# Enumeration of edges in some lattices of paths 

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#### Abstract

We enumerate the edges in the Hasse diagram of several lattices arising in the combinatorial context of lattice paths. Specifically, we will consider the case of Dyck, Grand Dyck, Motzkin, Grand Motzkin, Schröder and Grand Schröder lattices. Finally, we give a general formula for the number of edges in an arbitrary Young lattice (which can be interpreted in a natural way as a lattice of paths).


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## 1 Introduction

Fixed a Cartesian coordinate system in the discrete plane $\mathbb{Z} \times \mathbb{Z}$, for any class of lattice paths starting from the origin and ending at a same point on the $x$-axis, we can define a partial order by declaring that a path $\gamma_{1}$ is less than or equal to another path $\gamma_{2}$ whenever $\gamma_{1}$ lies weakly below $\gamma_{2}$. In some cases, the resulting poset has a structure of distributive lattice [9]. This is true, for instance, for the more common classes of paths: Dyck and Grand Dyck paths, Motzkin and Grand Motzkin paths, Schröder and Grand Schröder paths, in correspondence of which we have Dyck and Grand Dyck lattices, Motzkin and Grand Motzkin lattices, Schröder and Grand Schröder lattices. In all these cases, considering the paths according to their length or semi-length, we have a sequence of distributive lattices. Another important class is given by the (finite) Young lattices, which can be interpreted in a very natural way as lattices of paths. In particular, the Dyck lattices are isomorphic to the Young lattices associated with a staircase shape, and the Grand Dyck lattices are isomorphic to the Young lattices associated with a rectangular shape.

In this paper, we enumerate the edges of the Hasse diagram for all distributive lattices recalled above. For the classical paths, using the method of path decomposition, we obtain the generating series with respect to certain parameters and then (applying Proposition 1) we obtain the edge series. For the Young lattices, we obtain a general formula for the number of edges in the Hasse diagram of $\mathcal{Y}_{\lambda}$ which is valid for an arbitrary partition $\lambda$.

We also introduce the Hasse index of a poset as the density index of the associated Hasse diagram (the quotient between the number of edges and the number of vertices) and we prove that the Hasse index of all sequences of lattices of classical paths is always related (equal, asymptotical equivalent, or asymptotically quasi-equivalent) to the Hasse index of Boolean lattices.

The enumeration of the edges in the Hasse diagram of a poset $P$ can be considered as a specialization of the more general case of enumerating all saturated chains in $P$, which is clearly much more complicated. In [8], we obtain a general formula for counting saturated chains of any finite length $k$ in any Dyck lattice.

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## 2 Background

### 2.1 Lattice paths

In this paper, we will consider the following classes of paths.

1. The class $\mathcal{G D}$ of Grand Dyck paths, i.e. the class of all lattice paths starting from the origin, ending on the $x$-axis, and consisting of up steps $U=(1,1)$ and down steps $D=(1,-1)$. The class $\mathcal{D}$ of Dyck paths $[6,19]$, consisting of all Grand Dyck paths never going below the $x$-axis.
2. The class $\mathcal{G} \mathcal{M}$ of Grand Motzkin paths, i.e. the class of all lattice paths starting from the origin, ending on the $x$-axis, and consisting of up steps $U=(1,1)$, down steps $D=(1,-1)$ and horizontal steps $H=(1,0)$. The class $\mathcal{M}$ of Motzkin path [19], consisting of all Grand Motzkin paths never going below the $x$-axis.
3. The class $\mathcal{G S}$ of Grand Schröder paths (or central Delannoy paths), i.e. the class of all lattice paths starting from the origin, ending on the $x$-axis, and consisting of up steps $U=(1,1)$, down steps $D=(1,-1)$ and double horizontal steps $H=(2,0)$. The class $\mathcal{S}$ of Schröder path, consisting of all Grand Schröder paths never going below the $x$-axis.

The length of a path is the number of its steps, and the semi-length of a path of even length is half of the number of its steps. In general, for a class $\mathcal{P}$ containing paths of every length (as in the case of Grand Motzkin paths and Motzkin paths) we denote by $\mathcal{P}_{n}$ the set of all paths in $\mathcal{P}$ having length $n$. If the class $\mathcal{P}$ contains only paths of even length (as in the case of Grand Dyck paths, Dyck paths, Grand Schröder paths and Schröder paths) we denote by $\mathcal{P}_{n}$ the set of all paths in $\mathcal{P}$ having semi-length $n$. Sometimes, we write • for the empty path (consisting of zero steps).

For a class $\mathcal{P}$ containing paths never going below the $x$-axis, we define the class $\overline{\mathcal{P}}$ of reflected paths as the set of all paths obtained by reflecting about the $x$-axis the paths in $\mathcal{P}$. In particular, we will consider the class $\overline{\mathcal{D}}$ of reflected Dyck paths, the class $\overline{\mathcal{M}}$ of reflected Motzkin paths, and the class $\overline{\mathcal{S}}$ of reflected Schröder paths. Moreover, we say that a reflected path $\gamma \in \overline{\mathcal{P}}$ is subelevated when $\gamma=D \gamma^{\prime} U$, for any $\gamma^{\prime} \in \overline{\mathcal{P}}$.

A path can always be considered as a word on the alphabet given by the set of possible steps. A factor of a path $\gamma$ (considered as a word) is a word $\alpha$ such that $\gamma=\gamma^{\prime} \alpha \gamma^{\prime \prime}$. We write $\omega_{\alpha}(\gamma)$ for the number of all occurrences of the word $\alpha$ as a factor of $\gamma$. In particular, if $\alpha$ is a path starting and ending at the same level, then we write $\omega_{\alpha}^{*}(\gamma)$ for the number of all occurrences of the word $\alpha$ as a factor of $\gamma$ not on the $x$-axis. For instance, if $\gamma=U U D H D H U H D$, then $\omega_{H}(\gamma)=3$ and $\omega_{H}^{*}(\gamma)=2$ (since the second horizontal step lies on the $x$-axis).

### 2.2 Enumeration and asymptotics

For simplicity, we recall some well known enumerative properties of the paths considered in Subsection 2.1 that will be used in the rest of the paper.

The Grand Dyck paths are enumerated by the central binomial coefficients $\binom{2 n}{n}$ [17, A000984], and the Dyck paths are enumerated by the Catalan numbers $C_{n}=\binom{2 n}{n} \frac{1}{n+1}$ [17, A000108]. These numbers have generating series

$$
B(x)=\sum_{n \geq 0}\binom{2 n}{n} x^{n}=\frac{1}{\sqrt{1-4 x}}, \quad C(x)=\sum_{n \geq 0} C_{n} x^{n}=\frac{1-\sqrt{1-4 x}}{2 x}
$$

The trinomial coefficient $\binom{n ; 3}{k}$ is defined as the coefficient of $x^{k}$ in the expansion of $\left(1+x+x^{2}\right)^{n}$ [5] [17, A027907]. The Grand Motzkin paths are enumerated by the central trinomial
coefficients $\binom{n ; 3}{n}$ [17, A002426], and the Motzkin paths are enumerated by the Motzkin numbers [17, A001006]. These numbers have generating series

$$
T(x)=\sum_{k \geq 0}\binom{n ; 3}{n} x^{n}=\frac{1}{\sqrt{1-2 x-3 x^{2}}}, \quad M(x)=\sum_{k \geq 0} M_{n} x^{n}=\frac{1-x-\sqrt{1-2 x-3 x^{2}}}{2 x^{2}}
$$

The Grand Schröder paths are enumerated by the central Delannoy numbers $d_{n}$ [17, A001850], and the Schröder paths are enumerated by the large Schröder numbers $r_{n}$ [17, A006318]. These numbers have generating series

$$
d(x)=\sum_{n \geq 0} d_{n} x^{n}=\frac{1}{\sqrt{1-6 x+x^{2}}}, \quad r(x)=\sum_{n \geq 0} r_{n} x^{n}=\frac{1-x-\sqrt{1-6 x+x^{2}}}{2 x}
$$

Given two sequences $a_{n}$ and $b_{n}$ (where $b_{n}$ is definitively non zero), we recall that the notation $a_{n} \sim b_{n}$ means that $a_{n} / b_{n} \rightarrow 1$ as $n \rightarrow+\infty$. Moreover, we recall that the Darboux theorem [2, p. 252] says that: given a complex number $\xi \neq 0$ and a complex function $f(x)$ analytic at the origin, if $f(x)=(1-x / \xi)^{-\alpha} \psi(x)$ where $\psi(x)$ is a series with radius of convergence $R>|\xi|$ and $\alpha \notin\{0,-1,-2, \ldots\}$, then

$$
\left[x^{n}\right] f(x) \sim \frac{\psi(\xi)}{\xi^{n}} \frac{n^{\alpha-1}}{\Gamma(\alpha)}
$$

where $\Gamma(z)$ is Euler's Gamma function. Using such a theorem, it is possible to obtain the following asymptotic expansions:

$$
\begin{equation*}
\binom{n ; 3}{n} \sim \frac{3^{n}}{2} \sqrt{\frac{3}{n \pi}}, \quad M_{n} \sim \frac{3^{n+1}}{2 n} \sqrt{\frac{3}{n \pi}}, \quad d_{n} \sim \frac{(1+\sqrt{2})^{2 n+1}}{2 \sqrt{\sqrt{2} n \pi}}, \quad r_{n} \sim \frac{(1+\sqrt{2})^{2 n+1}}{n \sqrt{2 \sqrt{2} n \pi}} \tag{1}
\end{equation*}
$$

Finally, we recall the following elementary expansions

$$
\frac{x^{r}}{(1-x)^{s+1}}=\sum_{n \geq 0}\binom{n-r+s}{s} x^{n} \quad \text { and } \quad \frac{1}{(1-x)^{s}}=\sum_{n \geq 0}\left(\binom{s}{n}\right) x^{n}
$$

where $\left.\binom{n}{k}\right)=\frac{n(n+1) \cdots(n+k-1)}{k!}$ are the multiset coefficients.

### 2.3 Edge enumeration

The Hasse diagram $\mathcal{H}(P)$ of a finite poset $P$ is a (directed) graph representing the poset, where the vertices are the elements of $P$ and the adjacency relation is the cover relation. The number of vertices is $|P|$. We will denote with $\ell(P)$ the number of all edges in $\mathcal{H}(P)$. If $\Delta x$ is the set of all elements covering $x$ and $\nabla x$ is the set of all elements covered by $x$, then we have

$$
\ell(P)=\sum_{x \in P}|\Delta x|=\sum_{x \in P}|\nabla x|
$$

Moreover, if we consider the polynomials

$$
\Delta(P ; q)=\sum_{x \in P} q^{|\Delta x|} \quad \text { and } \quad \nabla(P ; q)=\sum_{x \in P} q^{|\nabla x|}
$$

then we have at once the identities

$$
\begin{equation*}
\ell(P)=\left[\partial_{q} \Delta(P ; q)\right]_{q=1} \quad \text { and } \quad \ell(P)=\left[\partial_{q} \nabla(P ; q)\right]_{q=1} \tag{2}
\end{equation*}
$$

where $\partial_{q}$ denotes the partial derivative with respect to $q$. The edge generating series of a sequence of posets $\mathscr{P}=\left\{P_{0}, P_{1}, P_{2}, \ldots\right\}$ is the ordinary generating series $\ell_{\mathscr{P}}(x)$ of the numbers $\ell\left(P_{n}\right)$. Similarly, the $\Delta$-series and $\nabla$-series associated with the sequence $\mathscr{P}$ are the generating series $\Delta_{\mathscr{P}}(q ; x)$ and $\nabla_{\mathscr{P}}(q ; x)$ of the polynomials $\left.\Delta_{( } P_{n} ; q\right)$ and $\nabla\left(P_{n} ; q\right)$, respectively. From identities (2), we have at once

Proposition 1 The edge generating series for the sequence of posets $\mathscr{P}=\left\{P_{0}, P_{1}, P_{2}, \ldots\right\}$ can be obtained from the associated $\Delta$-series and $\nabla$-series as follows

$$
\ell_{\mathscr{P}}(x)=\left[\partial_{q} \Delta_{\mathcal{P}}(q ; x)\right]_{q=1} \quad \text { and } \quad \ell_{\mathscr{P}}(x)=\left[\partial_{q} \nabla_{\mathcal{P}}(q ; x)\right]_{q=1} .
$$

The density index $i(G)$ of a graph $G$ is the quotient between the number of edges and the number of vertices, i.e. $i(G)=|E(G)| /|V(G)|$. This index has been considered in the study of topologies for the interconnection of parallel multicomputers, especially in the attempt of finding alternative topologies to the classical one given by the Boolean cube [12, 14], and in many other circumstances (as, for instance, in [4] or in [13]). Here, we define the Hasse index $i(P)$ of a poset $P$ as the density index of its Hasse diagram, i.e. $i(P)=\ell(P) /|P|$. For instance, the Hasse index of a Boolean lattice $B_{n}$ is $i\left(B_{n}\right)=\ell\left(B_{n}\right) /\left|B_{n}\right|=n / 2$, since $\left|B_{n}\right|=2^{n}$ and $\ell\left(B_{n}\right)=n 2^{n-1}$. We say that the Hasse index of a sequence of posets $\mathscr{P}=\left\{P_{0}, P_{1}, P_{2}, \ldots\right\}$ is Boolean when $i\left(P_{n}\right)=n / 2$, is asymptotically Boolean when $i\left(P_{n}\right) \sim n / 2$ as $n \rightarrow+\infty$, and is asymptotically quasi Boolean when there exists a small non-negative constant $c$ such that $i\left(P_{n}\right) \sim(1 / 2 \pm c) n$ as $n \rightarrow+\infty$. Here, we can assume $c \leq 1 / 10$.

Let $\mathcal{G P}$ be a class of lattice paths (the Grand paths) starting from the origin, ending on the $x$-axis consisting of steps of some kind (and respecting possible restrictions). Then let $\mathcal{P}$ be the class of all paths in $\mathcal{G P}$ never going below the $x$-axis. We say that the class $\mathcal{P}$ is Hasse-tamed if the Hasse index of the associated posets of paths is asymptotically equivalent to the Hasse index of the associated posets of Grand paths, i.e. $i\left(\mathcal{G} \mathcal{P}_{n}\right) \sim i\left(\mathcal{P}_{n}\right)$ as $n \rightarrow+\infty$. In all main examples we will consider, the property of being Hasse-tamed is true. However, there are also classes of paths without such a property, as in the case of the Fibonacci paths considered in Section 6.

In the rest of the paper, given a class $\mathcal{P}$ of paths, we write $\ell_{\mathcal{P}}(x)$ for the edge generating series $\ell_{\mathscr{P}}(x)$ associated with the sequence $\mathscr{P}$ of posets generated by all paths in $\mathcal{P}$.

For convenience, we report in Table 1 the first few values of the number of edges for the various lattices we will consider in the paper. Moreover, we observe that they appear in [17] as follows: $\ell\left(\mathcal{F}_{n}\right)$ form sequence A001629, $\ell\left(\mathcal{G} \mathcal{F}_{n}\right)$ form sequence A095977, $\ell\left(\mathcal{D}_{n}\right)$ form sequence A002054, $\ell\left(\mathcal{G} \mathcal{D}_{n}\right)$ form sequence $\mathrm{A} 002457, \quad \ell\left(\mathcal{M}_{n}\right)$ form sequence $\mathrm{A} 025567, \quad \ell\left(\mathcal{G} \mathcal{M}_{n}\right) / 2$ form sequence A132894, $\ell\left(\mathcal{G S}_{n}\right) / 2$ form sequence A108666.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\ell\left(\mathcal{F}_{n}\right)$ | 0 | 0 | 1 | 2 | 5 | 10 | 20 | 38 | 71 | 130 | 235 |
| $\ell\left(\mathcal{G} \mathcal{F}_{n}\right)$ | 0 | 0 | 2 | 4 | 14 | 32 | 82 | 188 | 438 | 984 | 2202 |
| $\ell\left(\mathcal{D}_{n}\right)$ | 0 | 0 | 1 | 5 | 21 | 84 | 330 | 1287 | 5005 | 19448 | 75582 |
| $\ell\left(\mathcal{G} \mathcal{D}_{n}\right)$ | 0 | 1 | 6 | 30 | 140 | 630 | 2772 | 12012 | 51480 | 218790 | 923780 |
| $\ell\left(\mathcal{M}_{n}\right)$ | 0 | 0 | 1 | 4 | 13 | 40 | 120 | 356 | 1050 | 3088 | 9069 |
| $\ell\left(\mathcal{G \mathcal { M }}_{n}\right)$ | 0 | 0 | 2 | 8 | 30 | 104 | 350 | 1152 | 3738 | 12000 | 38214 |
| $\ell\left(\mathcal{S}_{n}\right)$ | 0 | 1 | 6 | 34 | 190 | 1058 | 5894 | 32898 | 184062 | 1032322 | 5803270 |
| $\ell\left(\mathcal{G} \mathcal{S}_{n}\right)$ | 0 | 2 | 16 | 114 | 768 | 5010 | 32016 | 201698 | 1257472 | 7777314 | 47800080 |

Table 1: Number of edges in some lattices of paths.

## 3 Dyck and Grand Dyck lattices

Proposition 2 For any Dyck path $\gamma$, we have $|\Delta \gamma|=\omega_{D U}(\gamma)$ and $|\nabla \gamma|=\omega_{U D}^{*}(\gamma)$. Similarly, for any Grand Dyck path $\gamma$, we have $|\Delta \gamma|=\omega_{D U}(\gamma)$ and $|\nabla \gamma|=\omega_{U D}(\gamma)$.

Proof. In a Dyck lattice, a path $\gamma$ is covered by all paths that can be obtained from $\gamma$ by replacing a valley $D U$ with a peak $U D$, and covers all paths that can be obtained from $\gamma$ by
replacing a peak $U D$ (not at level 0 ) with a valley $D U$. In a Grand Dyck lattice, the situation is similar.

Proposition 3 The generating series for the class of Dyck paths with respect to semi-length (marked by $x$ ) and valleys (marked by q) is

$$
\begin{equation*}
f(q ; x)=\frac{1-(1-q) x-\sqrt{1-2(1+q) x+(1-q)^{2} x^{2}}}{2 q x} \tag{3}
\end{equation*}
$$

Proof. Any non-empty Dyck path $\gamma$ decomposes uniquely as $\gamma=U \gamma^{\prime} D \quad\left(\right.$ with $\left.\gamma^{\prime} \in \mathcal{D}\right)$ or as $\quad \gamma=U \gamma^{\prime} D \gamma^{\prime \prime} \quad\left(\right.$ with $\left.\gamma^{\prime}, \gamma^{\prime \prime} \in \mathcal{D}, \quad \gamma^{\prime \prime} \neq \bullet\right)$. Hence, we have the identity $f(q ; x)=$ $1+x f(q ; x)+q x f(q ; x)(f(q ; x)-1)$ whose solution is series (3).

Notice that series (3) is essentially the generating series of Narayana numbers [17, A001263] and that this statistic is well known (see, for instance, [6]).

Theorem 4 The edge generating series for Dyck lattices is

$$
\begin{equation*}
\ell_{\mathcal{D}}(x)=\frac{1-3 x-(1-x) \sqrt{1-4 x}}{2 x \sqrt{1-4 x}} \tag{4}
\end{equation*}
$$

Moreover, for every $n \in \mathbb{N}, n \geq 2$, the number of edges in $\mathcal{D}_{n}$ is

$$
\begin{equation*}
\ell\left(\mathcal{D}_{n}\right)=\frac{1}{2}\binom{2 n}{n} \frac{n-1}{n+1}=\binom{2 n-1}{n-2} \tag{5}
\end{equation*}
$$

In particular, the Hasse index of Dyck lattices is asymptotically Boolean.
Proof. By Proposition 2, the $\Delta$-series for Dyck lattices is series (3). So, by applying Proposition 1, we can obtain series (4). Moreover, since $\ell_{\mathcal{D}}(x)=(1+B(x)) / 2-C(x)$, we obtain identity (5). Finally, since $\left|\mathcal{D}_{n}\right|=C_{n}$, we have $i\left(\mathcal{D}_{n}\right)=\ell\left(\mathcal{D}_{n}\right) /\left|\mathcal{D}_{n}\right|=(n-1) / 2$ for every $n \geq 1$, and $i\left(\mathcal{D}_{n}\right) \sim n / 2$.

Proposition 5 The generating series for the class of Grand Dyck paths with respect to semi-length (marked by $x$ ) and valleys (marked by q) is

$$
\begin{equation*}
F(q ; x)=\frac{1}{\sqrt{1-2(1+q) x+(1-q)^{2} x^{2}}} \tag{6}
\end{equation*}
$$

Proof. Let $\overline{\mathcal{D}}$ be the class of reflected Dyck paths (i.e. Grand Dyck paths never going above the $x$-axis), and let $G(q ; x)$ be the corresponding generating series. Since every non empty path $\gamma \in \overline{\mathcal{D}}$ uniquely decomposes as $\gamma=D U \gamma^{\prime}$ (with $\gamma^{\prime} \in \overline{\mathcal{D}}$ ) or as $\gamma=D \gamma^{\prime} U \gamma^{\prime \prime}$ (with $\gamma^{\prime}, \gamma^{\prime \prime} \in \overline{\mathcal{D}}$, $\left.\gamma^{\prime} \neq \bullet\right)$, we have the identity $G(q ; x)=1+q x G(q ; x)+x(G(q ; x)-1) G(q ; x)$, whose unique solution is

$$
G(q ; x)=\frac{1+(1-q) x-\sqrt{1-2(1+q) x+(1-q)^{2} x^{2}}}{2 x}
$$

Now, let $U(q ; x)$ be the generating series for the class of Grand Dyck paths starting with an up step and let $D(q ; x)$ be the generating series for the class of Grand Dyck paths starting with a down step. Since every path $\gamma \in \mathcal{G D}$ uniquely decomposes as product of paths of the form $U \gamma D$ (with $\gamma \in \mathcal{D}$ ) and $D \gamma U$ (with $\gamma^{\prime} \in \overline{\mathcal{D}}$ ), we have the linear system

$$
\left\{\begin{array}{l}
F(q ; x)=1+U(q ; x)+D(q ; x) \\
U(q ; x)=x F(q ; x)(1+q U(q ; x)+D(q ; x)) \\
D(q ; x)=q x F(q ; x)+x(G(q ; x)-1) F(q ; x)
\end{array}\right.
$$

from which it is straightforward to obtain identity (6).

Theorem 6 The edge generating series for Grand Dyck lattices is

$$
\begin{equation*}
\ell_{\mathcal{G D}}(x)=\frac{x}{(1-4 x)^{3 / 2}} \tag{7}
\end{equation*}
$$

Moreover, the number of edges in $\mathcal{G} \mathcal{D}_{n}$ is

$$
\begin{equation*}
\ell\left(\mathcal{G D}_{n}\right)=\binom{2 n}{n} \frac{n}{2} \tag{8}
\end{equation*}
$$

In particular, the Hasse index of a Grand Dyck lattice is Boolean.
Proof. Proposition 2 implies that (6) is the $\Delta$-series for Grand Dyck lattices. So, by applying Proposition 1, we obtain series (7). Then, by expanding this series, we have at once identity (8). Finally, since $\left|\mathcal{G} \mathcal{D}_{n}\right|=\binom{2 n}{n}$, we have $i\left(\mathcal{G D}_{n}\right)=\ell\left(\mathcal{G} \mathcal{D}_{n}\right) /\left|\mathcal{G} \mathcal{D}_{n}\right|=n / 2$.

Theorems 4 and 6 immediately imply
Proposition 7 The class of Dyck lattices is Hasse-tamed: $i\left(\mathcal{D}_{n}\right) \sim i\left(\mathcal{G} \mathcal{D}_{n}\right) \sim n / 2$.

## 4 Motzkin and Grand Motzkin lattices

Proposition 8 For any Motzkin path $\gamma$, we have $|\Delta \gamma|=\omega_{H U}(\gamma)+\omega_{D H}(\gamma)+\omega_{D U}(\gamma)+\omega_{H H}(\gamma)$ and $|\nabla \gamma|=\omega_{U H}(\gamma)+\omega_{H D}(\gamma)+\omega_{U D}(\gamma)+\omega_{H H}^{*}(\gamma)$. Similarly, for any Grand Motzkin path $\gamma$, we have $|\Delta \gamma|=\omega_{H U}(\gamma)+\omega_{D H}(\gamma)+\omega_{D U}(\gamma)+\omega_{H H}(\gamma)$ and $|\nabla \gamma|=\omega_{U H}(\gamma)+\omega_{H D}(\gamma)+\omega_{U D}(\gamma)+$ $\omega_{H H}(\gamma)$.

Proof. In a Motzkin lattice, a path $\gamma$ is covered by all paths that can be obtained from $\gamma$ by replacing i) a factor $H U$ with a factor $U H$, or ii) a factor $D H$ with a factor $H D$, or iii) a valley $D U$ with a double horizonal step $H H$, or iv) a double horizonal step $H H$ with a peak $U D$. The number of paths covered by $\gamma$ can be obtained in a similar way. In a Grand Motzkin lattice we have a similar situation.

Proposition 9 The generating series for the class of Motzkin paths with respect to length (marked by x) and to factors $H U, D H, D U$ and $H H$ (marked by q) is

$$
\begin{equation*}
f(q ; x)=\frac{1-q x-(1-q) x^{2}-\sqrt{(1+x)\left(1-(1+2 q) x-\left(1-q^{2}\right) x^{2}+(1-q)^{2} x^{3}\right)}}{2 q x^{2}} \tag{9}
\end{equation*}
$$

Proof. Let $h(q ; x)$ be the generating series for the class of Motzkin paths starting with an horizontal step and let $u(q ; x)$ be the generating series for the class of Motzkin paths starting with an up step. Since any non-empty Motzkin path $\gamma$ decomposes uniquely as $\gamma=H \gamma^{\prime}$ (with $\gamma^{\prime} \in \mathcal{M}$ ), or as $\gamma=U \gamma^{\prime} D \gamma^{\prime \prime}$ (with $\gamma^{\prime}, \gamma^{\prime \prime} \in \mathcal{M}$ ), it is straightforward to obtain the linear system

$$
\left\{\begin{array}{l}
f(q ; x)=1+h(q ; x)+u(q ; x) \\
h(q ; x)=x(1+q h(q ; x)+q u(q ; x)) \\
u(q ; x)=x^{2} f(q ; x)(1+q h(q ; x)+q u(q ; x))
\end{array}\right.
$$

and consequently, solving such a system, to obtain identity (9).
The numbers generated by series (9) are essentially sequence A110470 in [17].
Theorem 10 The edge generating series for Motzkin lattices is

$$
\begin{equation*}
\ell_{\mathcal{M}}(x)=\frac{(1+x)\left(1-2 x-x^{2}-(1-x) \sqrt{1-2 x-3 x^{2}}\right)}{2 x^{2} \sqrt{1-2 x-3 x^{2}}} \tag{10}
\end{equation*}
$$

Moreover, the number of edges in $\mathcal{M}_{n}$ can be expressed in one of the following ways

$$
\begin{align*}
& \ell\left(\mathcal{M}_{n}\right)=\binom{n ; 3}{n}-M_{n}+\binom{n-1 ; 3}{n-1}-M_{n-1} \quad(n \geq 1)  \tag{11}\\
& \ell\left(\mathcal{M}_{n}\right)=\binom{n ; 3}{n-2}+\binom{n-1 ; 3}{n-3} \quad(n \geq 3)  \tag{12}\\
& \ell\left(\mathcal{M}_{n}\right)=\frac{2}{n} \sum_{k=0}^{\lfloor n / 2\rfloor}\binom{n}{k}\binom{n-k}{k} \frac{k(n-k)}{k+1} \quad(n \geq 1) \tag{13}
\end{align*}
$$

In particular, we have the asymptotic expansions

$$
\begin{equation*}
\ell\left(\mathcal{M}_{n}\right) \sim \frac{2 \cdot 3^{n}}{\sqrt{3 n \pi}} \quad \text { and } \quad i\left(\mathcal{M}_{n}\right)=\frac{\ell\left(\mathcal{M}_{n}\right)}{\left|\mathcal{M}_{n}\right|} \sim \frac{4}{9} n \tag{14}
\end{equation*}
$$

and the Hasse index of the Motzkin lattices is asymptotically quasi Boolean.
Proof. By Proposition 8, the $\Delta$-series for Motzkin lattices is series (9). So, by Proposition 1, we obtain series (10). It is easy to see that $\ell_{\mathcal{M}}(x)=(1+x)(T(x)-M(x))$, and consequently to obtain the first identity (11). Now, by Cauchy integral formula, we have

$$
\ell\left(\mathcal{M}_{n}\right)=\left[x^{n}\right] \ell_{\mathcal{M}}(x)=\frac{1}{2 \pi \mathrm{i}} \oint \ell_{\mathcal{M}}(z) \frac{\mathrm{d} z}{z^{n+1}} .
$$

With the substitution $z=\frac{w}{1+w+w^{2}}$, we have $\mathrm{d} z=\frac{1-w^{2}}{\left(1+w+w^{2}\right)^{2}} \mathrm{~d} w$ and

$$
\begin{aligned}
& \ell\left(\mathcal{M}_{n}\right)=\frac{1}{2 \pi \mathrm{i}} \oint(1+w)^{2}\left(1+w+w^{2}\right)^{n-1} \frac{\mathrm{~d} w}{w^{n-1}}= \\
& \quad=\left[x^{n-2}\right](1+x)^{2}\left(1+x+x^{2}\right)^{n-1}=\left[x^{n-2}\right]\left(1+x+x^{2}\right)^{n}+\left[x^{n-3}\right]\left(1+x+x^{2}\right)^{n-1}
\end{aligned}
$$

from which we have identity (12).
From the identity

$$
\ell_{\mathcal{M}}(x)=(1+x)(T(x)-M(x))=\frac{1+x}{1-x} B\left(\frac{x^{2}}{(1-x)^{2}}\right)-\frac{1+x}{1-x} C\left(\frac{x^{2}}{(1-x)^{2}}\right)
$$

we have the expansion

$$
\begin{aligned}
\ell_{\mathcal{M}}(x) & =(1+x) \sum_{k \geq 0}\binom{2 k}{k} \frac{x^{2 k}}{(1-x)^{2 k+1}}-(1+x) \sum_{k \geq 0}\binom{2 k}{k} \frac{1}{k+1} \frac{x^{2 k}}{(1-x)^{2 k+1}} \\
& =(1+x) \sum_{n \geq 0}\left[\sum_{k \geq 0}\binom{n}{2 k}\binom{2 k}{k} \frac{k}{k+1}\right] x^{n}
\end{aligned}
$$

from which it is straightforward to obtain identity (13).
Finally, using identity (11) and the asymptotic expansions reported in (1) for the central trinomial coefficients and for the Motzkin numbers, we can obtain the first asymptotic equivalence in (14). Then, using once again the asymptotic expansion for the Motzkin numbers in (1), we also obtain the second asymptotic equivalence in (14). Since $4 / 9 \simeq 0.44$, the Hasse index is asymptotically quasi boolean.

Proposition 11 The generating series for the class of Grand Motzkin paths with respect to semilength (marked by x) and to factors $H U, D H, D U$ and HH (marked by q) is

$$
\begin{equation*}
F(q ; x)=\frac{1+(1-q) x}{\sqrt{(1+x)\left(1-(1+2 q) x-\left(1-q^{2}\right) x^{2}+(1-q)^{2} x^{3}\right)}} \tag{15}
\end{equation*}
$$

Proof. Let $\overline{\mathcal{M}}$ be the class of reflected Motzkin paths (i.e. Grand Motzkin paths never going above the $x$-axis). Then, let $X(q ; x)$ be the generating series for the class of Grand Motzkin paths starting with a step $X \in\{H, U, D\}$. Any non-empty Grand Motzkin path $\gamma$ decomposes uniquely as $\gamma=H \gamma^{\prime}$ (with $\gamma^{\prime} \in \mathcal{G} \mathcal{M}$ ), as $\gamma=U \gamma^{\prime} D \gamma^{\prime \prime}$ (with $\gamma^{\prime} \in \mathcal{M}$ and $\gamma^{\prime \prime} \in \mathcal{G} \mathcal{M}$ ), or as $\gamma=D \gamma^{\prime} U \gamma^{\prime \prime}$ (with $\gamma^{\prime} \in \overline{\mathcal{M}}$ and $\gamma^{\prime \prime} \in \mathcal{G \mathcal { M }}$ ). From this decomposition, we can obtain the linear system

$$
\left\{\begin{array}{l}
f(q ; x)=1+H(q ; x)+U(q ; x)+D(q ; x) \\
H(q ; x)=x(1+q H(q ; x)+q U(q ; x)+D(q ; x)) \\
U(q ; x)=x^{2} f(q ; x)(1+q H(q ; x)+q U(q ; x)+D(q ; x)) \\
D(q ; x)=\bar{D}(q ; x) f(q ; x)
\end{array}\right.
$$

where $f(q ; x)$ is series (9) and $\bar{D}(q ; x)$ is the generating series for the class of subelevated reflected Motzkin paths.

Now, let $\bar{f}(q ; x), \bar{h}(q ; x)$, and $\bar{d}(q ; x)$ be the generating series for the classes of reflected Motzkin paths with no restriction, starting with a horizonal step and starting with a down step, respectively. Moreover, let $\bar{H}(q ; x)$ be the generating series for the class of reflected Motzkin paths starting and ending with a horizontal step. Any non-empty reflected Motzkin path $\gamma$ decomposes uniquely as $\gamma=H \gamma^{\prime}$ (with $\gamma^{\prime} \in \overline{\mathcal{M}}$ ), or as $\gamma=D \gamma^{\prime} U \gamma^{\prime \prime} \quad$ (with $\gamma^{\prime}, \gamma^{\prime \prime} \in \overline{\mathcal{M}}$ ). From this decomposition it is possible to obtain the linear system

$$
\left\{\begin{array}{l}
\bar{f}(q ; x)=1+\bar{h}(q ; x)+\bar{d}(q ; x) \\
\bar{h}(q ; x)=x(1+q \bar{h}(q ; x)+\bar{d}(q ; x)) \\
\bar{d}(q ; x)=\bar{D}(q ; x) \bar{f}(q ; x) \\
\bar{D}(q ; x)=x^{2}\left(q+q^{2} x+2 q \bar{h}(q ; x) \bar{D}(q ; x)+q^{2} \bar{H}(q ; x)+\bar{D}(q ; x)+\bar{D}(q ; x)^{2} \bar{f}(q ; x)\right) \\
\bar{H}(q ; x)=x^{2}\left(q+q^{2} x+2 q \bar{h}(q ; x) \bar{D}(q ; x)+q^{2} \bar{H}(q ; x)+\bar{D}(q ; x)+\bar{D}(q ; x)^{2} \bar{f}(q ; x)\right)
\end{array}\right.
$$

By solving both these systems, it is straightforward to obtain series (15).

Theorem 12 The edge generating series for Grand Motzkin lattices is

$$
\begin{equation*}
\ell_{\mathcal{G M}}(x)=\frac{2 x^{2}}{(1-3 x) \sqrt{1-2 x-3 x^{2}}} \tag{16}
\end{equation*}
$$

Moreover, we have the identities

$$
\begin{align*}
& \ell\left(\mathcal{G M}_{n+2}\right)=2 \sum_{k=0}^{n}\binom{k ; 3}{k} 3^{n-k}  \tag{17}\\
& \ell\left(\mathcal{G M}_{n+2}\right)=\frac{2}{4^{n}} \sum_{k=0}^{n}\binom{2 k}{k}\binom{2 n-2 k}{n-k}(2 k+1) 3^{k}(-1)^{n-k}  \tag{18}\\
& \ell\left(\mathcal{G M}_{n+2}\right)=2 \sum_{k=0}^{n}\binom{n+1}{k+1}\binom{2 k}{k}(-1)^{k} 3^{n-k} \tag{19}
\end{align*}
$$

and the asymptotic equivalences

$$
\begin{equation*}
\ell\left(\mathcal{G M}_{n}\right) \sim 2 \cdot 3^{n-2} \sqrt{\frac{3 n}{\pi}} \quad \text { and } \quad i\left(\mathcal{G M}_{n}\right) \sim \frac{4}{9} n \tag{20}
\end{equation*}
$$

In particular, the Hasse index of Grand Motzkin lattices is asymptotically quasi Boolean.
Proof. Proposition 8 implies that (15) is the $\Delta$-series for Grand Motzkin lattices. So, by applying Proposition 1, we obtain series (16). Since $\frac{\ell_{\mathcal{G \mathcal { M }}}(x)}{x^{2}}=\frac{2}{1-3 x} T(x)$, we have at once identity (17).

Moreover, we also have

$$
\frac{1}{2} \frac{\ell_{\mathcal{G} \mathcal{M}}(x)}{x^{2}}=\frac{1}{(1-3 x)^{3 / 2}} \frac{1}{\sqrt{1+x}}=\sum_{n \geq 0}\binom{2 n}{n} \frac{3^{n}}{4^{n}}(2 n+1) x^{n} \cdot \sum_{n \geq 0}\binom{2 n}{n} \frac{(-1)^{n}}{4^{n}} x^{n}
$$

from which we obtain identity (18). Finally, we have

$$
\begin{aligned}
& \frac{1}{2} \frac{\ell_{\mathcal{G M}}(x)}{x^{2}}=\frac{1}{(1-3 x) \sqrt{(1-3 x)^{2}+4 x(1-3 x)}}=\frac{1}{(1-3 x)^{2}} \frac{1}{\sqrt{1+\frac{4 x}{1-3 x}}}= \\
& \quad=\sum_{k \geq 0}\binom{2 k}{k}(-1)^{k} \frac{x^{k}}{(1-3 x)^{k+2}}=\sum_{k \geq 0}\binom{2 k}{k}(-1)^{k} \sum_{n \geq 0}\binom{n+1}{k+1} 3^{n-k} x^{n}
\end{aligned}
$$

from which we obtain identity (19).
From series (16), we have (for $n$ sufficiently large) the identity

$$
\ell\left(\mathcal{G} \mathcal{M}_{n}\right)=2\left[x^{n-2}\right](1+x)^{-1 / 2}\left(1-\frac{x}{1 / 3}\right)^{-3 / 2}
$$

Now, by applying the Darboux theorem, we have

$$
\ell\left(\mathcal{G}_{n}\right) \sim 2 \frac{\psi(\xi)}{\xi^{n-2}} \frac{(n-2)^{\alpha-1}}{\Gamma(\alpha)}
$$

where $\xi=1 / 3, \quad \psi(x)=(1+x)^{-1 / 2}$ and $\alpha=3 / 2$. Since $\psi(\xi)=\sqrt{3} / 2$ and $\Gamma(\alpha)=\Gamma(3 / 2)=$ $\sqrt{\pi} / 2$, we can obtain the asymptotic equivalence (20). Moreover, using the asymptotic expansion for the central trinomial coefficients given in (1), we have $i\left(\mathcal{G M}_{n}\right) \sim \frac{4}{9} n$. Since $4 / 9 \simeq 0.44$, the Hasse index is asymptotically quasi Boolean.

Theorems 10 and 12 immediately imply
Proposition 13 The class of Motzkin lattices is Hasse-tamed: $i\left(\mathcal{M}_{n}\right) \sim i\left(\mathcal{G} \mathcal{M}_{n}\right) \sim \frac{4}{9} n$.

## 5 Schröder lattices and Grand Schröder lattices

Proposition 14 For any Schröder path $\gamma$, we have $|\Delta \gamma|=\omega_{H}(\gamma)+\omega_{D U}(\gamma)$ and $|\nabla \gamma|=$ $\omega_{H}^{*}(\gamma)+\omega_{U D}(\gamma)$. Similarly, for any Grand Schröder path $\gamma$, we have $|\Delta \gamma|=\omega_{H}(\gamma)+\omega_{D U}(\gamma)$ and $|\nabla \gamma|=\omega_{H}(\gamma)+\omega_{U D}(\gamma)$.

Proof. In a Schröder lattice, a path $\gamma$ is covered by all paths that can be obtained by replacing a horizontal step $H$ with a peak $U D$, or a valley $D U$ with a horizontal step $H$, and covers all paths that can be obtained by replacing a horizontal step $H$ (not a level 0 ) with a peak $U D$ or a peak $U D$ with a horizontal step $H$. In a Grand Schröder lattice we have an analogous situation.

Proposition 15 The generating series for the class of Schröder paths with respect to semi-length (marked by $x$ ) and horizontal steps $H$ and valleys DU (marked by q) is

$$
\begin{equation*}
f(q ; x)=\frac{1-x-\sqrt{1-2(1+2 q) x+(1-2 q)^{2} x^{2}}}{2 q x(1+(1-q) x)} \tag{21}
\end{equation*}
$$

Proof. Let $h(q ; x)$ be the generating series for the class $\mathcal{S}$ of all Schröder paths starting with a horizontal step and let $u(q ; x)$ be the generating series for the class of all Schröder paths starting
with an up step. Since any non-empty Schröder path $\gamma$ decomposes uniquely as $\gamma=H \gamma^{\prime}$ (with $\gamma^{\prime} \in \mathcal{S}$ ) or $\gamma=D \gamma^{\prime} U \gamma^{\prime \prime}$ (with $\gamma^{\prime}, \gamma^{\prime \prime} \in \mathcal{S}$ ), we obtain the linear system

$$
\left\{\begin{array}{l}
f(q ; x)=1+h(q ; x)+u(q ; x) \\
h(q ; x)=q x f(q ; x) \\
u(q ; x)=x f(q ; x)(1+h(q ; x)+q u(q ; x))
\end{array}\right.
$$

By solving for $f$, we can obtain series (21).
The mirror triangle generated by series (21) is sequence A090981 in [17].
Theorem 16 The edge generating series for Schröder lattices is

$$
\begin{equation*}
\ell_{\mathcal{S}}(x)=\frac{(1-x)\left(1-4 x+x^{2}-(1-x) \sqrt{1-6 x+x^{2}}\right)}{2 x \sqrt{1-6 x+x^{2}}} \tag{22}
\end{equation*}
$$

Moreover, the number of edges in $\mathcal{S}_{n}$ can be expressed in one of the forms

$$
\begin{align*}
& \ell\left(\mathcal{S}_{n}\right)=d_{n}-r_{n}-d_{n-1}+r_{n-1}  \tag{23}\\
& \ell\left(\mathcal{S}_{n}\right)=\sum_{k=0}^{n}\left(\binom{2 k}{n-k}\right)\binom{2 k}{k} \frac{k}{k+1} . \tag{24}
\end{align*}
$$

Finally, we have the asymptotic equivalences

$$
\begin{equation*}
\ell\left(\mathcal{S}_{n}\right) \sim \frac{(1+\sqrt{2})^{2 n}}{\sqrt{\sqrt{2} n \pi}} \quad \text { and } \quad i\left(\mathcal{S}_{n}\right)=\frac{\ell\left(\mathcal{S}_{n}\right)}{\left|\mathcal{S}_{n}\right|} \sim(2-\sqrt{2}) n \tag{25}
\end{equation*}
$$

In particular, the Hasse index of Schröder lattices is asymptotically quasi Boolean.
Proof. Proposition 14 implies that (21) is the $\Delta$-series for Schröder lattices. So, by applying Proposition 1, we obtain series (22). It is easy to see that $\ell_{\mathcal{S}}(x)=(1-x)(d(x)-r(x))$. This implies at once identity (23). Moreover, expanding $\ell_{\mathcal{S}}(x)=(1-x)(d(x)-r(x))$ as follows

$$
\ell_{\mathcal{S}}(x)=\frac{1}{\sqrt{1-\frac{4 x}{(1-x)^{2}}}}-\frac{1-\sqrt{1-\frac{4 x}{(1-x)^{2}}}}{2 \frac{x}{(1-x)^{2}}}=\sum_{k \geq 0}\binom{2 k}{k} \frac{x^{2}}{(1-x)^{2 k}}-\sum_{k \geq 0}\binom{2 k}{k} \frac{1}{k+1} \frac{x^{2}}{(1-x)^{2 k}}
$$

it is straightforward to obtain identity (24). Finally, using identity (23) and the asymptotics in (1), we obtain equivalences (25). Since $2-\sqrt{2} \simeq 0.58$, the Hasse index is asymptotically quasi Boolean.

Proposition 17 The generating series for the class of Grand Schröder paths with respect to semilength (marked by $x$ ) and horizontal steps $H$ and valleys $D U$ (marked by $q$ ) is

$$
\begin{equation*}
g(q ; x)=\frac{1}{\sqrt{1-2(1+2 q) x+(1-2 q)^{2} x^{2}}} \tag{26}
\end{equation*}
$$

Proof. Let $\overline{\mathcal{S}}$ be the class of reflected Schröder paths (i.e. Grand Schröder paths never going above the $x$-axis), and let $\bar{f}(q ; x)$ be the corresponding generating series. Since any non-empty reflected Schröder path $\gamma$ decomposes uniquely as $\gamma=H \gamma^{\prime}$ (with $\gamma^{\prime} \in \overline{\mathcal{S}}$ ) or $\gamma=D \gamma^{\prime} U \gamma^{\prime \prime}$ (with $\gamma^{\prime}, \gamma^{\prime \prime} \in \overline{\mathcal{S}}$ ), we obtain the linear system

$$
\left\{\begin{array}{l}
\bar{f}(q ; x)=1+\bar{h}(q ; x)+\bar{d}(q ; x) \\
\bar{h}(q ; x)=q x \bar{f}(q ; x) \\
\bar{d}(q ; x)=x(q+\bar{f}(q ; x)-1) \bar{f}(q ; x),
\end{array}\right.
$$

where $\bar{h}(q ; x)$ and $\bar{d}(q ; x)$ are the generating series for the reflected Schröder paths starting with a horizontal step and with a down step, respectively. By solving for $\bar{f}(q ; x)$, we obtain

$$
\bar{f}(q ; x)=\frac{1+x-2 q x-\sqrt{(1+x-2 q x)^{2}-4 x}}{2 x}
$$

Since any non-empty Grand Schröder path $\gamma$ decomposes uniquely as $\gamma=H \gamma^{\prime}$ (with $\gamma^{\prime} \in \mathcal{G S}$ ), or as $\gamma=U \gamma^{\prime} D \gamma^{\prime \prime}$ (with $\gamma^{\prime} \in \mathcal{S}$ and $\gamma^{\prime \prime} \in \mathcal{G S}$ ), or as $\gamma=D \gamma^{\prime} U \gamma^{\prime \prime}$ (with $\gamma^{\prime} \in \overline{\mathcal{S}}$ and $\gamma^{\prime \prime} \in \mathcal{G S}$ ), we have the system

$$
\left\{\begin{array}{l}
g(q ; x)=1+h(q ; x)+u(q ; x)+d(q ; x) \\
h(q ; x)=q x g(q ; x) \\
u(q ; x)=x f(q ; x)(1+h(q ; x)+q u(q ; x)+d(q ; x)) \\
d(q ; x)=x(q+\bar{f}(q ; x)-1) g(q ; x)
\end{array}\right.
$$

where $h(q ; x), u(q ; x)$ and $d(q ; x)$ are the generating series for the Grand Schröder paths starting with a horizontal step, an up step and with a down step, respectively. By solving for $g(q ; x)$, it is straightforward to obtain series (26).

Theorem 18 The edge generating series for Grand Schröder lattices is

$$
\begin{equation*}
\ell_{\mathcal{G S}}(x)=\frac{2\left(x-x^{2}\right)}{\left(1-6 x+x^{2}\right)^{3 / 2}} \tag{27}
\end{equation*}
$$

Moreover, we have the identities

$$
\begin{align*}
& \ell\left(\mathcal{G S}_{n}\right)=2 \sum_{k=0}^{n}\binom{n+k}{2 k}\binom{2 k}{k}(n-k)  \tag{28}\\
& \ell\left(\mathcal{G} \mathcal{S}_{n}\right)=\sum_{k=0}^{\lfloor n / 2\rfloor}\binom{n}{k}\binom{n-k}{k} \frac{(n-2 k)(n+k+2)}{k+1} 2^{k+1} 3^{n-2 k-2} . \tag{29}
\end{align*}
$$

and the asymptotic equivalences

$$
\begin{equation*}
\ell\left(\mathcal{G S}_{n}\right) \sim \sqrt{\frac{n}{2 \sqrt{2} \pi}}(1+\sqrt{2})^{2 n} \quad \text { and } \quad i\left(\mathcal{G} \mathcal{S}_{n}\right)=\frac{\ell\left(\mathcal{G} \mathcal{S}_{n}\right)}{\left|\mathcal{G S}_{n}\right|} \sim(2-\sqrt{2}) n \tag{30}
\end{equation*}
$$

In particular, the Hasse index of Grand Schröder lattices is asymptotically quasi Boolean.
Proof. By Proposition 14, the $\Delta$-series for Grand Schröder lattices is series (26). So, by Proposition 1, we obtain series (27). By expanding this series as follows

$$
\ell_{\mathcal{G S}}(x)=\frac{2 x(1-x)}{\left((1-x)^{2}-4 x\right)^{3 / 2}}=\frac{2 x}{(1-x)^{3}} \frac{1}{\left(1-\frac{4 x}{(1-x)^{2}}\right)^{3 / 2}}=2 \sum_{k \geq 0}\binom{2 k}{k}(2 k+1) \frac{x^{k+1}}{(1-x)^{2 k+2}}
$$

we can obtain identity (28). Formula (29) can be obtained in a similar way (just consider the identity $\left.1-6 x+x^{2}=(1-3 x)^{2}-8 x^{2}\right)$. Finally, using the Darboux theorem, we can obtain the first equivalence in (30). Then, using the asymptotics in (1), we can also obtain the second equivalence in (30).

Theorems 16 and 18 immediately imply
Proposition 19 The class of Schröder lattices is Hasse-tamed: $i\left(\mathcal{S}_{n}\right) \sim i\left(\mathcal{G S}_{n}\right) \sim(2-\sqrt{2}) n$.

## 6 Fibonacci and Grand Fibonacci posets

All classes of paths considered in the previous sections have the property of being Hasse-tamed. Here, we will consider a simple class of paths for which such a property is not true.

A Fibonacci path is a Motzkin path confined in the horizontal strip $[0,1]$ with horizontal steps only on the $x$-axis. A Grand Fibonacci path is a Grand Motzkin path confined in the horizontal strip $[-1,1]$ with horizontal steps only on the $x$-axis. Let $\mathcal{F}$ be the class of Fibonacci posets (i.e. Fibonacci semilattices [10]), and let $\mathcal{G \mathcal { F }}$ be the class of Grand Fibonacci posets.

Let $F_{n}$ and $L_{n}$ be the Fibonacci and Lucas numbers, respectively. By Binet formulas, we have the identities $F_{n}=\left(\varphi^{n}-\widehat{\varphi}\right) / \sqrt{5}$ and $L_{n}=\varphi^{n}+\widehat{\varphi}$, and the asymptotics $F_{n} \sim \varphi^{n} / \sqrt{5}$ and $L_{n} \sim \varphi^{n}$, where $\varphi=(1+\sqrt{5}) / 2$ and $\widehat{\varphi}=(1-\sqrt{5}) / 2$.

Theorem 20 The edge generating series for the Fibonacci posets is

$$
\begin{equation*}
\ell_{\mathcal{F}}(x)=\frac{x^{2}}{\left(1-x-x^{2}\right)^{2}} \tag{31}
\end{equation*}
$$

Moreover, we have the explicit formula

$$
\begin{equation*}
\ell\left(\mathcal{F}_{n}\right)=\frac{n L_{n}-F_{n}}{5} \tag{32}
\end{equation*}
$$

and the asymptotic equivalences

$$
\begin{equation*}
\ell\left(\mathcal{F}_{n}\right) \sim \frac{n}{5} \varphi^{n} \quad \text { and } \quad i\left(\mathcal{F}_{n}\right) \sim \frac{n}{\sqrt{5} \varphi} \tag{33}
\end{equation*}
$$

In particular, the Fibonacci posets are not asymptotically quasi boolean.
Proof. In a Fibonacci poset, a path $\gamma$ is covered by all paths that can be obtained from $\gamma$ by replacing a horizontal double step $H H$ (necessarily on the $x$-axis) with a peak $U D$. So, we have $|\Delta \gamma|=\omega_{H H}(\gamma)$. Let $f(q ; x)$ be the generating series for Fibonacci paths with respect to horizontal double steps and length. Similarly, let $h(q ; x)$ and $u(q ; x)$ be the generating series for Fibonacci paths starting with a horizontal step or with an up step, respectively. Since any non-empty Fibonacci path $\gamma$ decomposes uniquely as $\gamma=H \gamma^{\prime}$ or as $\gamma=U D \gamma^{\prime}$, where $\gamma^{\prime}$ is an arbitrary Fibonacci path in both cases, it is easy to obtain the identities: $f(q ; x)=1+h(q ; x)+u(q ; x)$, $h(q ; x)=x(1+q h(q ; x)+u(q ; x)), \quad u(q ; x)=x^{2} f(q ; x)$. The solution of this linear system is straightforward and yields the series

$$
f(q ; x)=\frac{1+(1-q) x}{1-q x-x^{2}-(1-q) x^{3}} .
$$

Now, by applying Proposition 1, we can obtain series (31). Expanding this series we obtain identity (32). From this identity and the identity $i\left(\mathcal{F}_{n}\right)=\frac{n L_{n}-F_{n}}{5 F_{n+1}}$, we obtain at once asymptotics (33). Finally, since $1 /(\sqrt{5} \varphi) \simeq 0.276$, the Fibonacci posets are not asymptotically quasi boolean.

Theorem 21 The edge generating series for the Grand Fibonacci posets is

$$
\begin{equation*}
\ell_{\mathcal{G F}}(x)=\frac{2 x^{2}}{\left(1-x-2 x^{2}\right)^{2}} . \tag{34}
\end{equation*}
$$

Moreover, we have the explicit formula

$$
\begin{equation*}
\ell\left(\mathcal{G} \mathcal{F}_{n}\right)=\frac{(3 n-1) 2^{n+1}+(-1)^{n} 2(3 n+1)}{27} \tag{35}
\end{equation*}
$$

and the asymptotic equivalences

$$
\begin{equation*}
\ell\left(\mathcal{G F}_{n}\right) \sim \frac{n}{9} 2^{n+1} \quad \text { and } \quad i\left(\mathcal{G} \mathcal{F}_{n}\right) \sim \frac{n}{3} \tag{36}
\end{equation*}
$$

In particular, the Grand Fibonacci posets are not asymptotically quasi boolean.

Proof. In a Grand Fibonacci poset, a path $\gamma$ is covered by all paths that can be obtained from $\gamma$ by replacing a horizontal double step $H H$ (necessarily on the $x$-axis) with a peak $U D$, or by replacing a valley $D U$ touching the line $y=-1$ with a horizontal double step $H H$ on the $x$-axis. Then, let $F(q ; x)$ be the generating series for Grand Fibonacci paths with respect to horizontal double steps and valleys touching the line $y=-1$ (marked by $q$ ) and length (marked by $x$ ). Similarly, let $H(q ; x), U(q ; x)$ and $D(q ; x)$ be the generating series for these paths starting with a horizontal step, with an up step or with a down step, respectively. Since any non-empty Grand Fibonacci path $\gamma$ decomposes uniquely as $\gamma=H \gamma^{\prime}$, as $\gamma=U D \gamma^{\prime}$, or as $\gamma=D U \gamma^{\prime}$, where $\gamma^{\prime}$ is an arbitrary Grand Fibonacci path in each case, it is easy to obtain the identities: $F(q ; x)=1+H(q ; x)+U(q ; x)+D(q ; x), H(q ; x)=x(1+q H(q ; x)+U(q ; x)+D(q ; x))$, $U(q ; x)=x^{2} F(q ; x), \quad D(q ; x)=q x^{2} F(q ; x)$. The solution of this linear system is straightforward and yields the series

$$
F(q ; x)=\frac{1+(1-q) x}{1-q x-(1+q) x^{2}-\left(1-q^{2}\right) x^{3}}
$$

Now, by applying Proposition 1, we can obtain series (34). Decomposing this series as sum of partial fractions, it is easy to obtain identity (35). From this identity, we can obtain the first relation in (36). Finally, notice that the generating series for the vertices of the Grand Fibonacci posets is $g(x)=F(1 ; x)=1 /\left(1-x-2 x^{2}\right)$ and that consequently the number of vertices is $g_{n}=\left(2^{n+1}+(-1)^{n}\right) / 3 \sim 2^{n+1} / 3$. So, we also have the second relation in (36).

Theorems 20 and 21 immediately imply
Proposition 22 The class of Fibonacci lattices is not Hasse-tamed.

## 7 Young lattices

A partition of a non-negative integer $n$ is a sequence $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}\right)$ of positive integers such that $\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{k}>0$ and $|\lambda|=\lambda_{1}+\lambda_{2}+\cdots+\lambda_{k}=n$. The $\lambda_{i}$ 's are the parts of $\lambda$. The (Ferrers) diagram of $\lambda$ is a left-justified array of squares (or dots) with exactly $\lambda_{i}$ squares in the $i$-th row. Partitions can be ordered by magnitude of parts [1]: if $\alpha=\left(a_{1}, \ldots, a_{h}\right)$ and $\beta=\left(b_{1}, \ldots, b_{k}\right)$, then $\alpha \leq \beta$ whenever $h \leq k$ and $a_{i} \leq b_{i}$ for every $i=1,2, \ldots, h$. If $\alpha \leq \beta$ the diagram of $\alpha$ is contained in the diagram of $\beta$. The resulting poset is an infinite distributive lattice, called Young lattice. The Young lattice $\mathcal{Y}_{\lambda}$ generated by a partition $\lambda$ is the principal ideal generated by $\lambda$ in $\mathcal{Y}$, i.e. the set of all integer partitions $\alpha$ such that $\alpha \leq \lambda$. Any Young lattice $\mathcal{Y}_{\lambda}$ can be considered as a lattice of paths. Indeed, it is isomorphic to the lattice of all paths made of horizontal steps $X=(1,0)$ and vertical steps $Y=(0,1)$, confined in the region defined by the diagram $\Phi_{\lambda}$ of $\lambda$. Moreover, as we noticed in [7], Dyck lattices are isomorphic to the dual of the Young lattices associated with the staircase partitions $(n, n-1, \ldots, 2,1)$.

Here we will give a general formula for the number of edges in any Young lattice $\mathcal{Y}_{\lambda}$. To do that we need the following definitions. We say that a cell of a Young diagram is a corner cell whenever the diagram has no cells below and no cells on the right of such a cell. The boundary of the diagram $\Phi_{\lambda}$ is the set $\partial \Phi_{\lambda}$ of all its corner cells.

In a Ferrers diagram $\Phi_{\lambda}$, we denote by $(i, j)$ the cell of $\Phi_{\lambda}$ lying at the cross of the $i$-th row with the $j$-th column. For instance, $(1,1)$ is the cell of $\Phi_{\lambda}$ in the top left corner. Given a cell $(i, j)$ of $\Phi_{\lambda}$, we have the partition $\lambda_{i j}^{\nwarrow}$ obtained by intersecting of the first $i$ rows and the first $j$ columns of $\Phi_{\lambda}$ (the light colored cells in Figure 1(a)), the partition $\lambda_{i j}$ obtained by deleting all cells $\left(i^{\prime}, j^{\prime}\right) \neq(i, j)$ such that $i^{\prime} \geq i$ and $j^{\prime} \geq j$ (the white cells in Figure 1(a)), the partition $\lambda_{i j}^{\Pi}$ obtained by intersecting the first $i-1$ rows and the last $\lambda_{1}-j$ columns of $\Phi_{\lambda}$ (the dark cells above $(i, j)$ in Figure 1(a)), and finally the partition $\lambda_{i j}^{\swarrow}$ obtained by intersecting the last $k-i$ rows and the first $j-1$ columns of $\Phi_{\lambda}$ (the dark cells below $(i, j)$ in Figure 1(a)).

(a)

(b)

Figure 1: (a) Diagram of the partition $\lambda=(12,10,10,8,6,6,6,2,1)$, with the cell $(i, j)=(3,5)$ marked by a square. (b) Diagram of the partition $\lambda_{i j}=(12,10,5,4,4,4,4,2,1)$. In both pictures, the diagrams of the partitions $\lambda_{i j}^{\nearrow}=(7,5)$ and $\lambda_{i j}^{\swarrow}=(4,4,4,4,2,1)$ are in dark color.

Theorem 23 The number of edges in the Hasse diagram of the Young lattice $\mathcal{Y}_{\lambda}$ is

$$
\begin{equation*}
\ell\left(\mathcal{Y}_{\lambda}\right)=\sum_{(i, j) \in \Phi_{\lambda}}\left|\mathcal{Y}_{\lambda_{i j}^{\kappa}}\right|\left|\mathcal{Y}_{\lambda_{i j}^{\nearrow}}\right| \tag{37}
\end{equation*}
$$

Proof. Let $\mu \in \mathcal{Y}_{\lambda}$. The partitions covered by $\mu$ are exactly the partitions obtained by removing a cell from the boundary of $\mu$. Hence $|\nabla \mu|=\left|\partial \Phi_{\mu}\right|$ and

$$
\ell\left(\mathcal{Y}_{\lambda}\right)=\sum_{\mu \in \mathcal{Y}_{\lambda}}|\nabla \mu|=\sum_{\mu \in \mathcal{Y}_{\lambda}}\left|\partial \Phi_{\mu}\right|=\sum_{(i, j) \in \Phi_{\lambda}} C_{i j}
$$

where $C_{i j}$ is the number of all partitions $\mu \in \mathcal{Y}_{\lambda}$ such that $(i, j) \in \partial \Phi_{\mu}$.
Since $\lambda_{i j}^{K}$ and $\lambda_{i j}^{\Downarrow}$ are respectively the minimal and the maximal partition in $\mathcal{Y}_{\lambda}$ admitting $(i, j)$ as a corner cell, a partition $\mu$ has $(i, j)$ as a corner cell if and only if $\mu \in\left[\lambda_{i j}^{\nwarrow}, \lambda_{i j}^{\downarrow}\right]$. This implies that the partition $\mu$ is equivalent to a pair $\left(\mu_{1}, \mu_{2}\right)$ of partitions, where $\Phi_{\mu_{1}} \subseteq \Phi_{\lambda_{i j}^{\swarrow j}}$ and $\Phi_{\mu_{2}} \subseteq \Phi_{\lambda_{i j}^{\nearrow}}$. So $C_{i j}=\left|\mathcal{Y}_{\lambda_{i j}^{\swarrow}}\right|\left|\mathcal{Y}_{\lambda_{i j}^{\nearrow}}\right|$, and this yields identity (37).

To illustrate Theorem 23, we consider the particularly interesting case of the Young lattices $L(m, n)=\mathcal{Y}_{\lambda} \quad($ for $m, n \geq 1)$ obtained for the partition $\lambda=(n, \ldots, n)$ with $m$ parts (see, for instance, $[18,15])$. Since $L(n, n)=\mathcal{G} \mathcal{D}_{n}$, the next theorem generalizes Theorem 6 .

Theorem 24 The number of edges in the Hasse diagram of the Young lattice $L(m, n)$ is

$$
\begin{equation*}
\ell(L(m, n))=\binom{m+n-1}{n} n=\frac{(m+n-1)!}{(m-1)!(n-1)!} . \tag{38}
\end{equation*}
$$

Proof. For every cell $(i, j) \in \Phi_{\lambda}$, we have $\lambda_{i j}^{\nearrow}=(n-j)^{i-1}$ and $\lambda_{i j}^{\swarrow}=(j-1)^{m-i}$. Hence, by formula (37), we have

$$
\ell(L(m, n))=\sum_{i=1}^{m} \sum_{j=1}^{n}|L(n-j, i-1)||L(j-1, m-i)| .
$$

Since $|L(m, n)|=\binom{m+n}{n}$, we have

$$
\ell(L(m, n))=\sum_{i=1}^{m} \sum_{j=1}^{n}\binom{m-i+j-1}{j-1}\binom{n+i-j-1}{i-1}
$$

The generating series of these numbers is

$$
\begin{aligned}
L(x, y) & =\sum_{m, n \geq 1} \ell(L(m, n)) x^{m} y^{n} \\
& =\sum_{i, j \geq 1}\left[\sum_{m \geq 1}\binom{m-i+j-1}{j-1} x^{m}\right]\left[\sum_{n \geq 1}\binom{n+i-j-1}{i-1} y^{n}\right] \\
& =\sum_{i, j \geq 0}\left[\sum_{m \geq 0}\binom{m-i+j}{j} x^{m+1}\right]\left[\sum_{n \geq 0}\binom{n+i-j}{i} y^{n+1}\right] \\
& =\sum_{i, j \geq 0} \frac{x^{i+1}}{(1-x)^{j+1}} \frac{y^{j+1}}{(1-y)^{i+1}}=\sum_{i \geq 0} \frac{x^{+1 i}}{(1-y)^{i+1}} \sum_{j \geq 0} \frac{y^{j+1}}{(1-x)^{j+1}} \\
& =\frac{x y}{(1-x-y)^{2}}=x y \frac{\partial}{\partial y} \frac{1}{1-x-y}=\sum_{m, n \geq 0}\binom{m+n-1}{n} n x^{m} y^{n}
\end{aligned}
$$

Hence, we have identity (38).

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