^* = \Phi(x^*)$ and $\phi \in X$ such that $\mathrm{Ker}(\Phi'(x^*)) = \mathbb{R}\phi$, $\Psi \in Y^*$ such that $R(\Phi'(x^*)) = \mathrm{Ker}(\Psi)$ and consider $z \in Z$ such that $\Psi(z) = 1$, where $Y = Z \oplus \mathrm{Ker}(\Psi)$. Suppose*

$$\Psi(\Phi''(x^*)[\phi,\phi]) > 0.$$

Then there exist $\varepsilon^*, \delta > 0$ such that the equation

$$\Phi(x) = y^* + \varepsilon z, \quad x \in B_\delta(x^*),$$

has exactly two solutions for each $0 < \varepsilon < \varepsilon^*$ and no solutions for all $-\varepsilon^* < \varepsilon < 0$. Moreover, there exists $\sigma > 0$ such that the solutions can be parametrized with a parameter $t \in (-\sigma, \sigma)$, $t \mapsto x(t)$ is a C^1 map and

$$x(t) = x^* + t\phi + \mathrm{o}(\sqrt{\varepsilon}) \quad \text{with } t = \pm\sqrt{\frac{2\varepsilon}{\Psi(\Phi''(x^*)[\phi,\phi])}}. \tag{3-2}$$

Let us now set the framework that will allow us to apply the previous results. Given $k > N$, consider $X = \{w \in W^{2,k}(\Omega) : w = 0 \text{ on } \partial\Omega\}$, $Y = L^k(\Omega)$ and $U = \{w \in X : w > 0 \text{ in } \Omega \text{ and } \partial_\nu w < 0 \text{ on } \partial\Omega\}$. Take $\Phi : X \times \mathbb{R}^2 \to L^k(\Omega) \times \mathbb{R}^2$ defined by

$$\Phi(u,\mu,\lambda) = \left(\Delta u - \lambda u + \mu u^p,\ \int_\Omega u^2\,dx - 1,\ \int_\Omega |\nabla u|^2\,dx\right). \tag{3-3}$$

Remark 3.3. Note that $\Phi \in C^2(U, Y)$. This is immediate when $p \geq 2$ while for $1 < p < 2$ it can be proved, for instance, along the lines of [Ortega and Verzini 2004, Lemma 4.1].

We start with the following result:

Lemma 3.4. *Let $\alpha_n \to \lambda_1(\Omega)^+$, and suppose there exists (u_n, μ_n, λ_n) such that $\Phi(u_n, \mu_n, \lambda_n) = (0, 0, \alpha_n)$ with $u_n \geq 0$. Then $u_n \to \varphi_1$ in $H_0^1(\Omega)$, $\mu_n \to 0$ and $\lambda_n \to -\lambda_1(\Omega)$. In particular,*

$$\Phi(u, \mu, \lambda) = (0, 0, \lambda_1(\Omega)), \quad u \geq 0, \quad \textit{if and only if } (u, \mu, \lambda) = (\varphi_1, 0, -\lambda_1(\Omega)).$$

Proof. As u_n is bounded in $H_0^1(\Omega)$, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Moreover, $\int_\Omega u^2\, dx = 1$, $u \geq 0$, and by the Poincaré inequality, $\lambda_1(\Omega) \leq \int_\Omega |\nabla u|^2 \leq \liminf \int_\Omega |\nabla u_n|^2\, dx = \lambda_1(\Omega)$, whence $u = \varphi_1$ and the whole sequence u_n converges strongly to φ_1 in $H_0^1(\Omega)$. By Lemma 2.5, we have that μ_n and λ_n are bounded. Denote by μ_∞ and λ_∞ limits of subsequences of each. Then

$$-\Delta\varphi_1 + \lambda_\infty \varphi_1 = \mu_\infty \varphi_1^p,$$

which shows that $\mu_\infty = 0$ and $\lambda_\infty = -\lambda_1(\Omega)$.

Lemma 3.5. *The point $(\varphi_1, 0, -\lambda_1(\Omega)) \in U$ is ordinary singular for Φ. More precisely, for $L := \Phi'(\varphi_1, 0, -\lambda_1(\Omega)) : X \times \mathbb{R}^2 \to L^k(\Omega) \times \mathbb{R}^2$, we have:*

(i) $\mathrm{Ker}(L) = \mathrm{span}\{(\psi, 1, \int_\Omega \varphi_1^{p+1}\, dx)\} =: \mathrm{span}\{\phi\}$, *where $\psi \in X$ is the unique solution of*

$$-\Delta\psi - \lambda_1(\Omega)\psi = \varphi_1^p - \varphi_1 \int_\Omega \varphi_1^{p+1}\, dx \quad \textit{such that} \int_\Omega \psi\varphi_1\, dx = 0. \tag{3-4}$$

(ii) $R(L) = \mathrm{Ker}(\Psi)$ *with $\Psi : L^k(\Omega) \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\Psi(\xi, h, k) = k - \lambda_1(\Omega)h$.*

(iii) $\Psi(\Phi''(\varphi_1, 0, -\lambda_1(\Omega))[\phi, \phi]) > 0$.

Proof. (i) We recall that $-\Delta - \lambda_1(\Omega)\,\mathrm{Id}$ is a Fredholm operator of index 0 with

$$\mathrm{Ker}(-\Delta - \lambda_1(\Omega)\,\mathrm{Id}) = \mathrm{span}\{\varphi_1\},$$
$$R(-\Delta - \lambda_1(\Omega)\,\mathrm{Id}) = \left\{v \in L^k(\Omega) : \int_\Omega v\varphi_1\, dx = 0\right\}.$$

Therefore, by the Fredholm alternative, there exists a unique $\psi \in X$ solution of (3-4). Let us check that $\mathrm{Ker}(L) = \mathrm{span}\{(\psi, 1, \int_\Omega \varphi_1^{p+1}\, dx)\}$. We have

$$L(v, m, l) = \left(\Delta v + \lambda_1(\Omega)v - l\varphi_1 + m\varphi_1^p, 2\int_\Omega \varphi_1 v\, dx, 2\int_\Omega \nabla\varphi_1 \cdot \nabla v\, dx\right);$$

thus, $(v, m, l) \in \mathrm{Ker}(L)$ if and only if $l = m\int_\Omega \varphi_1^{p+1}$, $\int_\Omega \varphi_1 v\, dx = \int_\Omega \nabla\varphi_1 \cdot \nabla v\, dx = 0$ and

$$-\Delta v - \lambda_1(\Omega)v = m\left(\varphi_1^p - \varphi_1 \int_\Omega \varphi_1^{p+1}\, dx\right) \quad \text{for some } m \in \mathbb{R}.$$

By the uniqueness of ψ in (3-4), we obtain $v = m\psi$.

(ii) Let us prove that $R(L) = \{(\xi, h, \lambda_1(\Omega)h) : \xi \in L^k(\Omega),\ h \in \mathbb{R}\}$. Recalling the expression for L found in (i), it is clear that $L(v, m, l) = (\xi, h, k)$ implies $k = \lambda_1(\Omega)h$. As for the other inclusion, given any $\xi \in L^k(\Omega)$, let $w \in X$ be the solution of

$$-\Delta w - \lambda_1(\Omega)w = \varphi_1 \int_\Omega \xi\varphi_1\, dx - \xi \quad \text{with} \int_\Omega w\varphi_1\, dx = 0,$$

which exists and is unique again by the Fredholm alternative. Then $L(h\varphi_1/2 + w, 0, \int_\Omega \xi\varphi_1\, dx) = (\xi, h, \lambda_1(\Omega)h)$.

(iii) We have that

$$\Phi''(\varphi_1, 0, -\lambda_1(\Omega))[\phi, \phi] = 2\Big(p\varphi_1^{p-1}\psi - \psi \int_\Omega \varphi_1^{p+1}\, dx,\ \int_\Omega \psi^2\, dx,\ \int_\Omega |\nabla\psi|^2\, dx\Big)$$

with ϕ and ψ defined in (i). Hence,

$$\Psi(\Phi''(\varphi_1, 0, \lambda_1(\Omega))[\phi, \phi]) = \int_\Omega 2(|\nabla\psi|^2 - \lambda_1(\Omega)\psi^2)\, dx > 0 \tag{3-5}$$

since ψ satisfies (3-4).

Proposition 3.6. *There exists ε^* such that the equation*

$$\Phi(u, \mu, \lambda) = (0, 0, \lambda_1(\Omega) + \varepsilon), \quad (u, \mu, \lambda) \in U \times \mathbb{R}^2,$$

has exactly two positive solutions for each $0 < \varepsilon < \varepsilon^*$ (one with $\mu > 0$ and one with $\mu < 0$). Moreover, such solutions satisfy the asymptotic expansion

$$(u, \mu, \lambda) = (\varphi_1, 0, -\lambda_1(\Omega)) \pm \sqrt{\frac{\varepsilon}{\int_\Omega \varphi_1^p \psi\, dx}}\Big(\psi, 1, \int_\Omega \varphi_1^{p+1}\, dx\Big) + o(\sqrt{\varepsilon}),$$

where ψ is defined in (3-4). In addition, the L^{p+1}-norm of one of the solutions is equal to $m_{\lambda_1(\Omega)+\varepsilon}$ and the other is equal to $M_{\lambda_1(\Omega)+\varepsilon}$.

Proof. We apply Theorem 3.2 with Φ defined in (3-3), $x^* = (\varphi_1, 0, -\lambda_1(\Omega))$ and $z = (0, 0, 1)$. By the previous lemma, x^* is ordinary singular for Φ, and, moreover, using the notation therein, $\Psi(\Phi''(x^*)[\phi, \phi]) > 0$ and $\Psi(z) = 1$. Therefore, the assumptions of Theorem 3.2 are satisfied, and there exist $\varepsilon^*, \delta > 0$ such that the problem

$$\Phi(u, \mu, \lambda) = (0, 0, \lambda_1(\Omega) + \varepsilon), \quad (u, \mu, \lambda) \in B_\delta(\varphi_1, 0, -\lambda_1(\Omega)),$$

has exactly two solutions for each $0 < \varepsilon < \varepsilon^*$, which can be parametrized using a map $t \mapsto (u(t), \mu(t), \lambda(t))$ of class C^1 in $U \times \mathbb{R}^2$. The asymptotic expansion is obtained by combining (3-2) with the fact (see (3-5))

$$\Psi(\Phi''(\varphi_1, 0, \lambda_1(\Omega))[\phi, \phi]) = 2\int_\Omega \varphi_1^p \psi\, dx.$$

Finally, by possibly choosing a smaller ε^*, $(u(t), \mu(t), \lambda(t))$ are the unique positive solutions in $U \times \mathbb{R}^2$ for $0 < \varepsilon < \varepsilon^*$, as a consequence of Lemma 3.4, and the statement concerning $\int_\Omega u(t)^{p+1}\, dx$ follows from Lemma 2.3 and Proposition 2.4.

Remark 3.7. From the proof of Proposition 3.6, we deduce an alternative proof of [Fukuizumi et al. 2012, Theorem 17(ii)]; namely, we can show that

$$(\mu^2)'(\lambda_1(\Omega)^+) > 0.$$

This result is relevant when facing stability issues; see Corollary 6.2 ahead.

4. Asymptotics as $\alpha \to +\infty$

In this section, we consider the case when α is large in order to conclude the proof of Theorem 1.11. Since in that case the problems M_α and m_α exhibit different asymptotics, here we only address the study of M_α, and we postpone to Appendix B the complete description of the minimizers corresponding to m_α.

Define, for any $\mu, \lambda \in \mathbb{R}$, the action functional associated to (2-1), namely $J_{\mu,\lambda} : H_0^1(\Omega) \to \mathbb{R}$:

$$J_{\mu,\lambda}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\, dx - \frac{\mu}{p+1}\int_\Omega |u|^{p+1}\, dx. \tag{4-1}$$

Lemma 4.1. *For every $\mu > 0$ and $\lambda \in \mathbb{R}$, we have that*

$$u \in \widetilde{\mathcal{U}}_\alpha,\ \int_\Omega |u|^{p+1}\, dx = M_\alpha \quad \Longrightarrow \quad J_{\mu,\lambda}(u) = \inf_{\widetilde{\mathcal{U}}_\alpha} J_{\mu,\lambda}.$$

Proof. By the definition of M_α,

$$\frac{\mu}{p+1}M_\alpha = \sup_{w\in\widetilde{\mathcal{U}}_\alpha}\left\{\frac{\mu}{p+1}\int_\Omega |w|^{p+1}\, dx + \frac{1}{2}\left(\alpha - \int_\Omega |\nabla w|^2\, dx\right) + \frac{\lambda}{2}\left(1 - \int_\Omega w^2\, dx\right)\right\},$$

and hence,

$$J_{\mu,\lambda}(u) = \frac{\alpha+\lambda}{2} - \frac{\mu}{p+1}M_\alpha = \inf_{w\in\widetilde{\mathcal{U}}_\alpha} J_{\mu,\lambda}(w).$$

Lemma 4.2. *Fix $\alpha > \lambda_1(\Omega)$, and let $(u, \mu, \lambda) \in \widetilde{\mathcal{U}}_\alpha \times \mathbb{R}^+ \times (-\lambda_1(\Omega), +\infty)$ be any triplet associated to M_α as in Lemma 2.3. Then the Morse index of $J''_{\mu,\lambda}(u)$ is either* 1 *or* 2.

Proof. If (u, μ, λ) is a triplet associated to M_α, then $\mu > 0$ by Proposition 2.4. Equation (2-1) implies

$$J''_{\mu,\lambda}(u)[u, u] = -(p-1)\mu\int_\Omega u^{p+1}\, dx < 0,$$

so that the Morse index is at least 1. Next we claim that, for such (u, μ, λ),

$$J''_{\mu,\lambda}(u)[\phi, \phi] \geq 0 \quad \text{for every } \phi \in H_0^1(\Omega) \text{ with } \int_\Omega \nabla u \cdot \nabla\phi\, dx = \int_\Omega u\phi\, dx = 0,$$

which implies that the Morse index is at most 2. Indeed, any such ϕ belongs to the tangent space of $\widetilde{\mathcal{U}}_\alpha$ at u; hence, there exists a C^∞ curve $\gamma(t)$ satisfying, for some $\varepsilon > 0$,

$$\gamma : (-\varepsilon, \varepsilon) \to \widetilde{\mathcal{U}}_\alpha, \quad \gamma(0) = u,\ \gamma'(0) = \phi.$$

Lemma 4.1 implies that $J_{\mu,\lambda}(\gamma(t)) - J_{\mu,\lambda}(\gamma(0)) \geq 0$. Hence,

$$0 \leq J_{\mu,\lambda}(\gamma(t)) - J_{\mu,\lambda}(u) = J'_{\mu,\lambda}(u)[\phi]t + J''_{\mu,\lambda}(u)[\phi,\phi]\frac{t^2}{2} + J'_{\mu,\lambda}(u)[\gamma''(0)]\frac{t^2}{2} + \mathrm{o}(t^2).$$

Finally, (2-1) implies that $J'_{\mu,\lambda}(u) \equiv 0$, which concludes the proof.

Lemma 4.3. *Let $\alpha_n \to +\infty$, and let $u_n \in H_0^1(\Omega)$, $u_n > 0$, satisfy*

$$-\Delta u_n + \lambda_n u_n = \mu_n u_n^p \quad \text{in } \Omega, \qquad \int_\Omega |\nabla u_n|^2\,dx = \alpha_n, \qquad \int_\Omega u_n^2\,dx = 1$$

for some $\mu_n > 0$ and $\lambda_n > -\lambda_1(\Omega)$. Then $\lambda_n \to +\infty$.

Proof. Set

$$L_n := \|u_n\|_{L^\infty(\Omega)} = u_n(x_n).$$

Since $\Delta u_n(x_n) \leq 0$, from the equation for u_n, we obtain $\mu_n L_n^p - \lambda_n L_n \geq 0$, i.e.,

$$-\frac{\lambda_1(\Omega)}{\mu_n L_n^{p-1}} < \frac{\lambda_n}{\mu_n L_n^{p-1}} \leq 1$$

(recall that $\lambda > -\lambda_1(\Omega)$). In particular, since $\mu_n L_n^{p-1} \geq \mu_n \int_\Omega u_n^{p+1}\,dx \geq \alpha_n + \lambda_n \to +\infty$, we have (up to subsequences)

$$\frac{\lambda_n}{\mu_n L_n^{p-1}} \to \lambda^* \in [0, 1]. \tag{4-2}$$

In order to prove that $\lambda_n \to +\infty$, it only remains to show that $\lambda^* \neq 0$. To this aim, we define

$$v_n(x) := \frac{1}{L_n} u_n\left(x_n + \frac{x}{(\mu_n L_n^{p-1})^{1/2}}\right)$$

so that v_n satisfies

$$-\Delta v_n + \frac{\lambda_n}{\mu_n L_n^{p-1}} v_n = v_n^p \quad \text{in } \Omega_n := (\mu_n L_n^{p-1})^{1/2}(\Omega - x_n).$$

Using (4-2) and reasoning as in [Gidas and Spruck 1981b, pp. 887–889], we have that $v_n \to v$ in $(W^{2,p} \cap C^{1,\beta})_{\mathrm{loc}}(\mathbb{R}^N)$ for every $\beta \in (0, 1)$. Moreover, $v \geq 0$, $v(0) = 1$ and

$$-\Delta v + \lambda^* v = v^p \quad \text{in } H,$$

where H is either $\mathbb{R}^N$ or a half-space of $\mathbb{R}^N$ and $v = 0$ on ∂H in case H is the half-space. Since $v \not\equiv 0$, the nonexistence results in [Gidas and Spruck 1981a] imply that $\lambda^* > 0$, and this concludes the proof.

Next, we use some results from [Esposito and Petralla 2011] in order to show that a suitable rescaling of the solutions converges to the function $Z_{N,p}$ defined in Remark 1.1. Such results rely on pointwise estimates that take fundamental inspiration from the monograph [Druet et al. 2004] (see also [Druet et al. 2012]).

Lemma 4.4. *With the same assumptions as the previous lemma, suppose moreover that the Morse index of $J''_{\mu_n,\lambda_n}(u_n)$ is equal to $k \in \mathbb{N}$ for every n. Then u_k admits k local maxima $P_n^i \in \Omega$, $i = 1, \ldots, k$, such that, defining*

$$v_{i,n}(x) = \left(\frac{\mu_n}{\lambda_n}\right)^{1/(p-1)} u_n\left(\frac{x}{\sqrt{\lambda_n}} + P_n^i\right) \tag{4-3}$$

for $x \in \Omega_{i,n} = \sqrt{\lambda_n}(\Omega - P_i^n)$, we have

$$v_{i,n} \to Z_{N,p} \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{as } n \to +\infty \text{ for every } i.$$

As a consequence, for every $q \geq 1$,

$$\left(\frac{\mu_n}{\lambda_n}\right)^{q/(p-1)} \lambda_n^{N/2} \int_\Omega u_n^q \, dx \to k \int_{\mathbb{R}^N} Z_{N,p}^q \, dx \quad \text{as } n \to +\infty. \tag{4-4}$$

Proof. Since $\lambda_n \to +\infty$ by the previous lemma, we can apply [Esposito and Petralla 2011, Theorem 3.2] to $U_n := \mu_n^{1/(p-1)} u_n$, inferring the existence of k local maxima P_n^i, $i = 1, \ldots, k$, such that, for every $i \neq j$,

$$\sqrt{\lambda_n}\,\mathrm{dist}(P_n^i, \partial\Omega) \to +\infty, \qquad \sqrt{\lambda_n}|P_n^i - P_n^j| \to +\infty, \tag{4-5}$$

and for some $C, \gamma > 0$, the following pointwise estimate holds:

$$u_n(x) = \mu_n^{-1/(p-1)} U_n(x) \leq C\left(\frac{\lambda_n}{\mu_n}\right)^{1/(p-1)} \sum_{i=1}^k e^{-\gamma\sqrt{\lambda_n}|x - P_n^i|} \quad \text{for all } x \in \Omega.$$

Furthermore, since $v_{i,n}$ solves $-\Delta v_{i,n} + v_{i,n} = v_{i,n}^p$ in $\Omega_{i,n}$, [Esposito and Petralla 2011, Theorem 3.1] yields that $v_{i,n} \to Z_{N,p}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$, so the only thing that remains to be proved is estimate (4-4).

To this aim, let $R > 0$ be fixed and $r_n = R/\sqrt{\lambda_n}$. Then, if n is sufficiently large, (4-5) implies that, for every $i \neq j$,

$$B_{r_n}(P_n^i) \subset \Omega, \qquad B_{r_n}(P_n^i) \cap B_{r_n}(P_n^j) = \varnothing.$$

We obtain

$$\begin{aligned}
\left|\left(\frac{\mu_n}{\lambda_n}\right)^{q/(p-1)} \lambda_n^{N/2} \int_\Omega u_n^q \, dx - \sum_{j=1}^k \int_{B_R(0)} v_{j,n}^q \, dx\right| &= \left(\frac{\mu_n}{\lambda_n}\right)^{q/(p-1)} \lambda_n^{N/2} \left|\int_\Omega u_n^q \, dx - \sum_{j=1}^k \int_{B_{r_n}(P_n^j)} u_n^q \, dx\right| \\
&= \left(\frac{\mu_n}{\lambda_n}\right)^{q/(p-1)} \lambda_n^{N/2} \int_{\Omega \setminus \bigcup_{j=1}^k B_{r_n}(P_n^j)} u_n^q \, dx \\
&\leq C\lambda_n^{N/2} \sum_{i=1}^k \int_{\Omega \setminus \bigcup_{j=1}^k B_{r_n}(P_n^j)} e^{-q\gamma\sqrt{\lambda_n}|x - P_n^i|} \, dx \\
&\leq C\lambda_n^{N/2} \sum_{i=1}^k \int_{\mathbb{R}^N \setminus B_{r_n}(P_n^i)} e^{-q\gamma\sqrt{\lambda_n}|x - P_n^i|} \, dx \\
&= Ck \int_{\mathbb{R}^N \setminus B_R(0)} e^{-q\gamma|y|} \, dy \leq C_1 e^{-C_2 R}
\end{aligned}$$

for some positive C_1 and C_2. As $n \to +\infty$, we have, up to subsequences,

$$\left| \lim_n \left(\frac{\mu_n}{\lambda_n} \right)^{q/(p-1)} \lambda_n^{N/2} \int_\Omega u_n^q \, dx - k \int_{B_R(0)} Z_{N,p}^q \, dx \right| \le C_1 e^{-C_2 R},$$

and (4-4) follows by taking $R \to +\infty$.

Finally, the previous lemma allows us to study the asymptotic behavior of μ as $\alpha \to +\infty$.

Lemma 4.5. *With the same assumptions as the previous lemma, we have that*

(1) *if* $1 < p < 1 + 4/N$, *then* $\mu_n \to +\infty$,

(2) *if* $p = 1 + 4/N$, *then* $\mu_n \to k^{2/N} \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^{4/N}$ *and*

(3) *if* $1 + 4/N < p < 2^* - 1$, *then* $\mu_n \to 0$.

Furthermore,

$$\frac{\alpha_n}{\lambda_n} \to \frac{N(p-1)}{N+2-p(N-2)}.$$

Proof. Exploiting (4-4) with $q = 2$ and $q = p+1$ as well as the relations $\|u_n\|_{L^2}^2 = 1$, $\|\nabla u_n\|_{L^2}^2 = \alpha_n$ and $\alpha_n + \lambda_n = \mu_n \|u_n\|_{L^{p+1}}^{p+1}$, we can write

$$\begin{aligned} \mu_n^{2/(p-1)} \lambda_n^{N/2-2/(p-1)} &\to k \int_{\mathbb{R}^N} Z_{N,p}^2 \, dx, \\ \mu_n^{(p+1)/(p-1)} \lambda_n^{N/2-(p+1)/(p-1)} \int_\Omega u_n^{p+1} \, dx &\to k \int_{\mathbb{R}^N} Z_{N,p}^{p+1} \, dx, \\ \frac{\alpha_n}{\lambda_n} \mu_n^{2/(p-1)} \lambda_n^{N/2-2/(p-1)} &\to k \int_{\mathbb{R}^N} |\nabla Z_{N,p}|^2 \, dx. \end{aligned} \tag{4-6}$$

Now, since $\lambda_n \to +\infty$ (Lemma 4.3) and the exponent $N/2 - 2/(p-1)$ is negative, zero or positive respectively in the subcritical, critical and supercritical cases, the first relation in (4-6) immediately provides the properties for μ_n.

On the other hand, dividing the third relation by the first one, we have

$$\frac{\alpha_n}{\lambda_n} \to \frac{\|\nabla Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2}{\|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2} = \frac{N(p-1)}{N+2-p(N-2)}.$$

The explicit evaluation of this constant can be obtained by the relations

$$\begin{cases} \|\nabla Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2 + \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2 = \|Z_{N,p}\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}, \\ \frac{N-2}{2} \|\nabla Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2 + \frac{N}{2} \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2 = \frac{N}{p+1} \|Z_{N,p}\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}, \end{cases}$$

i.e., by testing the equation for $Z_{N,p}$ either with $Z_{N,p}$ itself or with $x \cdot \nabla Z_{N,p}$ (recall that $Z_{N,p}$ decays exponentially at ∞). The second relation is the well-known Pohozaev identity; see for instance [Berestycki and Lions 1983, §2].

End of the proof of Theorem 1.11. The fact that M_α is achieved by a triplet (u, μ, λ) with $\mu > 0$ and $\lambda > -\lambda_1(\Omega)$ is a consequence of Lemma 2.3 and Proposition 2.4. Lemma 3.4 implies the asymptotic behavior as $\alpha \to \lambda_1(\Omega)^+$ while the results as $\alpha \to +\infty$ follow from Lemmas 4.3, 4.4 and 4.5, recalling that u has Morse index k, with k being either 1 or 2, by Lemma 4.2. The only thing that remains to be proved is that both in Lemma 4.4 and in Lemma 4.5(2) k must be equal to 1; in other words, we are left to show that, if u_n achieves M_{α_n}, with α_n large, then its Morse index must be 1 (and not 2).

For easier notation, in the following, we write $Z = Z_{N,p}$. Since u_n achieves M_{α_n}, from (4-6), we infer (up to subsequences)

$$\mu_n \sim k^{(p-1)/2} \|Z\|_{L^2}^{p-1} \lambda_n^{1-N(p-1)/4}, \qquad \alpha_n \sim \frac{\|\nabla Z\|_{L^2}^2}{\|Z\|_{L^2}^2} \lambda_n$$

and

$$\frac{M_{\alpha_n}}{\alpha_n^{N(p-1)/4}} = \frac{\int_\Omega u_n^{p+1}\,dx}{\alpha_n^{N(p-1)/4}} \sim k\|Z\|_{L^{p+1}}^{p+1} \frac{\lambda_n^{(p+1)/(p-1)-N/2}}{\mu_n^{(p+1)/(p-1)} \alpha_n^{N(p-1)/4}}$$

$$\to k^{-(p-1)/2} \frac{\|Z\|_{L^{p+1}}^{p+1}}{\|\nabla Z\|_{L^2}^{N(p-1)/2} \|Z\|_{L^2}^{p+1-N(p-1)/2}}, \tag{4-7}$$

where either $k = 1$ or $k = 2$. On the other hand, let us fix $x_0 \in \Omega$ and $\eta \in C_0^\infty(\Omega)$ such that $\eta(x) = 1$ around x_0. It is always possible to find a sequence $a_n \to 0^+$ such that

$$w_n(x) := \eta(x) Z_{N,p}\Big(\frac{x - x_0}{a_n}\Big), \ \widetilde{w}_n := \frac{w_n}{\|w_n\|_{L_2}} \quad \text{satisfy} \int_\Omega |\nabla \widetilde{w}_n|^2\,dx = \alpha_n$$

(indeed $\alpha_n \to +\infty$ and $\int_\Omega |\nabla \widetilde{w}_n|^2\,dx$ is of order a_n^{-2} as $a_n \to 0$). Then direct calculation yields

$$\frac{M_{\alpha_n}}{\alpha_n^{N(p-1)/4}} \geq \frac{\int_\Omega \widetilde{w}_n^{p+1}\,dx}{\alpha_n^{N(p-1)/4}} \to \frac{\|Z\|_{L^{p+1}}^{p+1}}{\|\nabla Z\|_{L^2}^{N(p-1)/2} \|Z\|_{L^2}^{p+1-N(p-1)/2}},$$

which, together with (4-7), forces $k = 1$.

Remark 4.6. The previous argument shows that, when α is large, M_α is achieved by a single-peak solution having Morse index 1. This was actually suggested to us by the anonymous referee in his/her report. This also implies the sharper estimate for the asymptotics of μ:

$$\mu_\alpha \sim C\alpha^{1-N(p-1)/4},$$

where C is a constant depending only on N and p (through $Z_{N,p}$).

5. Least energy solutions in the ball

From now on, we will focus on the case

$$\Omega := B_1.$$

To start with, we collect in the following theorem some well-known results about uniqueness and nondegeneracy of positive solutions of (1-2) on the ball:

Theorem 5.1 [Gidas et al. 1979; Kwong 1989; Kwong and Li 1992; Korman 2002; Aftalion and Pacella 2003]. *Let* $\lambda \in (-\lambda_1(B_1), +\infty)$ *and* $\mu > 0$ *be fixed. Then the problem*

$$-\Delta u + \lambda u = \mu u^p \quad \textit{in } B_1, \qquad u = 0 \quad \textit{on } \partial B_1$$

admits a unique positive solution u, *which is nondegenerate*, *radially symmetric and decreasing with respect to the radial variable* $r = |x|$.

Proof. The existence easily follows from the mountain pass lemma. The radial symmetry and monotonicity of positive solutions is a direct consequence of [Gidas et al. 1979].

The uniqueness in the case $\lambda > 0$ was proved by Kwong [1989] for $N \geq 2$. For $\lambda \in (-\lambda_1(B_1), 0)$, the uniqueness in dimension $N \geq 3$ was proved by Kwong and Li [1992, Theorem 2] (see also [Zhang 1992]) whereas in dimension $N = 2$ it was proved by Korman [2002, Theorem 2.2]. The case $\lambda = 0$ is treated in Section 2.8 of [Gidas et al. 1979].

As for the nondegeneracy, for $\lambda > 0$, this follows from [Aftalion and Pacella 2003, Theorem 1.1] since we know that u has Morse index 1 as it is a mountain pass solution for $J_{\mu,\lambda}$ (recall that such a functional is defined as in (4-1)). As for $\lambda \in (-\lambda_1(B_1), 0]$, we could not find a precise reference, and for this reason, we present here a proof, following some ideas of [Kabeya and Tanaka 1999].

Assume by contradiction that u is a degenerate solution for some $\lambda \in (-\lambda_1(B_1), 0]$. This means that there exists a solution $0 \neq w \in H_0^1(B_1)$ of

$$-\Delta w + \lambda w = p u^{p-1} w;$$

hence, $w \in H^1_{0,\mathrm{rad}}(B_1)$ and $J''_{\mu,\lambda}(u)[w, \xi] = 0$ for all $\xi \in H_0^1(B_1)$. Moreover, we have that $J''_{\mu,\lambda}(u)[u, u] = -(p-1)\mu \int_{B_1} u^{p+1}\,dx < 0$, and thus,

$$J''_{\mu,\lambda}(u)[h, h] \leq 0 \quad \text{for all } h \in H := \operatorname{span}\{u, w\}.$$

For $\delta > 0$, consider the perturbed functional

$$I_\delta(w) = \int_{B_1} \left(\frac{|\nabla w|^2}{2} + \frac{\lambda + \delta u^{p-1}}{2} w^2 - \frac{\mu + \delta}{p+1} (w^+)^{p+1} \right) dx. \tag{5-1}$$

On the one hand, this functional satisfies, for every $h \in H \setminus \{0\}$,

$$\begin{aligned} I''_\delta(u)[h, h] &= J''_{\mu,\lambda}(u)[h, h] + \int_{B_1} (\delta u^{p-1} h^2 - p\delta u^{p-1} h^2)\,dx \\ &\leq -(p-1)\delta \int_{B_1} u^{p-1} h^2\,dx < 0. \end{aligned} \tag{5-2}$$

On the other hand, I_δ has a mountain pass geometry for δ sufficiently small; hence, it has a critical point of mountain pass type. Every nonzero critical point of I_δ is positive (by the maximum principle), and it solves

$$\begin{cases} -\Delta w = V_\delta(r) w + (\mu + \delta) w^p & \text{in } B_1, \\ w > 0 & \text{in } B_1, \\ w \in H_0^1(B_1) \end{cases}$$

for $V_\delta(r) := -\lambda - \delta u^{p-1}$. Now this problem has a unique radial solution, which is u itself, which is in contradiction to (5-2). The uniqueness of this perturbed problem follows from [Korman 2002, Theorem 2.2] in case $\lambda < 0$ (in fact, $V_\delta(r) > 0$ and $\frac{d}{dr}[r^{2n(1/2-1/(p+1))} V_\delta(r)] \geq 0$) while in case $\lambda = 0$ we can reason exactly as in [Felmer et al. 2008, Proposition 3.1] (the proof there is for the annulus, but the argument also works in the case of a ball).

Remark 5.2. As we already mentioned, the Morse index of $u > 0$ as a critical point of $J_{\mu,\lambda}$ is 1. Recalling the definition of I_δ in (5-1), we have that also the Morse index of $I''_\delta(u)$ is 1 at least if $\lambda > -\lambda_1(B_1)$ and if $\delta > 0$ is small enough. When $\lambda < 0$, this was shown in the proof of the previous result, where we have dealt also with the case $\lambda = 0$. The proof for $\lambda > 0$ is the same as in the latter case.

Given $k > N$, as before, let us take $X = \{w \in W^{2,k}(B_1) : w = 0 \text{ on } \partial B_1\}$. Let us introduce the map $F : X \times \mathbb{R}^3 \to L^k(B_1) \times \mathbb{R}^2$ defined by

$$F(u, \mu, \lambda, \alpha) = \left(\Delta u - \lambda u + \mu u^p, \int_{B_1} u^2 \, dx - 1, \int_{B_1} |\nabla u|^2 \, dx - \alpha \right)$$

and its null set restricted to positive u,

$$\mathcal{S} = \{(u, \mu, \lambda, \alpha) \in X \times \mathbb{R}^3 : u > 0, \ F(u, \mu, \lambda, \alpha) = (0, 0, 0)\}.$$

It is immediate to check that $\mathcal{S} \cap \{\alpha \leq \lambda_1(B_1)\} = \{(\varphi_1, 0, -\lambda_1(B_1), \lambda_1(B_1))\}$ so that

$$\mathcal{S}^\pm := \mathcal{S} \cap \{\pm\mu > 0\} \subset \{\alpha > \lambda_1(B_1)\}.$$

We are going to show that $\mathcal{S}^+$ can be parametrized in a smooth way on α, thus proving the part of Theorem 1.12 regarding focusing nonlinearities. As we mentioned, the (easier) study of $\mathcal{S}^-$ is postponed to Appendix B. In view of the application of the implicit function theorem, we have the following:

Lemma 5.3. *Let $(u, \mu, \lambda, \alpha) \in \mathcal{S}^+$. Then the linear bounded operator*

$$F_{(u,\mu,\lambda)}(u, \mu, \lambda, \alpha) : X \times \mathbb{R}^2 \to L^k(B_1) \times \mathbb{R}^2$$

is invertible.

Proof. The lemma is a direct consequence of the Fredholm alternative and of the closed graph theorem once we show that the operator above is injective. Let us suppose by contradiction the existence of $(v, m, l) \neq (0, 0, 0)$ such that $F_{(u,\mu,\lambda)}(u, \mu, \lambda, \alpha)[v, m, l] = (0, 0, 0)$. This explicitly gives

$$\begin{aligned} &-\Delta u + \lambda u = \mu u^p, && \int_{B_1} u^2 \, dx = 1, && \int_{B_1} |\nabla u|^2 \, dx = \alpha, \\ &-\Delta v + \lambda v + l u = p\mu u^{p-1} v + m u^p, && \int_{B_1} uv \, dx = 0, && \int_{B_1} \nabla u \cdot \nabla v \, dx = 0. \end{aligned} \tag{5-3}$$

By testing the two differential equations by v, we obtain

$$\int_{B_1} u^p v \, dx = 0, \qquad \int_{B_1} |\nabla v|^2 \, dx + \lambda \int_{B_1} v^2 \, dx = p\mu \int_{B_1} u^{p-1} v^2 \, dx \tag{5-4}$$

so that

$$J''_{\mu,\lambda}(u)[u,u] < 0, \qquad J''_{\mu,\lambda}(u)[u,v] = 0, \qquad J''_{\mu,\lambda}(u)[v,v] = 0.$$

This implies that $J''_{\mu,\lambda}(u)[h,h] \le 0$ for every $h \in H = \operatorname{span}\{u, v\}$. By defining I_δ as in (5-1), for $\delta > 0$ small, we obtain $I''_\delta(u)[h,h] < 0$ for every $0 \ne h \in H$. Since H has dimension 2 ($v = cu$ would imply $c\int_\Omega u^2 = 0$), this contradicts Remark 5.2.

Proposition 5.4. *$\mathcal{S}^+$ is a smooth curve, parametrized by a map*

$$\alpha \mapsto (u(\alpha), \mu(\alpha), \lambda(\alpha)), \quad \alpha \in (\lambda_1(B_1), +\infty).$$

In particular, $u(\alpha)$ is the unique maximizer of M_α (as defined in (1-3)).

Proof. To start with, Lemma 2.3 and Proposition 2.4 imply that, for every fixed $\alpha^* > \lambda_1(B_1)$, there exists at least a corresponding point in $\mathcal{S}^+$. If $(u^*, \mu^*, \lambda^*, \alpha^*)$ denotes any such point (not necessarily related to M_{α^*}), then by Lemma 5.3, it can be continued, by means of the implicit function theorem, to an arc $(u(\alpha), \mu(\alpha), \lambda(\alpha))$, defined on a maximal interval $(\underline{\alpha}, \overline{\alpha}) \ni \alpha^*$, chosen in such a way that $\mu(\alpha) > 0$ on this interval. Since $u(\alpha)$ solves the equation, standard arguments involving the maximum principle and Hopf lemma allow one to obtain that $u(\alpha) > 0$ (recall that we are using the $W^{2,k}$-topology) along the arc, which consequently belongs to $\mathcal{S}^+$. We want to show that $(\underline{\alpha}, \overline{\alpha}) = (\lambda_1(B_1), +\infty)$.

Let us assume by contradiction $\underline{\alpha} > \lambda_1(\Omega)$. For $\alpha_n \to \underline{\alpha}^+$, Lemma 2.5 implies that, up to a subsequence,

$$u_n \rightharpoonup \bar{u} \quad \text{in } H^1_0(\Omega), \qquad \lambda_n \to \bar{\lambda}, \qquad \mu_n \to \bar{\mu}.$$

Thus,

$$-\Delta \bar{u} + \bar{\lambda}\bar{u} = \bar{\mu}\bar{u}^p \quad \text{in } \Omega,$$

and the convergence $u_n \to \bar{u}$ is actually strong in $H^2(\Omega)$. Then $\int_\Omega |\nabla \bar{u}|^2\,dx = \underline{\alpha} > \lambda_1(\Omega)$ so that $\bar{\mu} > 0$. Thus, Lemma 5.3 allows us to reach a contradiction with the maximality of $\underline{\alpha}$, and therefore, $\underline{\alpha} = \lambda_1(\Omega)$. Analogously, we can show that $\overline{\alpha} = +\infty$.

Once we know $\mathcal{S}^+$ is the disjoint union of smooth curves, each parametrized by $\alpha \in (\lambda_1(B_1), +\infty)$, it only remains to show that the curve of solutions is indeed unique. Suppose by contradiction that, for $\alpha_n \to \lambda_1(B_1)$, there exist $(u_1(\alpha_n), \mu_1(\alpha_n), \lambda_1(\alpha_n)) \ne (u_2(\alpha_n), \mu_2(\alpha_n), \lambda_2(\alpha_n))$ for every n. Then by Lemma 3.4, both triplets converge to $(\varphi_1, 0, -\lambda_1(B_1))$ in contradiction to Proposition 3.6.

Corollary 5.5. *Writing*

$$\frac{d}{d\alpha}(u(\alpha), \mu(\alpha), \lambda(\alpha)) = (v(\alpha), \mu'(\alpha), \lambda'(\alpha)),$$

we have

$$-\Delta v + \lambda' u + \lambda v = p\mu u^{p-1} v + \mu' u^p, \quad v \in H^1_0(B_1)$$

and

$$\int_{B_1} uv\,dx = 0, \qquad \int_{B_1} \nabla u \cdot \nabla v\,dx = \tfrac{1}{2}, \tag{5-5}$$

$$\mu \int_{B_1} u^p v\,dx = \tfrac{1}{2}, \qquad \mu' \int_{B_1} u^{p+1}\,dx = \lambda' - \frac{p-1}{2}. \tag{5-6}$$

Proof. Direct computations (by differentiating $F(u(\alpha), \mu(\alpha), \lambda(\alpha), \alpha) = 0$ and testing the differential equations by u and v) give the result.

In the following, we address the study of the monotonicity properties of the map

$$\alpha \mapsto (u(\alpha), \mu(\alpha), \lambda(\alpha))$$

introduced above, v always denoting the derivative of u with respect to α:

Lemma 5.6. *We have $\lambda'(\alpha) > 0$ for every $\alpha > \lambda_1(B_1)$.*

Proof. Let $(h, k) \in \mathbb{R}^2$, and let us consider the quadratic form

$$J''_{\mu,\lambda}(u)[hu + kv, hu + kv] =: ah^2 + 2bhk + ck^2.$$

Using Corollary 5.5, we obtain

$$\begin{aligned} a = J''_{\mu,\lambda}(u)[u, u] &= \int_{B_1} \left[|\nabla u|^2 + \lambda u^2 - p\mu u^{p+1}\right] dx = -(p-1)\mu \int_{B_1} u^{p+1}\, dx, \\ b = J''_{\mu,\lambda}(u)[u, v] &= \int_{B_1} \left[\nabla u \cdot \nabla v + \lambda uv - p\mu u^p v\right] dx = -\frac{p-1}{2}, \\ c = J''_{\mu,\lambda}(u)[v, v] &= \int_{B_1} \left[|\nabla v|^2 + \lambda v^2 - p\mu u^{p-1} v^2\right] dx = \frac{\mu'}{2\mu}. \end{aligned}$$

Since $J''_{\mu,\lambda}(u)$ has (large) Morse index equal to 1 (Remark 5.2) and $a < 0$, we have that $b^2 - ac > 0$, i.e.,

$$\mu' \int_{B_1} u^{p+1}\, dx > -\frac{p-1}{2}.$$

The lemma follows by comparing to (5-6).

Lemma 5.7. *If $\omega_N = |\partial B_1|$, then*

$$\mu' \int_{B_1} u^{p+1}\, dx = \frac{p+1}{2(p-1)} \left[\left(-p + 1 + \frac{4}{N}\right) - \frac{4\omega_N}{N} u_r(1) v_r(1)\right].$$

Proof. Recall that both u and v are radial. Since $\int_{B_1} u^2\, dx = 1$, the standard Pohozaev identity gives

$$\left(\frac{N}{2} - 1\right) \int_{B_1} |\nabla u|^2\, dx + \frac{1}{2} \int_{\partial B_1} |\nabla u|^2 (x \cdot \nu)\, d\sigma + \frac{\lambda N}{2} = \frac{\mu N}{p+1} \int_{B_1} |u|^{p+1}\, dx.$$

Inserting the information that u is radial and the equalities $\alpha = \int_{B_1} |\nabla u|^2\, dx$ and $\alpha + \lambda = \mu \int_{B_1} u^{p+1}\, dx$, we obtain

$$\lambda = \frac{2}{N} \frac{p+1}{p-1} \alpha - \alpha - \frac{\omega_N}{N} \frac{p+1}{p-1} u_r(1)^2.$$

Differentiating with respect to α, we have

$$\lambda' = \frac{2}{N} \frac{p+1}{p-1} - 1 - \frac{2\omega_N}{N} \frac{p+1}{p-1} u_r(1) v_r(1).$$

The result follows by recalling relation (5-6).

The following crucial lemma shows that, if p is subcritical or critical, then μ is an increasing function of α:

Lemma 5.8. *If $p \le 1+4/N$, then $\mu'(\alpha) > 0$ for every $\alpha > \lambda_1(B_1)$.*

Proof. The proof goes by contradiction: suppose that $\mu'(\bar{\alpha}) \le 0$ for some $\bar{\alpha} > \lambda_1(B_1)$. In the rest of the proof, all quantities are evaluated at such $\bar{\alpha}$.

Step 1. Let $v := \frac{d}{d\alpha}u|_{\alpha=\bar{\alpha}}$; then $v_r(1) < 0$ in case $p < 1+4/N$ and $v_r(1) \le 0$ if $p = 1+4/N$. This is an immediate consequence of Lemma 5.7, since $u_r(1) < 0$ by the Hopf lemma.

Step 2. We claim that, if r is sufficiently close to 1^-, then $v(r) > 0$. Since $v(1) = 0$, this is obvious if $v_r(1) < 0$. Hence, it only remains to consider the case $p = 1+4/N$ and $v_r(1) = 0$.

From the equation for v written in the radial coordinate

$$-v_{rr} - \frac{N-1}{r}v_r + \lambda v + \lambda' u = p\mu u^{p-1}v + \mu' u^p, \quad r \in (0,1),$$

we know (by letting $r \to 1^-$) that $v_{rr}(1) = 0$. Differentiating both sides of the above equation, we can write

$$-v_{rrr} + \frac{N-1}{r^2}v_r - \frac{N-1}{r}v_{rr} + \lambda v_r + \lambda' u_r = p(p-1)\mu u^{p-2}u_r v + p\mu u^{p-1}v_r + p\mu' u^{p-1}u_r;$$

now, if $p \ge 2$, the limit as $r \to 1^-$ yields

$$-v_{rrr}(1) + \lambda' u_r(1) = 0.$$

On the other hand, if $p < 2$, the same identity holds since by the l'Hôpital's rule

$$\lim_{r\to 1^-} u^{p-2}u_r v = \lim_{r\to 1^-} \frac{u_{rr}v + u_r v_r}{(2-p)u^{1-p}u_r} = \frac{u_{rr}(1)v(1) + u_r(1)v_r(1)}{(2-p)u_r(1)}u(1)^{p-1} = 0.$$

Thus, $v_{rrr}(1) < 0$ by Lemma 5.6, and the claim follows.

Step 3. Let $\bar{r} := \inf\{r : v > 0 \text{ in } (r,1)\}$ ($\bar{r} > 0$ since $\int_{B_1} uv\,dx = 0$). We claim that $v \le 0$ in $B_{\bar{r}}$. If not, there would be $0 \le r_1 < r_2 \le \bar{r}$ with the property that $v > 0$ in (r_1, r_2) and $r_i v(r_i) = 0$. Defining

$$v_1 := v|_{B_{r_2}\setminus B_{r_1}}, \qquad v_2 := v|_{B_1 \setminus B_{\bar{r}}},$$

we have that $v_i \in H_0^1(B_1)$ and $v_i \ge 0$ for $i = 1, 2$, and v_1 and v_2 are linearly independent. One can use the equation for v in order to evaluate

$$J''_{\mu,\lambda}(u)[v, v_i] = \int_{B_1} (\nabla v \cdot \nabla v_i + (\lambda - p\mu u^{p-1})vv_i)\,dx = \int_{B_1} (\mu' u^p v_i - \lambda' u v_i)\,dx < 0$$

and obtain

$$J''_{\mu,\lambda}(u)[t_1v_1 + t_2v_2, t_1v_1 + t_2v_2] < 0 \quad \text{whenever } t_1^2 + t_2^2 \ne 0$$

in contradiction to the fact that the Morse index of u is 1 (Remark 5.2).

Step 4. Once we know that $v \le 0$ in B_r and that $v > 0$ in $B_1 \setminus B_r$, we can combine the first equations in (5-5) and (5-6), together with the fact that u is monotone decreasing with respect to r, to write

$$\begin{aligned}\frac{1}{2\mu} &= \int_{B_1} u^p v\,dx = \int_{B_1\setminus B_{\bar r}} u^p v\,dx + \int_{B_{\bar r}} u^p v\,dx \\ &\le \Big(\max_{B_1\setminus B_{\bar r}} u^{p-1}\Big) \int_{B_1\setminus B_{\bar r}} uv\,dx + \Big(\min_{B_{\bar r}} u^{p-1}\Big) \int_{B_{\bar r}} uv\,dx \\ &= u^{p-1}(\bar r) \int_{B_1\setminus B_{\bar r}} uv\,dx + u^{p-1}(\bar r) \int_{B_{\bar r}} uv\,dx = 0,\end{aligned}$$

a contradiction.

Remark 5.9. When $1+4/N < p < 2^*-1$, Lemma 4.5 implies $\mu(+\infty) = 0$. Since also $\mu(\lambda_1(B_1)^+) = 0$, we deduce that μ' must change sign in the supercritical regime. Numerical experiments suggest that this should happen only once so that μ should have a unique global maximum and be strictly monotone elsewhere; see Remark 6.4 ahead.

We are ready to prove the existence of least energy solutions for (1-2).

Proof of Theorem 1.5. Recalling Definition 1.2, let $\rho > 0$ be fixed, and let $U \in \mathcal{P}_\rho$. Then

$$\int_{B_1} U^2\,dx = \rho, \quad U > 0, \qquad -\Delta U + \lambda U = U^p$$

for some λ. Then, setting $u = \rho^{-1/2} U$, direct calculations yield

$$\int_{B_1} u^2\,dx = 1, \quad u > 0, \qquad -\Delta u + \lambda u = \rho^{(p-1)/2} u^p.$$

Writing $\int_{B_1} |\nabla u|^2\,dx = \alpha$, this amounts to saying that $(u, \rho^{(p-1)/2}, \lambda, \alpha) \in \mathcal{S}^+$. Equivalently,

$$U \in \mathcal{P}_\rho \iff \rho = \mu^{2/(p-1)}, \quad U = \mu^{1/(p-1)} u \quad \text{for some } (u, \mu, \lambda, \alpha) \in \mathcal{S}^+.$$

We divide the end of the proof into three cases.

Case 1: $1 < p < 1+4/N$. By Lemmas 4.5 and 5.8 and Proposition 5.4, we have that, for every ρ, there exists exactly one point in $\mathcal{S}^+$ satisfying $\mu^{2/(p-1)} = \rho$.

Case 2: $p = 1+4/N$. The same as the previous case, taking into account that, by Lemma 4.5, $\mathcal{P}_{\mu^{2/(p-1)}}$ is not empty if and only if $\mu < \|Z_{N,p}\|^{p-1}_{L^2(\mathbb{R}^N)}$.

Case 3: $1+4/N < p < 2^*-1$. Since in this case $\mu(\lambda_1(B_1)) = \mu(+\infty) = 0$ (by Lemma 4.5), then

$$\mu^* = \max_{(\lambda_1(B_1),+\infty)} \mu$$

is well defined and achieved. Furthermore, $\mathcal{P}_{\mu^{2/(p-1)}}$ is empty for $\mu > \mu^*$, and it contains at least two points for $0 < \mu < \mu^*$. It remains to prove that, if $0 < \rho \le \rho^* = (\mu^*)^{(p-1)/2}$, then e_ρ is achieved. This is immediate whenever $\mathcal{P}_\rho$ is finite. Otherwise, let $u_n = u(\alpha_n)$, with $\mu(\alpha_n) = \rho^{(p-1)/2}$, denote a minimizing sequence. Then Lemma 4.5 implies that α_n is bounded, and by continuity, the same is true for λ_n. We deduce that, up to subsequences, $u_n \to u^* \in \mathcal{P}_\mu$, and $J_{\mu,0}(u^*) = e_\rho$.

Remark 5.10. By comparing Theorem 1.5 and Proposition A.1, we have that, when $p \le 1+4/N$ and positive least energy solutions exist, the condition $U > 0$ may be safely removed from Definition 1.2 without altering the problem (in fact, also the condition $-\Delta U + \lambda U = U^{p+1}$ for some λ is not necessary). On the other hand, in other cases, it is essential. For instance, when p is critical, then the set of not necessarily positive solutions with fixed mass

$$\mathscr{P}'_\rho = \{U \in H^1_0(B_1) : \mathfrak{Q}(U) = \rho,\ \text{there exists } \lambda \text{ such that } -\Delta U + \lambda U = U^p\}$$

is not empty also when $\rho \ge \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$, as illustrated in [Fibich and Merle 2001, Figure 1].

6. Stability results

In this section, we discuss orbital stability of standing wave solutions $e^{i\lambda t}U(x)$ for the NLS (1-1). We recall that such solutions are called orbitally stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $\Phi_0 \in H^1_0(B_1, \mathbb{C})$ is such that $\|\Phi_0 - U\|_{H^1_0(B_1,\mathbb{C})} < \delta$ and $\Phi(t, x)$ is the solution of (1-1) with $\Phi(0, \cdot) = \Phi_0$ in some interval $[0, t_0)$, then $\Phi(t, \cdot)$ can be continued to a solution in $0 \le t < \infty$ and

$$\sup_{0<t<\infty} \inf_{s\in\mathbb{R}} \|\Phi(t, \cdot) - e^{i\lambda s}U\|_{H^1_0(B_1,\mathbb{C})} < \varepsilon;$$

otherwise, they are called unstable. To do this, we lean on the following result, which expresses in our context the abstract theory developed in [Grillakis et al. 1987]:

Proposition 6.1 [Fukuizumi et al. 2012, Proposition 5]. *Let us assume local existence as in Theorems 1.7 and 1.8, and let R_λ be the unique positive solution of* (1-2).

- *If $\partial_\lambda \|R_\lambda\|^2_{L^2} > 0$, then $e^{i\lambda t}R_\lambda$ is orbitally stable.*
- *If $\partial_\lambda \|R_\lambda\|^2_{L^2} < 0$, then $e^{i\lambda t}R_\lambda$ is unstable.*

Corollary 6.2. *Let $(u(\alpha), \mu(\alpha), \lambda(\alpha), \alpha) \in \mathscr{S}^+$ with $U(\alpha) = \mu^{1/(p-1)}(\alpha)u(\alpha)$ denoting the corresponding solution of* (1-2) (*with $\lambda = \lambda(\alpha)$*).

- *If $\mu'(\alpha) > 0$, then $e^{i\lambda(\alpha)t}U(\alpha)$ is orbitally stable.*
- *If $\mu'(\alpha) < 0$, then $e^{i\lambda(\alpha)t}U(\alpha)$ is unstable.*

Proof. Taking into account Proposition 5.4 and Lemma 5.6, and reasoning as in the proof of Theorem 1.5, we have that $R_{\lambda(\alpha)} = \mu^{1/(p-1)}(\alpha)u(\alpha)$ so that

$$\partial_\lambda \|R_\lambda\|^2_{L^2} = \frac{(\mu^{2/(p-1)})'(\alpha)}{\lambda'(\alpha)} = \frac{2\mu^{(3-p)/(p-1)}(\alpha)}{(p-1)\lambda'(\alpha)}\mu'(\alpha).$$

We recall that μ' may be negative only when p is supercritical. This case is enlightened by the following lemma:

Lemma 6.3. *Let $p > 1+4/N$, and consider the map $\alpha \mapsto (u(\alpha), \mu(\alpha), \lambda(\alpha))$ defined as in Proposition 5.4. If $\alpha_1 < \alpha_2$ are such that*

$$\mu(\alpha) > \mu(\alpha_1) = \mu(\alpha_2) =: \bar{\mu} \quad \textit{for every } \alpha \in (\alpha_1, \alpha_2),$$

then

$$J_{\bar{\mu},0}(u(\alpha_1)) < J_{\bar{\mu},0}(u(\alpha_2)).$$

Proof. Writing $M(\alpha) = M_\alpha = \int_{B_1} u^{p+1}(\alpha)\,dx$, we have that

$$2J_{\bar{\mu},0}(\alpha_i) = \alpha_i - \frac{2\bar{\mu}}{p+1} M(\alpha_i).$$

Now, (5-6) yields $M'(\alpha) = (p+1)\int_{B_1} u^p v\,dx = (p+1)/(2\mu(\alpha))$, where as usual $v := \frac{d}{d\alpha}u$. The Lagrange theorem applied to M forces the existence of $\alpha^* \in (\alpha_1, \alpha_2)$ such that

$$\frac{M(\alpha_2) - M(\alpha_1)}{\alpha_2 - \alpha_1} = \frac{p+1}{2\mu(\alpha^*)} < \frac{p+1}{2\bar{\mu}},$$

which is equivalent to the desired statement.

We are ready to give the proofs of our stability results.

Proof of Theorems 1.7 and 1.8. The proof in the subcritical and critical cases is a direct consequence of Lemma 5.8 and Corollary 6.2 (recall that in this case there is a full correspondence between least energy solutions and least action ones). To show Theorem 1.7(2), we prove stability for any $\rho > 0$ such that $\bar{\mu} = \rho^{(p-1)/2}$ is a regular value of the map $\alpha \mapsto \mu(\alpha)$, the conclusion following by the Sard lemma. Recalling that $\mu(\lambda_1(B_1)) = \mu(+\infty) = 0$, we have that, if $\bar{\mu}$ is regular, then its counterimage $\{\alpha : \mu(\alpha) = \bar{\mu}\}$ is the union of a finite number of pairs $\{\alpha_{i,1}, \alpha_{i,2}\}$, each of which satisfies the assumptions of Lemma 6.3, and moreover, $\mu'(\alpha_{i,1}) > 0 > \mu'(\alpha_{i,2})$. Since such a counterimage is in 1-to-1 correspondence with $\mathcal{P}_\rho$ and

$$\mathscr{E}(U(\alpha_{i,j})) = \mathscr{E}(\bar{\mu}^{1/(p-1)} u(\alpha_{i,j})) = \bar{\mu}^{2/(p-1)} J_{\bar{\mu},0}(u(\alpha_{i,j})),$$

we deduce from Lemma 6.3 that the least energy solution corresponds to $\alpha_{i,1}$, for some i, and the conclusion follows again by Corollary 6.2.

Remark 6.4. In the supercritical case $p > 1 + 4/N$, we expect orbital stability for every $\rho \in (0, \rho^*)$ and instability for $\rho = \rho^*$. Indeed, in case $N = 3$ and $p = 3$, we have plotted numerically the graph of $\mu(\alpha)$ in Figure 1. The picture suggests that μ has a unique local maximum μ^*, associated to the maximal value of the mass $\rho^* = (\mu^*)^{(p-1)/2}$. For any $\mu < \mu^*$, we have exactly two solutions, and the least energy one corresponds to $\mu'(\alpha) > 0$; hence, it is associated with an orbitally stable standing wave. For $\mu = \mu^*$, we have exactly one solution; in that case, the abstract theory developed in [Grillakis et al. 1987] predicts the corresponding standing wave to be unstable.

Appendix A: Gagliardo–Nirenberg inequalities

It is proved in [Weinstein 1983] that the sharp Gagliardo–Nirenberg inequality

$$\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \le C_{N,p} \|u\|^{p+1-N(p-1)/2}_{L^2(\mathbb{R}^N)} \|\nabla u\|^{N(p-1)/2}_{L^2(\mathbb{R}^N)} \tag{A-1}$$

holds for every $u \in H^1(\mathbb{R}^N)$ and that the best constant $C_{N,p}$ is achieved by (any rescaling of) $Z_{N,p}$.

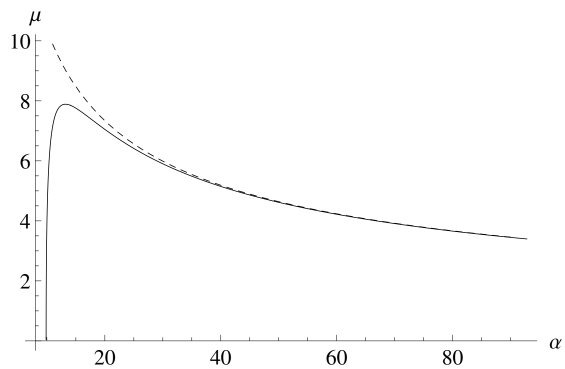

Figure 1. Numerical graph of $\alpha \mapsto \mu(\alpha)$ in the supercritical case $N = 3$ and $p = 3$ (continuous line) and of the map $\alpha \mapsto \alpha^{-1/2} \cdot \sqrt{3} \int_{\mathbb{R}^3} Z_{3,3}^2 \, dx$ (dashed line). The latter is the theoretical asymptotic expansion of $\mu(\alpha)$ as $\alpha \to +\infty$ as predicted by Lemmas 4.4 and 4.5.

When dealing with $H_0^1(\Omega)$, $\Omega \neq \mathbb{R}^N$, one can prove that the identity holds with the same best constant: in fact, one inequality is trivial, and the other is obtained by constructing a suitable competitor of the form $u(x) = (hZ_{N,p}(kx) - j)^+$, for suitable h, k and j, and exploiting the exponential decay of Z. Contrary to the previous case, now such a constant cannot be achieved; otherwise, we would contradict [Weinstein 1983]. This is related to the maximization problem (1-3) since

$$C_{N,p} = \sup_{H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_{L^{p+1}(\Omega)}^{p+1}}{\|u\|_{L^2(\Omega)}^{p+1-N(p-1)/2} \|\nabla u\|_{L^2(\Omega)}^{N(p-1)/2}} = \sup_{\alpha \geq \lambda_1(\Omega)} \frac{M_\alpha}{\alpha^{N(p-1)/4}}.$$

By the above considerations, we deduce that

$$M_\alpha < C_{N,p}\alpha^{N(p-1)/4} \quad \text{for every } \alpha, \qquad \lim_{\alpha \to +\infty} \frac{M_\alpha}{\alpha^{N(p-1)/4}} = C_{N,p} \tag{A-2}$$

in perfect agreement with the estimates at the end of Section 4.

For the reader's convenience, we deduce the following well-known result:

Proposition A.1. *Let $\rho > 0$ be fixed. The infimum*

$$\inf\{\mathscr{E}(U) : U \in H_0^1(\Omega) \text{ and } \mathscr{Q}(U) = \rho\}$$

(i) *is achieved by a positive function if either $1 < p < 1+4/N$ or $p = 1+4/N$ and $\rho < \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2$, and*

(ii) *equals $-\infty$ if either $1 + 4/N < p < 2^* - 1$ or $p = 1 + 4/N$ and $\rho > \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2$.*

Proof. As usual, writing $u = \rho^{-1/2}U$ and $\bar{\mu} = \rho^{(p-1)/2}$, we have that the above minimization problem is equivalent to

$$\inf\{J_{\bar{\mu},0}(u) : u \in H_0^1(\Omega) \text{ and } \|u\|_{L^2(\Omega)} = 1\},$$

where $J_{\mu,\lambda}$ is defined in (4-1). In turn, this problem can be written as

$$\inf_{\alpha\geq\lambda_1(\Omega)} \tfrac{1}{2}\alpha - \frac{\bar{\mu}}{p+1} M_\alpha.$$

The proposition follows from (A-2), recalling that, when $p = 1+4/N$,

$$C_{N,p} = \left(1+\frac{2}{N}\right)\left(\int_{\mathbb{R}^N} Z_{N,p}^2\,dx\right)^{-2/N}$$

by the Pohozaev identity.

Appendix B: The defocusing case $\mu < 0$

In this case, it is not necessary to restrict to spherical domains; therefore, in this appendix, we consider a generic smooth, bounded domain Ω. As in Section 5, we work in the space $X = \{w \in W^{2,k}(\Omega) : w = 0 \text{ on } \partial\Omega\}$, for some $k > N$, and with the map $F : X \times \mathbb{R}^3 \to L^k(\Omega) \times \mathbb{R}^2$ defined by

$$F(u, \mu, \lambda, \alpha) = \left(\Delta u - \lambda u + \mu u^p,\ \int_\Omega u^2\,dx - 1,\ \int_\Omega |\nabla u|^2 - \alpha\right).$$

We aim to provide a full description of the set

$$\mathcal{S}^- = \{(u, \mu, \lambda, \alpha) \in X \times \mathbb{R}^3 : u > 0,\ \ \mu < 0,\ \ F(u, \mu, \lambda, \alpha) = (0, 0, 0)\},$$

thus concluding the proof of Theorem 1.12.

Lemma B.1. *Let $(u, \mu, \lambda, \alpha) \in \mathcal{S}^-$. Then the linear bounded operator*

$$F_{(u,\mu,\lambda)}(u, \mu, \lambda, \alpha) : X \times \mathbb{R}^2 \to L^k(\Omega) \times \mathbb{R}^2$$

is invertible.

Proof. As in the proof of Lemma 5.3, it is sufficient to prove injectivity.

As in that proof, we assume the existence of a nontrivial (v, m, l) such that (5-3) and (5-4) hold. Since $\partial_\nu u < 0$ on $\partial\Omega$, we can test the equation for u by $v^2/u \in H_0^1(\Omega)$, obtaining

$$\begin{aligned}
\int_\Omega (\mu u^{p-1} v^2 - \lambda v^2)\,dx &= \int_\Omega \nabla u \cdot \nabla\left(\frac{v^2}{u}\right) dx = \int_\Omega \nabla u \cdot \left(2\frac{v}{u}\nabla v - \frac{v^2}{u^2}\nabla u\right) dx \\
&= -\int_\Omega \left|\frac{v}{u}\nabla u - \nabla v\right|^2 dx + \int_\Omega |\nabla v|^2\,dx \\
&\leq \int_\Omega (p\mu u^{p-1} v^2 + m u^p v - luv - \lambda v^2)\,dx \\
&= \int_\Omega (p\mu u^{p-1} v^2 - \lambda v^2)\,dx.
\end{aligned}$$

Therefore, with $\mu < 0$ and $p > 1$, we must have $v \equiv 0$. Finally, by testing the equation for v by u, we deduce that $l = m \int_\Omega u^{p+1}\,dx$, concluding the proof.

Proposition B.2. *$\mathcal{S}^-$ is a smooth curve, and it can be parametrized by a unique map*

$$\alpha \mapsto (u(\alpha), \mu(\alpha), \lambda(\alpha)), \quad \alpha \in (\lambda_1(\Omega), +\infty).$$

In particular, $u(\alpha)$ is the unique minimizer associated to m_α (as defined in (1-3)*). Furthermore, $\mu'(\alpha) < 0$ and $\lambda'(\alpha) < 0$ for every α.*

Proof. One can use Lemma B.1 and reason as in the proof of Proposition 5.4 in order to prove that $\mathcal{S}^-$ consists of a unique, smooth curve parametrized by $\alpha \in (\lambda_1(\Omega), +\infty)$ so that $u(\alpha)$ must achieve m_α. Moreover, all the relations contained in Corollary 5.5 are true also in this case.

In order to show the monotonicity of μ and λ, we remark that one can also prove, in a standard way, that u is the global unique minimizer of the related functional $J_{\mu,\lambda}$, which is bounded below and coercive since $\mu < 0$. Since u is nondegenerate (by virtue of Lemma B.1), we obtain that $J''_{\mu,\lambda}(u)[w, w] > 0$ for every nontrivial w. But then one can reason as in the proof of Lemma 5.6: using the corresponding notation, we have that in this case both $c > 0$ and $b^2 - ac < 0$. This, together with (5-6), concludes the proof.

Remark B.3. By the above results, it is clear that $\mathcal{S}^-$ may be parametrized also with respect to λ (or μ). Under this perspective, uniqueness and continuity for the case $p = 3$ were proved in [Berger and Fraenkel 1970] (for the problem without mass constraint).

We conclude by showing some asymptotic properties of $\mathcal{S}^-$ as $\alpha \to +\infty$ (the case $\alpha \to \lambda_1(\Omega)^+$ has been considered in Section 3). Such properties are well known in the case $p = 3$ since they have been studied in a different context (among others, we cite [Berger and Fraenkel 1970; Bethuel et al. 1993; André and Shafrir 1998; Serfaty 2001]) and the proof can be adapted to general p.

Proposition B.4. *Under the notation of Proposition B.2, we have that, as $\alpha \to +\infty$, $\mu \to -\infty$ and $\lambda \to -\infty$. Furthermore, if $\partial\Omega$ is smooth, then*

$$u \to |\Omega|^{-1/2} \text{ strongly in } L^{p+1}(\Omega), \qquad \frac{\lambda}{\mu} \to |\Omega|^{-(p-1)/2}, \qquad \frac{\alpha}{\lambda} \to 0$$

as $\alpha \to +\infty$.

Proof. Since we know that μ is decreasing and that for each $\mu < 0$ there exists a solution, we must have $\mu(\alpha) \to -\infty$. Moreover, $\lambda \leq -\alpha \to -\infty$.

Next we are going to show that, under the assumption that $\partial\Omega$ is smooth,

$$\int_\Omega u^{p+1} \to |\Omega|^{-(p-1)/2}. \tag{B-1}$$

To this aim, notice that, by the uniqueness proved in the previous proposition, u satisfies

$$J_{\mu,0}(u) = \min\left\{ J_{\mu,0}(\varphi) : \varphi \in H_0^1(\Omega),\ \int_\Omega \varphi^2\,dx = 1 \right\}.$$

For $x \in \Omega$, setting $d(x) := \operatorname{dist}(x, \partial\Omega)$, we construct a competitor function for the energy $J_{\mu,0}(u)$ as

$$\varphi_\mu(x) = \begin{cases} k^{-1}|\Omega|^{-1/2} & \text{if } d(x) \geq (-\mu)^{-1/2}, \\ k^{-1}|\Omega|^{-1/2}(-\mu)^{1/2} d(x) & \text{if } 0 \leq d(x) \leq (-\mu)^{-1/2}, \end{cases}$$

where k is such that $\|\varphi_\mu\|_{L^2(\Omega)} = 1$. With the aid of the coarea formula, and using the fact that $\partial\Omega$ is smooth, it is possible to check that $k = 1 + \mathrm{O}((-\mu)^{-1/2})$, and thus,

$$\int_\Omega |\nabla\varphi_\mu|^2\,dx = \mathrm{O}(\sqrt{-\mu}), \qquad \int_\Omega (\varphi_\mu^q - |\Omega|^{-q/2})\,dx = \mathrm{O}((-\mu)^{-1/2}) \tag{B-2}$$

for every $q > 1$. By rewriting $J_{\mu,0}$ in the form

$$J_{\mu,0}(\varphi) = \int_\Omega \left\{ \frac{|\nabla\varphi|^2}{2} - \frac{\mu}{p+1}(|\varphi|^{p+1} - |\Omega|^{-(p+1)/2}) \right\} dx - \frac{\mu}{p+1}|\Omega|^{-(p-1)/2},$$

and by using the estimates (B-2) with $q = p+1$, we obtain

$$J_{\mu,0}(u) \le J_{\mu,0}(\varphi_\mu) = \mathrm{O}(\sqrt{-\mu}) - \frac{\mu}{p+1}|\Omega|^{-(p-1)/2}$$

so that

$$0 \le \int_\Omega (u^{p+1} - |\Omega|^{-(p+1)/2})\,dx \le \mathrm{O}((-\mu)^{-1/2}) \to 0$$

(by using Lemma 2.1(iv)) so that (B-1) is proved.

Now, for each L^2-normalized φ, we rewrite $J_{\mu,0}(\varphi)$ as

$$J_{\mu,0}(\varphi) = \int_\Omega \left\{ \frac{|\nabla\varphi|^2}{2} - \frac{\mu}{p+1}(|\varphi|^{(p+1)/2} - |\Omega|^{-(p+1)/4})^2 \right\} dx$$
$$- \frac{2\mu}{p+1}|\Omega|^{-(p+1)/4} \int_\Omega (|\varphi|^{(p+1)/2} - |\Omega|^{-(p+1)/4})\,dx - \frac{\mu}{p+1}|\Omega|^{-(p-1)/2}.$$

Reasoning as before (using this time (B-2) for $q = (p+1)/2$), one shows that

$$\int_\Omega (|u|^{(p+1)/2} - |\Omega|^{-(p+1)/4})^2\,dx + 2|\Omega|^{-(p+1)/2} \int_\Omega (|u|^{(p+1)/2} - |\Omega|^{-(p+1)/2})\,dx \le \mathrm{O}((-\mu)^{-1/2}).$$

If $p \ge 3$, by the Hölder inequality, we have that the second integral in the left-hand side above is nonnegative while for $p < 3$ it tends to 0 as $\alpha \to +\infty$. The latter statement is a consequence of both the Hölder and interpolation inequalities, which yield

$$\int_\Omega u^{(p+1)/2}\,dx \le |\Omega|^{(3-p)/4}, \qquad \|u\|_{L^{(p+1)/2}(\Omega)} \ge \|u\|_{L^{p+1}(\Omega)}^{(p-3)/(p-1)},$$

as well as of (B-1). Thus, we have concluded that

$$u^{(p+1)/2} \to |\Omega|^{-(p+1)/4} \quad \text{in } L^2(\Omega).$$

In particular, up to a subsequence, $u \to |\Omega|^{-1/2}$ a.e., and there exists $h \in L^2$ (independent of α) so that $|u|^{(p+1)/2} \le h$. We can now conclude by applying Lebesgue's dominated convergence theorem.

To proceed with the proof, notice that, from the equality $\alpha + \lambda = \mu \int_\Omega u^{p+1}\,dx$ and Lemma 2.1(iv), we deduce

$$\lambda \le \mu|\Omega|^{-(p-1)/2}. \tag{B-3}$$

On the other hand, we have

$$-\lambda \le (p+1) J_{\mu,0}(u) \le (p+1) J_{\mu,0}(\varphi_\mu) \le C(-\mu)^{1/2} - \mu|\Omega|^{-(p-1)/2}.$$

Dividing the last inequality by $-\mu$ and letting $\mu \to -\infty$, we obtain

$$\limsup \frac{\lambda}{\mu} \le |\Omega|^{-(p-1)/2},$$

which together with (B-3) provides the convergence of μ.

The last part of the statement is obtained by combining the previous asymptotics with the identity $\alpha/\mu = -\mu + \int_\Omega u^{p+1}\, dx$.

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