SHAPE OPTIMIZATION FOR STOKES FLOW: A REFERENCE DOMAIN APPROACH

Ivan Fumagalli, Nicola Parolini and Marco Verani[1]

Introduction

Optimal control for partial differential equations [24] is a challenging field of applied mathematics, thanks to its combination of sophisticated theoretical tools and interesting engineering applications. Among optimal control problems, shape optimization [10, 30] has recently undergone a renewal of interest, mainly due to the wide range of industrial and real world applications, like fluid dynamics [17] and structural mechanics [1], and to the increased computational power avail-able for numerical simulations. Shape optimization aims at finding the solution of problems of the following general form:

$$\min_{\Omega \in \mathcal{O}} J(\Omega, S(\Omega)), \quad \text{subject to a differential problem } L(S(\Omega)) = 0 \text{ in } \Omega,$$

where J is a cost functional, defined on a suitable set $\mathcal{O}$ of admissible domains, L is a differential operator and S is the operator mapping an admissible domain $\Omega \in \mathcal{O}$ to the corresponding solution of the differential problem $L(S(\Omega)) = 0$ in Ω.

This kind of problems has been widely discussed in the literature, employing different techniques in the description of the set $\mathcal{O}$, generally considered as a proper subset of finite (see, e.g., [3, 5]) or infinite (see, e.g., [10, 30]) dimensional spaces. The present paper belongs to the latter category, as the boundary of the admissible domains (or a subset of it) is described by the graph of a suitable control function. This approach has been widely adopted by many authors (see, e.g., [2, 4, 13, 18, 20–23]).

Concerning the numerical solution of shape optimization problems, a standard technique is represented by gradient type iterative algorithm, in which the state problem is solved on differently shaped domains at each iteration (see, e.g., [1, 11]). A critical point of this approach is the repeated

[1] MOX - Modellistica e Calcolo Scientifico, Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, via Bonardi 9, 20133 Milano, Italy, e-mail: `ivan1.fumagalli@mail.polimi.it`, `nicola.parolini@polimi.it`, `marco.verani@polimi.it`

deformation of the computational mesh, leading to an increase of the computational effort and to the possible generation of highly skewed mesh elements. In order to avoid such problems, in this work the reference-domain approach introduced in [22] is followed, mapping the actual domain and the whole optimization problem onto a reference domain Ω_0. Exploiting this mapping, a priori estimates for the discretization error of the optimization problem are derived, and these results are assessed through numerical tests. Discretization of shape optimization problems and convergence issues have been discussed in other works, such as [8,9,20,21]. However, to the best of our knowledge, only [14,22] provide a convergence rate for the discretization error for the Poisson equation. In this paper we obtain similar convergence results for the Stokes problem; this seems to be the first convergence result for shape optimization problems governed by this class of equations.

The present paper is organized as follows. In Section 1, we present the shape optimization problem governed by Stokes equations, and we reformulate it on the reference domain. Within this framework, the existence of an optimal solution to the minimization problem is proved. Finally, we consider first order optimality conditions and we provide a boundary-integral expression for them. Section 2 is devoted to the proof of a priori error estimates for the numerical discretization error of the optimization problem. Finally, in Section 3 we present some numerical tests, assessing the theoretical results. In Appendix A, we discuss the regularity assumptions needed by the a priori estimates, whereas in Appendix B some technical results are proved.

1. The optimal control problem

The aim of the present paper is to study a shape optimization problem governed by Stokes equations, which reads as follows

$$\min_{q \in Q^{ad}} J(q, \mathbf{u}, p) \text{ subject to the following generalized Stokes system:}$$

$$\begin{cases}
\eta \mathbf{u} - \operatorname{div}(\nu \nabla \mathbf{u}) + \nabla p = \mathbf{f}, & \text{in } \Omega_q, \\
\operatorname{div} \mathbf{u} = 0, & \text{in } \Omega_q, \\
\mathbf{u} = 0, & \text{on } \Gamma_q, \\
\nu \partial_{\mathbf{n}} \mathbf{u} - p\mathbf{n} = \mathbf{g}_N, & \text{on } \Gamma_1, \\
\partial_{\mathbf{n}} u_x = 0, \quad u_y = 0, & \text{on } \Gamma_2, \\
\mathbf{u} = \mathbf{g}_D, & \text{on } \Gamma_3,
\end{cases} \tag{1}$$

where J is a given cost functional to be optimized, $\mathbf{u} = (u_x, u_y)$ and p are the so-called state variables and q is the control function (belonging to the admissible set Q^{ad}) that identifies the domain Ω_q.

In particular, the control function $q : I = (0, 1) \to \mathbb{R}$ describes the lower part Γ_q of the boundary of domain $\Omega_q = \{(x, y) \in \mathbb{R}^2 \mid x \in I , \ y \in (q(x), 1)\}$. As shown in Fig. 1(left), the boundary of Ω_q is partitioned as $\partial\Omega_q = \Gamma_q \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

In order to avoid domain degeneration, we fix $\varepsilon \in (0, 1)$ a priori, and we introduce the following
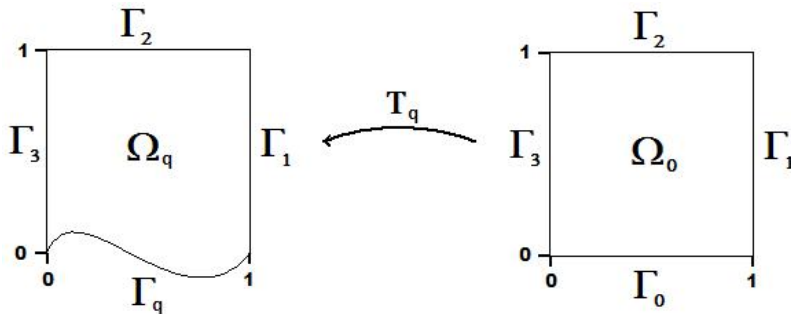

Figure 1. Physical and reference domains

intermediate set of admissible controls

$$\overline{Q}^{ad} = \{q \in H^3(I) \cap H_0^1(I) \colon q(x) \leqslant 1 - \varepsilon, \ \forall x \in I\}. \tag{2}$$

In the following, it will be useful to have the admissible controls in a bounded set, so we fix a constant $C > 0$ and reduce $\overline{Q}^{ad}$ to the following set:

$$Q^{ad} = \{q \in \overline{Q}^{ad} \colon \|q\|_{H^3(I)} \leqslant C\}. \tag{3}$$

From the above definition, it follows that all the feasible domains Ω_q are contained in a bounded, convex, hold-all domain $\widehat{\Omega} \subset \mathbb{R}^2$.

The weak formulation of problem (1) reads:

Find $\mathbf{u} = \widehat{\mathbf{u}} + \widetilde{\mathcal{R}}\mathbf{g}_D$, $\widehat{\mathbf{u}} \in V_q$ *and* $p \in P_q$ *such that*

$$\begin{cases} a_q(\widehat{\mathbf{u}}, \mathbf{v}) + b_q(\mathbf{v}, p) = F_q(\mathbf{v}), & \forall \ \mathbf{v} \in V_q, \\ \qquad\qquad b_q(\widehat{\mathbf{u}}, \pi) = -b_q(\widetilde{\mathcal{R}}\mathbf{g}_D, \pi), & \forall \ \pi \in P_q, \end{cases} \tag{4}$$

where

$$V_q = \{\mathbf{v} \in [H^1(\Omega_q)]^2 \colon \mathbf{v} = (v_x, v_y) = \mathbf{0} \ on \ \Gamma_3 \cup \Gamma_q \ and \ v_y = 0 \ on \ \Gamma_2\},$$
$$P_q = L^2(\Omega_q),$$

and

$$a_q(\mathbf{u}, \mathbf{v}) = \int_{\Omega_q} \eta \mathbf{u}\mathbf{v} + \nu \nabla \mathbf{u} : \nabla \mathbf{v},$$

$$b_q(\mathbf{v}, \pi) = -\int_{\Omega_q} \pi \ \mathrm{div} \ \mathbf{v},$$

$$F_q(\mathbf{v}) = \int_{\Omega_q} \mathbf{f} \cdot \mathbf{v} - a_q(\widetilde{\mathcal{R}}\mathbf{g}_D, \mathbf{v}) + \int_{\Gamma_1} \mathbf{g}_N \cdot \mathbf{v} \, d\Gamma.$$

Data functions $\eta, \nu, \mathbf{f}$ are defined on the hold-all domain $\widehat{\Omega}$,[1] boundary data $\mathbf{g}_N, \mathbf{g}_D$ are defined on the fixed edges Γ_1, Γ_3, respectively, and $\widetilde{\mathcal{R}}\mathbf{g}_D$ is a continuous lifting of $\mathbf{g}_D$ on Ω_q.

Remark 1.1 (Well-posedness of the state problem)**.** Using classical results on Stokes problem (see, e.g., [16]), we can ensure the well-posedness of (1). About data functions, we have to assume what follows: [2]

- $\nu(\mathbf{x}) \geqslant \nu_0 > 0 \ \ \forall \mathbf{x} \in \widehat{\Omega}$,
- $\nu, \eta \in L^\infty(\widehat{\Omega})$,
- $\mathbf{f} \in [H^{-1}(\widehat{\Omega})]^2$, $\mathbf{g}_D \in [H^{1/2}(\Gamma_3)]^2$, $\mathbf{g}_N \in [H^{-1/2}(\Gamma_1)]^2$.

Under these conditions, the following stability estimate holds:

$$\|\widetilde{\mathbf{u}}\|_{V_q} + \|\widetilde{p}\|_{P_q} \leqslant c(\|\mathbf{f}\|_{[H^{-1}(\widehat{\Omega})]^2} + \|\mathbf{g}_D\|_{[H^{1/2}(\Gamma_3)]^2} + \|\mathbf{g}_N\|_{[H^{-1/2}(\Gamma_3)]^2}). \tag{5}$$

We remark that constant c in (5) is independent of q, since the inf-sup constant of the form b_q is lower-bounded, for any q, by the inf-sup constant related to the hold-all domain $\widehat{\Omega}$. Moreover, since the right-hand sides of (5) can be bounded by a data independent constant, also $\|\widetilde{\mathbf{u}}\|_{V_q}, \|\widetilde{p}\|_{P_q}$ are bounded, uniformly on q.

---

[1] If not necessary, no special notation will be used to point out whether the entire functions are to be considered, or their restrictions to Ω_q: the distinction will be inferable from the context.

[2] If a particular q is fixed, the conditions need only to be respected on Ω_q. However, in order to be free from dependence on the control, we formulate them on the hold-all domain $\widehat{\Omega}$.

Finally, we introduce the cost functional

$$J(q, \mathbf{u}, p) = \int_{\Omega_q} |\nabla \mathbf{u}|^2 + \alpha \|q''\|_{L^2(I)}^2 + \beta \left(\int_I q(x)dx - \overline{V} \right)^2,$$

representing the total energy dissipation of the Stokes flow, with a regularization term $\|q''\|_{L^2(I)}^2$ (as in [22]) and a volume penalty term, measuring the distance of the area under the graph of q from a fixed value $\overline{V}$. [3]

Let us introduce the state solution operator $\widetilde{S}(q)$, mapping each $q \in Q^{ad}$ to the corresponding solution $\widetilde{S}(q) = (\mathbf{u}, p)$ of (4), and the reduced cost functional, as follows:

$$\widetilde{j} : Q^{ad} \to \mathbb{R}, \quad \widetilde{j}(q) = J(q, \widetilde{S}(q)). \tag{6}$$

For convenience, it can be useful to define the following constants, whose existence is ensured by the fact that q belongs to $\hat{Q}^{ad}$:

$$d_1, d_2 > 0 \text{ such that } \|q''\|_{L^\infty(I)} \leqslant d_1, \ |q'(0)| \leqslant d_2.$$

Finally, we introduce the set of admissible control variations, namely:

$$\delta Q = \{\delta q \in H^3(I) \cap H_0^1(I) \colon q + \delta q \in Q^{ad}, \ \forall q \in Q^{ad}\}.$$

Remark 1.2. We point out that Q^{ad} is convex, closed and bounded in $H^3(I)$: boundedness is stated in (3), whereas closure and convexity are consequences of the fact that definitions (2) and (3) involve only constraints of the form $\zeta(q) \leqslant c$, where c is a constant and ζ is a semi-norm in $H^3(I)$. Hence, closure follows from the continuity of any semi-norm in a Banach space, and convexity holds thanks to the triangle inequality.

1.1. **Domain transformation**

In this section, we map the original problem (1) onto a reference-domain. The main advantage of this technique lays in the numerical solution of the optimization problem: solving the state problem on a reference domain avoids the need to deform the computational mesh at each step of the optimization algorithm.

Let us introduce the reference domain $\Omega_0 = (0,1)^2$, which is equivalent to the choice $q \equiv 0$. It follows that any admissible domain Ω_q can be seen as a transformation of Ω_0 by means of the map

$$T_q : \Omega_0 \to \Omega_q, \quad \text{with} \ \ T_q(x, y) = (\mathbb{1} + V_q)(x, y) = \begin{pmatrix} x \\ y + (1-y)q(x) \end{pmatrix}.$$

We denote by $(\cdot, \cdot)$ the L^2 inner product on Ω_0, while $(\cdot, \cdot)_I$ and $(\cdot, \cdot)_{\Omega_q}$ indicate the scalar product in $L^2(I)$ and $L^2(\Omega_q)$, respectively.

Remark 1.3 (Notation I)**.** We will use the following quantities depending on T_q:
  Map gradient: DT_q with $(DT_q)_{i,j} = \partial_{x_j}(T_q)_i \quad\quad i, j = 1, 2.$
  Map jacobian: $\gamma_q = det(DT_q)$.
  Laplacian-related matrix: $A_q = \gamma_q DT_q^{-1} DT_q^{-T}$.

Remark 1.4 (Notation II)**.** By the superscript $\cdot^q$ we denote the composition with the map T_q. On the other hand, whenever no doubt arises on which q is considered, the composition with the inverse map T_q^{-1} will be denoted by $\widetilde{\cdot}$.

We are now ready to state the variational problem (4) on pulled-back spaces V and P, that do not depend anymore on q:

---

[3]Volume constraints are typical of shape optimization for fluid dynamics: see, e.g., [26, 28].

Find $(\mathbf{u}, p) \in V \times P$, *such that*

$$\begin{cases} a(q)(\mathbf{u}, \mathbf{v}) + b(q)(\mathbf{v}, p) = F(q)(\mathbf{v}), & \forall \ \mathbf{v} \in V, \\ \qquad \qquad b(q)(\mathbf{u}, \pi) = G(q)(\pi), & \forall \ \pi \in P, \end{cases} \tag{7}$$

where

$$V = \{\mathbf{v} \in [H^1(\Omega_0)]^2 : \mathbf{v} = (v_x, v_y) = \mathbf{0} \ on \ \Gamma_3 \cup \Gamma_0 \ and \ v_y = 0 \ on \ \Gamma_2\},$$
$$P = L^2(\Omega_0),$$

and

$$a(q)(\mathbf{u}, \mathbf{v}) = \int_{\Omega_0} \left[\eta^q \mathbf{u} \cdot \mathbf{v} \gamma_q + \nu^q \ tr(\nabla \mathbf{u} A_q \nabla \mathbf{v}^T) \right] d\Omega,$$

$$b(q)(\mathbf{v}, \pi) = -\int_{\Omega_0} \pi \ tr(\nabla \mathbf{v} D T_q^{-1}) \gamma_q \ d\Omega,$$

$$F(q)(\mathbf{v}) = \int_{\Omega_0} \mathbf{f}^q \cdot \mathbf{v} \gamma_q \ d\Omega - a(q)(\mathcal{R}\mathbf{g}_D, \mathbf{v}) + \int_{\Gamma_1} \mathbf{g}_N \cdot \mathbf{v} \ d\Gamma,$$

$$G(q)(\pi) = -b(q)(\mathcal{R}\mathbf{g}_D, \mathbf{v}).$$

Remark 1.5 (Lifting). $\mathcal{R}\mathbf{g}_D$ represents a continuous lifting of the Dirichlet datum $\mathbf{g}_D$ onto Ω_0. However, as $\mathbf{g}_D$ is defined on Γ_3, where T_q is equal to the identity, it does not need to be mapped onto the reference domain. In general, $\mathcal{R}\mathbf{g}_D \neq \widetilde{\mathcal{R}}\mathbf{g}_D \circ T_q$, but this is not a problem, since in the following we are not making use of any explicit expression of the lifting.

Finally, we introduce the solution operator $S : Q^{ad} \to V \times P$, which maps an admissible control function to the solution of the transformed state problem (7). It follows that the original optimization problem can be reformulated as follows:
  Find $\bar{q} \in Q^{ad}$ *minimizing the functional* j *defined in* (8), *i.e.*

$$j(\bar{q}) = \min_{q \in Q^{ad}} j(q) = \min_{q \in Q^{ad}} J(q, S(q) \circ T_q^{-1}). \tag{8}$$

This is the formulation we will refer to on the rest of the paper.

1.2. **Well-posedness of the problem**

In this section, we analyze the well-posedness of the state problem (7) and the existence of an optimal solution to our minimization problem (8).

At first, we observe that matrix A_q belongs to $[L^\infty(\Omega_0)]^{2 \times 2}$, it is symmetric and positive definite, and its eigenvalues are lower-bounded by

$$\overline{\lambda} = 2 \left(1 + \frac{1 + (d_1 + d_2)^2}{\varepsilon} + \sqrt{\left(1 + \frac{1 + (d_1 + d_2)^2}{\varepsilon} \right)^2 - 4} \right)^{-1} > 0.$$

Under the same assumptions of Remark 1.1, the coercivity of the form $a(q)$ and the continuity of the functionals and forms involved in (7) are given by the following inequalities, holding for any

$\mathbf{u}, \mathbf{v} \in H_0^1(\Omega_0)$, $\pi \in L^2(\Omega_0)$, $q \in Q^{ad}$:

$$a(q)(\mathbf{v}, \mathbf{v}) \geqslant \nu_0 \overline{\lambda} \|\nabla \mathbf{v}\|^2 =: \alpha_c \|\nabla \mathbf{v}\|^2,$$

$$|a(q)(\mathbf{u}, \mathbf{v})| \leqslant (\|\eta\|_{L^\infty(\widehat{\Omega})} \|\gamma_q\|_\infty + \|\nu\|_{L^\infty(\widehat{\Omega})} \|A_q\|_\infty) \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \leqslant$$

$$\leqslant \left(\|\eta\|_{L^\infty(\widehat{\Omega})} (1 + d_1 + d_2) + \|\nu\|_{L^\infty(\widehat{\Omega})} \frac{1}{\lambda} \right) \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| =: M \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\|,$$

$$|b(q)(\mathbf{v}, \pi)| \leqslant \|\gamma_q DT_q^{-T}\|_\infty \|\nabla \mathbf{v}\| \|\pi\| \leqslant (1 + d_1 + d_2) \|\nabla \mathbf{v}\| \|\pi\| =: M_b \|\nabla \mathbf{v}\| \|\pi\|,$$

$$|F(q)(\mathbf{v})| \leqslant \|\gamma_q\|_\infty \|\mathbf{f}\|_{[L^2(\widehat{\Omega})]^2} \|\mathbf{v}\| + M c_{\mathcal{R}} \|\mathbf{g}_D\|_{H^{1/2}(\Gamma_3)} \|\nabla \mathbf{v}\| + \|\mathbf{g}_N\|_{[H^{-1/2}(\Gamma_1)]^2} c_{tr} \|\nabla \mathbf{v}\| \leqslant$$

$$\leqslant [c_{\widehat{\Omega}}(1 + d_1 + d_2) \|\mathbf{f}\|_{[L^2(\widehat{\Omega})]^2} + M c_{\mathcal{R}} \|\mathbf{g}_D\|_{H^{1/2}(\Gamma_3)} + \|\mathbf{g}_N\|_{[H^{-1/2}(\Gamma_1)]^2} c_{tr}] \|\nabla \mathbf{v}\| =$$

$$=: M_F \|\nabla \mathbf{v}\|,$$

where the constants $\alpha_c, M, M_b, M_F, M_G$ are independent of q.

The inf-sup condition for problem (7) reads
There exists a positive constant $\overline{\beta}$, independent of q, such that

$$\forall \pi \in P \ \exists \mathbf{v} \in V \ : \ b(q)(\mathbf{v}, \pi) \geqslant \overline{\beta} \|\nabla \mathbf{v}\| \|\pi\|. \tag{10}$$

The validity of this property would allow to exploit the classical saddle-point-problem theory also for the transformed problem (7).

To prove (10), we start considering the inf-sup condition on Ω_0, with constant $\widehat{\beta} > 0$, namely:

$$\forall \pi \in P \ \exists \mathbf{v} \in V \text{ such that } b(0)(\mathbf{v}, \pi) = -\int_{\Omega_0} \pi \operatorname{div} \mathbf{v} \, d\Omega \geqslant \widehat{\beta} \|\pi\| \|\nabla \mathbf{v}\|. \tag{11}$$

Employing the definition of $b(q)$, the following holds for any $q \in Q^{ad}$

$$b(q)(\mathbf{v}, \pi) = -\int_{\Omega_0} \pi \nabla \mathbf{v} \cdot \gamma_q DT_q^{-T} d\Omega = -\int_{\Omega_0} \pi \nabla \mathbf{v} \cdot \left(\mathbb{1} + \gamma_q DT_q^{-T} - \mathbb{1} \right) d\Omega \geqslant$$

$$\geqslant -\int_{\Omega_0} \pi \operatorname{div} \mathbf{v} \, d\Omega - \left| \int_{\Omega_0} \pi \nabla \mathbf{v} \cdot \left(\gamma_q DT_q^{-T} - \mathbb{1} \right) d\Omega \right| \geqslant (\widehat{\beta} - \|\gamma_q DT_q^{-T} - \mathbb{1}\|_\infty) \|\pi\| \|\nabla \mathbf{v}\|,$$

being $\pi \in P$ and $\mathbf{v} \in V$ related through (10). As it holds

$$\gamma_q DT_q^{-T} - \mathbb{1} = \operatorname{cof}(DT_q) - \mathbb{1} = \begin{pmatrix} -q & -(1-y)q' \\ 0 & 0 \end{pmatrix},$$

we get

$$b(q)(\mathbf{v}, \pi) \geqslant (\widehat{\beta} - \|q\|_{W^{1,\infty}(I)}) \|\pi\| \|\nabla \mathbf{v}\|.$$

Therefore, requiring $\|q\|_{W^{1,\infty}(\overline{I})}$ to be strictly smaller than $\widehat{\beta}$ yields the validity of the inf-sup (10), uniformly on q.

It is easy to check (see, e.g., [12]) that on the domain Ω_0 the inf-sup constant $\widehat{\beta}$ in (11) satisfies $\widehat{\beta} \geqslant \frac{1}{4\sqrt{2}}$. Hence, in order to ensure the validity of (10) it is sufficient to require

$$\|q\|_{H^3(I)} \leqslant \frac{\xi}{4\sqrt{2}}, \quad \text{for some } \xi \in (0, 1), \tag{12}$$

in the definition of the set Q^{ad} of admissible controls.

Remark 1.6. We remark that condition (12) is representative of a class of sufficient conditions ensuring the validity of (10). Most likely, less stringent conditions can be found. However, real world shape optimization problems often deal with very smooth configurations, thus compatible with (12).

Bearing in mind the properties showed at the beginning of this section, we can finally employ the classical results of saddle-point theory to prove the following result (see, e.g., [7]):

Proposition 1.7. *Under condition* (12)*, for each* $q \in Q^{ad}$ *the pulled-back problem* (7) *admits unique solution, and the following inequality holds*

$$\|S(q)\|_{V \times P} \leqslant c(\mathbf{f}, \mathbf{g}_D, \mathbf{g}_N, \eta, \nu, \widehat{\Omega}),$$

where the constant c *is independent of* q.

Concluding this section, we prove the existence of an optimal solution to (8).

Theorem 1.8. *Let* Q^{ad} *be a non-empty, convex, closed and bounded subset of* $H^3(I)$ *and let* $S : Q^{ad} \to [H^1(\Omega_0)]^2 \times L^2(\Omega_0)$ *be the solution operator of problem* (7)*. Then, there exists a solution to the minimization problem* (8).

Proof. The proof follows standard ideas of calculus of variations. Hence, in the following we sketch the main steps of the proof. From Remark 1.2, we know that Q^{ad} is a closed, bounded and convex subset of $H^3(I)$. This set is also non-empty, since $q \equiv 0$ fulfills all its constraints.
Observing that $j(q) \geqslant 0$ for any $q \in Q^{ad}$ and that $Q^{ad} \neq \varnothing$, we have that a minimizing sequence $\{q_n\}_{n \in \mathbb{N}} \subset Q^{ad}$ exists, such that

$$\lim_{n \in \mathbb{N}} j(q_n) = \inf_{q \in Q^{ad}} j(q) =: \overline{j}.$$

Being Q^{ad} bounded in $H^3(I)$, the sequence $\{q_n\}$ is bounded itself, then there exists a subsequence $\{q_{n_k}\}$ and some $\overline{q} \in H^3(I)$ such that,

$$q_{n_k} \rightharpoonup \overline{q} \quad \text{in } H^3(I) \qquad \text{for } k \to \infty.$$

Being Q^{ad} closed and convex, the limit $\overline{q}$ belongs to Q^{ad}.
The next step to take is to show that we can take the limit also in the state variables sequence $\{S(q_{n_k})\} = \{(\mathbf{u}_k, p_k)\}$. For this purpose, following some ideas of the proof of Theorem 2.1 [18], we consider the physical counterpart of the sequence, $\{\widetilde{S}(q_{n_k})\} = \{S(q_{n_k}) \circ T_q^{-1}\}$, and the trivial extension to zero of its elements in $\widehat{\Omega} \supset \Omega_q$, denoted by $\{\widehat{S}(q_{n_k})\}$.
Thanks to the well-posedness of problem (1), uniformly on $q \in Q^{ad}$, the sequence $\{\widehat{S}(q_{n_k})\}$ is bounded in $\widehat{V} \times \widehat{P} = (H_0^1(\widehat{\Omega}) \cap V) \times (L^2(\widehat{\Omega}) \cap P)$. Hence, there exists a subsequence, for simplicity denoted by $\{S(q_l)\}$, and some $\overline{\widehat{S}} = (\overline{\widehat{\mathbf{u}}}, \overline{\widehat{p}}) \in \widehat{V} \times \widehat{P}$ such that,

$$\widehat{S}(q_l) = (\widehat{\mathbf{u}}_l, \widehat{p}_l) \rightharpoonup \overline{\widehat{S}} = (\overline{\widehat{\mathbf{u}}}, \overline{\widehat{p}}) \text{ in } \widehat{V} \times \widehat{P} \qquad \text{for } l \to \infty.$$

Now we have to prove that $\overline{S} = \overline{\widehat{S}}|_{\Omega_{\overline{q}}} \circ T_{\overline{q}}$ is the state solution corresponding to $\overline{q}$, i.e. $\overline{S} = S(\overline{q})$. This can be done transforming each term in problem (7) back on Ω_{q_l}, extending it on $\widehat{\Omega}$ and then passing to the limit for $l \to \infty$. As a paradigmatic example, we consider the viscosity term. Taking $\widehat{\mathbf{v}} \in [C_0^\infty(\widehat{\Omega})]^2$, it holds

$$\lim_{l \to \infty} \int_{\Omega_0} \nu^{q_l} \nabla \mathbf{u}_l A_{q_l} \cdot \nabla \widehat{\mathbf{v}} \, d\Omega = \lim_{l \to \infty} \int_{\Omega_{q_l}} \nu \nabla \widetilde{\mathbf{u}}_l \cdot \nabla \widehat{\mathbf{v}} \, d\Omega = \lim_{l \to \infty} \int_{\widehat{\Omega}} \nu \nabla \widehat{\mathbf{u}}_l \cdot \nabla \widehat{\mathbf{v}} \, d\Omega =$$

$$= \int_{\widehat{\Omega}} \nu \nabla \overline{\widehat{\mathbf{u}}} \cdot \nabla \widehat{\mathbf{v}} \, d\Omega = \int_{\Omega_0} \nu \nabla \overline{\mathbf{u}} A_{\overline{q}} \cdot \nabla \widehat{\mathbf{v}} \, d\Omega.$$

Finally, using dominate convergence theorem and the weak, lower semi-continuity of seminorms in a Banach space yields the weak, lower semi-continuity of functional j, allowing to conclude that

$$j(q_l) \to j(\overline{q}) = \overline{j} \qquad \text{for } l \to \infty.$$

Hence $\overline{q}$ turns out to be a solution of the optimization problem (8).

1.3. **Optimality conditions**

In this section, we inspect the first order optimality condition

$$j'(\overline{q})(\delta q) = 0 \qquad \forall \delta q \in \delta Q, \tag{13}$$

in order to obtain the Hadamard formula (see, e.g., [30]) for the gradient of functional j, useful for the analysis made in the following section and for numerical tests.

We first recall the expression of $\widetilde{j}$ defined in (6), as

$$\widetilde{j}(q) = \int_{\Omega_q} |\nabla \widetilde{\mathbf{u}}|^2 \, d\Omega + \alpha \|q''\|^2_{L^2(I)} + \beta \left(\int_I q(x)dx \; - \; \overline{V} \right)^2, \tag{14}$$

where $\widetilde{\mathbf{u}} : \Omega_q \to \mathbb{R}^2$, together with $\widetilde{p} : \Omega_q \to \mathbb{R}$, is the solution of Stokes problem (1).
The so-called shape-derivative of $(\widetilde{\mathbf{u}}, \widetilde{p})$ can be defined as the solution $(\widetilde{\delta \mathbf{u}}, \widetilde{\delta p})$ of the following problem (see, e.g., [26]):

$$\begin{cases} \eta \widetilde{\delta \mathbf{u}} - \mathrm{div}(\nu \nabla \widetilde{\delta \mathbf{u}}) + \nabla \widetilde{\delta p} = 0, & \text{in } \Omega_q, \\ \mathrm{div}(\widetilde{\delta \mathbf{u}}) = 0 & \text{in } \Omega_q, \\ \nu \partial_{\mathbf{n}} \widetilde{\delta \mathbf{u}} - \widetilde{\delta p} \mathbf{n} = \mathbf{0}, & \text{on } \Gamma_1, \\ \partial_{\mathbf{n}} \widetilde{\delta u} = 0, \quad \widetilde{\delta v} = 0, & \text{on } \Gamma_2, \\ \widetilde{\delta \mathbf{u}} = \mathbf{0}, & \text{on } \Gamma_3, \\ \widetilde{\delta \mathbf{u}} = -(V_{q,\delta q} \cdot \mathbf{n}) \partial_{\mathbf{n}} \widetilde{\mathbf{u}}, & \text{on } \Gamma_q, \end{cases} \tag{15}$$

where $V_{q,\delta q}$ is the vector field describing a transformation from Ω_q to $\Omega_{q+\delta q}$, given by

$$V_{q,\delta q}(x,y) = \begin{pmatrix} 0 \\ \frac{1-y}{1-q(x)} \delta q(x) \end{pmatrix}.$$

Differentiating the expression (14) along direction δq, one obtains

$$j'(q)(\delta q) = 2(\nabla \widetilde{\mathbf{u}}, \nabla \widetilde{\delta \mathbf{u}})_{\Omega_q} + \int_{\Omega_q} |\nabla \widetilde{\mathbf{u}}|^2 V_{q,\delta q} \cdot \mathbf{n} \, d\Gamma +$$

$$+ 2\alpha(q'', \delta q'')_I + 2\beta \left(\int_I q(x)dx - \overline{V} \right) \int_I \delta q(x)dx.$$

In order to make the dependence of $j'(q)(\delta q)$ on δq completely explicit, we introduce the adjoint state $(\widetilde{z}, \widetilde{s})$, solution of the following adjoint problem:

$$\begin{cases} -\mathrm{div}(\nu \nabla \widetilde{\mathbf{z}}) + \eta \widetilde{\mathbf{z}} + \nabla \widetilde{s} = -2\Delta \widetilde{\mathbf{u}}, & \text{in } \Omega_q, \\ \mathrm{div}\, \widetilde{\mathbf{z}} = 0, & \text{in } \Omega_q, \\ -\nu \partial_{\mathbf{n}} \widetilde{\mathbf{z}} + \widetilde{s} \mathbf{n} = -2\partial_{\mathbf{n}} \widetilde{\mathbf{u}}, & \text{on } \Gamma_1, \\ -\nu \partial_{\mathbf{n}} \widetilde{z}_x = -\nu \partial_{\mathbf{n}} \widetilde{z}_x + \widetilde{s}\, n_x = -2\partial_{\mathbf{n}} \widetilde{u}_x, \quad \widetilde{z}_y = 0, & \text{on } \Gamma_2, \\ \widetilde{\mathbf{z}} = \mathbf{0}, & \text{on } \Gamma_q \cup \Gamma_3. \end{cases} \tag{16}$$

Using both problems (15) and (16), and exploiting integration by parts and changes of variable from Ω_q to Ω_0, and from Γ_0 to I, we can prove the following result (see, e.g., [15]):

Lemma 1.9. *Given the functional $j(q)$ defined as in (8), its Gateaux-derivative in q along direction δq is given by*

$$j'(q)(\delta q) = 2\alpha(q'', \delta q'')_I + (\Psi(q), \delta q)_I \qquad \forall q \in Q^{ad}, \delta q \in \delta Q,$$

where $\Psi(q) : I \to \mathbb{R}$ *is defined as*

$$\Psi(q)(x) = 2\beta \left(\int_I q(t)dt \ - \ \overline{V} \right) +$$
$$+ \ [\nabla \tilde{\mathbf{u}} DT_q^{-1} DT_q^{-T} \mathbf{n}](x, q(x)) \cdot [(\nu \nabla \mathbf{z} - \tilde{\mathbf{u}}) DT_q^{-1} DT_q^{-T} \mathbf{n}](x, q(x)).$$

2. A priori error estimates

In this section, we aim at deriving some a priori estimates for the numerical discretization error of the main quantities involved in our problem, namely the control function q, the state variable $S(q)$ and the reduced cost functional $j(q)$.

At first, we are going to discuss some differentiability properties of the state solution operator S, under suitable assumptions. Then, we will introduce a discretization on the control space and derive corresponding error estimates. Afterwards, the discretization of the state problem will be studied. Finally, we will derive a convergence result for the complete shape optimization problem.

2.1. Solution operator properties

In order to provide some differentiability properties for the state solution operator and the cost functional, we begin by considering the following generalization of the Implicit Function Theorem to Banach spaces:

Theorem 2.1 ([22, Theorem 3.3]). *Let* $F \in C^k(X^{ad} \times Y, Z), k \geqslant 1$, *where* Y *and* Z *are Banach spaces and* X^{ad} *is an open subset of Banach space* X. *Suppose that* $\mathcal{F}(x^*, y^*) = 0$ *and* $\mathcal{F}'_y(x^*, y^*)$ *is continuously invertible. Then there exist neighbourhoods* Θ *of* x^* *in* X, Φ *of* y^* *in* Y *and a map* $g \in C^k(\Theta, Y)$ *such that* $\mathcal{F}(x, g(y)) = 0$ *for all* $x \in \Theta$. *Furthermore,* $F(x, y) = 0$ *for* $(x, y) \in \Theta \times \Phi$ *implies* $y = g(x)$.

As a direct consequence, we can prove the following result:

Corollary 2.2. *Let the following assumptions hold:*

$$\eta, \nu \in C^2(\overline{\widehat{\Omega}}), \qquad \mathbf{f} \in [C^2(\overline{\widehat{\Omega}})]^2.$$

Then, the solution operator S *is at least twice continuously Fréchet-differentiable.*

Proof. It is enough to use Theorem 2.1, with $X = H^2(I) \cap H_0^1(I), Y = V \times P, Z = Y^*$, the open set $X^{ad} = int(Q^{ad})$ and the map $\mathcal{F} : X^{ad} \times Y \to Z$ such that

$$\mathcal{F}(q; \mathbf{u}, p) = \begin{pmatrix} a(q)(\mathbf{u}, \cdot) + b(q)(\cdot, p) - F(q)(\cdot) \\ b(q)(\mathbf{u}, \cdot) - G(q)(\cdot) \end{pmatrix} \text{ for any } q \in int(Q^{ad}), \mathbf{u} \in V, p \in P.$$

The regularity of the map is a consequence of the regularity of the forms involved in its definition. It is easy to check that the operator S corresponds to the map g defined in Theorem 2.1, hence the regularity result for g holds for S as well. $\qquad \square$

Now, let us preliminarily collect some properties of the map T_q.

Proposition 2.3. *Given* $q \in Q^{ad}$, *the maps defined in Remark 1.3, depending on* T_q *and its derivatives, satisfy the following inequalities, for any admissible variation* $\delta q \in \delta Q$:

(1) $\|\gamma'_{q, \delta q}\|_\infty = \|\text{div}(V_{\delta q})\|_\infty \leqslant c\|\delta q\|_{L^\infty(I)} \leqslant \bar{c}\|\delta q\|_{H^1(I)}$,
(2) $\|V_{\delta q}\|_\infty \leqslant c\|\delta q\|_{H^1(I)}$,
(3) $\|\text{cof}(DV_{\delta q})\|_\infty = \|DV_{\delta q}\|_\infty \leqslant c\|\delta q\|_{H^2(I)}$,
(4) $\text{div}(\text{cof}(DV_{\delta q})) = (0, 0)^T$,
(5) $\|A'_{q, \delta q}\|_\infty \leqslant c\|\delta q\|_{H^2(I)}$,
(6) $\|\text{div}(A'_{q, \delta q})\|_\infty \leqslant c\|\delta q\|_{H^2(I)}$,
(7) $\|A'_{q, \delta q}\|_2 \leqslant c\|\delta q\|_{H^1(I)}$,
(8) $\|A''_{q, \delta q, \delta q}\|_\infty \leqslant c\|\delta q\|^2_{H^2(I)}$,

where the constants c and $\bar{c}$ are independent of q and δq.

Proof. The result simply follows from direct computation and the application of the Fundamental Theorem of Calculus.

The differentiability properties of the solution operator S are characterized in the following result.

Theorem 2.4. *The first and second variations of the solution operator S along the directions $\delta q, \tau q \in Q^{ad}$ are defined as follows:*

(1) $S'(q)(\delta q) = (\delta \mathbf{u}, \delta p) \in V \times P$, where $(\delta \mathbf{u}, \delta p)$ is the solution of

$$
\begin{cases}
a(q)(\delta \mathbf{u}, \mathbf{v}) + b(q)(\mathbf{v}, \delta p) = \dot{F}(q, \delta q)(\mathbf{v}) - \dot{a}(q, \delta q)(\mathbf{u}, \mathbf{v}) - \dot{b}(q, \delta q)(\mathbf{v}, p) & \forall\ \mathbf{v} \in V, \\
b(q)(\delta \mathbf{u}, \pi) = \dot{G}(q, \delta q)(\pi) - \dot{b}(q, \delta q)(\mathbf{u}, \pi) & \forall\ \pi \in P.
\end{cases}
\tag{17}
$$

(2) $S''(q)(\delta q, \tau q) = (\tau \delta \mathbf{u}, \tau \delta p) \in V \times P$, where $(\tau \delta \mathbf{u}, \tau \delta p)$ is the solution of

$$
\begin{cases}
a(q)(\tau \delta \mathbf{u}, \mathbf{v}) + b(q)(\mathbf{v}, \tau \delta p) = \\
\quad = \ddot{F}(q, \delta q, \tau q)(\mathbf{v}) - \ddot{a}(q, \delta q, \tau q)(\mathbf{u}, \mathbf{v}) - \ddot{b}(q, \delta q, \tau q)(\mathbf{v}, p)+ & \forall\ \mathbf{v} \in V, \\
\quad - \dot{a}(q, \delta q)(\tau \mathbf{u}, \mathbf{v}) - \dot{b}(q, \delta q)(\mathbf{v}, \tau p) - \dot{a}(q, \tau q)(\delta \mathbf{u}, \mathbf{v}) - \dot{b}(q, \tau q)(\mathbf{v}, \delta p) \\
b(q)(\tau \delta \mathbf{u}, \pi) = \ddot{G}(q, \delta q, \tau q)(\pi) - \ddot{b}(q, \delta q, \tau q)(\mathbf{u}, \pi)+ & \forall\ \pi \in P, \\
\quad - \dot{b}(q, \delta q)(\tau \mathbf{u}, \pi) - \dot{b}(q, \tau q)(\delta \mathbf{u}, \pi)
\end{cases}
\tag{18}
$$

with $(\tau \mathbf{u}, \tau p) = S'(q)(\tau q)$.

The forms and functionals employed in (17) and (18) are defined as follows:

$$
\dot{F}(q, \delta q)(\mathbf{v}) = \int_{\Omega_0} \left(\gamma'_{q,\delta q} \mathbf{f}^q \cdot \mathbf{v} + \gamma_q \nabla \mathbf{f}^q V_{\delta q} \cdot \mathbf{v} \right) d\Omega - \dot{a}(q, \delta q)(\mathcal{R}\mathbf{g}_D, \mathbf{v}),
$$

$$
\dot{G}(q, \delta q)(\pi) = -\dot{b}(q, \delta q)(\mathcal{R}\mathbf{g}_D, \pi),
$$

$$
\dot{a}(q, \delta q)(\mathbf{u}, \mathbf{v}) = \int_{\Omega_0} \left[\left(\gamma_q \nabla \eta^q \cdot V_{\delta q} + \eta^q \gamma'_{q,\delta q} \right) \mathbf{u} \cdot \mathbf{v} + \right.
$$
$$
\left. + \nabla \nu^q \cdot V_{\delta q} tr(\nabla \mathbf{u} A_q \nabla \mathbf{v}^T) + \nu^q tr(\nabla \mathbf{u} A'_{q,\delta q} \nabla \mathbf{v}^T) \right] d\Omega,
$$

$$
\dot{b}(q, \delta q)(\mathbf{v}, \pi) = -\int_{\Omega_0} \pi \nabla \mathbf{v} \cdot \mathrm{cof}(DV_{\delta q}) \, d\Omega,
$$

$$
\ddot{F}(q, \delta q, \tau q)(\mathbf{v}) = \int_{\Omega_0} [\gamma''_{q,\delta q \tau q} \mathbf{f}^q \cdot \mathbf{v} + \gamma'_{q,\delta q} \nabla \mathbf{f}^q V_{\tau q} \cdot \mathbf{v} + \gamma'_{q,\tau q} \nabla \mathbf{f}^q V_{\delta q} \cdot \mathbf{v}+
$$
$$
+ \gamma_q (\widetilde{\nabla}^2 \mathbf{f}^q V_{\tau q} + \nabla \mathbf{f}^q DV_{\tau q}) V_{\delta q} \cdot \mathbf{v}] d\Omega - \ddot{a}(q, \delta q)(\mathcal{R}\mathbf{g}_D, \mathbf{v}),
$$

$$
\ddot{G}(q, \delta q, \tau q)(\pi) = -\ddot{b}(q, \delta q, \tau q)(\mathbf{v}, \pi) = 0,
$$

$$
\ddot{a}(q, \delta q, \tau q)(\mathbf{u}, \mathbf{v}) = \int_{\Omega_0} \{ [\gamma'_{q,\tau q} \nabla \eta^q \cdot V_{\delta q} + (\nabla^2 \eta^q V_{\tau q} + DV_{\tau q}^T \nabla \eta^q) \cdot V_{\delta q} \gamma_q +
$$
$$
+ \eta^q \gamma''_{q,\delta q, \tau q}) \mathbf{u} \cdot \mathbf{v} + \gamma'_{q,\delta q} \nabla \eta^q \cdot V_{\tau q}] \mathbf{u} \cdot \mathbf{v}+
$$
$$
+ (\nabla^2 \nu^q V_{\delta q} + DV_{\tau q}^T \nabla \nu^q) \cdot V_{\delta q} \, tr(\nabla \mathbf{u} A_q \nabla \mathbf{v}^T)+
$$
$$
+ \nabla \nu^q \cdot V_{\delta q} \, tr(\nabla \mathbf{u} A'_{q,\tau q} \nabla \mathbf{v}^T) + \nu^q \, tr(\nabla \mathbf{u} A''_{q,\delta q, \tau q} \nabla \mathbf{v}^T)+
$$
$$
+ \nabla \nu^q \cdot V_{\tau q} \, tr(\nabla \mathbf{u} A'_{q,\delta q} \nabla \mathbf{v}^T) \} d\Omega,
$$

$$
\ddot{b}(q, \delta q, \tau q)(\mathbf{v}, \pi) = 0,
$$

with the differential operator $\widetilde{\nabla}^2$ acting as $\left(\widetilde{\nabla}^2\boldsymbol{\varphi}\right)_{ijk} = \left(\nabla^2\varphi_i\right)_{kj}$ and the over-signed dots denoting the partial Gateaux derivative w.r.t. the control q. Moreover, the following stability results hold:

$$
\begin{aligned}
\|S(q)\|_{V\times P} &\leqslant c, \\
\|S'(q)(\delta q)\|_{V\times P} &\leqslant c\|\delta q\|_{H^2(I)}, \\
\|S''(q)(\delta q, \delta q)\|_{V\times P} &\leqslant c\|\delta q\|_{H^2(I)}^2,
\end{aligned}
\tag{19}
$$

provided that the data satisfy the following regularity requirements:

$$
\eta, \nu \in W^{2,\infty}(\widehat{\Omega}), \quad \mathbf{f} \in [H^1(\widehat{\Omega})]^2.
$$

Proof. The weak problems defined in (17)-(18) can directly be obtained by differentiating the state problem (7) w.r.t. q. The stability results (19) follow from classical well-posedness results for saddle-point problems (see, e.g., [16]), combined with Proposition 2.3 (see [15] for details). $\quad\square$

Remark 2.5. We observe that the first derivative of the solution operator, $S'(q)(\delta q) = (\delta \mathbf{u}, \delta p)$, is the transformation of the shape derivative $(\widetilde{\delta\mathbf{u}}, \widetilde{\delta p})$ introduced in (15), since one can prove that $\delta\mathbf{u} = \widetilde{\delta\mathbf{u}} \circ T_q, \ \delta p = \widetilde{\delta p} \circ T_q$.

Hinging upon Theorem 2.4, we are now ready to compute the derivatives of j, as follows:

$$
j(q) = (\nabla\mathbf{u}\, A_q, \nabla\mathbf{u}) + \alpha\|q''\|_I^2 + \beta\left(\int_I q(x)dx \ - \ \overline{V}\right)^2,
\tag{20a}
$$

$$
\begin{aligned}
j'(q)(\delta q) = &(\nabla\mathbf{u}\, A'_{q,\delta q}, \nabla\mathbf{u}) + 2(\nabla\delta\mathbf{u}\, A_q, \nabla\mathbf{u}) + 2\alpha(\delta q'', q'')_I + \\
&+ 2\beta\left(\int_I q(x)dx - \overline{V}\right)\int_I \delta q(x)dx,
\end{aligned}
\tag{20b}
$$

$$
\begin{aligned}
j''(q)(\delta q, \tau q) = &(\nabla\mathbf{u}\, A''_{q,\delta q,\tau_q}, \nabla\mathbf{u}) + 2(\nabla\tau\mathbf{u}\, A'_{q,\delta q} + \nabla\delta\mathbf{u}\, A'_{q,\tau q}, \nabla\mathbf{u}) + \\
&+ 2(\nabla\delta\mathbf{u}\, A_q, \nabla\tau\mathbf{u}) + 2(\nabla\tau\delta\mathbf{u}\, A_q, \nabla\mathbf{u}) + \\
&+ 2\alpha(\delta q'', \tau q'')_I + 2\beta\int_I \delta q(x)dx\int_I \tau q(x)dx,
\end{aligned}
\tag{20c}
$$

where $\mathbf{u}, \delta\mathbf{u}, \tau\mathbf{u}, \tau\delta\mathbf{u}$ are the same as in Theorem 2.4. The continuity of the derivatives is an easy consequence of the regularity and symmetry of the matrix A_q and its derivatives.

2.2. **Control discretization**

Let $\{I_i = (x_{i-1}, x_i)\}_{i=1}^N$ be a partition of the domain I, with discretization parameter $\sigma = \max_{i\in\{1,\dots,N\}}|I_i|$. We can then define the discrete controls set as

$$
Q_\sigma^{ad} = Q^{ad} \cap Q_\sigma, \text{ with } Q_\sigma = \{q \in C^0(\overline{I})\colon q|_{I_i} \in \mathbb{P}_4(I_i), \ i \in \{1, \dots, N\}\}.
$$

The semi-discretized optimization problem reads as follows

$$
\min_{q_\sigma \in Q_\sigma^{ad}} j(q_\sigma) = J(q_\sigma, \widetilde{S}(q_\sigma)).
\tag{21}
$$

As $Q^{ad} \supseteq Q_\sigma^{ad}$, the minimization problem (21) inherits the existence and regularity properties holding for the original continuous optimization problem (8).

Let us denote by $\Pi_\sigma^4\colon L^2(I) \to Q_\sigma$ the classical polynomial interpolation operator and notice that $\Pi_\sigma^4(Q^{ad}) \subseteq Q_\sigma^{ad}$. Standard interpolation error estimates hold (see e.g. [6]): for $r \geqslant 1$, $0 \leqslant m \leqslant r+1$, it holds that

$$
|q - \Pi_\sigma^r q|_{H^m(I)} \leqslant c\sigma^{r+1-m}|q|_{H^{r+1}(I)} \quad \forall q \in H^{r+1}(I).
\tag{22}
$$

In this section, we aim at proving the following convergence result:

Proposition 2.6. *Let $\bar{q} \in Q^{ad}$ be the exact solution of* (8), *and $\bar{q}_\sigma$ the solution of the partially discretized problem* (21). *Then, assuming that the optimal control $\bar{q}$ belongs to $H^5(I)$, the following convergence error estimate holds:*

$$\|\bar{q} - \bar{q}_\sigma\|_{H^3(I)} \leqslant c\sigma^2 |\bar{q}|_{H^5(I)}.$$

Remark 2.7. We observe that Proposition 2.6 needs the optimal control $\bar{q}$ to be in $H^5(I)$. To achieve this regularity, there is no need to re-define the admissible controls set Q^{ad}, but it is sufficient to assume the validity of a regularity result for the classical Stokes problem. This assumption and the proof of the needed regularity on $\bar{q}$ are reported in Appendix A.

In order to prove Proposition 2.6, we need to collect some preliminary results that will be derived under the following two assumptions, already employed in [22].

Assumption 2.8 ([22, Assumption 1.5]). *For the optimal solution $\bar{q}$ of problem* (8), *the constraint $q \leqslant 1 - \varepsilon$ is not active, i.e.*

$$\exists \delta > 0 \ \text{such that} \ \bar{q}(x) \leqslant 1 - \varepsilon - \delta \quad \forall x \in I.$$

Assumption 2.9 ([22, Assumption 3.1]). *For any local minimum $\bar{q}$, we have*

$$j''(\bar{q})(\delta q, \delta q) > 0 \quad \forall \delta q \in \delta Q \backslash \{0\}.$$

We start by proving some regularity results for the solution operator S and its derivatives.

Lemma 2.10. *Let S be the solution operator of the transformed Stokes problem* (7). *If there exists some $k > 0$ such that data functions fulfill the regularity requests*

$$\eta, \nu \in C^k(\widehat{\Omega}), \qquad \mathbf{f} \in [C^k(\widehat{\Omega})]^2,$$

then S is at least k times continuously Fréchet differentiable.

Proof. The proof is the same as in Corollary 2.2, simply applying the Implicit Function Theorem in the form presented in Theorem 2.1.

Based on the previous result, we can prove the following:

Lemma 2.11. *Let $k \in \mathbb{N}$ and let data functions fulfill the following regularity requests:*

$$\eta, \nu \in C^{k+1}(\widehat{\Omega}), \qquad \mathbf{f} \in [C^{k+1}(\widehat{\Omega})]^2.$$

Then, for any $q, r \in Q^{ad}$ and $\delta q_1, \delta q_2, \ldots, \delta q_k \in \delta Q$, the following inequalities hold:

$$\|S^{(i)}(q)(\delta q_1, \ldots, \delta q_i) - S^{(i)}(r)(\delta q_1, \ldots, \delta q_i)\|_{V \times P} \leqslant c\|q - r\|_{H^2(I)} \prod_{j=1}^{i} \|\delta q_j\|_{H^2(I)}, \quad \text{for } i = 0, \ldots, k.$$

Proof. Let q and r be two control functions in Q^{ad} and $\delta q, \tau q$ admissible control variations. Applying Lemma 2.10 under the hypotheses of the present Lemma, we get

$$S \in C^{k+1}(int(Q^{ad}); V \times P).$$

Let us consider $k = 0$. As $S \in C^1$, given the control functions $q, r \in Q^{ad}$, the Mean Value Theorem ensures that

$$\exists \xi \in Q^{ad} \ \text{such that} \ S(q) - S(r) = S'(\xi)(q - r).$$

Being the Fréchet derivative $S'(\xi)$ a linear operator on the control variation, its continuity is equivalent to its boundedness, thus we get

$$\|S(q) - S(r)\|_{V \times P} = \|S'(\xi)(q - r)\|_{V \times P} \leqslant c\|q - r\|_{H^2(I)}.$$

In the general case $k > 0$, for each $i \in \{0, \ldots, k\}$ there exists $\xi_i \in Q^{ad}$ such that

$$S^{(i)}(q) - S^{(i)}(r) = S^{(i+1)}(\xi_i)(q - r), \tag{23}$$

where we remark that (23) is an equality between linear operators belonging to $\mathscr{L}_i := \mathscr{L}(\delta Q^i; V \times P)$. Observing that $S^{(i+1)}(\xi_i) \in \mathscr{L}_{i+1}$, we can proceed as before to obtain

$$\|S^{(i)}(q)(\delta q_1, \ldots, \delta q_i) - S^{(i)}(r)(\delta q_1, \ldots, \delta q_i)\|_{V \times P} =$$

$$= \|S^{(i)}(q) - S^{(i)}(r)\|_{\mathscr{L}_i} \prod_{j=1}^{i} \|\delta q_i\|_{H^2(I)} = \|S^{(i+1)}(\xi_i)(q - r)\|_{\mathscr{L}_i} \prod_{j=1}^{i} \|\delta q_i\|_{H^2(I)} \leqslant$$

$$\leqslant \|S^{(i+1)}(\xi_i)\|_{\mathscr{L}_{i+1}} \|q - r\|_{H^2(I)} \prod_{j=1}^{i} \|\delta q_i\|_{H^2(I)}.$$

Since $S^{(i+1)}$ is continuous, it is also bounded, so there exists a constant $c > 0$ such that $\|S^{(i+1)}(\xi)\|_{\mathcal{L}_{i+1}} \leqslant c$ for all $\xi \in \delta Q$. Hence the proof is complete.

The continuity of the solution operator S directly implies the continuity of the functional j, as stated in the following result.

Lemma 2.12. *For any $q, r \in Q^{ad}$ and any $\delta q \in H^2(I) \cap H_0^1(I)$, it holds that*
(a) $|j(q) - j(r)| \leqslant c\|q - r\|_{H^2(I)}$,
(b) $|j'(q)(\delta q) - j'(r)(\delta q)| \leqslant c\|q - r\|_{H^2(I)}\|\delta q\|_{H^2(I)}$,
(c) $|j''(q)(\delta q, \delta q) - j''(r)(\delta q, \delta q)| \leqslant c\|q - r\|_{H^2(I)}\|\delta q\|_{H^2(I)}^2$.

Proof. Let us fix $q, r \in Q^{ad}$. To simplify the notation, let $S(q) = (\mathbf{u}, p)$ and $S(r) = (\mathbf{z}, s)$. As the proofs of (a)-(c) are similar, we focus on (c), highlighting the most technical parts.
Bearing in mind the expression of j'' (see (20c)), we first focus on the following term:

$$| \left(\nabla \mathbf{u} A''_{q,\delta q,\delta q}, \nabla \mathbf{u}\right) - \left(\nabla \mathbf{z} A''_{r,\delta q,\delta q}, \nabla \mathbf{z}\right) | =$$
$$= | \left((\nabla \mathbf{u} - \nabla \mathbf{z}) A''_{q,\delta q,\delta q}, \nabla \mathbf{u} + \nabla \mathbf{z}\right) + \left(\nabla \mathbf{z}(A''_{q,\delta q,\delta q} - A''_{r,\delta q,\delta q}), \nabla \mathbf{z}\right) | \leqslant$$
$$\leqslant \|A''_{q,\delta q,\delta q}\|_\infty \left(\|\nabla \mathbf{u}\| + \|\nabla \mathbf{z}\|\right) \|\nabla \mathbf{u} - \nabla \mathbf{z}\| + \|\nabla \mathbf{z}\|^2 \|A''_{q,\delta q,\delta q} - A''_{r,\delta q,\delta q}\|_\infty \leqslant$$
$$\leqslant c\|\delta q\|_{H^2(I)}^2,$$

where the state variables have been bounded using Lemmas 1.7 and 2.4, while $\|A''_{q,\delta q,\delta q}\|_\infty$ has been handed employing Proposition 2.3.
Using the same results, it is easy to bound also the following term:

$$| \left(\nabla \delta \mathbf{u} \, A'_{q,\delta q}, \nabla \mathbf{u}\right) - \left(\nabla \delta \mathbf{z} \, A'_{r_{\delta q}}, \nabla \mathbf{z}\right) | =$$
$$= | \left((\nabla \delta \mathbf{u} - \nabla \delta \mathbf{z}) A'_{q,\delta q}, \nabla \mathbf{u}\right) + \left(\nabla \delta \mathbf{z}(A'_{q,\delta q} - A'_{r,\delta q}), \nabla \mathbf{u}\right) + \left(\nabla \delta \mathbf{z} \, A'_{r,\delta q}, \nabla \mathbf{u} - \nabla \mathbf{z}\right) |.$$

All the other terms entering in j'' can be treated in a similar way, to get (c). ∎

The results stated so far are sufficient to prove the following coercivity result on j.

Lemma 2.13. *If $\bar{q}$ is a local solution of (8), fulfilling Assumption 2.9, then there exist $\delta_1, \delta_2 > 0$ such that, if $\|\bar{q} - r\|_{H^2(I)} \leqslant \delta_1$ for $r \in Q^{ad}$, then*

$$j''(r)(\delta q, \delta q) \geqslant \frac{\delta_2}{2} \|\delta q\|_{H^3(I)}^2 \qquad \forall \delta q \in Q^{ad}.$$

The proof of Lemma 2.13 is the same as in [22, Lemma 3.14], replacing $H^2(I)$ with $H^3(I)$ and provided two more intermediate results, reported in Appendix B.

Now, we are ready to conclude this section with the proof of Proposition 2.6.

Proof (Proposition 2.6). The Mean Value Theorem and Lemma 2.13 imply the existence of some $t \in [0,1]$ such that, for $\xi = t\,\Pi_\sigma^4 \overline{q} + (1-t)\overline{q}_\sigma$, we have

$$
\begin{aligned}
\frac{\delta_2}{2}\|\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma\|_{H^3(I)}^2 &\leqslant j''(\xi)(\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma, \Pi_\sigma^4 \overline{q} - \overline{q}_\sigma) = \\
&= j'(\Pi_\sigma^4 \overline{q})(\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma) - j'(\overline{q}_\sigma)(\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma) = \\
&\stackrel{\mathrm{a}}{=} j'(\Pi_\sigma^4 \overline{q})(\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma) - j'(\overline{q})(\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma) \leqslant \\
&\stackrel{\mathrm{b}}{\leqslant} c\|\overline{q} - \Pi_\sigma^4 \overline{q}\|_{H^3(I)}\|\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma\|_{H^3(I)} \leqslant \\
&\stackrel{\mathrm{c}}{\leqslant} c\,\sigma^2 \|\overline{q}\|_{H^5(I)}\|\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma\|_{H^3(I)},
\end{aligned}
\tag{24}
$$

where we used:
   (a) $j'(\overline{q})(\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma) = j'(\overline{q}_\sigma)(\Pi_\sigma^4 \overline{q} - \overline{q}_\sigma) = 0$, due to Assumption 2.8 and then the first order optimal condition;
   (b) point b of Lemma 2.12 and the fact that $\|\cdot\|_{H^2(I)} \leqslant \|\cdot\|_{H^3(I)}$;
   (c) the interpolation error estimate (22).

From (24), we obtain

$$
\|\Pi_\sigma \overline{q} - \overline{q}_\sigma\|_{H^3(I)} \leqslant c\sigma^2 \|\overline{q}\|_{H^5(I)}.
$$

Finally, triangular inequality gives the thesis.

2.3. State discretization

Let $\mathcal{T}_h$ be a regular triangulation of Ω_0, with discretization parameter $h = \max_{K \in \mathcal{T}_h}|K|$. We can thus introduce the finite element spaces

$$
\begin{aligned}
X_h^r(\Omega_0) &= \{\varphi \in C^0(\overline{\Omega}_0) \ : \ \varphi|_K \in \mathbb{P}_r(K) \ \ \forall K \in \mathcal{T}_h\}, \\
V_h &= V \cap [X_h^2(\Omega_0)]^2, \\
P_h &= P \cap X_h^1(\Omega_0),
\end{aligned}
\tag{25}
$$

where $\mathbb{P}_r(K)$ is the space of polynomials on K having degree less than or equal to r.

Passing from the continuous to the discrete case, the variational forms involved in problem (7) preserve all their properties, with discrete inf-sup condition ensured by the following:

Proposition 2.14 (LBB condition)**.** *There exists a positive constant $\overline{\beta}$ such that*

$$
\forall \pi_h \in P_h \ \exists \mathbf{v}_h \in V_h \ : \ b(q)(\mathbf{v}_h, \pi_h) \geqslant \overline{\beta}\|\nabla \mathbf{v}_h\|\|\pi_h\|,
\tag{26}
$$

and $\overline{\beta}$ is independent from $q \in Q^{ad}$ and from $h \in [0, \widehat{h}]$, for a certain $\widehat{h} > 0$.

Proof. From FEM approximation of Stokes problem [16], we know that pair (V_h, P_h) is stable, i.e. there exists a constant $\widehat{\beta} > 0$ such that

$$
\forall \pi_h \in P_h \ \exists \mathbf{v}_h \in V_h \ : \ b(0)(\mathbf{v}_h, \pi_h) \geqslant \widehat{\beta}\|\nabla \mathbf{v}_h\|\|\pi_h\|,
\tag{27}
$$

with $\widehat{\beta}$ independent from $h \in [0, \widehat{h}]$.

In order to show that such discrete spaces fulfill inf-sup condition also for the transformed form $b(q)$, one can just follow the steps presented in section 1.2, with constant $\widehat{\beta}$ from (27). Indeed, no assumptions on the spaces V, P have been made there, apart from the validity of inf-sup condition for $b(0)$.

The finite element discretization of (7) reads as follows:

Find $(\mathbf{u}_h, p_h) \in V_h \times P_h$, such that
$$\begin{cases} a(q_\sigma)(\mathbf{u}_h, \mathbf{v}_h) + b(q_\sigma)(\mathbf{v}_h, p_h) = F(q_\sigma)(\mathbf{v}_h) & \forall\ \mathbf{v}_h \in V_h, \\ b(q_\sigma)(\mathbf{u}_h, \pi_h) = G(q_\sigma)(\pi_h) & \forall\ \pi_h \in P_h. \end{cases} \tag{28}$$

The well-posedness of (28) stems from the validity of (26).

The discrete state solution operator, resulting from problem (28), and the corresponding discrete cost functional, are defined as

$$S_h : Q^{ad} \to V_h \times P_h, \text{ with } S_h(q) = (\mathbf{u}_h, p_h), \qquad j_h : Q^{ad} \to \mathbb{R}, \text{ with } j(q) = J(q, S_h(q) \circ T_q^{-1}),$$

whereas the fully discretized shape optimization problem can be written as

$$\min_{q_\sigma \in Q_\sigma^{ad}} j_h(q_\sigma) = J(q_\sigma, S_h(q_\sigma) \circ T_{q_\sigma}^{-1}).$$

For future use, it is useful to explicitly write the problems defining the derivatives of S_h:

(1) $S_h'(q)(\delta q) = (\delta \mathbf{u}_h, \delta p_h) \in V_h \times P_h$, where $(\delta \mathbf{u}_h, \delta p_h)$ is the solution of
$$\begin{cases} a(q)(\delta \mathbf{u}_h, \mathbf{v}_h) + b(q)(\mathbf{v}_h, \delta p_h) = \dot{F}(q, \delta q)(\mathbf{v_h}) + \\ \qquad\qquad - \dot{a}(q, \delta q)(\mathbf{u}_h, \mathbf{v}_h) - \dot{b}(q, \delta q)(\mathbf{v}_h, p_h) & \forall\ \mathbf{v_h} \in V_h, \\ b(q)(\delta \mathbf{u}_h, \pi_h) = \dot{G}(q, \delta q)(\pi_h) - \dot{b}(q, \delta q)(\mathbf{u}_h, \pi_h) & \forall\ \pi_h \in P_h. \end{cases} \tag{29}$$

(2) $S_h''(q)(\delta q, \tau q) = (\tau \delta \mathbf{u}_h, \tau \delta p_h) \in V_h \times P_h$, where $(\tau \delta \mathbf{u}_h, \tau \delta p_h)$ is the solution of
$$\begin{cases} a(q)(\tau \delta \mathbf{u}_h, \mathbf{v}_h) + b(q)(\mathbf{v}_h, \tau \delta p_h) = \\ \quad = \ddot{F}(q, \delta q, \tau q)(\mathbf{v}_h) - \ddot{a}(q, \delta q, \tau q)(\mathbf{u}_h, \mathbf{v}_h) - \ddot{b}(q, \delta q, \tau q)(\mathbf{v}_h, p_h) + \\ \qquad - \dot{a}(q, \delta q)(\tau \mathbf{u}_h, \mathbf{v}_h) - \dot{b}(q, \delta q)(\mathbf{v}_h, \tau p_h) + \\ \qquad - \dot{a}(q, \tau q)(\delta \mathbf{u}_h, \mathbf{v}_h) - \dot{b}(q, \tau q)(\mathbf{v}_h, \delta p_h) & \forall\ \mathbf{v_h} \in V_h, \\ b(q)(\tau \delta \mathbf{u}_h, \pi_h) = \ddot{G}(q, \delta q, \tau q)(\pi_h) - \ddot{b}(q, \delta q, \tau q)(\mathbf{u}_h, \pi_h) + \\ \qquad - \dot{b}(q, \delta q)(\tau \mathbf{u}_h, \pi_h) - \dot{b}(q, \tau q)(\delta \mathbf{u}_h, \pi_h) & \forall\ \pi_h \in P_h, \end{cases} \tag{30}$$

with $(\tau \mathbf{u}_h, \tau p_h) = S_h'(q)(\tau q)$.

Like in the previous section, in order to study the convergence of the discrete quantities to their continuous counterparts, we introduce projection operators onto the discrete spaces. Since there will be no room for misunderstanding, to avoid redundant notation, all of them will be indicated by the same symbol Π_h^r, never minding if returning functions in V_h, P_h, or $V_h \times P_h$.

Referring to X_h^r, the following interpolation estimate is known (see e.g. [29], section 3.4.2), for $r \geqslant 1$, $m = 0, 1$:
$$|\varphi - \Pi_h^r \varphi|_{H^m(\Omega_0)} \leqslant ch^{r+1-m} |\varphi|_{H^{r+1}(\Omega_0)}. \tag{31}$$

The particular choice of $\mathbb{P}_2 - \mathbb{P}_1$ couple in the spaces defined in (25), leads us to assume the following regularity for the state variables and their shape derivatives:

Assumption 2.15. *For any $q \in Q^{ad}$, $\delta q \in \delta Q$, any of $S(q), S'(q)(\delta q), S''(q)(\delta q, \delta q)$ belong to $[H^3(\Omega_0)]^2 \times H^2(\Omega_0)$ and the following inequalities hold:*

$$\begin{aligned} \|S(q)\|_{[H^3(\Omega_0)]^2 \times H^2(\Omega_0)} &= & \|\mathbf{u}\|_{[H^3(\Omega_0)]^2} + \|p\|_{H^2(\Omega_0)} & \leqslant c_1, \\ \|S'(q)(\delta q)\|_{[H^3(\Omega_0)]^2 \times H^2(\Omega_0)} &= & \|\delta \mathbf{u}\|_{[H^3(\Omega_0)]^2} + \|\delta p\|_{H^2(\Omega_0)} & \leqslant c_2 \|\delta q\|_{H^3(I)}, \\ \|S''(q)(\delta q, \delta q)\|_{[H^3(\Omega_0)]^2 \times H^2(\Omega_0)} &= & \|\delta\delta \mathbf{u}\|_{[H^3(\Omega_0)]^2} + \|\delta\delta p\|_{H^2(\Omega_0)} & \leqslant c_3 \|\delta q\|_{H^3(I)}^2. \end{aligned}$$

Remark 2.16. In Appendix A we prove (Theorem A.5) the validity of Assumption 2.15, that involves suitable regularity assumptions on data of Stokes problem.

Assumption 2.15, together with (31), yields the following estimate:

$$
\begin{aligned}
\|\mathbf{u} - \Pi_h^2 \mathbf{u}\|_V + \|p - \Pi_h^1 p\|_P &\leqslant c_1 h^2, \\
\|\delta\mathbf{u} - \Pi_h^2 \delta\mathbf{u}\|_V + \|\delta p - \Pi_h^1 \delta p\|_P &\leqslant c_2 h^2 \|\delta q\|_{H^3(I)}, \\
\|\delta\delta\mathbf{u} - \Pi_h^2 \delta\delta\mathbf{u}\|_V + \|\delta\delta p - \Pi_h^1 \delta\delta p\|_P &\leqslant c_3 h^2 \|\delta q\|_{H^3(I)}^2,
\end{aligned}
\tag{32}
$$

which are crucial to obtain the following convergence result.

Remark 2.17. Under regularity Assumption 2.15, one can afford the optimal convergence rate for $\mathbb{P}_2 - \mathbb{P}_1$ discretization: lower regularity of the state variables would lead to a lower order on h in (32).

The interpolation error estimates are once again the basis upon which we build our convergence result, which reads as follows:

Lemma 2.18. *For any $q_\sigma \in Q_\sigma^{ad}, \delta q \in \delta Q$, the following convergence estimates hold:*

(a) $\|S(q_\sigma) - S_h(q_\sigma)\|_{V \times P} \leqslant ch^2$,
(b) $\|S'(q_\sigma)(\delta q) - S_h'(q_\sigma)(\delta q)\|_{V \times P} \leqslant ch^2 \|\delta q\|_{H^3(I)}$,
(c) $\|S''(q_\sigma)(\delta q, \delta q) - S_h''(q_\sigma)(\delta q, \delta q)\|_{V \times P} \leqslant ch^2 \|\delta q\|_{H^3(I)}^2$.

Proof. Since the discrete problems (28)-(30) fulfill the same properties as the continuous ones, we have that Theorem 2.4 on the boundedness of the continuous solution operator S is true also for the discrete operator S_h and its derivatives. Hinging upon this result and Assumption 2.15, we fix some $q_\sigma \in Q_\sigma^{ad}, \delta q \in \delta Q$ and proceed according to the following steps.

We first prove (a). From [16] and the independence of the continuity, coercivity and LBB constants from q_σ and h, we can obtain the classical convergence result for a saddle-point problem, i.e.,

$$
\|S(q_\sigma) - S_h(q_\sigma)\|_{V \times P} \leqslant c(\|\mathbf{u} - \Pi_h^2 \mathbf{u}\|_V + \|p - \Pi_h^1 p\|_P) \leqslant ch^2,
$$

with the last inequalities exploiting interpolation error estimate (32).

We now proceed to prove (b). We set $(\mathbf{u}, p) = S(q)$, $(\delta\mathbf{u}, \delta p) = S'(q)(\delta q)$, $(\delta\delta\mathbf{u}, \delta\delta p) = S''(q)(\delta q, \delta q)$, with subscript $\cdot_h$ denoting the correspondent discrete quantities, and we introduce the "intermediate derivative" $(\delta\widehat{\mathbf{u}}_h, \delta\widehat{p}_h)$, solution in $V_h \times P_h$ of the following problem: [4]

$$
\begin{cases}
a(q)(\delta\widehat{\mathbf{u}}_h, \mathbf{v}_h) + b(q)(\mathbf{v}_h, \delta\widehat{p}_h) = \dot{F}(q, \delta q)(\mathbf{v}_h) + \\
\qquad\qquad - \dot{a}(q, \delta q)(\mathbf{u}, \mathbf{v}_h) - \dot{b}(q, \delta q)(\mathbf{v}_h, p) & \forall\, \mathbf{v}_h \in V_h, \\
b(q)(\delta\widehat{\mathbf{u}}_h, \pi_h) = \dot{G}(q, \delta q)(\pi_h) - \dot{b}(q, \delta q)(\mathbf{u}_h, \pi_h) & \forall\, \pi_h \in P_h.
\end{cases}
\tag{33}
$$

Thanks to (33), we can separate the error due to the discretization of the problem on $S'(q)(\delta q)$ from the one that is inherited from the discretization of $S(q)$. Using triangular inequality yields

$$
\begin{aligned}
\|S'(q)(\delta q) - S_h'(q)(\delta q)\|_{V \times P} &\leqslant \\
\leqslant \|S'(q)(\delta q) - (\delta\widehat{\mathbf{u}}_h, \delta\widehat{p}_h)\|_{V \times P} &+ \|(\delta\widehat{\mathbf{u}}_h, \delta\widehat{p}_h) - S_h'(q)(\delta q)\|_{V \times P} = \\
= \|\delta\mathbf{u} - \delta\widehat{\mathbf{u}}_h\|_V + \|\delta\widehat{\mathbf{u}}_h - \delta\mathbf{u}_h\|_V &+ \|\delta p - \delta\widehat{p}_h\|_P + \|\delta\widehat{p}_h - \delta p_h\|_P.
\end{aligned}
\tag{34}
$$

Considering the first term in (34), we have that, for any $\mathbf{w}_h \in V_h$,

$$
\begin{aligned}
\alpha_c \|\nabla\delta\mathbf{u} - \nabla\delta\widehat{\mathbf{u}}_h\|^2 &\leqslant a(q_\sigma)(\delta\mathbf{u} - \delta\widehat{\mathbf{u}}_h, \delta\mathbf{u} - \delta\widehat{\mathbf{u}}_h) = \\
= a(q_\sigma)(\delta\mathbf{u} - \delta\widehat{\mathbf{u}}_h, \delta\mathbf{u} - \mathbf{w}_h) &+ a(q_\sigma)(\delta\mathbf{u} - \delta\widehat{\mathbf{u}}_h, \mathbf{w}_h - \delta\widehat{\mathbf{u}}_h) = \\
= a(q_\sigma)(\delta\mathbf{u} - \delta\widehat{\mathbf{u}}_h, \delta\mathbf{u} - \mathbf{w}_h) &- b(q_\sigma)(\mathbf{w}_h - \delta\widehat{\mathbf{u}}_h, \delta p - \delta\widehat{p}_h),
\end{aligned}
\tag{35}
$$

---

[4] The problem here introduced is a combination of problem (17) for $S'(q_\sigma)(\delta q)$ and its discrete counterpart (29): we solve a discrete problem in spaces V_h, P_h, with the first equation being the same as in (17), and the second one as in (29).

with the equality holding thanks to the fact that the first equations in (17) and (33) share the same right-hand side. Since (35) holds for every $\mathbf{w}_h \in V_h$, it still holds if we take the infimum w.r.t. $\mathbf{w}_h$. For the first term of the right-hand side we get

$$
\begin{aligned}
\inf_{\mathbf{w}_h \in V_h} a(q_\sigma)(\delta\mathbf{u} - \delta\widehat{\mathbf{u}}_h, \delta\mathbf{u} - \mathbf{w}_h) &\leqslant a(q_\sigma)(\delta\mathbf{u} - \delta\widehat{\mathbf{u}}_h, \delta\mathbf{u} - \Pi_h\delta\mathbf{u}) \leqslant \\
&\leqslant M\|\nabla\delta\mathbf{u} - \nabla\delta\widehat{\mathbf{u}}_h\|\|\nabla\delta\mathbf{u} - \nabla\Pi_h\delta\mathbf{u}\| \leqslant ch^2 M\|\nabla\delta\mathbf{u} - \nabla\delta\widehat{\mathbf{u}}_h\|\|\delta q\|_{H^3(I)},
\end{aligned}
\tag{36}
$$

where we employed the interpolation error estimate (32) and the boundedness of $\|\delta\mathbf{u}\|_{H^2(I)}$ (due to Assumption 2.15). Instead, taking $\mathbf{w}_h = \delta\widehat{\mathbf{u}}_h$ in the second term yields

$$
\inf_{\mathbf{w}_h \in V_h} \left[-b(q_\sigma)(\mathbf{w}_h - \delta\widehat{\mathbf{u}}_h, \delta p - \delta\widehat{p}_h) \right] \leqslant 0.
\tag{37}
$$

Using (36) and (37) in (38) and dividing both sides by $\alpha_c\|\nabla\delta\mathbf{u} - \nabla\delta\widehat{\mathbf{u}}_h\|$, we eventually obtain

$$
\|\nabla\delta\mathbf{u} - \nabla\delta\widehat{\mathbf{u}}_h\| \leqslant c\frac{M}{\alpha_c}h^2\|\delta q\|_{H^3(I)}.
\tag{38}
$$

The second term in (34) can be estimated using the problems (28) and (33), fulfilled by $\delta\mathbf{u}_h, \delta\widehat{\mathbf{u}}_h$, together with the coercivity of a and the continuity of the forms involved in such problems. We can thus obtain:

$$
\begin{aligned}
\alpha_c\|\nabla\delta\widehat{\mathbf{u}}_h - \nabla\delta\mathbf{u}_h\|^2 &\leqslant a(q_\sigma)(\delta\widehat{\mathbf{u}}_h - \delta\mathbf{u}_h, \delta\widehat{\mathbf{u}}_h - \delta\mathbf{u}_h) = \\
&= -\dot{a}(q_\sigma, \delta q)(\mathbf{u} - \mathbf{u}_h, \delta\widehat{\mathbf{u}}_h - \delta\mathbf{u}_h) - \dot{b}(q_\sigma, \delta q)(\delta\widehat{\mathbf{u}}_h - \delta\mathbf{u}_h, p - p_h) + \\
&\quad - b(q_\sigma)(\delta\widehat{\mathbf{u}}_h - \delta\mathbf{u}_h, \delta\widehat{p}_h - \delta p_h) \leqslant \\
&\leqslant c\|\delta q\|_{H^2(I)}(\|\nabla\mathbf{u} - \nabla\mathbf{u}_h\| + \|p - p_h\|)\|\nabla\delta\widehat{\mathbf{u}}_h - \nabla\delta\mathbf{u}_h\|,
\end{aligned}
\tag{39}
$$

where the last inequality holds because $b(q_\sigma)(\delta\widehat{\mathbf{u}}_h, \pi_h) = b(q_\sigma)(\delta\mathbf{u}_h, \pi_h) \quad \forall\pi_h \in P_h$. After dividing by $\|\nabla\delta\widehat{\mathbf{u}}_h - \nabla\delta\mathbf{u}_h\|$ both sides of (39), the right-hand side can be controlled as in the first point of the present Lemma, leading to

$$
\|\nabla\delta\widehat{\mathbf{u}}_h - \nabla\delta\mathbf{u}_h\| \leqslant ch^2\|\delta q\|_{H^3(I)}.
\tag{40}
$$

Now we have to deal with pressure error terms in (34): taking a generic $\pi_h \in P_h$, the first term can be split as follows:

$$
\|\delta p - \delta p_h\| \leqslant \|\delta p - \pi_h\| + \|\pi_h - \delta\widehat{p}_h\|.
\tag{41}
$$

We remark that, since inequality (41) holds for any $\pi_h \in P_h$, it holds also taking the infimum w.r.t. π_h. The infimum of the first term is directly controlled by $ch^2\|\delta q\|_{H^2(I)}$ thanks to the interpolation error estimate (32) and the boundedness of $\|\delta p\|_{H^2(\Omega_0)}$ asserted in Theorem A.5. The second term goes to zero when passing to the infimum, since $\delta p_h \in P_h$.

Finally, for the last term in (34) we exploit LBB condition (26) and proceed as follows:

$$
\begin{aligned}
\|\delta\widehat{p}_h - \delta p_h\| &\leqslant \sup_{\mathbf{v}_h \in V_h} \frac{b(q_\sigma)(\mathbf{v}_h, \delta\widehat{p}_h - \delta p_h)}{\widehat{\beta}\|\nabla\mathbf{v}_h\|} = \\
&= \sup_{\mathbf{v}_h \in V_h} \frac{-\dot{a}(q_\sigma, \delta q)(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - \dot{b}(q_\sigma, \delta q)(\mathbf{v}_h, p - p_h) - a(q_\sigma)(\delta\widehat{\mathbf{u}}_h - \delta\mathbf{u}_h, \mathbf{v}_h)}{\widehat{\beta}\|\nabla\mathbf{v}_h\|} \leqslant \\
&\leqslant \frac{1}{\widehat{\beta}}\left[c\|\delta q\|_{H^2(I)}(\|\nabla\mathbf{u} - \nabla\mathbf{u}_h\| + \|p - p_h\|) + M\|\nabla\delta\widehat{\mathbf{u}}_h - \nabla\delta\mathbf{u}_h\| \right].
\end{aligned}
$$

From estimate (40) and point (a) of the present lemma, we get the desired bound, i.e. $ch^2\|\delta q\|_{H^3(I)}$. Collecting the estimates for the four terms in (34) yields the validity of point (b).

Finally, we prove (c), employing the regularity result for $S''(q)(\delta q, \delta q)$ given at the third point of Assumption 2.15. The only difference from the previous point is the "intermediate derivative"

$(\delta\delta\breve{\mathbf{u}}_h, \delta\delta\breve{p}_h) \in V_h \times P_h$, defined as the solution of the following problem:

$$
\begin{cases}
a(q)(\delta\delta\breve{\mathbf{u}}_h, \mathbf{v}_h) + b(q)(\mathbf{v}_h, \delta\delta\breve{p}_h) = \\
\qquad = \ddot{F}(q, \delta q, \delta q)(\mathbf{v}_h) - \ddot{a}(q, \delta q, \delta q)(\mathbf{u}, \mathbf{v}_h) - \ddot{b}(q, \delta q, \delta q)(\mathbf{v}_h, p) + \qquad \forall\, \mathbf{v}_h \in V_h, \\
\qquad - 2\,\dot{a}(q, \delta q)(\delta\mathbf{u}, \mathbf{v}_h) - 2\,\dot{b}(q, \delta q)(\mathbf{v}_h, \delta p) \\
b(q)(\delta\delta\breve{\mathbf{u}}_h, \pi_h) = \ddot{G}(q, \delta q, \delta q)(\pi_h) - \ddot{b}(q, \delta q, \delta q)(\mathbf{u}_h, \pi_h) + \\
\qquad - 2\,\dot{b}(q, \delta q)(\delta\mathbf{u}_h, \pi_h) \qquad\qquad\qquad\qquad\qquad\qquad\quad \forall\, \pi_h \in P_h.
\end{cases}
$$

All the previous steps performed to estimate $S' - S_h'$ can be easily adapted to the present context.

A direct consequence of the previous lemma is the following convergence result for the discrete functional.

Lemma 2.19. $\forall q_\sigma \in Q_\sigma^{ad}, \delta q \in \delta Q$ *it holds*

(a) $|j(q_\sigma) - j_h(q_\sigma)| \leqslant ch^2$,
(b) $|j'(q_\sigma)(\delta q) - j_h'(q_\sigma)(\delta q)| \leqslant ch^2 \|\delta q\|_{H^3(I)}$,
(c) $|j''(q_\sigma)(\delta q, \delta q) - j_h''(q_\sigma)(\delta q, \delta q)| \leqslant ch^2 \|\delta q\|_{H^3(I)}^2$.

Proof. Let us fix a $q_\sigma \in Q_\sigma^{ad}$, $\delta q \in \delta Q$ and define $(\mathbf{u}, p) = S(q_\sigma)$, $(\delta\mathbf{u}, \delta p) = S'(q_\sigma)(\delta q)$, $(\delta\delta\mathbf{u}, \delta\delta p) = S''(q_\sigma)(\delta q, \delta q)$.

Let us first prove (a). It it easy to show that the following holds

$$
|j(q_\sigma) - j_h(q_\sigma)| = |((\nabla\mathbf{u} - \nabla\mathbf{u}_h)\, A_q, \nabla\mathbf{u} + \nabla\mathbf{u}_h)| \leqslant
$$
$$
\leqslant \|A_q\|_\infty (\|\nabla\mathbf{u}\| + \|\nabla\mathbf{u}_h\|)\|\nabla\mathbf{u} - \nabla\mathbf{u}_h\| \leqslant ch^2,
$$

where the last inequality employs the boundedness of $A_q, \nabla\mathbf{u}, \nabla\mathbf{u}_h$ and Lemma 2.18.

Now we prove (b), according to the following steps:

$$
|j'(q_\sigma)(\delta q) - j_h'(q_\sigma)(\delta q)| \leqslant |((\nabla\mathbf{u} - \nabla\mathbf{u}_h)\, A_{q,\delta q}', \nabla\mathbf{u} + \nabla\mathbf{u}_h)| +
$$
$$
+ 2|((\nabla\delta\mathbf{u} - \nabla\delta\mathbf{u}_h)\, A_q, \nabla\mathbf{u})| + 2|(\nabla\delta\mathbf{u}_h\, A_q, \nabla\mathbf{u} - \nabla\mathbf{u}_h)| \leqslant
$$
$$
\leqslant ch^2 \|\delta q\|_{H^3(I)}.
$$

Indeed, it holds $\|A_{q,\delta q}'\|_\infty \leqslant c\|\delta q\|_{H^2(I)}$ (see Proposition 2.3) while $\|\nabla\mathbf{u}\|$ and $\|\nabla\delta\mathbf{u}_h\|$ are controlled thanks to the continuous and discrete versions of Theorem 2.4, and the discretization error terms are bounded through Lemma 2.18.

Finally, we prove (c), as follows:

$$
|j''(q_\sigma)(\delta q, \delta q) - j_h''(q_\sigma)(\delta q, \delta q)| \leqslant |((\nabla\mathbf{u} - \nabla\mathbf{u}_h)\, A_{q,\delta q,\delta q}'', \nabla\mathbf{u} + \nabla\mathbf{u}_h)| +
$$
$$
+ 4|((\nabla\delta\mathbf{u} - \nabla\delta\mathbf{u}_h)\, A_{q,\delta q}', \nabla\mathbf{u})| + 4|(\nabla\delta\mathbf{u}_h\, A_{q,\delta q}', \nabla\mathbf{u} - \nabla\mathbf{u}_h)| +
$$
$$
+ 2|((\nabla\delta\mathbf{u} - \nabla\delta\mathbf{u}_h)\, A_q, \nabla\delta\mathbf{u} + \nabla\delta\mathbf{u}_h)| +
$$
$$
+ 2|((\nabla\delta\delta\mathbf{u} - \nabla\delta\delta\mathbf{u}_h)\, A_q, \nabla\mathbf{u})| + 2|(\nabla\delta\delta\mathbf{u}_h\, A_q, \nabla\mathbf{u} - \nabla\mathbf{u}_h)|.
$$

To bound the terms not involving $\delta\delta\mathbf{u}$ and $\delta\delta\mathbf{u}_h$, one can employ Proposition 2.3 to handle the matrix terms, together with similar techniques already used to prove (a) and (b). To bound the last two terms, we have to apply Lemma 2.18, point c, and Theorem A.5 in order to provide estimates for $\|\nabla\delta\delta\mathbf{u} - \nabla\delta\delta\mathbf{u}_h\|$ and $\|\nabla\delta\delta\mathbf{u}_h\|$.

Finally, collecting the previous results, we can prove the main result of this section.

Theorem 2.20 (A priori convergence estimates)**.** *Let Assumptions 2.8, 2.9 and 2.15 hold. Then, denoted by q a local solution of (8), there exists a sequence $\{q_{\sigma,h}\}_{\sigma,h>0}$ of local optimal solution of the discrete problem*

$$
\min_{q_\sigma \in Q_\sigma^{ad}} j_h(q_\sigma), \tag{42}
$$

such that

$$\|\bar{q} - \bar{q}_{\sigma,h}\|_{H^3(I)} = \mathcal{O}(\sigma^2 + h^2),$$
$$\|S(\bar{q}) - S_h(\bar{q}_{\sigma,h})\|_{V \times P} = \mathcal{O}(\sigma^2 + h^2),$$
$$|j(\bar{q}) - j_h(\bar{q}_{\sigma,h})| = \mathcal{O}(\sigma^2 + h^2).$$

Proof. Let $\bar{q}_\sigma, \bar{q}_{\sigma,h}$ denote the optimal controls for the semi-discrete problem (21) and the fully discretized problem (42), respectively. The Mean Value Theorem ensures the existence of $t \in (0,1)$ such that, with $\xi = t\bar{q}_\sigma + (1-t)\bar{q}_{\sigma,h}$, we have

$$j_h'(\bar{q}_\sigma)(\delta q_\sigma) - j_h'(\bar{q}_{\sigma,h})(\delta q_\sigma) = j_h''(\xi)(\delta q_\sigma, \bar{q}_\sigma - \bar{q}_{\sigma,h}). \tag{43}$$

Applying Lemma 2.13 and taking $\bar{q}_\sigma - \bar{q}_{\sigma,h}$ as a variation, we get:

$$
\begin{aligned}
\frac{\delta_2}{2}\|\bar{q}_\sigma - \bar{q}_{\sigma,h}\|^2_{H^3(I)} &\leqslant j''(\xi)(\bar{q}_\sigma - \bar{q}_{\sigma,h}, \bar{q}_\sigma - \bar{q}_{\sigma,h}) \leqslant \\
&\leqslant j_h''(\xi)(\bar{q}_\sigma - \bar{q}_{\sigma,h}, \bar{q}_\sigma - \bar{q}_{\sigma,h}) + \\
&\quad + |j''(\xi)(\bar{q}_\sigma - \bar{q}_{\sigma,h}, \bar{q}_\sigma - \bar{q}_{\sigma,h}) - j_h''(\xi)(\bar{q}_\sigma - \bar{q}_{\sigma,h}, \bar{q}_\sigma - \bar{q}_{\sigma,h})| \leqslant \\
&\leqslant j_h'(\bar{q}_\sigma)(\bar{q}_\sigma - \bar{q}_{\sigma,h}) - j_h'(\bar{q}_{\sigma,h})(\bar{q}_\sigma - \bar{q}_{\sigma,h}) + c_1 h^2 \|\bar{q}_\sigma - \bar{q}_{\sigma,h}\|^2_{H^3(I)},
\end{aligned}
\tag{44}
$$

where the last inequality is obtained by (43) and Lemma 2.19(c). Using the fact that $j_h'(\bar{q}_{\sigma,h})(\bar{q}_\sigma - \bar{q}_{\sigma,h}) = j'(\bar{q}_\sigma)(\bar{q}_\sigma - \bar{q}_{\sigma,h}) = 0$ in the right-hand side of (44) and then applying Lemma 2.19(b), we obtain:

$$
\begin{aligned}
\frac{\delta_2}{2}\|\bar{q}_\sigma - \bar{q}_{\sigma,h}\|^2_{H^2(I)} &\leqslant j_h'(\bar{q}_\sigma)(\bar{q}_\sigma - \bar{q}_{\sigma,h}) - j'(\bar{q}_\sigma)(\bar{q}_\sigma - \bar{q}_{\sigma,h}) + c_1 h^2 \|\bar{q}_\sigma - \bar{q}_{\sigma,h}\|^2_{H^3(I)} \leqslant \\
&\leqslant c_2 h^2 \|\bar{q}_\sigma - \bar{q}_{\sigma,h}\|_{H^3(I)} + c_1 h^2 \|\bar{q}_\sigma - \bar{q}_{\sigma,h}\|^2_{H^3(I)}.
\end{aligned}
$$

Therefore, for sufficiently small h, i.e. for

$$h \leqslant \left(\frac{\delta_2}{2c_1}\right)^{1/2},$$

the following convergence error estimate holds:

$$\|\bar{q} - \bar{q}_{\sigma,h}\|_{H^3(I)} \leqslant \|\bar{q} - \bar{q}_\sigma\|_{H^3(I)} + \|\bar{q}_\sigma - \bar{q}_{\sigma,h}\|_{H^3(I)} = \mathcal{O}(\sigma^2 + h^2).$$

This result yields the second point of thesis, since

$$\|S(\bar{q}) - S_h(\bar{q}_{\sigma,h})\|_{V \times P} \leqslant \|S(\bar{q}) - S(\bar{q}_{\sigma,h})\|_{V \times P} + \|S(\bar{q}_{\sigma,h}) - S_h(\bar{q}_{\sigma,h})\|_{V \times P}, \tag{45}$$

and the desired estimate for S_h follows from applying Lemmas 2.11 and 2.18 to the two terms at right-hand side of (45). An analogous argument, using Lemmas 2.12 and 2.19, yields the estimate for j_h.

3. Numerical results

In this section, we present two sets of numerical results. The numerical implementation has been carried out basing on the *FEniCS* project (see [25] and http://fenicsproject.org), and the optimal solution is obtained iteratively, using the following gradient method [27]:

> **Gradient method iteration**
>
> Given q_{old} from the previous iteration,
> set the descent step length ε to the initial value $\widehat{\varepsilon} > 0$. Then,
>   (1) solve state and adjoint problems in order to obtain $(\mathbf{u}, p), (\mathbf{z}, s)$
>   (2) build $\nabla j(q_{old})$
>   (3) project $\nabla j(q_{old})$ on the set of admissible variations, obtaining G
>   (4) restrict G on Γ_0 and then map it to I, to get g
>   (5) back-tracking: set $q_{new} = q_{old} - \widehat{\varepsilon} g$
>   while $j(q_{new}) > j(q_{old})$ and $\varepsilon > \varepsilon_{min}$ do:
>   (a) update: $q_{new} = q_{old} - \varepsilon g$
>   (b) $\varepsilon = \varepsilon/2$

In general, the functional gradient $\nabla j(q_{old})$, obtained as in Lemma 1.9, is not an admissible variation, since one cannot prove the existence of some $\varepsilon > 0$ such that $q = q_{old} - \varepsilon \nabla j(q_{old})$ satisfies

$$q(0) = q(1) = 0.$$

This is why in the gradient method the projection step (3) is required. The gradient $\nabla j(q_{old})$ is projected onto $H^1_{\partial\Omega_0 \backslash \Gamma_0}(\Omega_0)$ solving the following problem:

$$
\begin{cases}
-\Delta G + G = 0, & \text{in } \Omega_0, \\
G = 0, & \text{on } \partial\Omega_0 \backslash \Gamma_0, \\
-\partial_{\mathbf{n}} G = -\nabla j(q_{old}), & \text{on } \Gamma_0.
\end{cases}
$$

Then, step (4) of the algorithm reduces G, defined on Ω_0, to a function g belonging to the space of controls.

The results obtained by the application of the above algorithm to the shape optimization problem (8) are now presented and discussed. Two different functionals will be considered in the two test cases.

Remark 3.1. We remark that we use finite element discretization, with $\mathbb{P}_2 - \mathbb{P}_1$ pair for state velocity and pressure and with piecewise linear basis functions for the control. As we will see, even if the polynomial degree for controls is not as high as assumed in the derivation of a priori estimates, the numerical results comply the theoretical ones. In these numerical tests, we consider a unique discretization parameter, i.e. we set $\sigma = h$.

3.1. **Test case 1**

In this first test case, we take into account the following functional:

$$\widetilde{j}(q) = \int_{\Omega_q} |\nabla\widetilde{\mathbf{u}}|^2 \, d\Omega + \alpha \int_{\Gamma_q} d\Gamma + \beta \left(\int_I q(x) \, dx - \overline{V} \right)^2.$$

Its counterpart on the pulled-back formulation (7) reads

$$j(q) = \int_{\Omega_0} |\nabla\mathbf{u} DT_q^{-1}|^2 \, d\Omega + \alpha \int_I \sqrt{1 + (q'(x))^2} \, dx + \beta \left(\int_I q(x) \, dx - \overline{V} \right)^2. \tag{46}$$

The gradient of this functional is given by

$$\nabla j(q) = [\nabla\mathbf{u} DT_q^{-1} DT_q^{-T}\mathbf{n}] \cdot \left[(\nu\nabla\mathbf{z} - \mathbf{u}) DT_q^{-1} DT_q^{-T}\mathbf{n} \right] |_{\Gamma_0} +$$
$$- 2\alpha \frac{q''}{1 + (q')^2} + 2\beta \left(\int_I q(x) dx - \overline{V} \right).$$

The regularization term considered in (46) is often used in literature (see, e.g., [11, 26]) and it consists in the penalization of the perimeter of the moving portion Γ_q of the domain boundary.

This new term is simpler to handle than the curvature term $\|q''\|^2_{L^2(I)}$: indeed, using the original term would require the introduction of a further adjoint problem, to extract the Riesz representative in $L^2(I)$ of $\delta q \mapsto (q'', \delta q'')_I$. Moreover, the perimeter term can be supposed to generally act in the same way as the curvature term, since a shorter perimeter corresponds to less oscillations, and vice versa.
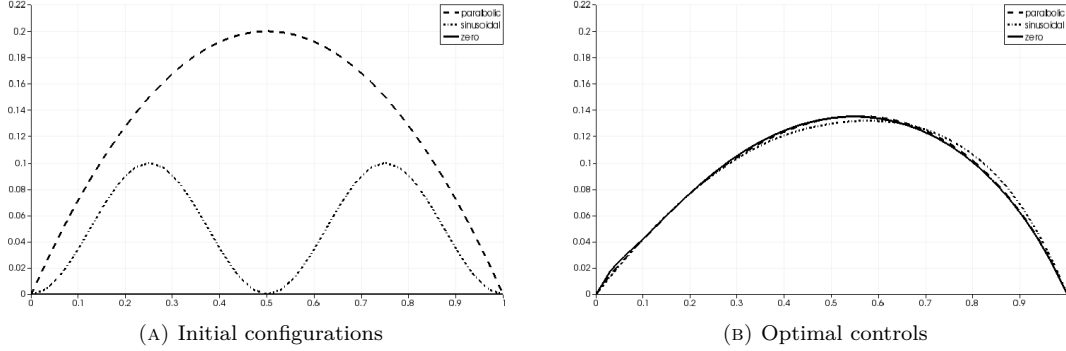

(A) Initial configurations

(B) Optimal controls

FIGURE 2. Independence of the optimal control from the initial configuration, for $\alpha = 10$, $\beta = 10\,000$, $\overline{V} = [0.7$ times the initial area of the parabolic case].

We first analyze the dependence of the optimal solution on the initial configuration. We considered three different initial solutions, defined by a parabolic function ($q(x) = 0.2[1 - 4(x - 0.5)^2]$), a sinusoidal function ($q(x) = 0.1\sin(2\pi x)^2$), and the flat function ($q(x) \equiv 0$). As shown in Fig. 2, the optimal control obtained are very close, starting from different initial controls. The final configurations in Fig. 2b are reached in less than 10 iterations, with $\widehat{\varepsilon} = 0.1, \varepsilon_{min} = 10^{-8}$, and the reaching of $\varepsilon \leqslant \varepsilon_{min}$ as the stop criterion on the iterations of the gradient method.

The dependence of the solution on the value of the penalty parameters has also been analyzed, starting from the parabolic configuration in Fig. 2a. Concerning parameter α, a minimum value has to be exceeded in order to prevent the gradient method from converging to a local, sub-optimal minimum. Indeed, Fig. 3a shows that for lower values of α, oscillating controls are found at the end of the optimization algorithm, though the value of the functional in such configurations is higher than the ones corresponding to $\alpha = 10, 1000$. Moreover, a maximum value must not be exceeded, otherwise the regularization parameter dominates too much in the total functional value, leading to a nearly flat optimal control. About parameter β, instead, we only need it to be greater than a minimum threshold, in order to sufficiently express the volume constraint. Under these considerations, Fig. 3 shows that the values $\alpha = 10, \beta = 10\,000$, considered in the previous test, are suitable for a proper expression of the two penalty terms.
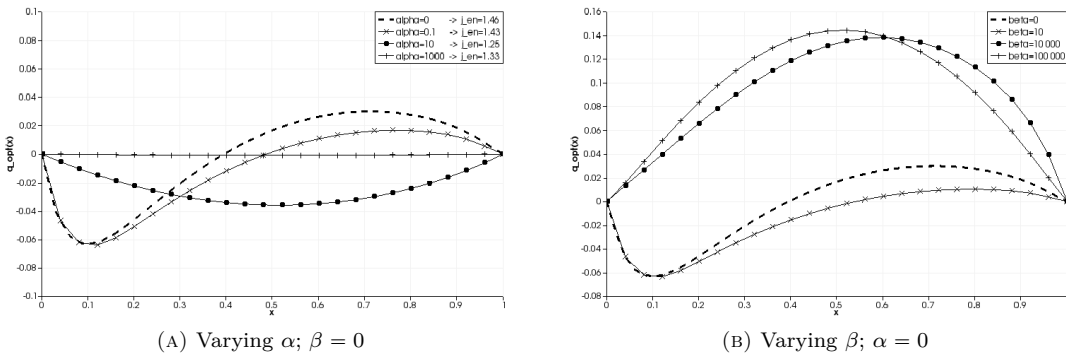

(A) Varying α; $\beta = 0$

(B) Varying β; $\alpha = 0$

FIGURE 3. Final controls obtained by the optimization algorithm for different values of the penalty parameters (j_{en} is the energetic term of the functional j)

3.2. **Test case 2**

In this section, we report a numerical convergence analysis, carried out to validate the a priori error estimates proved in Theorem 2.20. For this purpose, we would like to have an exact solution as a reference point. To this end, we take into account the following functional:

$$\widetilde{j}(q) = \int_{\Omega_q} |\nabla \widetilde{\mathbf{u}} - \nabla \widetilde{\mathbf{u}}_d|^2 \, d\Omega + \frac{\alpha}{2} \int_{\Gamma_q} d\Gamma,$$

with its pulled-back counterpart given by

$$j(q) = \int_{\Omega_0} (\nabla \mathbf{u} - \nabla \mathbf{u}_d) A_q (\nabla \mathbf{u} - \nabla \mathbf{u}_d) d\Omega + \frac{\alpha}{2} \int_I \sqrt{1 + (q'(x))^2} \, dx. \tag{47}$$

The velocity $\widetilde{\mathbf{u}}_d$ is obtained solving the Stokes problem on a domain Ω_{q_d}, identified by the given control function

$$q_d = 0.1 + 0.1 \cos(2\pi(x - 0.5)),$$

and $\mathbf{u}_d = \widetilde{\mathbf{u}}_d \circ T_{q_d}^{-1}$.

Indeed, if no penalty terms are active, the minimum for this functional is zero, and it is reached for $q = q_d$. The functional (47) is a slight generalization of the functional (6), and the theoretical results presented in the previous sections can be easily generalized to the new functional.

Following the steps of Section 1.3, we can derive an expression for the shape gradient in q:

$$\nabla j(q) = -2\alpha \frac{q''(x)}{1 + (q'(x))^2} +$$
$$+ \left[(\nabla \mathbf{u} - \nabla \mathbf{u}_d) DT_q^{-1} DT_q^{-T} \mathbf{n} \right] \cdot \left[(\nu \nabla \mathbf{z}_{\mathbf{u}_d} - \mathbf{u} + \mathbf{u}_d) DT_q^{-1} DT_q^{-T} \mathbf{n} \right],$$

where $\mathbf{z}_{\mathbf{u}_d}$ is the adjoint velocity variable, solution of a problem obtained from a minimal modification of (16), replacing any occurrence of $\widetilde{\mathbf{u}}$ with $\widetilde{\mathbf{u}} - \widetilde{\mathbf{u}}_d$.

Based on the functional defined in (47), different spatial convergence tests have been carried out, taking four specific values for perimeter penalty coefficient α, namely $\alpha = 0, 0.01, 0.1, 1$.

The results reported in Fig. 4 are in agreement with the a priori estimates of the convergence error proved in Theorem 2.20, since an approximately quadratic convergence order is obtained, for a broad spectrum of values of h. However, for $h \to 0$, the graphs in Fig. 4 show a sort of saturation bending. A reason for this can be found in the stopping criterion of the optimization algorithm and in the lower bound imposed on the descent step length, that introduce a finite error. This influence is amplified as α grows, to the point of polluting the convergence behaviour, hence we do not report results for $\alpha > 1$.

Conclusions

In this paper, we have studied a shape optimization problem, namely the minimization of the total energy dissipation for the low-Reynolds flow of a viscous, incompressible fluid, modeled by two-dimensional, steady Stokes equations. After the definition of the problem and the admissible set of control functions, we have reformulated the problem onto a reference domain, by means of a control-dependent map. The well-posedness of the transformed problem has been inspected, and particular attention has been devoted to the inf-sup condition for the form $b(q)$, obtaining a control-independent lower bound for the inf-sup constant. The existence of an optimal solution has also been proved, for the minimization problem at hand, and corresponding first order optimality conditions have been provided.

After the inspection of some differentiability properties of the state solution operator, a FEM discretization of the problem has been introduced. For this discretization, a priori error estimates have been derived, showing a quadratic convergence rate. To our best knowledge, this is the first result about convergence rates obtained for the discretization of Stokes problem in a shape optimization environment. Numerical tests have been performed to assess the validity of the theoretical results.

(A) $\alpha = 0$

(B) $\alpha = 0.01$

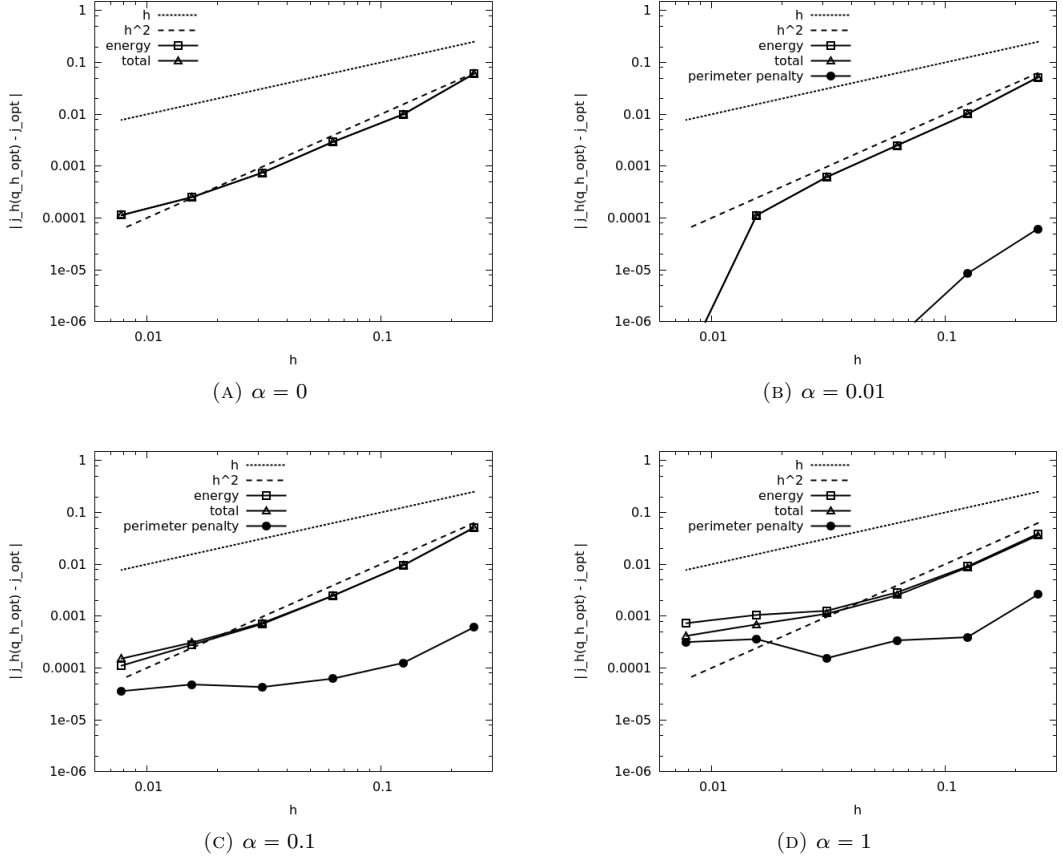(C) $\alpha = 0.1$

(D) $\alpha = 1$

FIGURE 4. Spatial convergence of discrete functional value $j_h(q_{h,opt})$ to its reference value $j(q_{opt})$. Each term of the functional is presented w.r.t. its corresponding term in $j(q_{opt})$, which is known for $\alpha = 0$, and obtained by Richardson extrapolation for $\alpha \neq 0$.

APPENDIX A. ADDITIONAL REGULARITY

In this Appendix we want to show a possible way to derive the regularity properties stated in Assumption 2.15, starting from suitable requests on data and a regularity result on Stokes problem with mixed boundary conditions.

At first, let us state a preliminary result about the transformation of norms defined on the reference domain (Ω_0) and on the physical one (Ω_q).

Lemma A.1. Let $k \in \mathbb{N}$ be fixed, $\varphi \in H^k(\Omega_0)$ and $q \in W^{k,\infty}(I)$. It holds that

$$\varphi \circ T_q^{-1} \in H^k(\Omega_q), \quad c_1 \|q\|_{W^{k,\infty}(I)} \|\varphi \circ T_q^{-1}\|_{H^k(\Omega_q)} \leqslant \|\varphi\|_{H^k(\Omega_0)} \leqslant c_2 \|q\|_{W^{k,\infty}(I)} \|\varphi \circ T_q^{-1}\|_{H^k(\Omega_q)}.$$

Vice versa, it holds that $\widetilde{\varphi} \in H^k(\Omega_q)$ *implies* $\widetilde{\varphi} \circ T_q \in H^k(\Omega_0)$, *together with similar inequalities.*

In connection with this lemma, we restrict a little the set of admissible controls. From now on, the definition of Q^{ad} will contain also the belonging of control functions q to $W^{3,\infty}(I)$ and the existence of a constant $c_\infty > 0$ such that

$$\|q\|_{W^{3,\infty}} \leqslant c_\infty \qquad \forall\, q \in Q^{ad}. \tag{48}$$

that is

$$Q^{ad} := \{q \in W^{3,\infty}(I) \cap H_0^1(I) \colon q(x) \leqslant 1 - \varepsilon, \ \forall x \in I, \ \text{ and } \ \|q\|_{W^{3,\infty}(I)} \leqslant c_\infty\}.$$

Thanks to the above definition and to Lemma A.1, when handling with functions belonging to $H^k(\Omega_0)$ or $H^k(\Omega_q)$ for $k \leqslant 3$, we can indifferently consider their norm in the physical domain Ω_q or in the reference domain Ω_0.

Now we take into account the state problem, and we assume the validity of the following regularity result for Stokes problem:

Assumption A.2. *Let Ω_q be an open, bounded set of $\mathbb{R}^2$ and let Γ_q be $C^{1,1}$ and $\partial\Omega_q \backslash \Gamma_q$ polygonal with $\partial\Omega_q$ having convex corners. Assume that data functions fulfill the following requests:*

$$\nu \in H^3(\widehat{\Omega}), \quad \eta \in H^2(\widehat{\Omega}), \quad \mathbf{f} \in [H^2(\widehat{\Omega})]^2, \quad \mathbf{g}_D \in [H^{7/2}(\Gamma_3)]^2, \quad \mathbf{g}_N \in [H^{5/2}(\Gamma_1)]^2,$$

and suitable compatibility conditions. Then, for the solution $(\widetilde{\mathbf{u}}, \widetilde{p})$ of (4), the following hold:

(a) $(\widetilde{\mathbf{u}}, \widetilde{p}) \in [H^3(\Omega_q)]^2 \times H^2(\Omega_q)$
(b) $\|\nabla^3 \widetilde{\mathbf{u}}\|_{\Omega_q} + \|\nabla^2 \widetilde{p}\|_{\Omega_q} \leqslant c(\eta, \nu, \mathbf{g}_D, \mathbf{g}_N, \mathbf{f}, \widehat{\Omega})$.

Remark A.3. Assumption A.2 can be proved by resorting to results presented in [19]. In that paper, weighted Sobolev spaces $H^{k,l}_\beta(\Omega)$ are considered, but the results can be brought back to classical Sobolev spaces exploiting the following inclusions: $H^k(\Omega_q) \subset H^{k,l}_\beta(\Omega_q) \subset H^{l-1}(\Omega_q)$, for any $k \geqslant l \geqslant 0$ and $\beta \in [0,1]^4$.

The last ingredient that we need in order to prove a regularity result for the solution of our transformed problem (7) is represented by additional regularity requests on data. Since we want regularity not only for the solution of (7), but also for its derivatives w.r.t. the control, namely $S'(q)(\delta q), S''(q)(\delta q, \delta q)$, we have to assume a slightly stronger regularity of data than that considered in Assumption A.2.

Assumption A.4. *Data functions have the following regularity:*

$$\nu \in H^5(\widehat{\Omega}), \quad \eta \in H^4(\widehat{\Omega}), \quad \mathbf{f} \in [H^4(\widehat{\Omega})]^2, \quad \mathbf{g}_D \in [H^{7/2}(\Gamma_3)]^2, \quad \mathbf{g}_N \in [H^{5/2}(\Gamma_1)]^2,$$

and suitable compatibility conditions hold on data.

We are now ready to state a regularity result for the state variables and their shape derivatives.

Theorem A.5. *Under Assumptions A.4, A.2, there exist three positive constants c_0, c_1, c_2, such that for any $q \in Q^{ad}$, with $\|q\|_{W^{3,\infty}(I)} \leqslant c_\infty$, and for any $\delta q, \tau q \in \delta Q$, and independently from them, it holds that*

$$S(q), S'(q)(\delta q), S''(q)(\delta q, \tau q) \ \in \ [H^3(\Omega_0)]^2 \times H^2(\Omega_0) \qquad and$$

(a) $\|S(q)\|_{[H^3(\Omega_0)]^2 \times H^2(\Omega_0)} \leqslant c_0$
(b) $\|S'(q)(\delta q)\|_{[H^3(\Omega_0)]^2 \times H^2(\Omega_0)} \leqslant c_1 \|\delta q\|_{H^3(I)}$
(c) $\|S''(q)(\delta q, \tau q)\|_{[H^3(\Omega_0)]^2 \times H^2(\Omega_0)} \leqslant c_2 \|\delta q\|_{H^3(I)} \|\tau q\|_{H^3(I)}$.

Proof. Let $q \in Q^{ad}$, consider solution $(\mathbf{u}, p) = S(q)$ of the transformed problem (7) and remind that its physical counterpart $(\widetilde{\mathbf{u}}, \widetilde{p}) = \widetilde{S}(q)$ is the solution of Stokes problem (4) on Ω_q.
Now, we can verify the hypotheses of Lemma A.2: Ω_q is surely an open bounded subset of $\mathbb{R}^2$; its boundary Γ_q is $C^{1,1}$ because it is the graph of the control function $q \in Q^{ad} \subset H^3(I) \subset C^{1,1}(\overline{\Omega_q})$ and for the same reason its terminal points cannot present a concave angle; the regularity of external force and boundary data, together with the compatibility conditions, are given by Assumption A.4. Then, Lemma A.2 holds and we have $(\widetilde{\mathbf{u}}, \widetilde{p}) \in [H^3(\Omega_q)]^2 \times H^2(\Omega_q)$ and $\|\nabla^3 \widetilde{\mathbf{u}}\|_{\Omega_q} + \|\nabla^2 \widetilde{p}\|_{\Omega_q} \leqslant c(\mathbf{f}, \mathbf{g}_D, \mathbf{g}_N, \widehat{\Omega})$. Finally, the results on $(\widetilde{\mathbf{u}}, \widetilde{p})$ directly transfer to $(\mathbf{u}, p)$, thanks to Lemma A.1.

For points (b) and (c), the proof is exactly the same, considering Assumption A.4 in order to control the more complex right-hand sides appearing dealing with $S'(q)(\delta q)$ and $S''(q)(\delta q, \delta q)$, with the aim of prove the validity of the hypotheses of Lemma A.2. The dependence on $\|\delta q\|_{H^3(I)}$ on the right-hand side comes out from the bounds of the coefficients, similar to those reported in Proposition 2.3.

So far we have obtained a regularity result for the state variables: now we want to show that the optimal control belongs to $H^5(I)$. Indeed, this regularity holds for any $q \in Q^{ad}$ satisfying the first order optimality condition, as stated in the following result:

Theorem A.6. *Let $\overline{q} \in Q^{ad}$ be such that optimality condition* (13) *holds in $\overline{q}$. Then it holds that $\overline{q} \in H^5(I)$.*

Proof. Let us take into account Hadamard formula for j', given by Lemma 1.9, i.e. $j'(q)(\delta q) = 2\alpha(q'', \delta q'')_I + (\Psi, \delta q)_I$. We start by noticing that $\beta(\int_I q(x)dx - \overline{V})$ is constant, then certainly belonging to $H^1(I)$.

The regularity Theorem A.5 can be applied to both the state variables $(\mathbf{u}, p)$ and the adjoint state variables $(\mathbf{z}, s)$. Then, thanks to the definition (48) and Lemma A.1, we get

$$(\widetilde{\mathbf{u}}, \widetilde{\mathbf{z}}, \widetilde{s}) = (\mathbf{u}, \mathbf{z}, s) \circ T_q \quad \in [H^3(\Omega_q)]^2 \times [H^3(\Omega_q)]^2 \times H^2(\Omega_q).$$

Taking the traces of $\nabla\widetilde{\mathbf{u}}, \nabla\widetilde{\mathbf{z}}, \widetilde{s}$ on boundary Γ_q and using its parametrization $\gamma : x \mapsto (x, q(x))$, we get

$$(\nabla\widetilde{\mathbf{u}}, \nabla\widetilde{\mathbf{z}}, \widetilde{s})(x, q(x)) \quad \in \quad [H^{3/2}(I)]^{2\times 2} \times [H^{3/2}(I)]^{2\times 2} \times H^{3/2}(I).$$

Thanks to this regularity, together with the continuous embedding $H^{3/2}(I) \hookrightarrow W^{1,4}(I)$, we can conclude that Ψ belongs to $H^1(I)$.

Now, taking a $\overline{q} \in Q^{ad}$ such that the optimality condition $j'(\overline{q})(\delta q) = 0$ holds, we get

$$\int_I \overline{q}'' \delta q'' dx = -\int_I \frac{1}{2\alpha} \Psi \delta q\, dx \qquad \forall \delta q \in C_0^\infty(\overline{I}). \tag{49}$$

Finally, we observe that (49) is equivalent to say that the fourth weak derivative of $\overline{q}$ is exactly $-\frac{1}{2\alpha}\Psi$. Being α a non-zero constant and belonging Ψ to $H^1(I)$, we get $\overline{q}^{(iv)} \in H^1(I)$. Since we already have $\overline{q} \in W^{3,\infty} \subset H^3(I)$ (see (48)), we obtain the thesis, i.e. $\overline{q} \in H^5(I)$.

APPENDIX B. RESULTS FOR THE COERCIVITY OF FUNCTIONAL j

In this appendix, we present two useful results for the proof of Lemma 2.13. The first concerns the sequential continuity of the state operator derivatives w.r.t. the variations of control.

Lemma B.1. *Let $q \in Q^{ad}$ and consider a sequence $\{\delta q_n\}_{n\in\mathbb{N}} \subset Q$. If there exists a $\delta q \in Q$ such that $\delta q_n \to \delta q$ in $C^1(\overline{I})$, then*
(a) $S'(q)(\delta q_n) \to S'(q)(\delta q)$ in $V \times P$
(b) $S''(q)(\delta q_n, \delta q_n) \to S''(q)(\delta q, \delta q)$ in $V \times P$

Proof. Because of the linearity and the well-posedness of problems (17), (18), it suffices to prove the convergence of the right-hand sides in $V' \times P'$: this is obtained from the continuity of $\dot{F}, \dot{a}, \dot{b},$ $\ddot{F}, \ddot{a}, \ddot{b}$ w.r.t. the variation δq.

We just give an example of the steps to be taken, processing a term from $\ddot{a}(q, \delta q\delta q)(\mathbf{u}, \mathbf{v})$:

$$\left|\left(\nabla\nu^q \cdot V_{\delta q_n}\nabla\mathbf{u}\, A'_{q,\delta q_n}, \nabla\mathbf{v}\right) - \left(\nabla\nu^q \cdot V_{\delta q}\nabla\mathbf{u}\, A'_{q,\delta q}, \nabla\mathbf{v}\right)\right| \leqslant$$

$$\leqslant \left|\left(\nabla\nu^q \cdot V_{\delta q_n}\nabla\mathbf{u}(A'_{q,\delta q_n} - A'_{q,\delta q}), \nabla\mathbf{v}\right)\right| + \left|\left(\nabla\nu^q \cdot (V_{\delta q_n} - V_{\delta q})\nabla\mathbf{u}\, A'_{q,\delta q}, \nabla\mathbf{v}\right)\right| \leqslant$$

$$\leqslant \|\nu\|_{W^{1,\infty}(\widehat{\Omega})}\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\| \left(\|A'_{q,\delta q_n} - A'_{q,\delta q}\|_\infty\|V_{\delta q_n}\|_\infty + \|V_{\delta q_n} - V_{\delta q}\|_\infty\|A'_{q,\delta q}\|_\infty\right)$$

The convergence of δq_n in $C^1(\overline{I})$ implies the uniform convergence of $A'_{q,\delta q_n} - A'_{q,\delta q}$ and $V_{\delta q_n} - V_{\delta q}$ to zero, as it can be seen from the definition of such quantities. Moreover, being $\{\delta q_n\}$ bounded in $C^1(\overline{I})$, Proposition 2.3 ensures that $\|V_{\delta q_n}\|_\infty, \|A'_{q,\delta q}\|_\infty$ are bounded themselves.

From Lemma B.1, using the Dominated Convergence Theorem and the compact embedding $H^2(I) \subset\subset C^1(\overline{I})$ yields a similar result for the derivatives of cost functional j:

Corollary B.2. *Let* $q \in Q^{ad}$ *and* $\{\delta q_n\}_{n \in \mathbb{N}} \subset \delta Q$ *such that there exists a* $\delta q \in \delta Q$ *for which* $\delta q_n \rightharpoonup \delta q$ *in* $H^2(I)$. *Then,*

$$j'(q)(\delta q_n) \underset{n \to \infty}{\longrightarrow} j(q)(\delta q), \qquad j''(q)(\delta q, \delta q) \leqslant \liminf_{n \to \infty} j''(q)(\delta q_n, \delta q_n).$$

References

[1] G. Allaire. *Conception optimale de structures*, volume 58 of *Mathématiques & Applications*. Springer-Verlag, Berlin, 2007.

[2] P. F. Antonietti, A. Borzì, and M. Verani. Multigrid shape optimization governed by elliptic PDEs. *SIAM J. Control Optim.*, 51(2):1417–1440, 2013.

[3] F. Ballarin, A. Manzoni, G. Rozza, and S. Salsa. Shape Optimization by Free-Form Deformation: Existence Results and Numerical Solution for Stokes Flows. *Journal of Scientific Computing*, pages 1–27, December 2013.

[4] D. Begis and R. Glowinski. Application de la méthode des éléments finis à l'approximation d'un problème de domaine optimal. Méthodes de résolution des problèmes approchés. *Appl. Math. Optim.*, 2(2):130–169, 1975/76.

[5] F. Bélahcène and J.-A. Desideri. Paramétrisation de Bézier adaptative pour l'optimisation de forme en Aérodynamique. Technical report, INRIA, Sophia Antipolis Cedex, 2003.

[6] S. C. Brenner and R Scott. *The mathematical theory of finite element methods*. Springer Texts in Applied Mathematics, 3 edition, 2008.

[7] F. Brezzi. On the existence uniqueness and approximation of saddle point problems arising from lagrangian multipliers. *Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique*, 8(2):129–151, 1974.

[8] D. Chenais and E. Zuazua. Controllability of an elliptic equation and its finite difference approximation by the shape of the domain. *Numer. Math.*, 95(1):63–99, 2003.

[9] D. Chenais and E. Zuazua. Finite-element approximation of 2D elliptic optimal design. *Journal de Mathématiques Pures et Appliquées*, 85(2):225–249, 2006.

[10] M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*, volume 22 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2011.

[11] G. Dogan, P. Morin, R. H. Nochetto, and M. Verani. Discrete gradient flows for shape optimization and applications. *Computer Methods in Applied Mechanics and Engineering*, 196(37-40):3898–3914, August 2007.

[12] R. G. Durán. An elementary proof of the continuity from $L_0^2(\Omega)$ to $H_0^1(\Omega)^n$ of Bogovskii's right inverse of the divergence. *Revista de la Unión Matemática Argentina*, 53(2):59–78, 2013.

[13] K. Eppler. Second derivatives and sufficient optimality conditions for shape functionals. *Control Cybernet.*, 29(2):485–511, 2000.

[14] K. Eppler, H. Harbrecht, and R. Schneider. On Convergence in Elliptic Shape Optimization. *SIAM Journal on Control and Optimization*, 46(1):61–83, January 2007.

[15] I. Fumagalli. *Shape optimization for Stokes flows: a reference-domain approach*. M.sc. thesis, Politecnico di Milano, 2013, `http://www.mate.polimi.it/biblioteca/?pp=view&id=537&collezione=tesi&L=i`.

[16] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986.

[17] M. D. Gunzburger. *Perspectives in flow control and optimization*, volume 5 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.

[18] M. D. Gunzburger, H. Kim, and S. Manservisi. On a shape control problem for the stationary Navier-Stokes equations. *Mathematical Modelling and Numerical Analysis*, 34(6):1233–1258, 2000.

[19] B. Guo and C. Schwab. Analytic regularity of Stokes flow on polygonal domains in countably weighted Sobolev spaces. *Journal of Computational and Applied Mathematics*, 190(1-2):487–519, June 2006.

[20] J. Haslinger and R. A. E. Mäkinen. *Introduction to shape optimization*, volume 7 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.

[21] J. Haslinger and P. Neittaanmäki. *Finite element approximation for optimal shape design: Theory and applications*. Wiley Chichester, 1988.

[22] B. Kiniger and B. Vexler. A priori error estimates for finite element discretizations of a shape optimization problem. *ESAIM Math. Model. Numer. Anal.*, 47(6):1733–1763, 2013.

[23] M. Laumen. Newton's method for a class of optimal shape design problems. *SIAM J. Optim.*, 10(2):503–533 (electronic), 2000.

[24] J.-L. Lions. *Optimal control of systems governed by partial differential equations.* Translated from the French by S. K. Mitter. Die Grundlehren der mathematischen Wissenschaften, Band 170. Springer-Verlag, New York, 1971.

[25] A. Logg, K.-A. Mardal, and G. N. Wells, editors. *Automated Solution of Differential Equations by the Finite Element Method.* Springer, Berlin, 2012.

[26] P. Morin, R. H. Nochetto, M. S. Pauletti, and M. Verani. Adaptive finite element method for shape optimization. *ESAIM: Control, Optimisation and Calculus of Variations*, 18(4):1122–1149, January 2012.

[27] J. Nocedal and S. J. Wright. *Numerical optimization*. Springer Series in Operations Research and Financial Engineering. Springer, New York, second edition, 2006.

[28] O. Pironneau. On optimum profiles in Stokes flow. *J. Fluid Mech.*, 59:117–128, 1973.

[29] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer, Series in Computational Mathematics, 2008.

[30] J. Sokolowsky and J.-P. Zolésio. *Introduction to Shape Optimization*. Springer Berlin Heidelberg, 1992.