Convergence of partial maps

Gerald Beer [a,*], Agata Caserta [b], Giuseppe Di Maio [b], Roberto Lucchetti [c]

[a] *Department of Mathematics, California State University L.A., 5151 State University Drive, Los Angeles, CA 90032, USA*

[b] *Dipartimento di Matematica, Seconda Università di Napoli, Viale Lincoln, 81100 Caserta, Italy*

[c] *Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy*

Abstract

Given metric spaces (X, d) and (Y, ρ), a partial map between X and Y is a pair (D, u), where D is a closed subset of X and $u : D \to Y$ is a function. We introduce a general convergence notion for nets of such partial functions. While our initial description is variational in nature, we show that this description amounts to bornological convergence of the associated net of graphs as defined by Lechicki, Levi and Spakowski [26] with respect to a natural bornology on $X \times Y$, and which places the work on continuous partial functions of Brandi, Ceppitelli, and Holá [12, 13,20,21] in a general framework.

1. Introduction

Let (X, d) and (Y, ρ) be metric spaces. By a *partial function* or a *partial map* from X to Y, we mean a pair (D, u) where D is a nonempty closed subset of X and $u : D \to Y$ is a function (not assumed continuous on D). We denote the space of all such maps by $\mathcal{P}[X, Y]$, while by $\mathcal{C}[X, Y]$ we mean those partial maps that are continuous on their respective domains.

Continuous partial functions were first considered by Kuratowski [24] where X was assumed compact so that for each $(D, u) \in \mathcal{C}[X, Y]$, the graph of u, which we denote by $\mathrm{Gr}(u)$, is compact as well. He topologized $\mathcal{C}[X, Y]$ by equipping the graph space with the classical Hausdorff metric topology [3], identifying partial functions with their graphs.

In the more general setting we have introduced, graphs of continuous partial functions, while not compact, will be closed subsets of $X \times Y$. Partial functions play a central role in mathematical economics, since they are typical utility functions for agents. Tastes of agents on a space X are usually represented by a preference relation on X, that is, a subset R of $X \times X$ where $(x, y) \in R$ means the agent prefers alternative x

* Corresponding author.
E-mail addresses: gbeer@calanet.calstatela.edu (G. Beer), agata.caserta@unina2.it (A. Caserta), giuseppe.dimaio@unina2.it (G. Di Maio), roberto.lucchetti@polimi.it (R. Lucchetti).

Received 25 September 2013
Available online 16 May 2014
Submitted by A. Dontchev

to y. Following the seminal work of Debreu [15], utility functions have been considered as more suitable mathematical tools to represent agents' preferences. Similarities of agents can be described by a convergence or topology on partial maps.

Having a dual representation of preferences – relations on one side equipped themselves with a convergence or topology and functions on the other – demands that one connects the two possible conceptions of similarity. This actually was the main motivation for Back to introduce the generalized compact open topology on utility functions [2]. By exploiting a result of Levin [27], Back showed that when X is locally compact and separable, classical Kuratowski convergence of preference relations can be expressed in terms of the convergence of suitable utility functions representing the preferences in his topology. Our motivation here is to give a new variational definition of convergence of partial maps that is compatible with Back's topology in the locally compact setting and that might be applicable beyond.

This paper proposes a new convergence on the set of the partial maps that can be described in different ways. While our initial description is variational in nature, it can also be described in terms of convergence of graphs and thus is consonant with the initial paper of Kuratowski. All of our descriptions involve bornologies, macroscopic structures employed over the last 25 years to describe convergence of nets or sequences of sets, the prototype being the now classical Attouch–Wets convergence, sometimes called bounded Hausdorff convergence (see, e.g., [1,3,7,28]).

Definition 1.1. A *bornology* $\mathcal{B}$ on a metric space (X, d) is a family of nonempty subsets of X covering X, that is stable under taking finite unions, and that is hereditary, i.e., stable under taking nonempty subsets.

The smallest bornology on X is the family of nonempty finite subsets of X, $\mathcal{F}$, and the largest is the family of all nonempty subsets of X, $\mathcal{P}_0(X)$. Other important bornologies are: the family $\mathcal{B}_d$ of the nonempty d-bounded subsets; the family $\mathcal{B}_{tb}$ of the nonempty d-totally bounded subsets; and the family $\mathcal{K}$ of nonempty subsets of X with compact closure. Bornologies in general topology were first considered by Hu [22]; their role in locally convex spaces is the subject of the monograph of Hogbe-Nlend [19].

Given a bornology $\mathcal{B}$ on (X, d), we can describe an associated convergence notion $\mathcal{P}(\mathcal{B})$ on $\mathcal{P}[X, Y]$: a rule that assigns to each net in $\mathcal{P}[X, Y]$ a (potentially empty) set of limits in $\mathcal{P}[X, Y]$ (for adequate information on nets for our purposes, see [23]). First, given a nonempty subset A of a metric space and $\epsilon > 0$, let A^ϵ denote the *ϵ-enlargement* of A, that is, the union of all open balls of radius ϵ whose centers run over A.

Definition 1.2. Let (X, d), (Y, ρ) be metric spaces, and let $\mathcal{B}$ be a bornology on X. Let Γ be a directed set and let $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$. We say that the net is $\mathcal{P}(\mathcal{B})$-*convergent* to (D, u), and write $(D, u) \in \mathcal{P}(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$, if for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $\gamma_0 \in \Gamma$ such that the following two conditions hold for all indices $\gamma \geq \gamma_0$:

(1) for each nonempty subset B_1 of B, $u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon$;
(2) for each nonempty subset B_1 of B, $u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\epsilon)]^\epsilon$.

Notice that in conditions (1) and (2), the inside enlargement in the last expression is taken in X while the outside enlargement is taken in Y. Even within $\mathcal{C}[X, Y]$, limits need not be unique and we will characterize bornologies for which uniqueness of limits occurs.

We will present analytical alternatives for conditions (1) and (2) that will be more manageable to check convergence in practice. But the most tangible and visual description of $\mathcal{P}(\mathcal{B})$-convergence is the following: for each $B \in \mathcal{B}$ and $\epsilon > 0$, eventually both $\text{Gr}(u_\gamma) \cap (B \times Y) \subset \text{Gr}(u)^\epsilon$ and $\text{Gr}(u) \cap (B \times Y) \subset \text{Gr}(u_\gamma)^\epsilon$. In this formulation, the enlargement is taken with respect to any metric compatible with the product uniformity. For definiteness, we choose the *box metric* defined by $(d \times \rho)((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), \rho(y_1, y_2)\}$. Readers familiar with the set convergence literature will immediately recognize this as bornological convergence of graphs in the sense of Lechicki, Levi and Spakowski [26]. Convergence of graphs of continuous functions in this sense, where the bornology on X is either $\mathcal{K}$ or $\mathcal{B}_d$, has been studied extensively by Brandi, Ceppitelli, and Holá (see, e.g., [12–14]).

Our focus will be primarily on $\mathcal{C}[X, Y]$ where the convergence is better behaved and where there is more interest. It is still better behaved when the partial functions are strongly uniformly continuous on the bornology in a sense consistent with the terminology of Beer and Levi [8]. This occurs when either (1) the bornology has a compact base, or (2) the partial functions have compact domains and the bornology has a closed base, and in these settings, we show that there are hit-and-miss topologies compatible with $\mathcal{P}(\mathcal{B})$-convergence, identifying continuous partial functions with their graphs. When (X, d) is locally compact and $\mathcal{B} = \mathcal{K}$, we prove directly that our hit-and-miss topology on $\mathcal{C}[X, Y]$ is the generalized compact-open topology as defined by Back [2] in his pioneering study of utility functions for a noncompact commodity space, thus producing an alternate proof of a result of Brandi, Ceppitelli, and Holá [13, Theorem 4.2] which implicitly asserts that convergence as we have defined it is compatible with the generalized compact-open topology (note: in their result, the target space was $\mathbb{R}^m$ rather than a general metric space).

At the end of the article, we look at uniformizability and metrizability of $\mathcal{P}(\mathcal{B})$-convergence. But we do not attempt to find general necessary and sufficient conditions for the convergence to be topological, even for continuous partial functions. Given that our convergence notion amounts to bornological convergence of nets of certain closed subsets of $X \times Y$, there is hope for success. Indeed, in the definitive study of Beer, Costantini, and Levi [4], shields were introduced and used to characterize when bornological convergence of nets of closed subsets is topological.

2. Notation and background material

All metric space will be assumed to have at least two points. In a metric space (X, d) we will write $B_d(x, \epsilon)$ for the open ball with center x and radius $\epsilon > 0$. Thus, if $A \subset X$, $A^\epsilon = \bigcup_{a \in A} B_d(a, \epsilon)$. For $A \subset X$, we denote its closure and interior by $\overline{A}$ and $\text{int}(A)$, respectively. We shall denote the nonempty closed subsets of X by $\text{CL}(X)$ and the nonempty compact subsets by $\text{K}(X)$.

We next carefully go over terminology and notation relative to bornologies and bornological convergence. As we will be working with bornologies on both X and $X \times Y$, we will use a neutral letter for our metric space in our discussion here.

Let $\mathcal{B}$ be a bornology on a metric space (Z, d). By a *base* $\mathcal{B}_0$ for the bornology $\mathcal{B}$, we mean a subfamily of $\mathcal{B}$ cofinal with respect to inclusion. Thus, for $\mathcal{B}_d$, a countable base is $\{B_d(z_0, n) : n \in \mathbb{N}\}$ where z_0 is a fixed but arbitrary point of the space. If the bornology contains a small ball around each point of Z, it is called *local*. In case for every $B \in \mathcal{B}$ there is $\delta > 0$ such that $B^\delta \in \mathcal{B}$, the bornology is called *stable under small enlargements*. Evidently the bornology $\mathcal{K}$ is stable under small enlargements if and only if it is local, i.e., the space is locally compact. As an example of a bornology that is local but not stable under small enlargements, let $Z = (0, 1]$, let $f(z) = \frac{1}{z} \sin \frac{1}{z}$, and let $\mathcal{B} = \{A \subset (0, 1] : f(A) \text{ is bounded}\}$.

Bornological convergence as defined in [26] is split into upper and lower bornological convergences, and we will utilize both halves. A net $\langle D_\gamma \rangle_{\gamma \in \Gamma}$ in $\mathcal{P}_0(Z)$ is called $\mathcal{B}^-$-convergent (*lower bornological convergent*) to $D \in \mathcal{P}_0(Z)$ if for every $B \in \mathcal{B}$ and $\epsilon > 0$, the following inclusion holds eventually:

$$D \cap B \subset D_\gamma^{\epsilon}.$$

We shall write $D \in \mathcal{B}^-\text{-lim } D_\gamma$ when this occurs. The net is declared $\mathcal{B}^+$-convergent (*upper bornological convergent*) to D if for every $B \in \mathcal{B}$ and $\epsilon > 0$, the following inclusion holds eventually:

$$D_\gamma \cap B \subset D^\epsilon.$$

In this circumstance, we shall write $D \in \mathcal{B}^+\text{-lim } D_\gamma$.

Naturally *two-sided bornological convergence* occurs when both upper and lower convergences occur, and we then write $D \in \mathcal{B}\text{-lim } D_\gamma$. Special cases include *convergence in Hausdorff distance* when $\mathcal{B} = \mathcal{P}_0(X)$ and *Attouch–Wets convergence* when $\mathcal{B} = \mathcal{B}_d$ [3]. The conditions under which this convergence is topologi-cal, uniformizable or pseudo-metrizable are now well-understood [4]. For the relation between bornological convergence of a net of sets and uniform convergence of the associated net of distance functionals on the bornology, the reader may consult [11]. We mention our favorite result in the theory which gives spe-cial weight to Attouch–Wets convergence: if the two-sided convergence for nets of nonempty closed sets is metrizable, then it is actually Attouch–Wets convergence under an equivalent remetrization, i.e., the initial bornology can be realized as $\mathcal{B}_\rho$ for some equivalent metric ρ [7].

We now introduce standard notation relative to hit-and-miss topologies on $\mathrm{CL}(Z)$ [3]. If E is a nonempty subset of (Z, d), we define E^-, E^+ and E^{++} by the these familiar formulas:

- $E^- := \{A \in \mathrm{CL}(Z) : E \cap A \neq \emptyset\}$,
- $E^+ := \{A \in \mathrm{CL}(Z) : A \subset E\}$,
- $E^{++} := \{A \in \mathrm{CL}(Z) : \exists \epsilon > 0 \text{ such that } A^\epsilon \subset E\}$.

We return to partial maps to end this section. The partial map $(\hat{D}, \hat{u}) \in \mathcal{P}[X, Y]$ is called an *extension* of the map $(D, u) \in \mathcal{P}[X, Y]$ if $D \subset \hat{D}$ and $\hat{u}(x) = u(x)$ for all $x \in D$. It is called a *restriction* of the map (D, u) if $\hat{D} \subset D$ and $\hat{u}(x) = u(x)$ for all $x \in \hat{D}$.

3. Upper and lower convergence

We start by describing the lower and upper halves of the convergence on partial functions discussed in the introduction.

Definition 3.1. Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{B}$ be a bornology on X. A net $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ in $\mathcal{P}[X, Y]$ is said to be $\mathcal{P}^-(\mathcal{B})$-*convergent* to (D, u) if for every $B \in \mathcal{B}$ and $\epsilon > 0$, $\exists \gamma_0 \in \Gamma$ such that $\forall \gamma \geq \gamma_0$, $\forall B_1 \subset B$, $u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon$.

If $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ is $\mathcal{P}^-(\mathcal{B})$-convergent to (D, u), we write $(D, u) \in \mathcal{P}^-(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$.

Definition 3.2. Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{B}$ be a bornology on X. A net $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ in $\mathcal{P}[X, Y]$ is said to be $\mathcal{P}^+(\mathcal{B})$-*convergent* to (D, u) if for every $B \in \mathcal{B}$ and $\epsilon > 0$, $\exists \gamma_0 \in \Gamma$ such that $\forall \gamma \geq \gamma_0$, $\forall B_1 \subset B$, $u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\epsilon)]^\epsilon$.

If $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ is $\mathcal{P}^+(\mathcal{B})$-convergent to (D, u), we write $(D, u) \in \mathcal{P}^+(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$.

The join of these convergences, meaning the convergence that ensues if we declare $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ con-vergent to (D, u) provided it is both lower and upper convergent to (D, u), is obviously our two-sided $\mathcal{P}(\mathcal{B})$-convergence of the Introduction.

We first observe that $\mathcal{P}^-(\mathcal{B})$-convergence implies lower bornological convergence of domains, while $\mathcal{P}^+(\mathcal{B})$-convergence implies upper bornological convergence of domains.

Proposition 3.1. *Let $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$ and let $\mathcal{B}$ be a bornology on X.*

(1) *If $(D, u) \in \mathcal{P}^-(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$, then $\forall B \in \mathcal{B}$, $\forall \epsilon > 0$, eventually $D \cap B \subset D_\gamma^\epsilon$;*
(2) *If $(D, u) \in \mathcal{P}^+(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$, then $\forall B \in \mathcal{B}$, $\forall \epsilon > 0$, eventually $D_\gamma \cap B \subset D^\epsilon$.*

Proof. We just verify statement (2). Fix $B \in \mathcal{B}$ and $\epsilon > 0$ and choose $\gamma_0 \in \Gamma$ such that $\gamma \geq \gamma_0 \Rightarrow \forall B_1 \subset B$, $u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\epsilon)]^\epsilon$. Fix $\gamma \geq \gamma_0$ and $x \in D_\gamma \cap B$. With $B_1 = \{x\}$, we get

$$u_\gamma(x) \in \left[u\left(D \cap \{x\}^\epsilon\right)\right]^\epsilon.$$

This means that for some $w \in D \cap B_d(x, \epsilon)$ we have $\rho(u_\gamma(x), u(w)) < \epsilon$. In particular, $x \in B_d(w, \epsilon) \subset D^\epsilon$, and as $x \in D_\gamma \cap B$ was arbitrary, we have $D_\gamma \cap B \subset D^\epsilon$ as required. □

We next reformulate Definition 3.1 and Definition 3.2 in ways more in the spirit of classical variational analysis.

Proposition 3.2. *The condition* $\forall B \in \mathcal{B}$, $\forall \epsilon > 0$, $\exists \gamma_0 \in \Gamma$ *such that* $\forall \gamma \geq \gamma_0$, $\forall B_1 \subset B$, $u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B)]^\epsilon$ *in Definition 3.1 is equivalent to the following condition:* $\forall B \in \mathcal{B}$, $\forall \epsilon > 0$, *eventually* $\sup_{z \in D \cap B} \inf_{x \in B_d(z,\epsilon) \cap D_\gamma} \rho(u_\gamma(x), u(z)) < \epsilon$.

Proof. Suppose $\sup_{z \in D \cap B} \inf_{x \in B_d(z,\epsilon) \cap D_\gamma} \rho(u_\gamma(x), u(z)) < \epsilon$. Immediately, if $B_1 \subset B$, we have

$$\sup_{z \in D \cap B_1} \inf_{x \in B_d(z,\epsilon) \cap D_\gamma} \rho\big(u_\gamma(x), u(z)\big) < \epsilon.$$

This means that for all $z \in D \cap B_1$, there exists $x \in D_\gamma \cap \{z\}^\epsilon \subset D_\gamma \cap B$ with $\rho(u_\gamma(x), u(z)) < \epsilon$ as required.

Conversely, suppose the condition of Definition 3.1 holds for $\epsilon/2$. With the choice of $B_1 = \{z\}$ as z runs over $\in D \cap B$, eventually

$$\sup_{z \in D \cap B} \inf_{x \in B_d(z,\epsilon) \cap D_\gamma} \rho\big(u_\gamma(x), u(z)\big) \leq \sup_{z \in D \cap B} \inf_{x \in B_d(z,\epsilon/2) \cap D_\gamma} \rho\big(u_\gamma(x), u(z)\big) \leq \epsilon/2 < \epsilon,$$

completing the proof.

Dually, we have the following result whose proof is left to the reader.

Proposition 3.3. *The condition* $\forall B \in \mathcal{B}$, $\forall \epsilon > 0$, $\exists \gamma_0 \in \Gamma$ *such that* $\forall \gamma \geq \gamma_0$, $\forall B_1 \subset B$, $u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B)]^\epsilon$ *in Definition 3.2 is equivalent to the following condition:* $\forall B \in \mathcal{B}$, $\forall \epsilon > 0$, *eventually,* $\sup_{z \in D_\gamma \cap B} \inf_{x \in B_d(z,\epsilon) \cap D} \rho(u(x), u_\gamma(z)) < \epsilon$.

We next make several qualitative observations about the $\mathcal{P}(\mathcal{B})$-convergence in $\mathcal{P}[X, Y]$ and its lower and upper halves.

- If for each $\gamma \in \Gamma$, $(D_\gamma, u_\gamma) = (D, u)$, then $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ is $\mathcal{P}(\mathcal{B})$-convergent to (D, u);
- If $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ is either $\mathcal{P}(\mathcal{B})^-$ or $\mathcal{P}(\mathcal{B})^+$-convergent to (D, u), then so is each subnet;
- Neither $\mathcal{P}(\mathcal{B})^-$ nor $\mathcal{P}(\mathcal{B})^+$-convergence is altered by replacing d and/or ρ by uniformly equivalent metrics;
- If $(D, u) \in \mathcal{P}^-(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$, then $(\hat{D}, \hat{u}) \in \mathcal{P}^-(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$ for any restriction $(\hat{D}, \hat{u})$ of (D, u). Dually, if $(D, u) \in \mathcal{P}^+(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$, then $(\hat{D}, \hat{u}) \in \mathcal{P}^+(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$ for any extension $(\hat{D}, \hat{u})$ of (D, u);
- If $(D, u) \in \mathcal{P}^-(\mathcal{B})\text{-lim}(D_\gamma, u_\gamma)$, then $(D, u) \in \mathcal{P}^-(\hat{\mathcal{B}})\text{-lim}(D_\gamma, u_\gamma)$ for any bornology $\hat{\mathcal{B}} \subset \mathcal{B}$. The same happens with upper convergences. Thus, the smaller the bornology, the coarser the convergence;
- Looking at all partial functions that assume only the value y_0 in some fixed target space Y allows us to recover, by means of the above definitions, the usual definitions of upper and lower bornological convergences in $\mathrm{CL}(X)$. Besides convergence in Hausdorff distance and Attouch–Wets convergence which we have already described, the choice of $\mathcal{B} = \mathcal{K}$ yields convergence with respect to the *Fell topology* τ_{Fell} on $\mathrm{CL}(X)$ [18] generated by all sets of the form V^- where V is open plus all sets of the form $(X \backslash K)^+$ where $K \in \mathrm{K}(X)$ [3, p. 141].

Identifying partial functions with their graphs in $X \times Y$, we next show that $\mathcal{P}(\mathcal{B})^-$ and $\mathcal{P}(\mathcal{B})^+$-convergence in $\mathcal{P}[X, Y]$ are bornological convergences in the sense of [26]. The bornology that is in play is the one having as a base $\{B \times Y : B \in \mathcal{B}\}$, which we denote by $\mathcal{B}^*$. Note that $E \in \mathcal{B}^*$ if and only if $\pi_X(E) \in \mathcal{B}$, where π_X is the usual projection map on $X \times Y$ onto X. The cases when $\mathcal{B} = \mathcal{B}_d$ and $\mathcal{B} = \mathcal{K}$ for $\mathcal{C}[X, Y]$ have already been studied by Brandi, Ceppitelli, and Holá [12–14]. This alternate understanding provides the easiest way for one to determine if an actual sequence or net of partial functions either upper or lower converges.

Given a partial function (D, u) we remind the reader that we denote its graph simply by $\mathrm{Gr}(u)$ when D is clearly understood.

Theorem 3.1. *Let $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ be a net of partial functions from (X, d) to (Y, ρ) and let $\mathcal{B}$ be a bornology on X. Then for $(D, u) \in \mathcal{P}[X, Y]$,*

(1) $\mathrm{Gr}(u) \in (\mathcal{B}^*)^-\text{-}\lim \mathrm{Gr}(u_\gamma)$ *if and only if* $(D, u) \in \mathcal{P}^-(\mathcal{B})\text{-}\lim(D_\gamma, u_\gamma)$*;*
(2) $\mathrm{Gr}(u) \in (\mathcal{B}^*)^+\text{-}\lim \mathrm{Gr}(u_\gamma)$ *if and only if* $(D, u) \in \mathcal{P}^+(\mathcal{B})\text{-}\lim(D_\gamma, u_\gamma)$*.*

Proof. We prove statement (2), leaving (1) to the reader. Suppose $(D, u) \in \mathcal{P}^+(\mathcal{B})\text{-}\lim(D_\gamma, u_\gamma)$. To verify bornological convergence of graphs, it suffices to work with basic sets in $\mathcal{B}^*$. Let $B \times Y$ be such a basic set where $B \in \mathcal{B}$. Let $\epsilon > 0$ be arbitrary and choose an index $\gamma_0 \in \Gamma$ so large that for all $\gamma \geq \gamma_0$ both (i) $D_\gamma \cap B \subset D^\epsilon$, and (ii) $\sup_{z \in D_\gamma \cap B} \inf_{x \in B_d(z,\epsilon) \cap D} \rho(u(x), u_\gamma(z)) < \epsilon$. Now fix $\gamma \geq \gamma_0$ and $(z, u_\gamma(z)) \in (B \times Y) \cap \mathrm{Gr}(u_\gamma)$ so that $z \in D_\gamma$. By (i) $B_d(z, \epsilon) \cap D \neq \emptyset$, and by (ii) for some x in the intersection, we have $\rho(u(x), u_\gamma(z)) < \epsilon$. As a result, $(x, u(x)) \in \mathrm{Gr}(u)$ and

$$(d \times \rho)\big((x, u(x)), (z, u_\gamma(z))\big) < \epsilon,$$

and this yields $\mathrm{Gr}(u_\gamma) \cap (B \times Y) \subset \mathrm{Gr}(u)^\epsilon$ for $\gamma \geq \gamma_0$.

Conversely, suppose we have upper bornological convergence of graphs. Fix $B \in \mathcal{B}$ and $\epsilon > 0$. Choosing $\delta \in (0, \epsilon)$, there exists $\gamma_1 \in \Gamma$ such that for all $\gamma \geq \gamma_1$, we have

$$\mathrm{Gr}(u_\gamma) \cap (B \times Y) \subset \mathrm{Gr}(u)^\delta.$$

Fix $\gamma \geq \gamma_1$ and let $z \in D_\gamma \cap B$ be arbitrary. Clearly, $(z, u_\gamma(z)) \in (B \times Y) \cap \mathrm{Gr}(u_\gamma)$, so there exists $(x_0, u(x_0))$ in $\mathrm{Gr}(u)$ which is δ-close with respect to the box metric to $(z, u_\gamma(z))$, and we get $x_0 \in D$ with $d(z, x_0) < \delta < \epsilon$. At the same time, we get $\rho(u(x_0), u_\gamma(z)) < \delta$ so that

$$\inf_{x \in B_d(z,\epsilon) \cap D} \rho\big(u(x), u_\gamma(z)\big) < \delta,$$

and then taking the supremum over $z \in D_\gamma \cap B$ yields at most $\delta < \epsilon$. □

Corollary 3.1. *Let (X, d) and (Y, ρ) be metric spaces and let $\mathcal{B}$ be a bornology on X. Then $\mathcal{P}(\mathcal{B})$-convergence on $\mathcal{P}[X, Y]$ agrees with $\mathcal{B}^*$-convergence in the hyperspace of graphs of partial functions.*

We give three examples for sequences of real-valued continuous functions that are each globally defined.

Example 3.1. Let $X = (0, 1]$ and let $Y = \mathbb{R}$. Let $u : (0, 1] \to \mathbb{R}$ be defined by $u(x) = \frac{1}{x} \sin \frac{1}{x}$ and for each $n \in \mathbb{N}$, define u_n by

$$u_n(x) = \begin{cases} u(x) & \text{if } \frac{1}{n\pi} \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

With respect to the bornology $\mathcal{B} = \mathcal{P}_0(X)$, we have $\mathcal{P}^+(\mathcal{B})$-convergence but not $\mathcal{P}^-(\mathcal{B})$-convergence. However if we replace $\mathcal{P}_0(X)$ by $\mathcal{K}$, we get lower convergence as well.

Example 3.2. Let $X = [0, 1]$ and let $Y = \mathbb{R}$. Let u be the zero function on $[0, 1]$ and for each $n \in \mathbb{N}$, define u_n by

$$u_n(x) = \begin{cases} 1 - nx & \text{if } x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

With respect to the bornology $\mathcal{B} = \mathcal{P}_0(X)$, we clearly have $\mathcal{P}^-(\mathcal{B})$-convergence but not $\mathcal{P}^+(\mathcal{B})$-convergence; in fact, upper convergence fails even when $\mathcal{B} = \mathcal{F}$. This example shows that pointwise convergence is not ensured by $\mathcal{P}^-(\mathcal{B})$-convergence, even when the bornology is as large as possible.

Example 3.3. Again, let $X = [0, 1]$ and let $Y = \mathbb{R}$. For each $n \in \mathbb{N}$ by the Tietze extension theorem [23, p. 242], we can find a continuous function $u_n : [0, 1] \to [0, 1]$ such that for $j = 0, 1, 2, \ldots, n$, $u_n(j/n) = 0$ if j is even while $u_n(j/n) = 1$ if j is odd. By the intermediate value theorem, with $\mathcal{B} = \mathcal{P}_0(X)$, $\langle u_n \rangle$ is $\mathcal{P}(\mathcal{B})$-convergent to any function $u : [0, 1] \to [0, 1]$ whose graph is dense in the unit square (no such u can be continuous). The same of course is true of any smaller bornology on X. This shows that limits can fail to be unique.

One might suspect that $\mathcal{P}(\mathcal{B})$-limits are unique in $\mathcal{C}[X, Y]$, but this is not always the case. What does occur is anticipated by [26, Proposition 4.5] that focuses on convergence of nets of closed sets.

Proposition 3.4. *Let $\mathcal{B}$ be a bornology on (X, d). Then $\mathcal{P}(\mathcal{B})$-limits are unique in $\mathcal{C}[X, Y]$ for each metric target space (Y, ρ) if and only if $\mathcal{B}$ is local.*

Proof. If $\mathcal{B}$ is local, then so is $\mathcal{B}^*$ so that by [26, Proposition 4.5], with respect to $\mathcal{B}^*$-convergence on $\mathrm{CL}(X \times Y)$, limits are unique. Thus limits are unique in $\mathcal{C}[X, Y]$ because elements of $\mathcal{C}[X, Y]$ have closed graphs. Conversely, suppose the bornology fails to be local. There exists $x_0 \in X$ such that for each $n \in \mathbb{N}$ and each $B \in \mathcal{B}$, there exist $x(n, B) \in B_d(x_0, \frac{1}{n}) \backslash B$. Take $x_1 \neq x_0$ in X and let $y_0 \in Y$. Directing $\mathbb{N}$ in the usual way and $\mathcal{B}$ by inclusion and giving $\mathbb{N} \times \mathcal{B}$ the product direction, assign to (n, B) the continuous partial function mapping both $x(n, B)$ and x_1 to y_0. Then two distinct limits of the net $\langle (\{x(n, B), x_1\}, y_0) \rangle$ are (i) the partial function with domain $\{x_1\}$ sending x_1 to y_0, and (ii) the partial function with domain $\{x_0, x_1\}$ sending both points to y_0. □

As we have seen from Example 3.3, in a space equipped with a local bornology, a net of continuous partial functions can have multiple discontinuous limits, but clearly at most one with closed graph. On the other hand we have this notable result.

Proposition 3.5. *Let $\mathcal{B}$ be a local bornology on (X, d), and suppose the net of partial functions $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ each with values in (Y, ρ) is $\mathcal{P}(\mathcal{B})$-convergent to $(D, u) \in \mathcal{C}[X, Y]$. Then the net can $\mathcal{P}(\mathcal{B})$-converge to no other partial function of any kind.*

Proof. Suppose (C, v) is another partial function to which the net converges. Coincidence of D and C is a consequence of [26, Proposition 4.5]. Now suppose there is $x \in D$ with $u(x) \neq v(x)$. By continuity there exists $\epsilon > 0$ so small that $B_d(x, \epsilon) \in \mathcal{B}$, and

$$\big(u(D \cap B_d(x, 2\epsilon))\big)^\epsilon \cap B_\rho\big(v(x), \epsilon\big) = \emptyset.$$

By lower convergence, applied to (D, v), eventually there is $x_\gamma \in B_d(x, \epsilon)$ such that $u_\gamma(x_\gamma) \in B_\rho(v(x), \epsilon)$. By the definition of upper convergence, applied to the limit (D, u) where $B_1 = B_d(x, \epsilon)$, eventually we have

$$u_\gamma(x_\gamma) \in u_\gamma\big(D_\gamma \cap B_d(x, \epsilon)\big) \subset u\big(D \cap \big(B_d(x, \epsilon)\big)^\epsilon\big)^\epsilon \subset \big(u\big(D \cap B_d(x, 2\epsilon)\big)\big)^\epsilon,$$

a contradiction. This ends the proof. □

There is more than one way to define pointwise convergence for a net of partial functions, so that when all domains coincide, we recover conventional pointwise convergence (see, e.g., [29]). We adopt one that is as strong as any.

Definition 3.3. Let $\langle (D_\gamma, u_\gamma)\rangle_{\gamma \in \Gamma}$ be a net of partial functions and let (D, u) be a partial function. We say that $\langle (D_\gamma, u_\gamma)\rangle_{\gamma \in \Gamma}$ *converges pointwise* to (D, u) if whenever $x \in D_\gamma$ for a cofinal subset $\Gamma_0 \subset \Gamma$, then $x \in D$ and $u(x) = \lim_{\gamma \in \Gamma_0} u_\gamma(x)$.

A weaker requirement than pointwise convergence as we have defined it for partial functions is to require that x be residually in the domains to guarantee that x be in D. Alternatively, we may change our definition in a different direction by not insisting that membership of x to a cofinal set of domains forces membership of x to D, rather, simply insisting that $u(x) = \lim_{\gamma \in \Gamma_0} u_\gamma(x)$ whenever x happens to be in D.

Proposition 3.6. *Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{B}$ be a bornology on X. Suppose the net $\langle (D_\gamma, u_\gamma)\rangle_{\gamma \in \Gamma}$ is in $\mathcal{P}[X, Y]$.*

(a) *If the net is $\mathcal{P}^+(\mathcal{B})$-convergent to $(D, u) \in \mathcal{C}[X, Y]$, then it is pointwise convergent to (D, u);*
(b) *If the net is $\mathcal{P}^-(\mathcal{B})$-convergent to $(D, u) \in \mathcal{P}[X, Y]$, whenever $V \subset X \times Y$ is open and $\mathrm{Gr}(u) \cap V \neq \emptyset$, then eventually $\mathrm{Gr}(u_\gamma) \cap V \neq \emptyset$.*

Proof. For (a), suppose for each γ in a cofinal set of indices Γ_0, we have $x \in D_\gamma$. By Proposition 3.1, we have $x \in D$ because D is closed. Choose $\delta < \epsilon/2$ such that if $d(w, x) < \delta$, then $\rho(u(x), u(w)) < \epsilon/2$. By $(\mathcal{B}^*)^+$-convergence of graphs, since for each $\gamma \in \Gamma_0$

$$\big(x, u_\gamma(x)\big) \in \big(\{x\} \times Y\big) \cap \mathrm{Gr}(u_\gamma),$$

eventually in Γ_0 there exists $w_\gamma \in D$ with

$$(d \times \rho)\big(\big(x, u_\gamma(x)\big), \big(w_\gamma, u(w_\gamma)\big)\big) < \delta.$$

Applying the triangle inequality for ρ, we get $\rho(u_\gamma(x), u(x)) < \epsilon$ eventually in Γ_0. Statement (b) is immediate from statement (1) of Theorem 3.1 and the inclusion $\mathcal{F} \subset \mathcal{B}$. □

It follows from [13, Proposition 5.1] and Theorem 3.1 that $\mathcal{P}(\mathcal{K})$-convergence of a net in $\mathcal{C}[X, Y]$ to a continuous partial function guarantees convergence in the Fell topology of the associated net of graphs. We also note that for sequences of closed subsets – in particular, sequences of graphs of continuous functions – Fell convergence agrees with classical Kuratowski–Painlevé convergence. For nets, the latter convergence is usually stronger, but the two agree when X is locally compact (see [3, pp. 147–149] and [13, Proposition 5.2]).

To finish this section, we give a simple example showing Fell convergence of graphs of continuous partial functions does not ensure $\mathcal{P}(\mathcal{K})$-convergence, even if all functions are globally defined.

Example 3.4. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, equipped with the Euclidean metric. Here $\mathcal{K} = \mathcal{P}_0(X)$. Let $u : X \to \mathbb{R}$ be the zero function and for each $n \in \mathbb{N}$, let u_n be defined by

$$u_n(x) = \begin{cases} n & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, we have $\mathcal{P}(\mathcal{K})^-$-convergence (and thus convergence of graphs in the lower half of the Fell topology) but not $\mathcal{P}(\mathcal{K})^+$-convergence (note that we do have $\mathcal{P}(\mathcal{F})$-convergence). If K is a compact set in $X \times \mathbb{R}$ that misses $\mathrm{Gr}(u)$, then it misses $\mathrm{Gr}(u_n)$ as well for all $n > \max\{y : \exists x \text{ with } (x, y) \in K\}$. Thus we have convergence of graphs in the two-sided Fell topology.

4. Compactness, strong uniform continuity and convergence of partial functions

Presently, we will show that $\mathcal{P}(\mathcal{B})$-convergence of nets of continuous partial functions is topological provided $\mathcal{B}$ has a compact base. We actually display the topology as a hit-and-miss topology. For an alternate description of the topology in the special case $\mathcal{B} = \mathcal{K}$, we refer the reader to [13, Theorem 4.1].

But first we digress. We now define the notions of uniform continuity and strong uniform continuity for a partial map with respect to a bornology which reduce to those given by Beer and Levi in [8,9] for a globally defined function. In turn, this work has antecedents in [5].

Let us fix a partial map (D, u) and a bornology $\mathcal{B}$.

Definition 4.1. Let $\mathcal{B}$ be a bornology on (X, d) and $(D, u) \in \mathcal{P}[X, Y]$. We say that (D, u) is *uniformly continuous* relative to the bornology $\mathcal{B}$ if for every $B \in \mathcal{B}$ with $D \cap B \neq \emptyset$, the map

$$u : D \cap B \to Y$$

is uniformly continuous. We say that (D, u) is *strongly uniformly continuous* relative to $\mathcal{B}$ if for each $B \in \mathcal{B}$ and for each $\epsilon > 0$ there is $\delta > 0$ such that if $d(x, w) < \delta$ and $x, w \in D \cap B^\delta$, then $\rho(u(x), u(w)) < \epsilon$.

Remark 4.1. The following facts parallel results given in [8] and are easily proved:

- strong uniform continuity of (D, u) relative to $\mathcal{B}$ ensures $(D, u) \in \mathcal{C}[X, Y]$ and its uniform continuity relative to $\mathcal{B}$;
- uniform continuity of (D, u) relative to $\mathcal{B}$ ensures $(D, u) \in \mathcal{C}[X, Y]$ provided $\mathcal{K} \subset \mathcal{B}$;
- if (D, u) is uniformly continuous on D in the conventional sense, then (D, u) is strongly uniformly continuous relative to each bornology;
- if $\mathcal{B}$ has a compact base, then each $(D, u) \in \mathcal{C}[X, Y]$ is strongly uniformly continuous relative to $\mathcal{B}$;
- if $\mathcal{B}$ is stable under small enlargements, then (D, u) is strongly uniformly continuous relative to $\mathcal{B}$ if and only if it is uniformly continuous relative to $\mathcal{B}$.

Strong uniform continuity of the limit permits a different description of $\mathcal{P}^+(\mathcal{B})$-convergence.

Theorem 4.1. *Let $\mathcal{B}$ be a bornology on (X, d) and let (D, u) be strongly uniformly continuous relative to $\mathcal{B}$ with values in (Y, ρ). Then a net $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ in $\mathcal{P}[X, Y]$ is $\mathcal{P}^+(\mathcal{B})$-convergent to (D, u) if and only if for each $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $\delta > 0$ such that eventually,* $\sup_{z \in D_\gamma \cap B} \sup_{x \in B_d(z,\delta) \cap D} \rho(u(x), u_\gamma(z)) < \epsilon$.

Proof. In view of Proposition 3.3, we only need to prove that upper convergence of the net implies the "sup–sup" condition above.

Let $B \in \mathcal{B}$ and $\epsilon > 0$ be fixed. Let $\xi > 0$ be such that $2\xi < \epsilon$. By strong uniform continuity of u relative to $\mathcal{B}$, there exists $\sigma \in (0, \xi)$ such that $x, y \in D \cap B^\sigma$ and $d(x, y) < 2\sigma$ imply $\rho(u(x), u(y)) < \xi$. By Proposition 3.1 and Proposition 3.3, there exists $\gamma_0 \in \Gamma$ such that $D_\gamma \cap B \subset D^\sigma$ and

$$\sup_{z\in D_\gamma\cap B}\inf_{x\in B_d(z,\sigma)\cap D}\rho\big(u(x),u_\gamma(z)\big)<\sigma$$

hold whenever $\gamma\geq\gamma_0$.

We will show that the choice $\delta=\sigma$ does the job. Fix $\gamma\geq\gamma_0$. For every $z\in B\cap D_\gamma$, there exists $x_z\in B_d(z,\sigma)\cap D$ such that $\rho(u(x_z),u_\gamma(z))<\sigma<\xi$. Since $B_d(z,\sigma)\subset B^\sigma$, for every $x\in D\cap B_d(z,\sigma)$ we have by strong uniform continuity $\rho(u(x),u(x_z))<\xi$. Thus for every $x\in D\cap B_d(z,\sigma)$,

$$\rho\big(u(x),u_\gamma(z)\big)\leq\rho\big(u(x),u(x_z)\big)+\rho\big(u(x_z),u_\gamma(z)\big)<2\xi,$$

and thus $\sup_{z\in B\cap D_\gamma}\sup_{x\in B_d(z,\sigma)\cap D}\rho(u(x),u_\gamma(z))\leq 2\xi<\epsilon$. This concludes the proof. □

Theorem 4.2. *Let $\mathcal{B}$ be a bornology on (X,d) that is stable under small enlargements and let (D,u) be uniformly continuous relative to $\mathcal{B}$ with values in (Y,ρ). Then a net $\langle(D_\gamma,u_\gamma)\rangle_{\gamma\in\Gamma}$ in $\mathcal{P}[X,Y]$ is $\mathcal{P}(\mathcal{B})$-convergent to (D,u) if and only if both of the following conditions hold:*

(i) *for each $B\in\mathcal{B}$ and $\epsilon>0$, eventually $D\cap B\subset D_\gamma^\epsilon$;*
(ii) *for each $B\in\mathcal{B}$ and $\epsilon>0$, there exists $\delta>0$ such that eventually*

$$\sup_{z\in D_\gamma\cap B}\sup_{x\in B_d(z,\delta)\cap D}\rho\big(u(x),u_\gamma(z)\big)<\epsilon.$$

Proof. By Proposition 3.1 and Theorem 4.1, these conditions are necessary for $\mathcal{P}(\mathcal{B})$-convergence because (u, D) is strongly uniformly continuous relative to $\mathcal{B}$. For sufficiency, we need only show that given $B\in\mathcal{B}$ and $\epsilon>0$, eventually,

$$\sup_{z\in D\cap B}\inf_{x\in B_d(z,\epsilon)\cap D_\gamma}\rho\big(u_\gamma(x),u(z)\big)<\epsilon.$$

Choose $\delta<\epsilon$ so small that by (ii), for some $\gamma_0\in\Gamma$, whenever $\gamma\geq\gamma_0$,

$$(\diamondsuit)\qquad\sup_{z\in D_\gamma\cap B^\delta}\sup_{x\in B_d(z,\delta)\cap D}\rho\big(u(x),u_\gamma(z)\big)<\frac{\epsilon}{2},$$

and so that $B^\delta\in\mathcal{B}$. Then choose by (i) $\gamma_1\geq\gamma_0$ such that $\gamma\geq\gamma_1$ implies $D\cap B\subset D_\gamma^\delta$. Let $z\in D\cap B$ be arbitrary; then whenever $\gamma\geq\gamma_1$, there exists $x_{z,\gamma}\in D_\gamma\cap B^\delta$ with $d(z,x_{z,\gamma})<\delta$. In view of ($\diamondsuit$), this gives $\rho(u(z),u_\gamma(x_{z,\gamma}))<\frac{\epsilon}{2}$, and so

$$\sup_{z\in D\cap B}\inf_{x\in B_d(z,\epsilon)\cap D_\gamma}\rho\big(u_\gamma(x),u(z)\big)\leq\sup_{z\in D\cap B}\inf_{x\in B_d(z,\delta)\cap D_\gamma}\rho\big(u_\gamma(x),u(z)\big)\leq\frac{\epsilon}{2}<\epsilon$$

as required. □

In the remainder of the section, we show that two-sided convergence of continuous partial functions is topological in two separate contexts in which the functions are automatically strongly uniformly continuous with respect to the bornology. Our descriptions of the topologies show that convergence of partial functions in these particular contexts is not affected by replacing our metrics d and ρ by equivalent metrics.

Theorem 4.3. *Let $\mathcal{B}$ be a bornology with compact base on a metric space (X,d), and let (Y,ρ) be a second metric space. Then $\mathcal{P}(\mathcal{B})$-convergence on $\mathcal{C}[X,Y]$ is compatible with the hyperspace topology on graphs of partial functions generated by all sets of the form*

$$V^- := \{(D,u) : \mathrm{Gr}(u) \cap V \neq \emptyset\} \ \ V \textit{ open in } X \times Y,$$

$$((X \times Y)\backslash F)^+ := \{(D,u) : \mathrm{Gr}(u) \cap F = \emptyset\} \ \ F \in \mathrm{CL}(X \times Y) \textit{ and } \pi_X(F) \in \mathcal{B} \cap \mathrm{K}(X).$$

Proof. Suppose we have $\mathcal{P}(\mathcal{B})$-convergence of $\langle (D_\gamma, u_\gamma)\rangle_{\gamma\in\Gamma}$ to (D,u), that is, $\mathcal{B}^*$-convergence of $\langle \mathrm{Gr}(u_\gamma)\rangle_{\gamma\in\Gamma}$ to $\mathrm{Gr}(u)$. As recorded in Proposition 3.6, by lower convergence alone, if V is open and $\mathrm{Gr}(u)\cap V \neq \emptyset$, then $\mathrm{Gr}(u_\gamma)\cap V \neq \emptyset$ eventually. Suppose F is closed in $X\times Y$ where $\pi_X(F) \in \mathcal{B}\cap \mathrm{K}(X)$ and $\mathrm{Gr}(u)\cap F = \emptyset$. Now if $D\cap \pi_X(F) = \emptyset$, then there is positive gap between them, so that by $\mathcal{P}(\mathcal{B}^+)$-convergence and Proposition 3.1, eventually $D_\gamma \cap \pi_X(F) = \emptyset$. Thus eventually, $\mathrm{Gr}(u_\gamma)\cap F = \emptyset$ as well. Otherwise, $D\cap \pi_X(F)$ is a nonempty compact set and if $\hat{u}$ is the restriction of u to $D\cap\pi_X(F)$, by compactness of the graph, we can find $\epsilon > 0$ such that with respect to the box metric on the product, $\mathrm{Gr}(\hat{u})\cap F^\epsilon = \emptyset$.

We claim (♣): there exists $\delta < \frac{\epsilon}{2}$ such that for all $x \in (\pi_X(F))^\delta \cap D$ we have

$$B_{d\times\rho}\left((x, u(x)), \frac{\epsilon}{2}\right) \cap F = \emptyset.$$

Otherwise, for each $n \in \mathbb{N}$ there is $x_n \in (\pi_X(F))^{1/n}\cap D$ such that $B_{d\times\rho}((x_n, u(x_n)), \frac{\epsilon}{2}) \cap F \neq \emptyset$. Since $\pi_X(F)$ is compact and D is closed, $\langle x_n\rangle$ has a cluster point p in $D\cap\pi_X(F)$, and by continuity of (D,u), $B_{d\times\rho}((p,u(p)),\epsilon)\cap F \neq \emptyset$, and a contradiction is obtained.

For all γ sufficiently large, we have $\mathrm{Gr}(u_\gamma)\cap(\pi_X(F)\times Y) \subset \mathrm{Gr}(u)^\delta$, and it follows from (♣) that $\mathrm{Gr}(u_\gamma)\cap F = \emptyset$ for all such γ.

Conversely, suppose $\langle \mathrm{Gr}(u_\gamma)\rangle_{\gamma\in\Gamma}$ converges to $\mathrm{Gr}(u)$ in the stated hit-and-miss topology. To show $\mathcal{P}(\mathcal{B})$-convergence, it suffices to show that for each compact $B \in \mathcal{B}$ and $\epsilon > 0$, eventually both (i) $\mathrm{Gr}(u)\cap(B\times Y) \subset \mathrm{Gr}(u_\gamma)^\epsilon$ and (ii) $\mathrm{Gr}(u_\gamma)\cap(B\times Y)\subset \mathrm{Gr}(u)^\epsilon$.

Inclusion (i) follows from the fact that $\mathrm{Gr}(u)\cap(B\times Y)$ if nonempty is the graph of the restriction $\widetilde{u}$ of u to the compact set $B\cap D$. As $\mathrm{Gr}(\widetilde{u})$ is a compact set, it may be covered by a finite number of open $\frac{\epsilon}{2}$-balls with respect to the box metric whose centers lie in the graph of the restriction. If $\mathrm{Gr}(u_\gamma)$ were to hit each of these balls, then it is clear that $\mathrm{Gr}(u)\cap(B\times Y)\subset \mathrm{Gr}(u_\gamma)^\epsilon$.

For inclusion (ii), if $\mathrm{Gr}(u)\cap(B\times Y) = \emptyset$, take $F = B\times Y$. Then $(D,u) \in ((X\times Y)\backslash F)^+$ implies $(D_\gamma, u_\gamma)\in((X\times Y)\backslash F)^+$ eventually so $\mathrm{Gr}(u_\gamma)\cap(B\times Y) = \emptyset$ eventually.

Otherwise, by the assumption that Y contains at least two points, it is not hard to show using the uniform continuity of $\widetilde{u}$ that there exists $\delta < \epsilon$ so that

$$\pi_X\left((B\times Y)\backslash \mathrm{Gr}(\widetilde{u})^\delta\right) = B.$$

Put $F = (B\times Y)\backslash\mathrm{Gr}(\widetilde{u})^\delta$. Clearly, $(u_\gamma, D_\gamma) \in ((X\times Y)\backslash F)^+$ implies

$$\mathrm{Gr}(u_\gamma)\cap(B\times Y) \subset \mathrm{Gr}(u)^\delta \subset \mathrm{Gr}(u)^\delta,$$

and this establishes (ii). □

If $\mathcal{B}$ is a bornology with compact base, we now denote the topology of $\mathcal{P}(\mathcal{B})$-convergence on $\mathcal{C}[X, Y]$ by $\tau_{\mathcal{P}}(\mathcal{B})$. The topology $\tau_{\mathcal{P}}(\mathcal{K})$ is actually uniformizable when X is locally compact because the bornology is stable under small enlargements (see [13, Proposition 4.3] and Section 5 below).

In [2], Back introduced the so-called generalized compact-open topology on the set of all continuous real-valued partial maps with closed domains in a locally compact separable space X. This topology, also known as the Back topology, is widely studied in the literature for its several important applications in mathemat-ical economics [2], dynamic programming models [25,30] and for differential equations [12]. Holá [20,21], following [12,13], generalized it by substituting for X and the reals arbitrary Hausdorff spaces, and studied various topological properties of the function space.

Most important for our purposes, Brandi, Ceppitelli, and Holá [13, Theorem 4.2] proved that when $Y = \mathbb{R}^m$, $\mathcal{K}^*$-convergence of nets of graphs of continuous partial functions is compatible with the generalized compact-open topology if and only if X is locally compact. Here, we intend to establish this compatibility for a general metric target space in a very different way by showing directly that our hit-and-miss topology of Theorem 4.3 agrees with Back's topology.

In order to define the generalized compact-open topology on $\mathcal{C}[X,\ Y\ ]$, we introduce some standard notation. In what follows, G is an open subset of X, K is a compact subset of X, and I is an open possibly empty subset of Y.

- $[G] := \{(D,u) \in \mathcal{C}[X,Y] : D \cap G \neq \emptyset\}$;
- $[K : I] = \{(D,u) \in \mathcal{C}[X,Y] : u(D \cap K) \subset I\}$.

Definition 4.2. Let (X,d) and (Y,ρ) be metric spaces. The family of sets $[G]$, $[K : I]$ is a subbase for a topology on $\mathcal{C}[X,Y]$, denoted by τ_B, and called the *generalized compact-open topology* on $\mathcal{C}[X,Y]$.

Theorem 4.4. *Let (X,d) be a locally compact metric space and let (Y,ρ) be a metric space. Then on $\mathcal{C}[X,Y]$, the topology τ_B coincides with the topology $\tau_{\mathcal{P}}(\mathcal{K})$.*

Proof. The inclusion $\tau_B \subset \tau_{\mathcal{P}}(\mathcal{K})$ does not require local compactness of the domain. We show each subbasic open set in τ_B lies in $\tau_{\mathcal{P}}(\mathcal{K})$. Evidently whenever G is open in X we have $[G] = (G \times Y)^-$. Next suppose $K \subset X$ is compact and $I \subset Y$ is open. If $I = Y$, then $[K : I] = \mathcal{C}[X,Y]$ which is open in any topology on $\mathcal{C}[X,Y]$. Otherwise, put $F = K \times (Y \backslash I)$. Clearly, $\pi_X(F) = K \in \mathrm{K}(X)$ and $[K : I] = ((X \times Y)\backslash F)^+$.

For the reverse inclusion, suppose $(D,u) \in V^-$ where V is open in $X \times Y$. Choose $x \in D$ such that $(x, u(x)) \in V$. There exist $\epsilon > 0$ such that $B_{d\times\rho}((x,u(x)),\epsilon) \subset V$. Then with $K = \{x\}$ and $I = B_\rho(u(x),\epsilon)$, we have $(D,u) \in [K : I] \subset V^-$.

Finally, suppose $(D,u) \in ((X \times Y)\backslash F)^+$; denote the compact set $\pi_X(F)$ by C. If $D \cap C = \emptyset$, then

$$(D,u) \in [C;\emptyset] \subset \big((X \times Y)\backslash F\big)^+.$$

Otherwise, let $W = (X \times Y)\backslash F$, an open neighborhood of $\mathrm{Gr}(u)$. For each $x \in D \cap C \neq \emptyset$ we can find a ball $B_d(x,\delta)$ and an open neighborhood I of $u(x)$ such that $B_d(x,\delta) \times I \subset W$. Then by continuity and local compactness we can find a smaller ball with center x whose closure is compact and is mapped into I by u. By the compactness of $D \cap C$, we can find a finite subset $\{x_1, x_2, \dots, x_n\}$ of $D \cap C$ and balls $B_d(x_j,\epsilon_j)$ covering $D \cap C$ each with compact closure C_j for $j = 1,2,\dots,n$ in X such that $u(D \cap C_j) \subset I_j$ where I_j is open in Y and $(C_j \times I_j) \subset W$. Put $K = C \backslash \bigcup_{j=1}^n B_d(x_j,\epsilon_j)$, a possibly empty compact set. If K is empty, then

$$(D,u) \in \bigcap_{j=1}^n [C_j : I_j] \subset \big((X \times Y)\backslash F\big)^+.$$

Otherwise, as $K \cap D = \emptyset$,

$$(D,u) \in [K : \emptyset] \cap \bigcap_{j=1}^n [C_j : I_j] \subset \big((X \times Y)\backslash F\big)^+,$$

and this completes the proof. □

To end this section, we aim to show that $\mathcal{P}(\mathcal{B})$-convergence on the space of continuous partial maps with compact domains is topological provided the bornology has a closed base. This falls out of a more general result of interest about bornological convergence of nets of compact sets that was somehow overlooked in earlier studies [4,7,26], despite the fact the argument used in the proof of [3, Theorem 5.1.6] can be adapted to get this result. Clearly, the topology on $\mathrm{K}(X)$ we describe is a bornological modification of the classical Vietoris topology [3, p. 47]; in fact when $\mathcal{B} = \mathcal{P}_0(X)$, it *is* the Vietoris topology (see [3, Exercise 3.2.9]).

Theorem 4.5. *Let $\mathcal{B}$ be a bornology on a metric space (X, d) with a closed base. Then $\mathcal{B}$-convergence of nets of nonempty compact subsets is compatible with the topology τ on $\mathrm{K}(X)$ generated all sets of the form $\{K \in \mathrm{K}(X) : K \cap V \neq \emptyset\}$ where V is open in X plus all sets of the form $\{K \in \mathrm{K}(X) : K \cap F = \emptyset\}$ where $F \in \mathrm{CL}(X) \cap \mathcal{B}$.*

Proof. It is easy to show that $\mathcal{B}$-convergence of a net in $\mathrm{K}(X)$ ensures convergence in the topology τ, which we leave as an exercise to the reader. Conversely, suppose $\langle K_\gamma \rangle$ is a net in $\mathrm{K}(X)$ that is τ-convergent to $K \in \mathrm{K}(X)$. To check lower and upper bornological convergence, fix $B \in \mathcal{B} \cap \mathrm{CL}(X)$ and let $\epsilon > 0$. We first look at lower convergence $\mathcal{B}^-$.

If $K \cap B = \emptyset$, then $K \cap B \subset K^\epsilon$ for all γ. Otherwise by its compactness, we can find $\{x_1, x_2, \ldots, x_n\} \subset K \cap B$ with $K \cap B \subset \{x_1, x_2, \ldots, x_n\}^{\epsilon/2}$. By τ-convergence, eventually $\forall j \leq n$, $K_\gamma \cap B_d(x_j, \epsilon/2) \neq \emptyset$ and so $K \cap B \subset K_\gamma^\epsilon$ eventually.

For $\mathcal{B}^+$-convergence, if $B \cap (X \backslash K^\epsilon) = \emptyset$, then for all γ,

$$K_\gamma \cap B \subset B \subset K^\epsilon.$$

Otherwise $B_1 = B \cap (X \backslash K^\epsilon) \in \mathrm{CL}(X) \cap \mathcal{B}$ and $K \cap B_1 = \emptyset$. By τ-convergence, there exists γ_0 such that $\gamma \geq \gamma_0 \Rightarrow K_\gamma \cap B_1 = \emptyset$, and we compute for all such γ

$$\begin{aligned} K_\gamma \cap B &= (K_\gamma \cap B_1) \cup \left(K_\gamma \cap B \cap K^\epsilon\right) \\ &= \emptyset \cup \left(K_\gamma \cap B \cap K^\epsilon\right) \subset K^\epsilon, \end{aligned}$$

which establishes $\mathcal{B}^+$-convergence. □

Corollary 4.1. *Let $\mathcal{B}$ be a bornology with closed base on a metric space (X, d) and let (Y, ρ) be a second metric space. Put $\mathcal{C}_k[X, Y] := \{(D, u) \in \mathcal{C}[X, Y] : D$ is compact$\}$. Then $\mathcal{P}(\mathcal{B})$-convergence restricted to $\mathcal{C}_k[X, Y]$ is compatible with a topology on $\mathcal{C}_k[X, Y]$ generated by all sets of the form*

$$\left\{(D, u) \in \mathcal{C}_k[X, Y] : \mathrm{Gr}(u) \cap V \neq \emptyset\right\} \quad V \text{ open in } X \times Y,$$

plus all sets of the form

$$\left\{(D, u) \in \mathcal{C}_k[X, Y] : \mathrm{Gr}(u) \cap F = \emptyset\left\{F \in \mathrm{CL}(X \times Y) \cap \mathcal{B}^*.\right.\right.$$

Proof. The induced bornology $\mathcal{B}^*$ on $X \times Y$ also has a closed base, so that by Theorem 4.5, $\mathcal{B}^*$-convergence of nets of graphs of continuous partial functions that are defined on compact subsets of X can be described as above because each such function has compact graph. □

5. Uniformizability and metrizability of the space of partial maps

We shall assume throughout this section that the bornology $\mathcal{B}$ on X is stable under small enlargements. We will first show directly that $\mathcal{P}(\mathcal{B})$-convergence on $\mathcal{P}[X, Y]$ is compatible with a uniformizable topology that we denote by $\tau_{\mathcal{O}}(\mathcal{B})$.

Consider the following family of subsets of $\mathcal{P}[X,Y] \times \mathcal{P}[X,Y]$:

$$O(B,\epsilon) := \Big\{\big[(D,u),(C,v)\big] : \sup_{z\in C\cap B}\ \inf_{x\in B_d(z,\epsilon)\cap D} \rho\big(u(x),v(z)\big) < \epsilon \wedge \sup_{z\in D\cap B}\ \inf_{x\in B_d(z,\epsilon)\cap C} \rho\big(v(x),u(z)\big) < \epsilon\Big\}$$

where $B \in \mathcal{B}$ and $\epsilon > 0$ are arbitrary. If follows from the proof of Proposition 3.1 that $[(D, u), (C, v)] \in O(B, \epsilon) \Rightarrow C \cap B \subset D^{\epsilon}$ and $D \cap B \subset C^{\epsilon}$.

We want to prove that the family of all $O(B,\epsilon)$ is a base for a uniformity whenever $\mathcal{B}$ is stable under small enlargements.

Theorem 5.1. *Let $\mathcal{B}$ be a bornology on X stable under small enlargements. The family of all $O(B,\ \epsilon)$ where $B \in \mathcal{B}$ and $\epsilon > 0$ is a base for a uniformity on $\mathcal{P}[X,\ Y]$ and the topology $\tau_{\mathcal{O}}(\mathcal{B})$ determined by this uniformity is compatible with $\mathcal{P}(\mathcal{B})$-convergence.*

Proof. Trivially, any $O(B,\ \epsilon)$ for $B \in \mathcal{B}$, $\epsilon > 0$ is symmetric and contains the diagonal of $\mathcal{P}[X,\ Y] \times \mathcal{P}[X,\ Y]$. Moreover, for all $B_1, B_2 \in \mathcal{B}$ and $\epsilon_1, \epsilon_2 > 0$ there exist $B_3 \in \mathcal{B}$ and $\epsilon_3 > 0$ such that $O(B_3,\epsilon_3) \subset O(B_1,\epsilon_1) \cap O(B_2,\epsilon_2)$: take $B_3 = B_1 \cup B_2$, $\epsilon_3 \le \min\{\epsilon_1,\epsilon_2\}$.

Now, we claim for all $B \in \mathcal{B}$, $\epsilon > 0$ there exists $B_1 \in \mathcal{B}$ and $\epsilon_1 > 0$ such that $O(B_1,\epsilon_1) \circ O(B_1,\epsilon_1) \subset O(B,\ \epsilon)$. Since $\mathcal{B}$ is stable for small enlargements, we can find $\epsilon_1 < \epsilon/2$ such that $B_1 := B^{\epsilon_1} \in \mathcal{B}$. Let now $[(D,\ f), (C,\ h)] \in O(B_1,\epsilon_1)$ and $[(C,\ h), (L,\ g)] \in O(B_1,\epsilon_1)$. We want to prove that $[(D,\ f), (L,\ g)] \in O(B,\ \epsilon)$.

First, we claim that

$$\forall z \in L \cap B,\ \exists x \in B_d(z,\epsilon) : \quad \rho\big(g(z), f(x)\big) < 2\epsilon_1.$$

Let $z \in L \cap B$. Then there is $y \in C \cap B_d(z,\epsilon_1)$ such that $\rho(h(y), g(z)) < \epsilon_1$. Since $y \in C \cap B^{\epsilon_1}$ there is $x \in D \cap B_d(y,\epsilon_1) \subset B_d(z,\epsilon)$ such that $\rho(h(y), f(x)) < \epsilon_1$. Thus $\rho(g(z), f(x)) < 2\epsilon_1$.

The claim established and with $2\epsilon_1 < \epsilon$ in mind, it follows that

$$\sup_{z\in L\cap B}\ \inf_{x\in B_d(z,\epsilon)\cap D} \rho\big(f(x), g(z)\big) < \epsilon.$$

The other inequality is handled in the same way. Compatibility with our convergence for partial functions is obvious. □

In view of Proposition 3.4 the trace of $\tau_{\mathcal{O}}(\mathcal{B})$ on $\mathcal{C}[X,\ Y]$ is Hausdorff, while Example 3.3 shows that the induced topology on the entire space of partial functions need not be.

If we restrict our attention to $(\mathrm{CL}(X), \tau_{\mathcal{O}}(\mathcal{B}))$ by as usual considering only functions that assume the same constant value, the base of the uniformity on $\mathrm{CL}(X)$ we get is given by the following family of subsets of $\mathrm{CL}(X) \times \mathrm{CL}(X)$:

$$O(B,\ \delta) = \big\{(D, C) :\ \ D \cap B \subset C^{\delta},\ C \cap B \subset D^{\delta}\big\{$$

where $B \in \mathcal{B}$, $\delta > 0$, as identified in [7]. From this perspective, uniformizability of $\mathcal{P}(B)$-convergence on $\mathcal{P}[X,\ Y]$ also falls out of the uniformizability of $\mathcal{B}^*$-convergence on $\mathcal{P}_0(X \times Y)$, as $\mathcal{B}^*$ is stable under small enlargements if $\mathcal{B}$ is (the general uniformizability result for bornological convergence of nets of arbitrary subsets with respect to a bornology that is stable under small enlargements is implicit in [26] and is explicit as [7, Theorem 3.1]).

Since the topology $\tau_{\mathcal{O}}(\mathcal{B})$ is uniformizable, we are looking for necessary and sufficient conditions for the metrizability in the spirit of Corollary 3.9 in [7] that, for the convenience of the reader, we now state as a proposition.

Proposition 5.1. *(See [7].) Let $\mathcal{B}$ be a bornology on (X,d) that is stable under small enlargements. The following are equivalent:*

(1) *$\mathcal{B}$ has a countable base;*
(2) *$(\mathrm{CL}(X), \tau_{\mathcal{O}}(\mathcal{B}))$ is metrizable.*

Theorem 5.2. *Let $\mathcal{B}$ be a bornology on (X,d) that is stable under small enlargements. The following are equivalent:*

(1) *$\mathcal{B}$ has a countable base;*
(2) *$(\mathcal{P}[X,Y], \tau_{\mathcal{O}}(\mathcal{B}))$ is pseudo-metrizable;*
(3) *$(\mathcal{C}[X,Y], \tau_{\mathcal{O}}(\mathcal{B}))$ is metrizable.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{B}_0 = \{B_k : k \in \mathbb{N}\}$ be a countable base for $\mathcal{B}$ such that $B_k \subset B_{k+1}$ for every $k \in \mathbb{N}$. Clearly, $\{O(B_k, 1/n) : (k, n) \in \mathbb{N} \times \mathbb{N}\}$ is a countable family of entourages that forms a base for the defining uniformity, which gives pseudo-metrizability of $(\mathcal{P}(X, Y), \tau_{\mathcal{O}}(\mathcal{B}))$. (2) $\Rightarrow$ (3) As the defining uniformity is separated, we get metrizability. (3) $\Rightarrow$ (1) It follows from Proposition 5.1. □

Corollary 5.1. *(See [21].) The following are equivalent:*

(1) *X is locally compact and second countable;*
(2) *$(\mathcal{C}[X,Y], \tau_B)$ is metrizable;*
(3) *$(\mathcal{C}[X,Y], \tau_{\mathcal{O}}(\mathcal{K}))$ is metrizable;*
(4) *X is a hemicompact space.*

Corollary 5.2. *For every metric space (X, d), both $(\mathcal{C}[X, Y], \tau_{\mathcal{O}}(\mathcal{B}_d))$ and $(\mathcal{C}[X, Y], \tau_{\mathcal{O}}(\mathcal{P}_0(X)))$ are metrizable.*

The *proximal topology*, introduced in [6] and later given considerable attention in [3], is generated by all sets of the form V^- where V runs over the open subsets of X plus all sets of form $(X\backslash B)^{++}$ where $B \in \mathrm{CL}(X)$. We recall that the *bounded* (resp. *totally bounded*) *proximal topology* on $\mathrm{CL}(X)$ is generated by all sets of form V^- where V runs over the open subsets of X plus all sets of form $(X\backslash B)^{++}$ where B runs over $\mathcal{B}_d \cap \mathrm{CL}(X)$ (resp $\mathcal{B}_{tb} \cap \mathrm{CL}(X)$).

In [16], there is given the following definition which characterizes second countability of the totally bounded proximal topology σ_{TB}.

Definition 5.1. (See [16].) A metric space (X, d) is called *weakly globally hemitotally bounded* if there exists a sequence $\{B_n : n \in \mathbb{N}\}$ of totally bounded closed sets such that whenever V is open and B is totally bounded with $B \in V^{++}$, there exists a B_n with $B \subset B_n$ and $B_n \in V^{++}$.

Corollary 5.3. *(See [16].) For a metric space (X,d), and $\mathcal{B} = \mathcal{B}_{tb}$ the following are equivalent:*

(1) *X is weakly globally hemitotally bounded;*
(2) *$(\mathcal{C}[X,Y], \tau_{\mathcal{O}}(\mathcal{B}_{tb}))$ is metrizable.*

Proof. (2) $\Rightarrow$ (1) follows from Proposition 4.13 in [16]. □

Corollary 5.4. *(See [10,17].) For a metric space (X,d), the following are equivalent:*

(1) *each bounded set is totally bounded;*
(2) *$(\mathcal{C}[X,Y], \tau_{\mathcal{O}}(\mathcal{B}_d))$ (bounded proximal topology) is metrizable.*

Proof. (1) ⇒ (2) follows from Theorem 5.2 and Corollary 5.3.

(2) ⇒ (1) follows from Theorem 4.4 in [17]. □

Corollary 5.5. *(See [10,17].) For a metric space (X, d), the following are equivalent:*

(1) *X is totally bounded;*
(2) *$(\mathcal{C}[X, Y], \tau_{\mathcal{O}}(\mathcal{P}_0(X)))$ (proximal topology) is metrizable.*

Proof. (1) ⇒ (2) follows from Theorem 5.2 and Corollary 5.3.

(2) ⇒ (1) follows from Theorem 4.3 in [17]. □

References

[1] H. Attouch, R. Lucchetti, R. Wets, The topology of the ρ-Hausdorff distance, Ann. Mat. Pura Appl. 160 (1991) 303–320.
[2] K. Back, Concepts of similarity for utility functions, J. Math. Econom. 15 (1986) 129–142.
[3] G. Beer, Topologies on Closed and Closed Convex Sets, Kluwer Academic Press, 1993.
[4] G. Beer, C. Costantini, S. Levi, Bornological convergences and shields, Mediterr. J. Math. 10 (2013) 529–560.
[5] G. Beer, A. Di Concilio, Uniform continuity on bounded sets and the Attouch–Wets topology, Proc. Amer. Math. Soc. 112 (1991) 235–243.
[6] G. Beer, A. Lechicki, S. Levi, S. Naimpally, Distance functionals and suprema of hyperspace topologies, Ann. Mat. Pura Appl. 162 (1992) 367–381.
[7] G. Beer, S. Levi, Pseudometrizable bornological convergence is Attouch–Wets convergence, J. Convex Anal. 15 (2008) 439–453.
[8] G. Beer, S. Levi, Strong uniform continuity, J. Math. Anal. Appl. 350 (2009) 568–589.
[9] G. Beer, S. Levi, Uniform continuity, uniform convergence and shields, Set-Valued Var. Anal. 18 (2010) 251–275.
[10] G. Beer, R. Lucchetti, Well-posed optimization problems and a new topology for the closed subsets of a metric space, Rocky Mountain J. Math. 23 (1993) 1197–1220.
[11] G. Beer, S. Naimpally, J. Rodriguez-Lopez, $\mathcal{S}$-topologies and bounded convergences, J. Math. Anal. Appl. 339 (2008) 542–552.
[12] P. Brandi, R. Ceppitelli, A new graph topology connected with compact-open topology, Appl. Anal. 53 (1994) 185–196.
[13] P. Brandi, R. Ceppitelli, L. Holá, Topological properties of a new graph topology, J. Convex Anal. 6 (1999) 29–40.
[14] P. Brandi, R. Ceppitelli, L. Holá, Boundedly UC spaces and topologies on functions spaces, Set-Valued Anal. 16 (2008) 357–373.
[15] G. Debreu, The Theory of Value: An Axiomatic Analysis of Economic Equilibrium, Yale University Press, New Haven, London, 1959.
[16] G. Di Maio, L. Holá, A hypertopology determined by the family of bounded sets is the infimum of the upper Wijsman topologies, Questions Answers Gen. Topology 15 (1997) 51–66.
[17] G. Di Maio, L. Holá, E. Meccariello, Notes on hit and miss topologies, Rostock. Math. Kolloq. 52 (1998) 19–32.
[18] J. Fell, A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space, Proc. Amer. Math. Soc. 13 (1962) 472–476.
[19] H. Hogbe-Nlend, Bornologies and Functional Analysis, North-Holland, Amsterdam, 1977.
[20] L. Holá, Uniformizability of the generalized compact-open topology, Tatra Mt. Math. Publ. 14 (1998) 219–224.
[21] L. Holá, Complete metrizability of generalized compact-open topology, Topology Appl. 91 (1999) 159–167.
[22] S.T. Hu, Boundedness in a topological space, J. Math. Pures Appl. 228 (1949) 287–320.
[23] J. Kelley, General Topology, Van Nostrand, 1955.
[24] K. Kuratowski, Sur l'espace des fonctions partielles, Ann. Mat. Pura Appl. 40 (1955) 61–67.
[25] H.J. Langen, Convergence of dynamic programming models, Math. Oper. Res. 6 (1981) 493–512.
[26] A. Lechicki, S. Levi, A. Spakowski, Bornological convergences, J. Math. Anal. Appl. 297 (2007) 751–770.
[27] V. Levin, A continuous utility theorem for closed preorders on a σ-compact metrizable space, Soviet Math. Dokl. 28 (1983) 715–718.
[28] R. Lucchetti, Convexity and Well-Posed Problems, CMS Books Math., Springer, ISBN 0-387-28719-1, 2006.
[29] T. Orenshtein, B. Tsaban, Pointwise convergence of partial functions: the Gerlitz–Nagy problem, Adv. Math. 232 (2013) 311–326.
[30] W. Whitt, Continuity of Markov processes and dynamic programs, report, Yale University, New Haven, CT, 1975.