Some breathers and multi-breathers for FPU-type chains

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Abstract. We consider several breather solutions for FPU-type chains that have been found numerically. Using computer-assisted techniques, we prove that there exist true solutions nearby, and in some cases, we determine whether or not the solution is spectrally stable. Symmetry properties are considered as well. In addition, we construct solutions that are close to (possibly infinite) sums of breather solutions.

1. Introduction

We consider a system of interacting particles described by the equation

$$\omega^2 \ddot{q}_j = \phi'(q_{j+1} - q_j) - \phi'(q_j - q_{j-1}) - \psi'(q_j)\,, \qquad j \in \mathbb{Z}\,, \tag{1.1}$$

where $\phi(x) = \frac{1}{2}\phi_2 x^2 + \frac{1}{3}\phi_3 x^3 + \frac{1}{4}\phi_4 x^4$ and $\psi(x) = \frac{1}{2}\psi_2 x^2 + \frac{1}{4}\psi_4 x^4$, with ϕ_3 and ϕ_4 not both zero. If ϕ and ψ are given, then the parameter ω simply fixes a time scale.

The equation (1.1) with $\psi = 0$ is know as the Fermi-Pasta-Ulam (FPU) model: the α-model if $\phi_4 = 0$, or the β-model if $\phi_3 = 0$. Models of this type have been studied extensively in connection with the problem of equipartition of energy in systems with a large number of interacting particles. Recent surveys can be found in [6,8,10].

Our goal is to construct solutions that are periodic in time, and in some cases, to determine whether they are (spectrally) stable or not. By choosing the value ω appropriately, it suffices to consider solutions that are periodic with fundamental period 2π. We are interested in solutions that decrease rapidly in $|j|$, also referred to as "breathers", and in solutions that are close to sums of such breathers.

Most of the existing work on breather solutions involves numerical computations or other types of approximations. For simplicity, the computations often focus on solutions that have a reflection symmetry

$$\breve{q} = q\,, \qquad \breve{q}_j(t) = \varpi q_{\sigma - j}(t - \delta t)\,, \qquad j \in \mathbb{Z}\,, \quad t \in \mathbb{R}\,, \tag{1.2}$$

where σ is an integer, ϖ is one of $\pm$, and $\delta t \in \{0, \pi\}$. Breathers that have such a symmetry are commonly referred to as being site-centered if σ is even, or bond-centered if σ is odd. Due to the translation-invariance of the equation (1.1), it suffices to consider $\sigma \in \{0, 1\}$.

Mathematical results concerning breather solutions are based mostly on perturbation theory or variational methods, except for special choices of the potentials ϕ and ψ that admit simple solutions of a special form. A survey of rigorous results can be found in [9].

Numerical methods have typically a much larger scope and give more detailed information; but they do not guarantee that the findings are correct, say up to small errors. In this paper we give criteria that, if satisfied by an approximate solution $\bar{q}$, guarantee the

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existence of a true solution q nearby. For the solutions described in the theorem below, the error $\|q-\bar{q}\|_\infty$ is shown to be less that 2^{-46}. Here, and in what follows, $\|h\|_\infty = \sup_j \|h_j\|_\infty$ and $\|h_j\|_\infty = \sup_t |h_j(t)|$, for any bounded function $(j,t) \mapsto h_j(t)$ on $\mathbb{Z} \times \mathbb{R}$.

Theorem 1.1. *For each row in Table 1, the equation (1.1) with the given parameter values (ϕ, ψ, ω) admits a 2π-periodic solution q that is real analytic in t, decreases at least exponentially in $|j|$, and has norm $\|q\|_\infty > 1$. The entries τ and ϖ in Table 1 describe symmetry properties. The solution q is symmetric or antisymmetric with respect to time-reversal $t \mapsto -t$, depending on whether $\tau = 0$ or $\tau = 1$, respectively. An entry $\varpi = \pm$ indicates that q admits a reflection symmetry as described by (1.2), with $\delta t = 0$.*

The entries r and ρ in Table 1 define bounds on the domain of analyticity and decay rate of the solution; see Section 3. The remaining entries will be described below.

Our proof of this theorem relies on estimates that are verified by a computer. After writing (1.1) as a fixed point equation $G(q) = q$, we prove Theorem 1.1 by verifying that a Newton-type map associated with G is a contraction near $\bar{q}$. This strategy has been used in many computer-assisted proofs, including [7,13].

label	ϕ_2	ϕ_3	ϕ_4	ψ_2	ψ_4	ω	ϖ	σ	τ	r	ρ	ℓ	stab
1	2^{-20}	0	2	1/2	0	32/5	+	0	1	2	2	8	U
2	2^{-20}	0	2	1/2	0	32/5	−	0	1	2	2	16	S
3	1/8	0	1	1/2	0	32/5	+	1	1	2	2	15	S
4	1/8	0	1	1/2	0	32/5	−	1	1	2	2	13	S
5	−1	0	−27/16	1	1	32/5	−	1	1	5/4	5/4	19	U
6	−1	0	1	0	0	256/5	−	1	1	5/4	5/4	11	U
7	−1	0	1	0	0	256/5	+	1	1	5/4	5/4	11	U
8	−1/32	0	1	1	1/2	32/5	−	1	1	17/16	2	16	S
9	−1/32	0	1	1	1/2	32/5	+	0	1	17/16	2	16	ns
10	−1/32	0	1	1	1/2	32/5	+	0	1	17/16	2	16	nu
11	-2^{-20}	0	1	1	1/2	32/5	+	0	1	17/16	2	6	U
12	1	0	1	−1/2	0	32/5	none	1	1	9/8	9/8	21	U
13	1/4	2	4	8	0	64/9	−	1	0	9/8	9/8	15	S

Table 1. parameter values and properties of solutions

Our choice of parameters covers several different situations. Attracting potentials ϕ and ψ are used for the solutions 1-4 and 13. The solutions 1-4 cover the 4 possible reflection symmetries (bond-symmetry, bond-antisymmetry, site-symmetry, site-antisymmetry) with $\delta t = 0$. A globally repelling ϕ is used for solution 5. The solutions 6 and 7 correspond to typical mountain pass configuration for the functional $\mathbb{L}$ described below. For the solutions 8-11 we have a coercive potential ϕ, which is repelling in a neighborhood of 0, while ψ is attracting. The opposite is the case for solution 12. Here we chose a solution with four bumps with different fundamental time-periods. Finally, solution 13 is an even function of time (while our other solutions are odd) and has a nonzero time-average.

The solutions 11 and 12 are shown in Fig. 1, at sites $j \in \mathbb{Z}$ where $\|q_j\|_\infty > 2^{-10}\|q\|_\infty$. The values for non-integer j are obtained by linear interpolation. Analogous graphs for our other solutions, as well as animations, can be found in [14].

Remark 1. If ϕ and ψ both even, then $-q$ solves the equation (1.1) whenever q does. This applies to our solutions 1-12.

In order to discuss the stability of these solutions, we write the second order equation (1.1) for q as a first order equation for the pair $\mathfrak{u} = \left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$, where $p = \dot{q}$. The resulting equation is in fact Hamiltonian, with the Hamiltonian function given by

$$\mathbb{H}(q,p) = \sum_j \tfrac{1}{2}(p_j)^2 + V(q)\,, \quad V(q) = \omega^{-2}\sum_j \Bigl[\phi(q_{j+1}-q_j) + \psi(q_j)\Bigr]\,. \tag{1.3}$$

In other words, $\dot{q}_j = \partial_{p_j}\mathbb{H}$ and $\dot{p}_j = -\partial_{q_j}\mathbb{H}$. The corresponding time-t map $\mathfrak{u}(0) \mapsto \mathfrak{u}(t)$ will be denoted by Θ_t. Since we are interested in breather solutions, it suffices to consider initial conditions $\mathfrak{u}(0)$ whose components $q(0)$ and $p(0)$ belong to $\mathcal{H} = \ell^2(\mathbb{Z})$. Let now $\mathfrak{u}$ be a 2π-periodic orbit in $\mathcal{H}^2$. Then each Θ_t is well-defined and differentiable in some open neighborhood of $\mathfrak{u}(0)$ in $\mathcal{H}^2$. Define $\Phi(t)$ to be the derivative $D\Psi_t(\mathfrak{u}(0))$. We say that the orbit $\mathfrak{u}$ is spectrally stable if the spectrum of $\Phi(2\pi)$ belongs to the closed unit disk. In fact, we can replace "unit disk" by "unit circle", since the spectrum is invariant under $z \mapsto \bar{z}$ and $z \mapsto z^{-1}$, due to the Hamiltonian nature of the flow.

The spectrum of the time-2π map for trivial solution $\mathfrak{u} = 0$ is easily seen to be the set of all complex numbers $e^{2\pi i z}$ for which z^2 is real and belongs to the interval bounded by $\omega^{-2}\psi_2$ and $\omega^{-2}[\psi_2 + 4\phi_2]$. It is not hard to see that this set Σ^e also constitutes the essential spectrum of the time-2π map $\Phi(2\pi)$ for our breather solution.

Remark 2. If $\mathfrak{u}$ is an orbit for the flow generated by $\mathbb{H}$, then so is $t \mapsto \mathfrak{u}(t-c)$, for any constant c. This implies e.g. that $\dot{\mathfrak{u}}$ is an eigenvector of $\Phi(2\pi)$ with eigenvalue 1.

Theorem 1.2. *Consider the solution described in Theorem 1.1, associated with one of the rows of Table 1. If the entry in the last column is a* "S" *or* "U"*, then the solution is spectrally stable or unstable, respectively.*

An entry "ns" or "nu" in Table 1 means that the solution appears to be spectrally stable or unstable, respectively, based on numerical results. We did not succeed in validating these results, due to a limited ability of our methods to deal with continuous spectrum. In particular, solution 9 appears to have an eigenvalue with the "wrong" Krein signature embedded in the continuous spectrum. This is a notoriously difficult situation.

The first step in our proof of Theorem 1.2 is to show that it suffices to work with truncated systems whose time-2π maps $\Phi_n(2\pi)$ are essentially matrices. This result should be of independent interest. The spectrum of $\Phi(2\pi)$ outside Σ^e consist of isolated eigenvalues with finite multiplicities. In Theorem 2.13, we show that these eigenvalues are approximated by the eigenvalues of $\Phi_n(2\pi)$ for large n.

For the part of Theorem 1.2 that deals with spectral stability, we use ideas from [13], where spectral (in)stability was proved for some solutions of a periodically perturbed wave equation. The proof of instability is complicated by the presence of the above-mentioned eigenvalue 1. As a consequence, our instability results are restricted to cases where the continuous spectrum of $\Phi(2\pi)$ is either very narrow (solutions 1 and 11) or includes a real interval (solutions 5, 6, 7, and 12). In the first case, linear instability is the result of eigenvalues outside the unit disk. By standard results on invariant manifolds, this implies e.g. that the solutions 1 and 11 are truly (not just linearly) unstable.

Solution 3 and its spectrum are shown in Fig. 2. Spectrum that is not marked with dots lies in the red and black arcs. The color indicates the Krein signature: red $\mapsto$ positive, black $\mapsto$ negative. Blue dashes mark the primary separating values; see Section 4. Spectral plots for some of our other solutions can be found in [14].

For each of the solutions q described in Theorem 1.1, there exists a finite lattice interval $\mathcal{J} = \{j \in \mathbb{Z} : \sigma - \ell \le j - J \le \ell\}$ which we call the "approximate support" of q. The value of ℓ is given in Table 1.

Theorem 1.3. *Consider a fixed choice of parameters $(\phi, \psi, \omega, \sigma, \tau, r, \rho)$ from Table 1, excluding rows 5 and 13. Let $m \mapsto q^m$ be a sequence (finite or infinite) of solutions of the equation (1.1), associated with these parameter values, as described in Theorem 1.1 and Remark 1. By considering translates, we assume now that the approximate supports $\mathcal{J}_m$ of these solutions are mutually disjoint. Assume in addition that the distance between any two adjacent approximate supports is even. Then there exists a solution q of (1.1), with the property that $\|q_j - q_j^m\|_\infty < 2^{-45}$ whenever $j \in \mathcal{J}_m$ for some m, and $\|q_j\|_\infty < 2^{-50}$ whenever $j \notin \mathcal{J}_m$ for all m.*

The idea of the proof is of course to use that the breathers q^m interact very little if they are placed sufficiently far apart. This idea has been used e.g. in [4,11,12] to construct and analyze extended solutions for the FPU model and other lattice systems. Notice however that the notion of "sufficiently far" in Theorem 1.3 is specific and very mild.

Our proof of this theorem is based again on a contraction mapping argument. Here we have to work with ℓ^∞ type spaces; but since the interactions have finite range, the estimates that are needed are not much stronger than what is required for our proof of Theorem 1.1. Nevertheless, these estimates fail for the solutions 5 and 13. Instead of treating these cases differently, we chose to exclude them from Theorem 1.3, just for simplicity.

We expect that a multi-breather q is spectrally stable only in very special cases. Assuming that the sets $\mathcal{J}_m$ are placed sufficiently far apart, each of the breathers q^m will have to be spectrally stable. But this is not sufficient, since a system of N non-interacting breathers has an eigenvalue 1 with multiplicity $2N$. Under the influence of a small interaction, all but 2 of these eigenvalues can move away from 1. It may be possible to keep these eigenvalues on the unit circle by using time-translates $q^m(\,\cdot\, - c_m)$ in the construction of q, with properly chosen constants c_m. But this is outside the scope of our current methods.

As mentioned earlier, FPU-type models are accessible to variational methods as well. Time-periodic solutions of (1.1) with period 2π can be found as critical points of the Lagrangian functional

$$\mathbb{L}(q) = \int_0^{2\pi} \left[\sum_j \tfrac{1}{2}(\dot q_j)^2 - V(q) \right] dt\,, \tag{1.4}$$

defined on a suitable Hilbert space of functions $q : \mathrm{S}^1 \to \ell^2$. We refer to [2,3] for early results and to [9] for a survey. The coefficients ϕ_i and ψ_i determine the geometry of the functional. In particular, the choice $\psi = 0$ and $\phi_2 < 0 < \phi_4$ yields a mountain pass geometry. This is the first (and simplest) case that was considered with critical point theory, and it is the only case for which the existence of multi-breather solutions has been proved variationally [4].

The remaining part of this paper is organized as follows. In Section 2 we consider the time-2π map for an infinite chain and its spectral approximation by time-2π maps $\Phi_n(2\pi)$ for chains of length $2n$. Section 3 is devoted to the task of proving Theorems 1.1 and 1.3. Our proof of these theorems requires estimates on approximate solutions. The same is true for our proof of Theorem 1.2, which is given in Section 4. The instability proof (for solutions 1 and 11) uses a perturbation argument. The stability proof (for solutions 2, 3, 4, 8, and 13) uses Krein signatures, and a monotonicity argument from [13], to control the eigenvalues of $\Phi_n(2\pi)$. The estimates that are needed in Sections 3 and 4 are proved with the aid of a computer; a rough description is given in Section 5, and for details we refer to the source code of our programs [14].

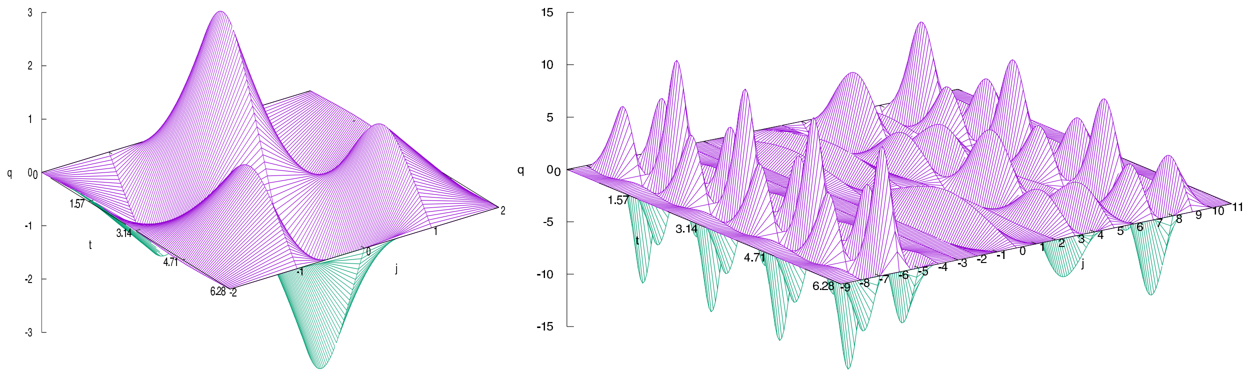

Figure 1. Solutions 11 and 12.

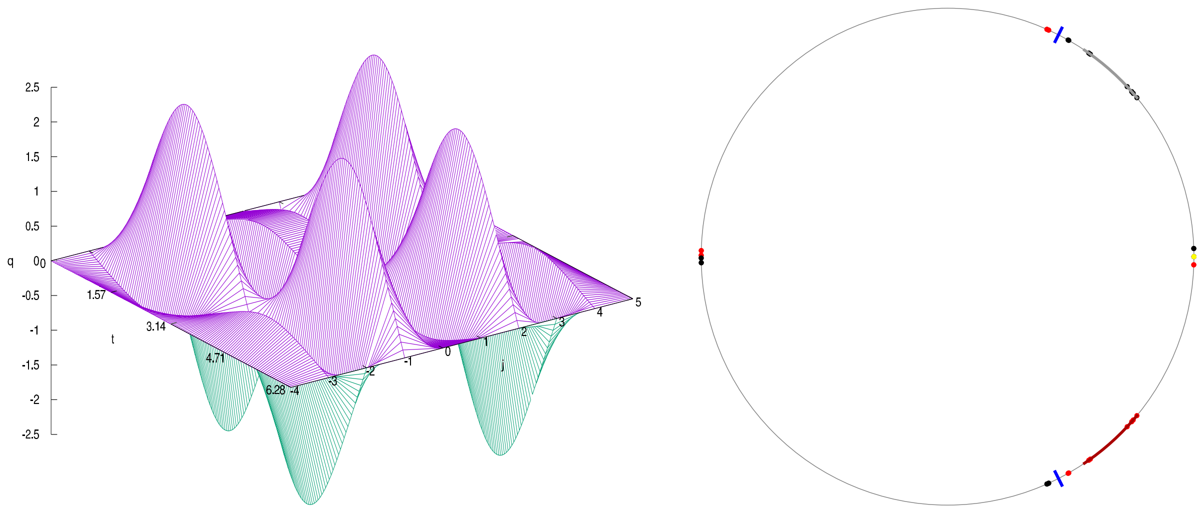

Figure 2. Solution 3 and its spectrum.

2. Infinite chains and approximations

After introducing some notation, we discuss spectral properties of operators in a class that includes the linearized time-2π maps for exponentially decreasing time-periodic breathers.

2.1. Notation

By a chain q we mean a real-valued function $j \mapsto q_j$ on $\mathbb{Z}$. Complex-valued functions will be considered only for spectral theory. If f is any real-valued function on $\mathbb{R}$, then $f(q)$ denotes the chain with values $f(q)_j = f(q_j)$. We say that a chain q is site-centered (bond-centered) if q has a symmetry (1.2) with $\sigma = 0$ ($\sigma = 1$). This restriction to $\sigma = 0$ and $\sigma = 1$ is motivated mainly by computational simplicity.

Given $\sigma \in \{0, 1\}$, we set

$$(\nabla_\sigma q)_j = q_{j+\sigma} - q_{j+\sigma-1}\,, \tag{2.1}$$

for every chain q and every integer j. This defines a continuous linear operator ∇_σ on $\mathcal{H} = \ell^2(\mathbb{Z})$. Its adjoint is given by $\nabla_\sigma^* = -\nabla_{1-\sigma}$. The equation (1.1) can now be written as

$$\ddot{q} = -K(q)\,, \qquad K(q) = \omega^{-2}\bigl[\nabla_\sigma^*\phi'(\nabla_\sigma q) + \psi'(q)\bigr]\,. \tag{2.2}$$

The corresponding first-order equation is

$$\dot{\mathfrak{u}} = Y(\mathfrak{u})\,, \qquad \mathfrak{u} = \begin{bmatrix} q \\ p \end{bmatrix}, \qquad Y(\mathfrak{u}) = \begin{bmatrix} 0 & \mathrm{I} \\ -K(q) & 0 \end{bmatrix}. \tag{2.3}$$

We note that the operator K is independent of the choice of σ. Our reason for considering two distinct versions of the lattice gradient is that ∇_0 maps site-centered chains to bond-centered chains, while ∇_1 maps bond-centered chains to site-centered chains.

Let now $\mathfrak{u}$ be a fixed 2π-periodic orbit for the flow (2.3), and define

$$\alpha = \omega^{-2}\phi''(\nabla_\sigma q)\,, \qquad \beta = \omega^{-2}\psi''(q)\,. \tag{2.4}$$

Consider an orbit $\mathfrak{u} + \mathfrak{v}$ close to $\mathfrak{u}$. To first order in $\mathfrak{v}$, we have

$$\dot{\mathfrak{v}} = X(\mathfrak{u})\mathfrak{v}\,, \qquad X(\mathfrak{u}) \stackrel{\text{def}}{=} DY(\mathfrak{u}) = \begin{bmatrix} 0 & \mathrm{I} \\ -H(q) & 0 \end{bmatrix}, \tag{2.5}$$

where $H(q)$ is the linear operator

$$H(q)v \stackrel{\text{def}}{=} DK(q)v = \nabla_\sigma^*\alpha\nabla_\sigma v + \beta v\,, \qquad v \in \mathcal{H}\,. \tag{2.6}$$

If Θ_t denotes the time-t map for the flow (2.3), then the time-t map for the flow (2.5) is given by the linear operator $\Phi(t) = D\Theta_t(\mathfrak{u}(0))$. Our goal is to analyze the spectrum of the operator $\Phi(2\pi)$.

Notice that $H(q(t))$ is 2π-periodic in t and self-adjoint for each t. Furthermore, far from the origin, where q is close to zero, $H(q)$ is well approximated by the operator

$$H(0) = \bar{\alpha}\nabla_\sigma^*\nabla_\sigma + \bar{\beta}\,. \tag{2.7}$$

Here, $\bar{\alpha} = \omega^{-2}\phi_2$ and $\bar{\beta} = \omega^{-2}\psi_2$.

In what follows, we consider operators of the type (2.6) and (2.7), where α and β can be more general 2π-periodic curves in $\mathcal{H}$. But we will assume that $\tilde{\alpha} = \alpha - \bar{\alpha}$ and $\tilde{\beta} = \beta - \bar{\beta}$ decrease exponentially. To simplify notation, the remaining part of this section is formulated for ∇_0 only.

2.2. Some spaces

In order to discuss the spectrum of $\Phi(2\pi)$, we will need certain constructions for several different Hilbert spaces. To avoid undue repetition, we start by considering a fixed but arbitrary Hilbert space $\mathfrak{h}$. The inner product in $\mathfrak{h}$ will be denoted by $\langle\,.\,,.\,\rangle$. Here, and in what follows, an inner product is always assumed to be linear in its second argument, and antilinear in the first argument. Consider the vector space $\mathfrak{c}_0(\mathbb{Z},\mathfrak{h})$ of all sequences $v:\mathbb{Z}\to\mathfrak{h}$ with the property that $v_j = 0$ for all but finitely many $j\in\mathbb{Z}$. For every $\rho\ge 0$, we define $\mathfrak{H}_\rho(\mathfrak{h})$ to be the Hilbert space obtained as the closure of $\mathfrak{c}_0(\mathbb{Z},\mathfrak{h})$ with respect to the norm

$$\|v\|_\rho = \sqrt{\langle v,v\rangle_\rho}\,,\qquad \langle v,v'\rangle_\rho = \sum_{j\in\mathbb{Z}} \cosh(2j\rho)\langle v_j, v'_j\rangle\,. \tag{2.8}$$

The equation

$$\bigl(U(\rho)v\bigr)_j = \cosh(2\rho j)^{1/2} v_j\,,\qquad v\in\mathfrak{H}_0(\mathfrak{h})\,,\quad j\in\mathbb{Z}\,, \tag{2.9}$$

defines a unitary operator $U(\rho)$ from $\mathfrak{H}_\rho(\mathfrak{h})$ to $\mathfrak{H}_0(\mathfrak{h})$. The Fourier transform $\mathcal{F}v$ of a function $v\in\mathfrak{H}_0(\mathfrak{h})$ is given by

$$\mathcal{F}v = \tilde{v}\,,\qquad \tilde{v}(\varphi) = \sum_{j\in\mathbb{Z}} v_j e^{-i\varphi j}\,. \tag{2.10}$$

The Fourier transform $\mathcal{F}$ defines a unitary operator from $\mathfrak{H}_0(\mathfrak{h}) = \ell^2(\mathbb{Z},\mathfrak{h})$ to the Hilbert space $\mathrm{L}^2(\mathcal{I},\mathfrak{h})$ with the inner product

$$\langle g,f\rangle_{\mathrm{L}^2} = \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} \bigl\langle g(\varphi), f(\varphi)\bigr\rangle\,d\varphi\,,\qquad \mathcal{I} = [-\pi,\pi]\,. \tag{2.11}$$

Consider now $v\in\mathfrak{H}_\rho(\mathfrak{h})$ with $\rho>0$. Then the sum in (2.10) extends $\tilde{v}$ to an analytic function on the interior of the strip

$$S_\rho = \bigl\{\varphi\in\mathbb{C} : |\operatorname{Im}\varphi| \le \rho\bigr\}\,. \tag{2.12}$$

Furthermore, the function $\varphi\mapsto\tilde{v}(\varphi+i\rho)$ belongs to $\mathrm{L}^2(\mathcal{I},\mathfrak{h})$, and

$$\|v\|_\rho^2 = \tfrac{1}{2}\|\tilde{v}(\,.+i\rho)\|_{\mathrm{L}^2}^2 + \tfrac{1}{2}\|\tilde{v}(\,.-i\rho)\|_{\mathrm{L}^2}^2\,. \tag{2.13}$$

We will also need to approximate functions in $\mathfrak{H}_\rho(\mathfrak{h})$ by functions that are supported in $\mathbb{Z}_n = \{j\in\mathbb{Z} : -n<j\le n\}$ for some positive integer n. To this end, define a projection $P_n:\mathfrak{H}_0(\mathfrak{h})\to\mathfrak{H}_0(\mathfrak{h})$ by setting

$$(P_n v)_j = \begin{cases} v_j & \text{if } j\in\mathbb{Z}_n,\\ 0 & \text{otherwise,}\end{cases}\qquad v\in\mathfrak{H}_0(\mathfrak{h})\,,\quad j\in\mathbb{Z}\,. \tag{2.14}$$

The following proposition is an immediate consequence of the fact that, if $0 \le \rho \le \varrho$, then $\cosh(2\varrho n)\|(\mathrm{I}-P_n)v\|_\rho \le \cosh(2\rho n)\|(\mathrm{I}-P_n)v\|_\varrho$.

Proposition 2.1. *Let $0 \le \rho < \varrho$. Assume that $v \in \mathfrak{H}_\varrho(\mathfrak{h})$ is nonzero and satisfies a bound $\|v\|_\varrho \le C\|v\|_\rho$. If n is sufficiently large, so that $\cosh(2\varrho n) > C^2\cosh(2\rho n)$, then*

$$\|(\mathrm{I}-P_n)v\|_\rho \le \left(\frac{\cosh(2\rho n)(C^2-1)}{\cosh(2\varrho n)-C^2\cosh(2\rho n)}\right)^{1/2} \|P_n v\|_\rho\,. \tag{2.15}$$

The subspace $P_n\mathfrak{H}_\rho(\mathfrak{h})$ of $\mathfrak{H}_\rho(\mathfrak{h})$ can be identified with the Hilbert space $\mathfrak{H}_{\rho,n}(\mathfrak{h})$ of all functions $v:\mathbb{Z}_n\to\mathfrak{h}$, equipped with the inner product

$$\langle v,v'\rangle_\rho = \sum_{j\in\mathbb{Z}_n}\cosh(2\rho j)\langle v_j,v'_j\rangle\,. \tag{2.16}$$

Notice that $\mathfrak{H}_{\rho,n}(\mathfrak{h})$ agrees with $\mathfrak{H}_{0,n}(\mathfrak{h})$ as a vector space, for any $\rho>0$. The Fourier transform $\tilde v=\mathcal{F}_n v$ of a function $v\in\mathfrak{H}_{0,n}(\mathfrak{h})$ is defined as in (2.10), but with the sum ranging over $j\in\mathbb{Z}_n$ only. The Fourier transform $\mathcal{F}_n$ defines a unitary operator from $\mathfrak{H}_{0,n}(\mathfrak{h})=\ell^2(\mathbb{Z}_n,\mathfrak{h})$ to the Hilbert space $\ell^2(\mathcal{I}_n,\mathfrak{h})$ with the inner product

$$\langle g,f\rangle_{\ell^2}=\frac{1}{|\mathcal{I}_n|}\sum_{\varphi\in\mathcal{I}_n}\bigl\langle g(\varphi),f(\varphi)\bigr\rangle\,,\qquad \mathcal{I}_n=\frac{\pi}{2n}\mathbb{Z}_n\,. \tag{2.17}$$

Notation 2.2. *The operator norm of a continuous linear operator L on $\mathfrak{H}_\rho(\mathfrak{h})$ or $\mathfrak{H}_{\rho,n}(\mathfrak{h})$ will be denoted by $\|L\|_\rho$.*

2.3. The flow and its spectrum

Let $\mathcal{H}_\rho=\mathfrak{H}_\rho(\mathbb{C})$, using the inner product $\langle z,z'\rangle=\bar z z'$ on $\mathbb{C}$. Here, and in what follows, ρ is a fixed but arbitrary nonnegative real number, unless specified otherwise.

Consider the space $\mathcal{H}_\rho^2$ of all pairs $\mathfrak{v}=\left[\begin{smallmatrix}v\\ \nu\end{smallmatrix}\right]$ with components $v,\nu\in\mathcal{H}_\rho$. We will identify $\mathcal{H}_\rho^2$ with the space $\mathfrak{H}_\rho(\mathbb{C}^2)$ of sequences $j\mapsto\mathfrak{v}_j=\left[\begin{smallmatrix}v_j\\ \nu_j\end{smallmatrix}\right]$, using the inner product $\langle\mathfrak{v}_j,\mathfrak{v}'_j\rangle=\bar v_j v'_j+\bar\nu_j\nu'_j$ on $\mathbb{C}^2$. On $\mathcal{H}_\rho^2$ we consider the operator

$$X_0=\begin{bmatrix}0 & \mathrm{I}\\ -H_0 & 0\end{bmatrix},\qquad H_0=\bar\alpha\nabla_0^*\nabla_0+\bar\beta\,, \tag{2.18}$$

where $\bar\alpha$ and $\bar\beta$ are fixed but arbitrary real numbers. Notice that the operator $\mathcal{F}H_0\mathcal{F}^{-1}$ on $\mathcal{F}\mathcal{H}_\rho$ is multiplication by the function h_0,

$$h_0(\varphi)=2\bar\alpha\bigl[1-\cos(\varphi)\bigr]+\bar\beta\,. \tag{2.19}$$

So the spectrum of the operators $H_0:\mathcal{H}_\rho\to\mathcal{H}_\rho$ and $X_0:\mathcal{H}_\rho^2\to\mathcal{H}_\rho^2$ are given by R_ρ and $iR_\rho^{1/2}$, respectively, where

$$R_\rho=\overline{\mathrm{range}(h_0:S_\rho\to\mathbb{C})}\,,\qquad iR_\rho^{1/2}=\bigl\{z\in\mathbb{C}:-z^2\in R_\rho\bigr\}\,, \tag{2.20}$$

and where S_ρ is the strip defined in (2.12).

A straightforward computation shows that

$$e^{tX_0} = \begin{bmatrix} \cos(t\boldsymbol{y}) & \boldsymbol{y}^{-1}\sin(t\boldsymbol{y}) \\ -\boldsymbol{y}\sin(t\boldsymbol{y}) & \cos(t\boldsymbol{y}) \end{bmatrix}, \qquad \boldsymbol{y} = H_0^{1/2}. \tag{2.21}$$

The choice of square root does not matter as this point, since $\cos(t\boldsymbol{y})$ and $\boldsymbol{y}^{\pm 1}\sin(t\boldsymbol{y})$ are even functions of $\boldsymbol{y}$.

Let $(j,t) \mapsto \tilde{\alpha}_j(t)$ and $(j,t) \mapsto \tilde{\beta}_j(t)$ be two functions on $\mathbb{Z}\times\mathbb{R}$ with the following properties.

Condition 2.3. *The functions $\tilde{\alpha}_j$ and $\tilde{\beta}_j$ are continuous and 2π-periodic. Furthermore, the sequences $U(\rho_0)\tilde{\alpha}(t)$ and $U(\rho_0)\tilde{\beta}(t)$ are bounded for some $\rho_0 > 0$, uniformly in t. Here $U(\rho_0)$ denotes the multiplication operator as defined in (2.9).*

This defines two families of compact linear operators on $\mathcal{H}_\rho$ via pointwise multiplication: $\big(\tilde{\alpha}(t)v\big)_j = \tilde{\alpha}_j(t)v_j$ and $\big(\tilde{\beta}(t)v\big)_j = \tilde{\beta}_j(t)v_j$. Define also

$$H_1(t)v = \nabla_0^*\tilde{\alpha}(t)\nabla_0 v + \tilde{\beta}(t)v\,, \qquad t \in \mathbb{R}\,. \tag{2.22}$$

On $\mathcal{H}_\rho^2$ we consider the flow give by the equation

$$\frac{d}{dt}\mathfrak{v}(t) = X(t)\mathfrak{v}(t)\,, \qquad X(t) = \begin{bmatrix} 0 & \mathrm{I} \\ -H(t) & 0 \end{bmatrix}, \quad H(t) = H_0 + H_1(t)\,. \tag{2.23}$$

The corresponding time-t map $\mathfrak{v}(0) \mapsto \mathfrak{v}(t)$ will be denoted by $\Phi(t)$. Clearly, $\Phi(t)$ is bounded on $\mathcal{H}_\rho^2$ for each $t \in \mathbb{R}$. In order to get more detailed information on $\Phi(t)$, we use that Φ can be obtained by solving the Duhamel equation

$$\Phi(t) = e^{tX_0} + \int_0^t e^{(t-s)X_0}\big[X(s) - X_0\big]\Phi(s)\,ds\,. \tag{2.24}$$

Notice that $X(s) - X_0 = P_2^* H_1(s) P_1$, where $P_1 = [1\ 0]$ and $P_2 = [0\ 1]$ are the operators that assign to a vector $\mathfrak{v} = \left[\begin{smallmatrix} v \\ \nu \end{smallmatrix}\right]$ in $\mathcal{H}_\rho^2$ its component $v \in \mathcal{H}_\rho$ and $\nu \in \mathcal{H}_\rho$, respectively. Since H_1 is a continuous curve of compact linear operators, the integral in this equation defines a compact linear operator. Thus, $\Phi(t)$ is a compact perturbation of e^{tX_0}. In particular, the essential spectrum of $\Phi(2\pi)$ agrees with the essential spectrum of $e^{2\pi X_0}$ which is

$$\Sigma_\rho^e = \exp\big(2\pi i R_\rho^{1/2}\big)\,. \tag{2.25}$$

Notice that Σ_0^e is included in the union of the unit circle and the real line. If Σ_0^e does not cover the entire circle, then the complement of Σ_0^e is connected. The same holds for Σ_ρ^e, if $\rho > 0$ is chosen sufficiently small.

Theorem 2.4. *Let $\rho \geq 0$, and assume that the complement of Σ_ρ^e is connected. Then the spectrum of $\Phi(2\pi) : \mathcal{H}_\rho^2 \to \mathcal{H}_\rho^2$ outside Σ_ρ^e consists of eigenvalues with finite (algebraic) multiplicities. These eigenvalues can accumulate only at Σ_ρ^e.*

This theorem is a consequence of the following fact. A bounded linear operator with essential spectrum Σ^e has at most countable spectrum in the unbounded component of $\mathbb{C}\setminus$

Σ^e. The spectrum in this component consists of isolated eigenvalues with finite (algebraic) multiplicities. See e.g. the proof of Theorem 4.5.33 in [1].

2.4. Eigenvalues and eigenvectors

Multiplying both sides of the equation (2.24) from the left by e^{-tX_0}, we find that the family of operators $A(t) = e^{-tX_0}\Phi(t) - \mathrm{I}$ satisfies the equation

$$A(t) = -\int_0^t B(s)\bigl[\mathrm{I} + A(s)\bigr]\,ds\,, \tag{2.26}$$

where

$$\begin{aligned} B(s) &= e^{-sX_0}P_2^* H_1(s)P_1 e^{sX_0} \\ &= \begin{bmatrix} -\boldsymbol{y}^{-1}\sin(s\boldsymbol{y})H_1(s)\cos(s\boldsymbol{y}) & -\boldsymbol{y}^{-1}\sin(s\boldsymbol{y})H_1(s)\boldsymbol{y}^{-1}\sin(s\boldsymbol{y}) \\ \cos(s\boldsymbol{y})H_1(s)\cos(s\boldsymbol{y}) & \cos(s\boldsymbol{y})H_1(s)\boldsymbol{y}^{-1}\sin(s\boldsymbol{y}) \end{bmatrix}. \end{aligned} \tag{2.27}$$

The integral equation (2.26) for A can be solved by iteration,

$$A(t_0) = \sum_{n=1}^{\infty}(-1)^n \int_0^{t_0} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n B(t_1)B(t_2)\cdots B(t_n)\,. \tag{2.28}$$

Denote by ρ_0 the decay rate of the sequences $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$, as described in Condition 2.3. Then the following holds.

Proposition 2.5. *Assume that $0 < \rho < \rho_0$. Then $H_1(t)$ defines a compact linear operator from $\mathcal{H}_0$ to $\mathcal{H}_\rho$. Furthermore, $B(t)$ and $A(t)$ define compact linear operators from $\mathcal{H}_0^2$ to $\mathcal{H}_\rho^2$. The bounds are uniform in t, if t is restricted to a bounded interval.*

Proof. Using that $U(\rho)^{-1}\nabla U(\rho)$ is bounded on $\mathcal{H}_\rho$, and that $\tilde{\alpha}(t)U(\rho)$ and $\tilde{\beta}(t)U(\rho)$ are compact on $\mathcal{H}_\rho$, we see that

$$H_1(t)U(\rho) = \nabla^*\tilde{\alpha}(t)U(\rho)\bigl[U(\rho)^{-1}\nabla U(\rho)\bigr] + \tilde{\beta}(t)U(\rho) \tag{2.29}$$

defines a compact operator on $\mathcal{H}_\rho$. This proves the first claim.

Composing $\cos(t\boldsymbol{y}) : \mathcal{H}_0 \to \mathcal{H}_0$ with $H_1 : \mathcal{H}_0 \to \mathcal{H}_\rho$ and $\cos(t\boldsymbol{y}) : \mathcal{H}_\rho \to \mathcal{H}_\rho$, we see that

$$B_{1,1}(t) = \cos(t\boldsymbol{y})H_1(t)\cos(t\boldsymbol{y}) \tag{2.30}$$

defines a compact linear operator from $\mathcal{H}_0$ to $\mathcal{H}_\rho$. Similarly for the other components of the operator $B(t)$ defined in (2.27). Thus, $B(t)$ is compact as a linear operator from $\mathcal{H}_0^2$ to $\mathcal{H}_\rho^2$. By (2.28) the same is true for $A(t)$. **QED**

Corollary 2.6. *Let $0 < \rho < \rho_0$. Let $\mathfrak{v} \in \mathcal{H}_0^2$ be an eigenvector of $\Phi(2\pi)$ with eigenvalue $\lambda \in \mathbb{C} \setminus \Sigma_\rho^e$. Then $\mathfrak{v}$ belongs to $\mathcal{H}_\rho^2$, and*

$$\mathfrak{v} = \bigl[\lambda\mathrm{I} - e^{2\pi X_0}\bigr]^{-1} e^{2\pi X_0} A(2\pi)\mathfrak{v}\,. \tag{2.31}$$

Proof. By assumption we have $\lambda\mathfrak{v} = \Phi(2\pi)\mathfrak{v} = e^{2\pi X_0}\bigl[\mathrm{I} + A(2\pi)\bigr]\mathfrak{v}$, and thus

$$\bigl[\lambda\mathrm{I} - e^{2\pi X_0}\bigr]\mathfrak{v} = e^{2\pi X_0}A(2\pi)\mathfrak{v}\,. \tag{2.32}$$

By Proposition 2.5, the right hand side of this equation belongs to $\mathcal{H}^2_\rho$. And $\lambda\mathrm{I} - e^{2\pi X_0}$ has a bounded inverse on $\mathcal{H}^2_\rho$ since $\lambda \notin \Sigma^e_\rho$. This proves the claim. **QED**

For completeness, we state a partial generalization of Corollary 2.6, which follows from (the proof of) Proposition 4.3 in [13].

Proposition 2.7. *Let λ be an isolated eigenvalue of $\Phi(2\pi) : \mathcal{H}^2_0 \to \mathcal{H}^2_0$. Then each vector in the corresponding spectral subspace belongs to $\mathcal{H}^2_\rho$ for some $\rho > 0$.*

2.5. Resolvent estimates

Consider now the case where $\bar\beta$ and $\bar\beta + 4\bar\alpha$ are both nonnegative. Define $\boldsymbol{y} = H_0^{1/2}$ by using a square root function that is analytic in $\mathbb{C}\setminus(-\infty,0]$. Then a partial diagonalization of X_0 is given by

$$X_0 = \begin{bmatrix} 0 & \mathrm{I} \\ -\boldsymbol{y}^2 & 0 \end{bmatrix} = \Lambda \begin{bmatrix} -i\boldsymbol{y} & 0 \\ 0 & i\boldsymbol{y} \end{bmatrix} \Lambda^{-1}\,, \tag{2.33}$$

where

$$\Lambda = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{y}^{-1/2} & \boldsymbol{y}^{-1/2} \\ -i\boldsymbol{y}^{1/2} & i\boldsymbol{y}^{1/2} \end{bmatrix}\,, \qquad \Lambda^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{y}^{1/2} & i\boldsymbol{y}^{-1/2} \\ \boldsymbol{y}^{1/2} & -i\boldsymbol{y}^{-1/2} \end{bmatrix}\,. \tag{2.34}$$

Definition 2.8. *Let ρ_1 be the largest positive real number with the property that the range of $h_0 : S_\rho \to \mathbb{C}$ is contained in $\mathbb{C}\setminus(-\infty,0]$, whenever $\rho < \rho_1$. For simplicity, we assume from now on that $\rho_0 \le \rho_1$.*

Clearly, if $\rho < \rho_1$, then Λ and Λ^{-1} define bounded linear operators on $\mathcal{H}^2_\rho$.

Proposition 2.9. *Let $0 \le \rho < \rho_1$ and define $C = \|\Lambda\|_\rho \|\Lambda^{-1}\|_\rho$. Then*

$$C\bigl\|e^{2\pi X_0}\mathfrak{v} - z\mathfrak{v}\bigr\|_\rho \ge \operatorname{dist}\bigl(z, \Sigma^e_\rho\bigr)\|\mathfrak{v}\|_\rho\,, \qquad \mathfrak{v} \in \mathcal{H}^2_\rho\,, \quad z \in \mathbb{C}\,. \tag{2.35}$$

Proof. Let $u \in \mathcal{H}_\rho$, and consider the Fourier transform $\tilde u$ for arguments $\varphi + i\kappa$ with $\varphi, \kappa \in \mathbb{R}$ and $|\kappa| \le \rho$. On this domain we have the bound

$$\Bigl|\Bigl(e^{\pm 2\pi i h_0^{1/2}} - z\Bigr)\tilde u\Bigr| \ge \operatorname{dist}\bigl(z, \Sigma^e_\rho\bigr)|\tilde u|\,, \tag{2.36}$$

for any choice of the square root function. Using the identity (2.13), this implies the bound

$$\bigl\|\bigl(e^{\pm 2\pi i\boldsymbol{y}} - z\mathrm{I}\bigr)u\bigr\|_\rho \ge \operatorname{dist}\bigl(z, \Sigma^e_\rho\bigr)\|u\|_\rho\,. \tag{2.37}$$

By (2.33) we have

$$\begin{bmatrix} e^{-2\pi i\boldsymbol{y}} - z\mathrm{I} & 0 \\ 0 & e^{2\pi i\boldsymbol{y}} - z\mathrm{I} \end{bmatrix} = \Lambda^{-1}\bigl[e^{2\pi X_0} - z\mathrm{I}\bigr]\Lambda\,. \tag{2.38}$$

Let $\mathfrak{v} \in \mathcal{B}_\rho$ and $\mathfrak{v}' = \Lambda^{-1}\mathfrak{v}$. Then

$$\begin{aligned} \big\|\big(e^{-2\pi i\boldsymbol{y}} - z\mathrm{I}\big)v'\big\|_\rho^2 + \big\|\big(e^{2\pi i\boldsymbol{y}} - z\mathrm{I}\big)\nu'\big\|_\rho^2 &= \big\|\Lambda^{-1}\big[e^{2\pi X_0} - z\mathrm{I}\big]\Lambda\mathfrak{v}'\big\|_\rho^2 \\ &\le \big\|\Lambda^{-1}\big\|_\rho^2\big\|e^{2\pi X_0}\mathfrak{v} - z\mathfrak{v}\big\|_\rho^2 . \end{aligned} \tag{2.39}$$

Combining this bound with (2.37), we get

$$\begin{aligned} \big\|\Lambda^{-1}\big\|_\rho^2\big\|e^{2\pi X_0}\mathfrak{v} - z\mathfrak{v}\big\|_\rho^2 &\ge \operatorname{dist}\big(z, \Sigma_\rho^e\big)^2\|\mathfrak{v}'\|_\rho^2 \\ &\ge \operatorname{dist}\big(z, \Sigma_\rho^e\big)^2\|\Lambda\|^{-2}\|\mathfrak{v}\|_\rho^2 . \end{aligned} \tag{2.40}$$

This proves (2.35). **QED**

Consider the $2n$-dimensional spaces $\mathcal{H}_{\rho,n} = \mathfrak{H}_{\rho,n}(\mathbb{C})$ defined after Proposition 2.1. On $\mathcal{H}_{0,n}$ we define a self-adjoint operator $H_{0,n}$ via the quadratic form

$$\langle v, H_{0,n}v\rangle = \bar{\alpha}|v_{1-n} - v_n|^2 + \bar{\alpha}\sum_{j=1-n}^{n-1}|v_{j+1} - v_j|^2 + \bar{\beta}\langle v, v\rangle , \tag{2.41}$$

for $v \in \mathcal{H}_{0,n}$. Notice that $H_0 = \bar{\alpha}\nabla_0^*\nabla_0 + \bar{\beta}$ is invariant under translations. The operator $H_{0,n}$ is invariant under translations as well, in the sense that it maps u to v precisely when H_0 maps the $2n$-periodic extension of u to the $2n$-periodic extension of v. Consider also the operator X_0 defined in (2.18). An operator $X_{0,n} : \mathcal{H}_{0,n}^2 \to \mathcal{H}_{0,n}^2$ is defined analogously, with H_0 replaced by $H_{0,n}$.

Assume again that $\bar{\beta}$ and $\bar{\beta} + 4\bar{\alpha}$ are nonnegative. Consider the operator Λ defined in (2.34). An operator $\Lambda_n : \mathcal{H}_{0,n}^2 \to \mathcal{H}_{0,n}^2$ is defined analogously, using $\boldsymbol{y} = H_{0,n}^{1/2}$.

Proposition 2.10. *Let $0 \le \rho < \rho_1$. Then there exists $C > 0$ such that for all $n \ge 1$,*

$$C\big\|e^{2\pi X_0}\mathfrak{v} - z\mathfrak{v}\big\|_\rho \ge \operatorname{dist}\big(z, \Sigma_\rho^e\big)\|\mathfrak{v}\|_\rho , \qquad \mathfrak{v} \in \mathcal{H}_{\rho,n}^2 , \quad z \in \mathbb{C} . \tag{2.42}$$

This proposition is proved in the same way as Proposition 2.9. Uniformity in n follows from translation invariance: $H_{0,n}$ is diagonalized by the Fourier transform $\mathcal{F}_n$, and the Fourier multiplier is always the function h_0.

2.6. Spectral approximation

The goal is to approximate the spectrum of $\Phi(2\pi)$ by the spectrum of operators $\Phi_n(2\pi)$ that are essentially matrices. Define a self-adjoint operator $H_n(t)$ on $\mathcal{H}_{0,n}$ via the quadratic form

$$\big\langle v, H_n(t)v\big\rangle = \alpha_{1-n}(t)|v_{1-n} - v_n|^2 + \sum_{j=1-n}^{n-1}\alpha_j(t)|v_{j+1} - v_j|^2 + \langle v, \beta v\rangle , \tag{2.43}$$

where $\alpha_j = \tilde{\alpha}_j + \bar{\alpha}$ and $\beta_j = \tilde{\beta}_j + \bar{\beta}$ for all j. Let us now identify $\mathcal{H}_{0,n}$ with $P_n\mathcal{H}_0$, where P_n is the orthogonal projection defined in (2.14). Then $H_{0,n}$ and H_n extend canonically to $\mathcal{H}_0$ via the identities (2.41) and (2.43), respectively.

In the canonical way we also define the vector field $X_{0,n}$ associated with $H_{0,n}$, the time-dependent vector field X_n associated with H_n, and the time-2π map $\Phi_n(2\pi)$ associated with the vector field X_n. Notice that all these operators on $\mathcal{H}_0^2$ commute with P_n and act trivially on $(\mathrm{I}-P_n)\mathcal{H}_0^2$.

Proposition 2.11. *Let $0 \le \rho < \rho_0$. Then $\Phi_n(2\pi)\mathfrak{v} \to \Phi(2\pi)\mathfrak{v}$ for every $\mathfrak{v} \in \mathcal{H}_\rho$.*

Proof. Since the operator norms of $\Phi_n(2\pi) : \mathcal{H}_\rho \to \mathcal{H}_\rho$ are bounded uniformly in n, it suffices to prove that $\Phi_n(2\pi)\mathfrak{v} \to \Phi(2\pi)\mathfrak{v}$ for every $\mathfrak{v}$ in some dense subset of $\mathcal{H}_\rho$. To this end, choose $\rho < \varrho < \rho_0$, and let $\mathfrak{v}$ be a nonzero vector in $\mathcal{H}_\varrho$. Define $\mathfrak{w}(t) = \Phi(t)\mathfrak{v}$ and $\mathfrak{w}_n(t) = \Phi_n(t)\mathfrak{v}$. The difference $\mathfrak{w} - \mathfrak{w}_n$ is the solution of the equation

$$\frac{d}{dt}(\mathfrak{w} - \mathfrak{w}_n) = X\mathfrak{w} - X_n\mathfrak{w}_n = X(\mathfrak{w} - \mathfrak{w}_n) + (X - X_n)\mathfrak{w}_n\,, \tag{2.44}$$

with zero initial condition. Taking the inner product with $\mathfrak{w} - \mathfrak{w}_n$ yields the bound

$$\frac{1}{2}\frac{d}{dt}\|\mathfrak{w} - \mathfrak{w}_n\|_\rho^2 \le \|X\|_\rho\|\mathfrak{w} - \mathfrak{w}_n\|_\rho^2 + \|\mathfrak{w} - \mathfrak{w}_n\|_\rho\|(X - X_n)\mathfrak{w}_n\|_\rho\,, \tag{2.45}$$

or equivalently,

$$\frac{d}{dt}\|\mathfrak{w} - \mathfrak{w}_n\|_\rho \le \|X\|_\rho\|\mathfrak{w} - \mathfrak{w}_n\|_\rho + \|(X - X_n)\mathfrak{w}_n\|_\rho\,. \tag{2.46}$$

In order to estimate the last term in this equation, we use that $\Phi(t)$ and $\Phi_n(t)$ define bounded linear operators on $\mathcal{H}_\rho^2$, that $\Phi(t)^{-1}$ and $\Phi_n(t)^{-1}$ define bounded linear operators on $\mathcal{H}_\varrho^2$, and that their operator norms are bounded uniformly in k and in $t \in [0, 2\pi]$. This yields a bound

$$\|\mathfrak{w}_n\|_\rho \le C_1\|\mathfrak{v}\|_\rho \le C_1\|\mathfrak{v}\|_\varrho \le C_2\|\mathfrak{w}_n\|_\varrho\,, \tag{2.47}$$

for some constants C_1 and C_2. Here, and in what follows, a given bound on a quantity that depends on n and t is meant to hold uniformly in n and $t \in [0, 2\pi]$.

Using Proposition 2.1, we conclude from (2.47) that, for any given $\varepsilon > 0$, there exists $m > 0$ such that $\|(\mathrm{I} - P_m)\mathfrak{w}_n\|_\rho \le \varepsilon$ for all n. Furthermore, $\|(X - X_n)P_m\|_\rho \le \varepsilon$ if n is sufficiently large. Thus, we have a bound

$$\|(X - X_n)\mathfrak{w}_n\|_\rho \le \|(X - X_n)(\mathrm{I} - P_m)\mathfrak{w}_n\|_\rho + \|(X - X_n)P_m\mathfrak{w}_n\|_\rho \le \varepsilon_n\,, \tag{2.48}$$

with $\varepsilon_n \to 0$ as $n \to \infty$. Now pick $a > \|X\|$. Then (2.46) and (2.48) imply that

$$\frac{d}{dt}e^{-at}\|\mathfrak{w}(t) - \mathfrak{w}_n(t)\|_\rho \le \varepsilon_n\,. \tag{2.49}$$

As a result, we have

$$\|\Phi(2\pi)\mathfrak{v} - \Phi_n(2\pi)\mathfrak{v}\|_\rho = \|\mathfrak{w}(2\pi) - \mathfrak{w}_n(2\pi)\|_\rho \le \varepsilon_n 2\pi e^{2\pi a}\,. \tag{2.50}$$

This proves the claim. **QED**

After having shown that $\Phi_n(2\pi) \to \Phi(2\pi)$ pointwise, we consider the problem of convergence of eigenvalues and eigenvectors.

Proposition 2.12. *Let $0 < \rho < \rho_0$. Assume that there exists a sequence $k \mapsto n_k$ of positive integers, a converging sequence $k \mapsto \lambda_k$ of complex numbers, and a sequence $k \mapsto \mathfrak{v}_k$ of unit vectors in $\mathcal{H}^2_\rho$, such that $\|\Phi_{n_k}(2\pi)\mathfrak{v}_k - \lambda_k\mathfrak{v}_k\|_\rho \to 0$ as $k \to \infty$. If $\lambda = \lim_k \lambda_k$ does not belong to Σ^e_ρ, then λ is an eigenvalue of $\Phi(2\pi)$, and some subsequence of $k \mapsto \mathfrak{v}_k$ converges in $\mathcal{H}^2_\rho$ to an eigenvector of $\Phi(2\pi)$ for the eigenvalue λ.*

Proof. Assume that $\lambda \notin \Sigma^e_\rho$. We may also assume that $\lambda_k \notin \Sigma^e_\rho$ for all k. Let

$$E_k = \Phi_{n_k}(2\pi)\mathfrak{v}_k - \lambda_k\mathfrak{v}_k\,. \tag{2.51}$$

Using that $\Phi_{n_k}(2\pi) = e^{2\pi X_{0,n_k}}\bigl[\mathrm{I} + A_k(2\pi)\bigr]$ we have

$$\mathfrak{v}_k = \bigl[\lambda_k - e^{2\pi X_{0,n_k}}\bigr]^{-1} e^{2\pi X_{0,n_k}} A_k(2\pi)\mathfrak{v}_k + \bigl[\lambda_k - e^{2\pi X_{0,n_k}}\bigr]^{-1} E_k\,. \tag{2.52}$$

By Proposition 2.10, the operators $\bigl[\lambda_k - e^{2\pi X_{0,n_k}}\bigr]^{-1}$ are bounded on $\mathcal{H}^2_\rho$, uniformly in k. Thus, the last term in (2.52) converges to zero as $k \to \infty$. If $\rho < \varrho < \rho_0$, then $A_k(2\pi)\mathfrak{v}_k$ belongs to $\mathcal{H}^2_\varrho$ for each k, by Proposition 2.5. In fact, it is clear from the proof of Proposition 2.5 that the sequence $k \mapsto A_k(2\pi)\mathfrak{v}_k$ is bounded in $\mathcal{H}^2_\varrho$. Thus, some subsequence converges in $\mathcal{H}^2_\rho$. To simplify notation, we take this subsequence to be $k \mapsto A_k(2\pi)\mathfrak{v}_k$. Then (2.52) implies that the sequence $k \mapsto \mathfrak{v}_k$ converges in $\mathcal{H}^2_\rho$. Denote the limit by $\mathfrak{v}$. Consider now the inequality

$$\begin{aligned} \|\Phi(2\pi)\mathfrak{v} - \lambda\mathfrak{v}\|_0 \le{}& \|\Phi(2\pi)\mathfrak{v} - \Phi_{n_k}(2\pi)\mathfrak{v}\|_0 + \|\Phi_{n_k}(2\pi)[\mathfrak{v} - \mathfrak{v}_{n_k}]\|_0 \\ &+ \|\Phi_{n_k}(2\pi)\mathfrak{v}_k - \lambda_k\mathfrak{v}_k\|_0 + \|\lambda\mathfrak{v} - \lambda_k\mathfrak{v}_k\|_0\,. \end{aligned} \tag{2.53}$$

By Proposition 2.11, the first term on the right hand side tends to zero as $k \to \infty$. The other three terms on the right tend to zero trivially. This shows that $\mathfrak{v}$ is an eigenvector of $\Phi(2\pi)$ with eigenvalue λ. **QED**

Denote by Σ_n the spectrum of $\Phi_n(2\pi) : \mathcal{H}^2_0 \to \mathcal{H}^2_0$. It includes one (if $\bar\beta = 0$) or two trivial eigenvalues $\exp\bigl(\pm i\sqrt{\bar\beta}\,\bigr)$ with infinite multiplicity. The remaining part of Σ_n consists of eigenvalues whose multiplicities add up to at most $4n$.

Theorem 2.13. *Let $\lambda \in \mathbb{C} \setminus \Sigma^e_0$. Then λ is an eigenvalue of $\Phi(2\pi) : \mathcal{H}^2_0 \to \mathcal{H}^2_0$ if and only if there exists a sequence of points $\lambda_n \in \Sigma_n$ that accumulates at λ.*

Proof. The "if" part follows from Proposition 2.12, since we can choose a positive $\rho < \varrho$ such that λ lies outside Σ^e_ρ.

To prove the "only if" part, assume that λ is an eigenvalue of $\Phi(2\pi) : \mathcal{H}^2_0 \to \mathcal{H}^2_0$. By Corollary 2.6, the eigenvectors for this eigenvalue belong to $\mathcal{H}^2_\rho$, if $\rho > 0$ is chosen sufficiently small. We may assume that $\rho < \rho_0$.

Choose $r > 0$ such that the closure of the disk $D = \{z \in \mathbb{C} : |z - \lambda| < r\}$ does not intersect Σ_ρ^e and contains no eigenvalue of $\Phi(2\pi)$ besides λ. Then the spectral projection associated with D of the operator $A = \Phi(2\pi)$ is given by

$$\mathbb{P}(A, D) = \frac{1}{2\pi i} \int_{\partial D} (z\mathrm{I} - A)^{-1} \, dz \,, \tag{2.54}$$

where ∂D denotes the positively oriented boundary of D. The goal is to show that the corresponding projection for $A = \Phi_n(2\pi)$ is well defined and nontrivial, if n is sufficiently large. To this end, define

$$c = \liminf_{n \to \infty} \inf_{z \in \partial D} \inf_{\mathfrak{v} \in B} \|\Phi_n(2\pi)\mathfrak{v} - z\mathfrak{v}\|_\rho \,, \tag{2.55}$$

where B is the unit ball in $\mathcal{H}_\rho^2$. Assume for contradiction that $c = 0$. Then we can find an increasing sequence $k \mapsto n_k$, a sequence $k \mapsto \lambda_k \in \partial D$, and a sequence $k \mapsto \mathfrak{v}_k \in B$, such that $\|\Phi_{n_k}(2\pi)\mathfrak{v} - \lambda_k \mathfrak{v}\|_\rho \to 0$. By choosing a subsequence, if necessary, we can achieve $\lambda_k \to \lambda \in \partial D$. By Proposition 2.12, this implies that λ is an eigenvalue of $\Phi(2\pi)$. But this is impossible by our choice of D. The conclusion is that $c > 0$.

Thus, there exists $N > 0$ that

$$\|\Phi_n(2\pi)\mathfrak{v} - z\mathfrak{v}\|_\rho \geq \frac{c}{2}\|\mathfrak{v}\|_\rho \,, \qquad z \in \partial D \,, \quad n \geq N \tag{2.56}$$

for all $\mathfrak{v} \in \mathcal{H}_\rho^2$. This implies e.g. that the spectral projection (2.54) is well-defined for $A = \Phi_n(2\pi)$. Here, and in what follows, we assume that $n \geq N$. Let now $\mathfrak{v} \in \mathcal{H}_\rho$ be an eigenvector of $\Phi(2\pi)$ with eigenvalue λ. By the second resolvent identity we have

$$\bigl\langle \mathfrak{v}, \bigl[\mathbb{P}(\Phi_n(2\pi), D) - \mathbb{P}(\Phi(2\pi), D)\bigr]\mathfrak{v} \bigr\rangle_0 = \frac{1}{2\pi i} \int_{\partial D} f_n(z) \, dz \,, \tag{2.57}$$

where

$$f_n(z) = \bigl\langle \mathfrak{v}, \bigl(z\mathrm{I} - \Phi_n(2\pi)\bigr)^{-1} E_n(z) \bigr\rangle_0 \tag{2.58}$$

and

$$E_n(z) = \bigl[\Phi_n(2\pi) - \Phi(2\pi)\bigr]\bigl(z\mathrm{I} - \Phi(2\pi)\bigr)^{-1}\mathfrak{v} \,. \tag{2.59}$$

The goal is to take $n \to \infty$. The bound (2.56) implies that the sequence $n \mapsto f_n$ is bounded uniformly on ∂D. Furthermore, we have $\|E_n(z)\|_0 \to 0$ for each $z \in \partial D$, as a consequence of Proposition 2.11. By Proposition 2.10, this implies that $f_n \to 0$ pointwise on ∂D. And by the bounded convergence theorem, it follows that the integral in (2.57) tends to zero as $n \to \infty$.

Given that $\langle \mathfrak{v}, \mathbb{P}\bigl(\Phi(2\pi), D\bigr)\mathfrak{v}\rangle_0 = \|\mathfrak{v}\|_0^2 > 0$, we conclude that $\langle \mathfrak{v}, \mathbb{P}\bigl(\Phi_n(2\pi), D\bigr)\mathfrak{v}\rangle_0$ is nonzero for sufficiently large n. This shows that $\Phi_n(2\pi)$ has an eigenvalue in D, if n is sufficiently large. Since the radius $r > 0$ of D can be taken arbitrarily small, the assertion follows. **QED**

3. Existence of solutions

3.1. Localized solutions

In this section we prove Theorem 1.1, based on a technical lemma that will be proved later.

Adding $\psi_2 q$ on both sides of the equation (2.2) we obtain

$$\bigl(\omega^2\partial_t^2+\psi_2\bigr)q=-\nabla_\sigma^*\phi'(\nabla_\sigma q)-\tilde\psi'(q)\,,\qquad \tilde\psi(x)=\psi(x)-\tfrac{1}{2}\psi_2 x^2\,. \tag{3.1}$$

Formally, we can rewrite (3.1) as the fixed point equation $G(q)=q$, where

$$G(q)=-L^{-1}\bigl[\nabla_\sigma\phi'(\nabla_\sigma q)+\tilde\psi'(q)\bigr]\,,\qquad L=\omega^2\partial_t^2+\psi_2\,. \tag{3.2}$$

For the domain of G we use one of the spaces $\mathcal{B}^\sigma_{r,\rho}$ defined below.

Given a real number $r>1$, denote by $\mathcal{A}_r$ the Banach space of all 2π-periodic functions $g:\mathbb{R}\to\mathbb{R}$ that have a finite norm $\|g\|_r$,

$$g(t)=\sum_{k\ge 0}g_k^0\cos(kt)+\sum_{k\ge 1}g_k^1\sin(kt)\,,\qquad \|g\|_r=\sum_{k\ge 0}\bigl|g_k^0\bigr|r^k+\sum_{k\ge 1}\bigl|g_k^1\bigr|r^k\,. \tag{3.3}$$

The even and odd subspaces of $\mathcal{A}_r$ are denoted by $\mathcal{A}_r^0$ and $\mathcal{A}_r^1$, respectively. A straightforward computation shows that $\|fg\|_r\le\|f\|_r\|g\|_r$. Thus, $\mathcal{A}_r$ and $\mathcal{A}_r^0$ are Banach algebras under pointwise multiplication. Notice that the functions in $\mathcal{A}_r$ extend analytically to the complex domain $|\operatorname{Im}(t)|<\log(r)$.

Next, consider the vector space of all chains $j\mapsto v_j\in\mathcal{A}_r$ with only finitely many nonzero values v_j. Such chains will be called finite. Given $\sigma\in\{0,1\}$ and a real number $\rho>1$, we define $\mathcal{B}^\sigma_{r,\rho}$ to be the completion of this space with respect to the norm

$$\|v\|^\sigma_{r,\rho}=\sum_{j\in\mathbb{Z}}\|v_j\|_r\rho^{|2j-\sigma|}\,. \tag{3.4}$$

The even and odd (as functions of t) subspaces of $\mathcal{B}^\sigma_{r,\rho}$ are denoted by $\mathcal{B}^{\sigma,0}_{r,\rho}$ and $\mathcal{B}^{\sigma,1}_{r,\rho}$, respectively. A straightforward computation shows that

$$\|uv\|^\sigma_{r,\rho}\le\|u\|^\sigma_{r,\rho}\sup_j\|v_j\|_r\,,\qquad \|v_j\|_r\le\rho^{-|2j-\sigma|}\|v\|^\sigma_{r,\rho}\,. \tag{3.5}$$

In particular $\mathcal{B}^\sigma_{r,\rho}$ and $\mathcal{B}^{\sigma,0}_{r,\rho}$ are Banach algebras under pointwise multiplication $(uv)_j=u_jv_j$. Furthermore, chains in $\mathcal{B}^\sigma_{r,\rho}$ decrease exponentially. The operator norm of a continuous linear operator L on $\mathcal{B}^\sigma_{r,\rho}$ or $\mathcal{B}^{\sigma,\tau}_{r,\rho}$ will be denoted by $\|L\|^\sigma_{r,\rho}$.

Notice that $\mathcal{B}^0_{r,\rho}=\mathcal{B}^1_{r,\rho}$ as vector spaces. Our reason for choosing the σ-dependent norm (3.4) is that the reflection $q\mapsto\check q$ defined by (1.2) is an isometry for this norm.

In order to solve the fixed point problem for G, we first determine (numerically) a finite chain $\bar q$ that is an approximate fixed point of G, and a linear isomorphism A of $\mathcal{B}_{r,\rho}$ that is an approximate inverse of $\mathrm{I}-DG(\bar q)$. Then the map $\mathcal{F}$ defined by

$$\mathcal{F}(h)=G(q)-q+h\,,\qquad q=\bar q+Ah\,, \tag{3.6}$$

can be expected to be a contraction near the origin. Clearly, h is a fixed point of $\mathcal{F}$ if and only if q is a fixed point of G.

Consider now a fixed but arbitrary row in Table 1. Among other things, it specifies a domain parameter $\text{в} = (\sigma, \tau, r, \rho)$ identifying a space $\mathcal{B}^{\sigma;\tau}_{r,\rho}$, and a symmetry parameter ϖ. If $\varpi = \pm$, then we define $\mathcal{R}_{\sigma,\varpi}$ to be the reflection $q \mapsto \check{q}$ given by the equation (1.2). Otherwise, if $\varpi =$"none", then $\mathcal{R}_{\sigma,\varpi}$ is defined to be the identity map.

The following lemma is proved with the assistance of a computer, as described in Section 5.

Lemma 3.1. *For each set of parameters $(\phi, \psi, \omega, \varpi, \text{в}, \ell)$ given in Table 1, there exists a finite chain $\bar{q}$, a linear isomorphism A of $\mathcal{B}^{\sigma;\tau}_{r,\rho}$, and positive constants ε, K, δ satisfying $\varepsilon + K\delta < \delta$, such that the map $\mathcal{F}$ defined by (3.6) is analytic on $\mathcal{B}^{\sigma;\tau}_{r,\rho}$ and satisfies*

$$\|\mathcal{F}(0)\|^{\sigma}_{r,\rho} \le \varepsilon\,, \qquad \|D\mathcal{F}(h)\|^{\sigma}_{r,\rho} \le K\,, \qquad h \in B_{\delta,2\delta}\,, \tag{3.7}$$

with $B_{\delta,2\delta}$ as defined below. The support of $\bar{q}$ is the set $\{j \in \mathbb{Z} : \sigma - \ell < j < \ell\}$, and $Ah = h$ for every chain h that vanishes on this set. Furthermore, $\bar{q}$ is invariant under $\mathcal{R}_{\sigma,\varpi}$ and A commutes with $\mathcal{R}_{\sigma,\varpi}$.

The set $B_{\delta,2\delta}$ in the equation (3.7) is a special case of the following. Given real numbers $u, v > 0$, we define $B_{u,v}$ to be the set of all chains $h \in \mathcal{B}^{\sigma;\tau}_{r,\rho}$ with the property that

$$\sum_{j \le \sigma-\ell} \|h_j\|_r \rho^{|2j-\sigma|} \le v\,, \qquad \sum_{\sigma-\ell<j<\ell} \|h_j\|_r \rho^{|2j-\sigma|} \le u\,, \qquad \sum_{j \ge \ell} \|h_j\|_r \rho^{|2j-\sigma|} \le v\,. \tag{3.8}$$

Notice that $B_{\delta,2\delta}$ includes the closed ball in $\mathcal{B}^{\sigma;\tau}_{r,\rho}$ of radius δ, centered at the origin.

Proof of Theorem 1.1. First, we note that ϕ and ψ are even whenever $\tau = 1$. Thus, the right hand side of (3.1) belongs to $\mathcal{B} = \mathcal{B}^{\sigma;\tau}_{r,\rho}$ whenever $q \in \mathcal{B}$. Furthermore, $\psi_2 - (\omega k)^2$ is bounded away from zero for all integers $k \ge \tau$. This implies that $L : \mathcal{B} \to \mathcal{B}$ has a bounded inverse. Thus, G is well-defined on $\mathcal{B}$ and analytic (in fact polynomial). The same is true for the map $\mathcal{F}$, since A is bounded.

By the contraction mapping theorem, the given bounds imply that $\mathcal{F}$ has a unique fixed point $h_* \in \mathcal{B}^{\sigma;\tau}_{r,\rho}$ with norm $\le \delta$. Now $q_* = \bar{q} + Ah_*$ is a fixed point of G and thus satisfies the equation (2.2).

It is straightforward to check that G commutes with $\mathcal{R} = \mathcal{R}_{\sigma,\varpi}$. Since A commutes with $\mathcal{R}$ as well, the same is true for the map $\mathcal{F}$. Here, we have used also that $\bar{q}$ is invariant under $\mathcal{R}$. Thus, given that $h_* = \lim_{k\to\infty} \mathcal{F}^k(0)$, it follows that h_* and $q_* = \bar{q} + Ah_*$ are invariant under $\mathcal{R}$. **QED**

We note that an alternative to the map G considered here would be the map $\mathcal{G}$, defined by

$$\mathcal{G}(q) = -\mathcal{L}^{-1}\bigl[\nabla_\sigma \tilde{\phi}'(\nabla_\sigma q) + \tilde{\psi}''(q)\bigr]\,, \qquad \mathcal{L} = \omega^2\partial_t^2 + \phi_2 \nabla_\sigma^* \nabla_\sigma + \psi_2\,, \tag{3.9}$$

where $\tilde{\phi}(x) = \phi(x) - \frac{1}{2}\phi_2 x^2$. The inverse of $\mathcal{L}$ involves a lattice-convolution with an exponentially decreasing kernel (for suitable values of ϕ_2, ψ_2, and ω). This kernel can be computed explicitly, but its nonlocality complicates the analysis significantly; especially the construction of multi-breather solutions.

3.2. Combining solutions

In this subsection we give a general result that will be used later to prove Theorem 1.3. This part is independent of previous sections, which allows us to adapt the notation to the problem at hand.

In what follows, if we write a Banach space Y as a direct sum of subspaces,

$$Y = \bigoplus_k Y_k \,, \tag{3.10}$$

then the norm on this space is assumed to satisfy

$$\|y\|_{Y} = \sup_k \|y_k\|_{Y} \,, \qquad y_k = P_k y \,, \tag{3.11}$$

where P_k denotes the canonical projection from Y onto Y_k. Let now Y be a direct sum as in (3.10), where the index k runs over the set of all integers.

For each integer k, let f_k be a C^1 mapping on $Y_k \oplus Y_{k+1}$. We extend f_k to Y by setting $f_k(y) = f_k(y_k, y_{k+1})$. Define

$$F_n^-(y) = \sum_{k<n}{}' f_k(y) \,, \qquad F_n^+(y) = \sum_{k\ge n}{}' f_k(y) \,, \qquad F(y) = \sum_{k}{}' f_k(y) \,, \tag{3.12}$$

for all $y \in Y$.

Notation 3.2. *Here, and in what follows,* $\sum'$ *denotes a pointwise sum, meaning that its* m*-th component converges in* Y_m*, for each* m*.*

Notice that, for the sums in (3.12), each component $P_m \sum'_k f_k(y)$ is a sum of at most two nonzero terms.

For odd integers n, let $X_n \subset \bigoplus_{k\le n} Y_k$ and $Z_n \subset \bigoplus_{k\ge n} Y_k$ be subspaces that carry norms $\|.\|_{X_n}$ and $\|.\|_{Z_n}$, respectively, and that are complete for these norms. We also assume that $X_n \cap Z_n = Y_n$ and

$$\|y_n\|_{X_n} = \|y_n\|_{Z_n} = \|y_n\|_{Y} \,, \qquad y_n \in Y_n \qquad (n \text{ odd}). \tag{3.13}$$

In the remaining part of this subsection, we always assume that n is an <u>even</u> integer, unless specified otherwise.

Define $\mathcal{H}_n = X_{n-1} \oplus Y_n \oplus Z_{n+1}$. So a vector $h \in \mathcal{H}_n$ admits a unique representation $h = x_{n-1} + y_n + z_{n+1}$ with $x_{n-1} \in X_{n-1}$, $y_n \in Y_n$, and $z_{n+1} \in Z_{n+1}$. We will also use the notation $h = (x_{n-1}, y_n, z_{n+1})$.

Remark 3. We think of $h = (x_{n-1}, y_n, z_{n+1})$ as representing a chain with "center" y_n, left-tail x_{n-1}, and right-tail z_{n+1}. The idea is to take F locally of the form (3.6), with $\bar q$ and A depending on n. The goal is to show that F has a fixed point near $h = 0$.

We now impose conditions on the function F that can be checked separately for each of the spaces $\mathcal{H}_n$. Let ε, K, δ be positive integers satisfying $\varepsilon + K\delta < \delta$. Assume that

$$\|P_n F(0)\|_{Y_n} \le \varepsilon \,, \quad \|P_{n-1} F_n^-(0)\|_{X_{n-1}} \le \tfrac{1}{2}\varepsilon \,, \quad \|P_{n+1} F_n^+(0)\|_{Z_{n+1}} \le \tfrac{1}{2}\varepsilon \,. \tag{3.14}$$

In addition, assume that F_n^- defines a C^1 function on $X_{n-1} \oplus Y_n$, that F_n^+ defines a C^1 function on $Y_n \oplus Z_{n+1}$, and that

$$\|P_n DF(h)\|_{\mathcal{H}_n \to Y_n} \leq K\,, \tag{3.15}$$

whenever $\|h\|_{\mathcal{H}_n} \leq \delta$.

In order to formulate our last assumption, we write F_n^- as a function of two arguments, the first in X_{n-1} and the second in Y_n. Similarly, we write F_n^+ as a function of two arguments, the first in Y_n, and the second in Z_{n+1}. Assume that

$$\begin{aligned} \|P_{n-1} DF_n^-(x_{n-1}, y_n)\|_{X_{n-1}\oplus Y_n \to Y_{n-1}} &\leq \tfrac{1}{2}K\,, \\ \|P_{n+1} DF_n^+(y_n, z_{n+1})\|_{Y_n \oplus Z_{n+1} \to Y_{n+1}} &\leq \tfrac{1}{2}K\,, \end{aligned} \tag{3.16}$$

whenever $\|h\|_{\mathcal{H}_n} \leq \delta$.

Proposition 3.3. *Under the assumptions described above, which includes the condition $\varepsilon + K\delta < \delta$, the map $F : Y \to Y$ has a unique fixed point in the closed ball of radius δ in Y, centered at the origin.*

Proof. If n is odd, then

$$\|P_n F(0)\|_{Y_n} \leq \|P_n F_{n-1}^+(0)\|_{Y_n} + \|P_n F_{n+1}^-(0)\|_{Y_n} \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon\,, \tag{3.17}$$

by the second and third inequality in (3.14). Combining this with the first inequality in (3.14) yields

$$\|F(0)\|_Y = \sup_{n\in\mathbb{Z}} \|P_n F(0)\|_{Y_n} \leq \varepsilon\,. \tag{3.18}$$

Let now u be a fixed but arbitrary vector in Y with norm $\|u\| \leq \delta$. Our goal is to estimate $DF(u)$. Notice that

$$\begin{aligned} P_n DF(u)v = P_n D_{Y_{n-1}} f_{n-1}(u_{n-1}, u_n) v_{n-1} + P_n D_{Y_n} f_{n-1}(u_{n-1}, u_n) v_n \\ + P_n D_{Y_n} f_n(u_n, u_{n+1}) v_n + P_n D_{Y_{n+1}} f_n(u_n, u_{n+1}) v_{n+1}\,, \end{aligned} \tag{3.19}$$

for all $u, v \in Y$ and all integers n. Here, D_{Y_k} denote the partial derivative operator with respect to the component in Y_k.

Consider first the case where n is even. Setting $x_{n-1} = u_{n-1}$, $y_n = u_n$, and $z_{n+1} = u_{n+1}$, the vector $h = (x_{n-1}, y_n, z_{n+1})$ has norm $\|h\|_{\mathcal{H}_n} \leq \delta$. So by (3.19) and (3.15), we have

$$\|P_n DF(u)\|_{Y \to Y_n} \leq \|P_n DF(x_{n-1}, y_n, z_{n+1})\|_{\mathcal{H}_n \to Y_n} \leq K\,. \tag{3.20}$$

Next, consider the case when n is odd. Setting $y_{n-1} = u_{n-1}$, $x_n = z_n = u_n$, and $y_{n+1} = u_{n+1}$, we have

$$\begin{aligned} \|P_n DF(u)\|_{Y \to Y_n} \leq \|P_n DF_{n-1}^+(y_{n-1}, z_n)\|_{Y_{n-1}\oplus Z_n \to Y_n} \\ + \|P_n DF_{n+1}^-(x_n, y_{n+1})\|_{X_n \oplus Y_{n+1} \to Y_n} \leq \tfrac{1}{2}K + \tfrac{1}{2}K\,, \end{aligned} \tag{3.21}$$

by (3.19) and (3.16). Combining (3.20) and (3.21) yields

$$\|DF(u)\|_{Y\to Y} = \sup_{n\in\mathbb{Z}} \|P_n DF(u)\|_{Y\to Y_n} \le K\,. \tag{3.22}$$

The claim now follows from the contraction mapping theorem. **QED**

3.3. Multi-breather solutions

In this subsection, we consider a fixed but arbitrary choice of parameters $(\phi,\psi,\omega,\tau,\sigma,r,\rho)$ that is represented by one of the rows 1-4 or 6-12 of Table 1. The claim in Theorem 1.3 is that we can produce solutions that look like strings of breather solutions. To simplify notation, consider first the case of a bi-infinite string, indexed by $\mathbb{Z}$.

For any given integer m, we choose one of the maps $\mathcal{F}$ for the given parameters, as described in Lemma 3.1. This involves an approximate fixed point $\bar q$ of G and an operator A. We note that, if q is a possible choice for the finite chain $\bar q$ mentioned in Lemma 3.1, then $-q$ is an equally good choice. Here we allow either choice.

Let $A' = A - \mathrm{I}$. After choosing an integer J_m, we set $\bar q^m = \mathcal{T}_{J_m}\bar q$ and $A'_m = T_{J_m} A' T_{J_m}^{-1}$. Here T_J denotes translation by J, that is, $(T_J h)_j = h_{j-J}$ for all j. Using the positive integer ℓ from Table 1, define $\mathcal{J}_m = \{j\in\mathbb{Z} : \sigma-\ell \le j - J_m \le \ell\}$. This is the set that we referred to as the approximate support of the breather q^m in Theorem 1.3. It includes the support of $\bar q^m$.

We may assume that the sequence $m\mapsto J_m$ is increasing, and that $J_0 = 0$. Assuming furthermore that $\operatorname{dist}(\mathcal{J}_m,\mathcal{J}_{m+1})$ is positive and even for all m, we define

$$F(h) = G(h) + \sum_m{}' \Big[G(h+\bar q^m + A'_m h) - G(h) - \bar q^m - A'_m h\Big]\,. \tag{3.23}$$

The sum in this equation converges pointwise, at each integer $j\in\mathbb{Z}$, since $(A'_m h)_j = 0$ whenever j lies outside the support of $\bar q^m$.

In order to see how this fits into the framework discussed in the preceding subsection, consider a fixed term in this sum, indexed by m. Let $n = 2m$. Consider the translated space $\mathcal{B}_n = T_{J_m}\mathcal{B}^{\sigma;\tau}_{r,\rho}$ with norm $\|h\|_{\mathcal{B}_n} = \big\|T_{J_m}^{-1} h\big\|^{\sigma}_{r,\rho}$. To every chain $h\in\mathcal{B}_n$ we associate its left-tail $x\in\mathcal{B}_n$, center $y\in\mathcal{B}_n$, and right-tail $z\in\mathcal{B}_n$ by setting

$$x_j = \begin{cases} h_j & \text{if } j\le j_m^-,\\ 0 & \text{if } j > j_m^-,\end{cases}\qquad y_j = \begin{cases} h_j & \text{if } j_m^- < j < j_m^+,\\ 0 & \text{otherwise},\end{cases}\qquad z_j = \begin{cases} h_j & \text{if } j\ge j_m^+,\\ 0 & \text{if } j < j_m^+,\end{cases} \tag{3.24}$$

where $j_m^- = \frac12[\max(\mathcal{J}_{m-1}) + \min(\mathcal{J}_m)]$ and $j_m^+ = \frac12[\max(\mathcal{J}_m)+\min(\mathcal{J}_{m+1})]$. Denote by P_n the projection $h\mapsto y$ and set $Y_n = P_n\mathcal{B}_n$. The ranges of the projections $h\mapsto x$ and $h\mapsto z$ are denoted by X_{n-1} and Z_{n+1}, respectively. In addition, we define Y_{n-1} and Y_{n+1} to be the one-dimensional subspaces of $\mathcal{B}_n$ spanned by all chains supported in $\{j_m^-\}$ and $\{j_m^+\}$, respectively. On $\mathcal{H}_n = X_{n-1}\oplus Y_n\oplus Z_{n-1}$ we choose the norm

$$\|h\|_{\mathcal{H}_n} = \max\big\{\|x\|_{\mathcal{B}_n}, \|y\|_{\mathcal{B}_n}, \|z\|_{\mathcal{B}_n}\big\}\,. \tag{3.25}$$

The goal now is to apply Proposition 3.3. The following is meant to be a continuation of Lemma 3.1.

Lemma 3.4. *Consider one of the rows* 1-4 *or* 6-12 *in Table 1. In addition to the properties of $\mathcal{F}$ described in Lemma 3.1, the bounds (3.15) and (3.16) with $n = 0$ are satisfied for each $h \in B_{\delta,2\delta}$. Furthermore, $\delta < 2^{-50}$ and $\delta\|A\|^{\sigma}_{r,\rho} < 2^{-46}$.*

For the proof of these estimates, we refer to Section 5. Based on this lemma, we can now give a

Proof of Theorem 1.3. . Consider first the case of a bi-infinite string. Then we may assume that the index set is $\mathbb{Z}$, and that the sequence $m \mapsto J_m$ has the properties mentioned before (3.23). The goal is to verify the assumptions of Proposition 3.3. The parameters $(\phi, \psi, \omega, \sigma, r, \rho)$ are assumed to be fixed.

By translation invariance, it suffices to verify the conditions (3.14), (3.15), and (3.16) for $n = 0$. Due to the projections that appear in these conditions, F can be replaced by the map $\mathcal{F}$ associated with $\bar{q} = \bar{q}^0$ and $A = A_0$. Notice that the set $B_{\delta,2\delta}$ defined by (3.8) includes the closed ball in $\mathcal{H}_0$ of radius δ, centered at the origin. Thus, under our assumption that (3.15) and (3.16) hold for $h \in B_{\delta,2\delta}$, these bounds hold whenever $\|h\|_{\mathcal{H}_0} \le \delta$, as required by Proposition 3.3. The first inequality in (3.14) follow from the first inequality in (3.7). The other two inequalities in (3.14) are satisfied trivially in our case: $j_0^{\pm}$ is at a distance ≥ 2 from the support of any of the chains $\bar{q}^m$, so $P_{\pm 1}F_0^{\pm}(0) = 0$.

Proposition 3.3 now implies that F has a locally unique fixed point $h \in Y$. Clearly, the chain

$$q = h + {\sum_m}' \bigl[\bar{q}^m + A'_m h\bigr] \tag{3.26}$$

is a solution of the equation (2.2). Here we are using Notation 3.2.

Notice that $\bar{q}^m + A'_m h$ is supported in $\mathcal{J}_m$, for each m. For j in between those supports, we have $|q_j(t)| = |h_j(t)| \le \|h\|_Y \le \delta$. Consider now $j \in \mathcal{J}_0$. If h^0 denoted the fixed point of the map $\mathcal{F}$ associated with $\bar{q} = \bar{q}^0$ and $A = A_0$, and if $q^0 = \bar{q} + Ah^0$ denotes the corresponding solution of (1.1), then

$$\bigl|q_j - q_j^0\bigr| = \bigl|\bigl(A\bigl(h - h^0\bigr)\bigr)_j\bigr| \le \|A\|_{Y_0 \to Y_0} \bigl\|P_0\bigl(h - h^0\bigr)\bigr\|_{Y_0} \le 2\delta\|A\|^{\sigma}_{r,\rho}\,, \tag{3.27}$$

for all $j \in \mathcal{J}_0$. The same bound holds of course for $j \in \mathcal{J}_m$ and any m. This concludes the proof of Theorem 1.3 for the case of two-sided infinite strings $m \mapsto \bar{q}^m$. The proof for one-sided infinite strings and for finite strings is similar, so we omit it here. **QED**

4. Spectral estimates

Our goal is to reduce the proof of Theorem 1.2 to estimates on finite-dimensional systems.

4.1. Instability

We first consider the task of proving spectral instability. Let $\mathcal{H} = \ell^2(\mathbb{Z})$. The simplest cases are the solutions 5, 6, 7, and 12, where the set Σ_0^e defined by (2.25) includes a real interval containing the point 1. These solutions are spectrally unstable, since Σ_0^e is the essential spectrum of $\Phi(2\pi):\mathcal{H}^2\to\mathcal{H}^2$, as was described before (2.25).

In the other cases we use a perturbation argument, involving an approximation $\Phi_o(2\pi)$ for the map $\Phi(2\pi)$. First, we need a uniform bound. Let $(j,t)\mapsto\alpha_j(t)$ and $(j,t)\mapsto\beta_j(t)$ be bounded functions on $\mathbb{Z}\times\mathbb{R}$ that are continuous in the time variable t. Consider the flow on $\mathcal{H}^2$ given by the equation

$$\partial_t\mathfrak{v}=X\mathfrak{v}\,,\qquad \mathfrak{v}=\begin{bmatrix}v\\ \nu\end{bmatrix},\quad X=\begin{bmatrix}0 & \mathrm{I}\\ -H & 0\end{bmatrix},\quad Hv=\nabla_\sigma^*\alpha\nabla_\sigma v+\beta v\,. \tag{4.1}$$

Here αv and βv are defined by pointwise multiplication. Assume that we have enclosures $[\alpha_*^-,\alpha_*^+]\ni\alpha_j(t)$ and $[\beta_*^-,\beta_*^+]\ni\beta_j(t)$ that are valid for all j and all t. Define

$$c=\tfrac{1}{2}\max\bigl\{|4\alpha_*^-+\beta_*^--1|\,,|4\alpha_*^++\beta_*^+-1|\bigr\}\,. \tag{4.2}$$

Proposition 4.1. *Under the above-mentioned assumption, the time-t maps $\Phi(t)$ for the flow (4.1) satisfy the bounds $\|\Phi(t)\|\le e^{c|t|}$ for all t.*

Proof. Let $\mathfrak{v}=\mathfrak{v}(t)$ be a fixed but arbitrary solution of the equation (4.1). Using that $\|H-\mathrm{I}\|\le 2c$, we have

$$\begin{aligned}\partial_t\|\mathfrak{v}\|^2&=\bigl\langle\mathfrak{v},(X+X^*)\mathfrak{v}\bigr\rangle=\bigl\langle v,(\mathrm{I}-H)\nu\bigr\rangle+\bigl\langle\nu,(\mathrm{I}-H)v\bigr\rangle\\ &\le\|\mathrm{I}-H\|\|\mathfrak{v}\|^2\le 2c\|\mathfrak{v}\|^2\,,\end{aligned} \tag{4.3)}$$

and thus $\partial_t\|\mathfrak{v}\|\le c\|\mathfrak{v}\|$, for every $t\in\mathbb{R}$. By integration we obtain

$$\bigl\|\Phi(t)\mathfrak{v}(0)\bigr\|=\|\mathfrak{v}(t)\|\le e^{c|t|}\|\mathfrak{v}(0)\|\,. \tag{4.4}$$

This holds for arbitrary initial conditions $\mathfrak{v}(0)\in\mathcal{H}^2$. Thus, $\|\Phi(t)\|\le e^{c|t|}$ as claimed. **QED**

For simplicity, assume now that $\tilde\alpha=\alpha-\bar\alpha$ and $\tilde\beta=\beta-\bar\beta$ satisfy the Condition 2.3, with $\bar\beta$ and $\bar\beta+4\bar\alpha$ contained in $[0,1)$. Then the spectrum of $\Phi(2\pi)$ off the unit circle consists of isolated eigenvalues with finite multiplicities.

Consider another operator H^o of the same type, for sequences α^o and β^o that have the same asymptotic limits $\bar\alpha$ and $\bar\beta$.

Proposition 4.2. *Let $\alpha_*^\pm$ and $\beta_*^\pm$ be real numbers, such that $\{\alpha_j^o(t),\alpha_j(t)\}\subset[\alpha_*^-,\alpha_*^+]$ and $\{\beta_j^o(t),\beta_j(t)\}\subset[\beta_*^-,\beta_*^+]$ holds for all j and all t. Let μ be a real number larger than 1. Assume that the time-2π map $\Phi_o(2\pi)$ associated with H^o has an odd number of eigenvalues (counting multiplicities) in the half-plane $\mathrm{Re}(z)>\mu$, and that μ is not an eigenvalue of $\Phi_o(2\pi)$. If in addition,*

$$2\pi e^{2\pi c}\bigl\|H(t)-H^o(t)\bigr\|\bigl\|(\Phi_o(2\pi)-\mu)^{-1}\bigr\|<1 \tag{4.5}$$

for all $t\in[0,2\pi]$, with c given by (4.2), then $\Phi(2\pi)$ has an odd number of eigenvalues in the half-plane $\mathrm{Re}(z)>\mu$.

Proof. For $0\le\kappa\le 1$ define $H^\kappa=(1-\kappa)H^o+\kappa H$. Denote by A_κ the time-2π map associated with H^κ. Our goal is to show that

$$\bigl\|(A_\kappa-A_0)(A_0-\mu)^{-1}\bigr\|<1\,,\qquad 0\le\kappa\le 1\,. \tag{4.6}$$

Then each $A_\kappa - \mu = \bigl[\mathrm{I} + (A_\kappa - A_0)(A_0-\mu)^{-1}\bigr](A_0-\mu)$ has a bounded inverse, implying that no A_κ has an eigenvalue μ. Since the eigenvalues of A_κ off the unit circle depend continuously on κ and come in complex-conjugate pairs, this implies that each operator A_κ has an odd number of eigenvalues in the half-plane $\mathrm{Re}(z) > \mu$. So the claim made in Proposition 4.2 follows form the bound (4.6).

What we need now is a bound on $A_\kappa - A_0$. Denote by $\Phi^\kappa_{t,s}$ the flow-map for H^κ from time s to time t. These maps satisfy the equation

$$\Phi^\kappa_{t,r} = \Phi^0_{t,r} + \int_r^t \Phi^0_{t,s} P_2^* \bigl[H^\kappa(s) - H^o(s)\bigr] P_1 \Phi^\kappa_{s,r}\, ds\,, \tag{4.7}$$

where $P_1 = [1\ 0]$ and $P_2 = [0\ 1]$ are the the operators from $\mathcal{H}^2$ to $\mathcal{H}$ that are described after (2.24). By Proposition 4.1 we have $\|\Phi^\kappa_{t,s}\| \le e^{c|t-s|}$. Taking norms in (4.7) we get

$$\bigl\|\Phi^\kappa_{t,r} - \Phi^0_{t,r}\bigr\| \le \int_r^t e^{c(t-s)} \bigl\|H^\kappa(s) - H^o(s)\bigr\| e^{c(s-r)}\, ds\,. \tag{4.8}$$

In particular,

$$\|A_\kappa - A_0\| \le 2\pi\kappa e^{2\pi c} \sup_{0\le t\le 2\pi} \bigl\|H(t) - H^o(t)\bigr\|\,. \tag{4.9}$$

When combined with the assumption (4.5), this yields the desired bound (4.6). **QED**

Our choice of H^o will be described in Subsection 4.4.

4.2. Separating sets and monotonicity

Next, we consider the task of proving spectral stability. We adapt an approach that was introduced in [13]. Roughly speaking, the goal is to find two simple approximations $H^{\pm 1}$ for the operator H defined in (2.6), such that $H^{-1} \lll H \lll H^1$. If we can control the time-2π maps $\Phi_s(2\pi)$ associated with the family of operators $H^s = \frac{1-s}{2}H^{-1} + \frac{1+s}{2}H^1$, in a way that will be explained below, then we can also control the time-2π map $\Phi(2\pi)$ associated with H. And as described at the end of this subsection, we can reduce this to a finite-dimensional problem. Here, and in what follows, the parameter s always ranges over the interval $[-1,1]$.

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Consider the Hilbert space $\mathcal{H}^2$ of all pairs $\mathfrak{v} = \bigl[{v \atop \nu}\bigr]$, equipped with the inner product $\langle \mathfrak{v}, \mathfrak{v}'\rangle = \langle v, v'\rangle + \langle \nu, \nu'\rangle$. A linear operator on $\mathcal{H}^2$ is said to be symplectic if it is “unitary” for the quadratic form

$$G(\mathfrak{v}, \mathfrak{v}') = i\langle v, \nu'\rangle - i\langle \nu, v'\rangle\,, \qquad \mathfrak{v}, \mathfrak{v}' \in \mathcal{H}^2\,. \tag{4.10}$$

We are interested in the parameter-dependence of eigenvalues that lie on the unit circle. Let $s \mapsto A_s$ be a continuous curve of symplectic operators on $\mathcal{H}^2$. Let $\mathfrak{v}_s$ be eigenvector of A_s with eigenvalue λ_s, both depending continuously on s. If λ_s lies on the unit circle and is simple, for some value $s = s_0$, then the same is true for s near s_0. The reason is that, by symplecticity, the spectrum of A_s is invariant under complex conjugation

$z \mapsto \bar{z}$ and under inversion $z \mapsto z^{-1}$. More can be said by using the the Krein signature of $\mathfrak{v}_s$, which is defined to be the sign of

$$G(\mathfrak{v}_s, \mathfrak{v}_s) = -2\,\mathrm{Im}\langle v_s, \nu_s\rangle\,. \tag{4.11}$$

It is straightforward to check that $G(\mathfrak{v}_s, \mathfrak{v}_s)$ vanishes unless λ_s lies on the unit circle. According to Krein theory, the only way that λ_s can move off the unit circle, as s is varied, is for λ_s to collide with an eigenvalue (for an eigenvector) of opposite Krein signature. This motivates the following

Definition 4.3. *Let $\Lambda = \mathbb{S} \cup (0, \infty)$, where $\mathbb{S}$ denoted the unit circle in $\mathbb{C}$. Consider a finite subset Z of $\mathbb{S} \setminus \{1\}$ that contains at least two points. This set defines a partition of $\Lambda \setminus Z$ into connected sets. The set containing 1 will be referred to as the "cross". The other sets in this partition are subsets of $\mathbb{S}$ and will be referred to as "arcs". Given a symplectic operator A, We say that Z is a separating set for A, if all eigenvalues of A that lie in the same arc have the same (nonzero) Krein signature. Furthermore, we impose that the cross contains at most two eigenvalues of A, and that Z contains no eigenvalues of A.*

We note that the separating sets defined in [13] were allowed to contain the point 1. But we only considered partitions of $\mathbb{S} \setminus Z$, since we did not allow bifurcations at 1. Here, the cross associated with Z is needed to control a pair of eigenvalues near 1, independently of whether these eigenvalues lie on $\mathbb{S}$ or not. Recall that the true system has an eigenvalue 1, and by symplecticity, this eigenvalue has an even multiplicity. When considering finite-dimensional approximations, this eigenvalue can split into multiple eigenvalues near 1.

Using the above-mentioned fact about the Krein signature, say in the form of Proposition 2.9 in [13], we immediately obtain the following.

Proposition 4.4. *Let Z be a finite subset of $\mathbb{S} \setminus \{1\}$ that does not contain any eigenvalues of A_s for any s. Assume that one of the operators A_s has the following property: Z is a separating set for A_s, and all eigenvalues of A_s belong to Λ. Then each of the operators A_s has this property.*

Let $(s, t) \mapsto H^s(t)$ be a continuous family of linear operators on $\mathcal{H}$, indexed by $[-1, 1] \times \mathbb{R}$. Consider the flow on $\mathcal{H}^2$ defined by the equation

$$\partial_t \mathfrak{v}(t) = X_s(t)\mathfrak{v}(t)\,, \qquad X_s(t) = \begin{bmatrix} 0 & \mathrm{I} \\ -H^s(t) & 0 \end{bmatrix}. \tag{4.12}$$

Assume in addition that each operator $H^s(t)$ is self-adjoint. Then a straightforward computation shows that the time-t maps $\Phi_s(t)$ for this flow are symplectic. In what follows, we also assume that $H^s(t)$ is 2π-periodic in t. Then the map $\Phi_s(2\pi)$ is of particular interest.

The monotonicity property that we mentioned earlier can be stated roughly as follows. Assume that $\frac{d}{ds} H^s(t)$ is positive for all s and all t. Then the eigenvalues of $\Phi_s(2\pi)$ that have negative (positive) Krein index move (counter)clockwise on $\mathbb{S}$, as s is increased.

Formally, this follows from an explicit computation [13]. To make this statement more precise, we need to avoid collisions of eigenvalues of opposite Krein signatures. And for simplicity, we restrict now to affine families

$$H^s = H^0 + sD\,, \qquad s \in [-1, 1]\,, \tag{4.13}$$

that are strongly increasing, in the sense that $D \gg 0$. Here, and in what follows, if $C = C(t)$ and $D = D(t)$ are curves of self-adjoint linear operators on $\mathcal{H}$, then we define $D \gg C$ or $C \ll D$ to mean that there exists $\varepsilon > 0$ such that $D(t) - C(t) - \varepsilon \mathrm{I}$ is a positive operator for all t.

An eigenvalue of $\Phi_s(2\pi)$ that lies on the unit circle will be written as $\lambda = e^{2\pi i \eta}$. The number η will be referred to as a Floquet number for $\Phi_s(2\pi)$.

Proposition 4.5. (monotonicity) *Assume that the family of operators $A_s = \Phi_s(2\pi)$ satisfies the hypotheses of Proposition 4.4. Consider the eigenvalues of A_s that lie in the arcs determined by Z. Then the corresponding Floquet numbers $\eta_1, \eta_2, \ldots$ can be labeled in such a way that each η_k is a real analytic function of the parameter s. Furthermore, if the Krein signature of λ_k is positive (negative) then $\frac{d}{ds}\eta_k$ is negative (positive).*

This proposition is a consequence of Proposition 4.4, and of Lemma 3.6 in [13].

Consider now a situation where $H^{-1} \ll H \ll H^1$, as mentioned at the beginning of this subsection. Since we can interpolate first between H^{-1} and H, and then between H and H^1, Proposition 4.5 suggests that each Floquet number for $\Phi(2\pi)$ can be bounded from above and below by the corresponding Floquet numbers of H^{-1} and H^1. This is indeed the case, but the following suffices for our purpose.

Proposition 4.6. *Assume that $H^{-1} \ll H \ll H^1$. Let Z be a finite subset of $\mathbb{S} \setminus \{1\}$ that does not contain any eigenvalues of $\Phi_s(2\pi)$, for any s. Assume that, for some s, the operator $A = \Phi_s(2\pi)$ has the following property: Z is a separating set for A, and all eigenvalues of A lie on Λ. Then $A = \Phi(2\pi)$ has the same property.*

The proof of this proposition is similar to the proof of Corollary 3.8 in [13], so we omit it here.

Remark 4. By Proposition 2.12, it suffices to consider the operators H_k defined by (2.43), if k is chosen sufficiently large. The results of this subsection will be applied with $H = H_k$. Choosing $\mathcal{H} = \mathcal{H}_{0,n}$ with $n > k$, the time-2π map $\Phi(2\pi)$ leaves $\mathcal{H}^2$ invariant, and its spectrum does not depend on n.

4.3. Verifying separation

Motivated by Proposition 4.4, consider the task of verifying that Z does not contain any eigenvalues of $A_s = \Phi_s(2\pi)$ for any s. It is worth noting that this task simplifies if we first verify that Z is a separating set for A_{-1}, and that all eigenvalues of A_{-1} lie on Λ. To see why, notice that every arc Γ defined by Z can be assigned a signature: the signature of the eigenvalues of A_{-1} that lie in Γ. We may assume that Z is "minimal", in the sense that adjacent arcs have opposite signatures. Consider now a point $z \in Z$, and let Γ be an arc that has z in its boundary. As s is increased, starting from -1, the eigenvalues of A_s in Γ all move either toward z, or they all move away from z. In the first case, we call z a "primary" point of Z. In the second case, any eigenvalue that could possibly enter Γ through z must have the same signature as Γ, so z lies on the boundary of the cross. Thus, if we verify that the eigenvalues of A_s avoid all primary points of Z, as s is increased from

-1 to 1, and that A_1 has the same number of eigenvalues in the cross as A_{-1}, then all points in Z are being avoided.

We describe now a method for proving that a given point $e^{2\pi i\eta}$ on the unit circle is not an eigenvalue of any of the operators $\Phi_s(2\pi)$. Consider first a fixed value of the parameter s. Assume that H^s is a self-adjoint linear operator on $\mathcal{H}$ that depends continuously and 2π-periodically on time t. Let $\mathfrak{v}$ be an eigenvector of $\Phi_s(2\pi)$ with eigenvalue $\lambda = e^{2\pi i\eta}$, and let v be the first component of $\mathfrak{v}$. Then the function $w = e^{-i\eta t}v$ is 2π-periodic and satisfies the equation $(\partial_t + i\eta)^2 w = -H^s w$, or equivalently,

$$M_s(\eta)w = 0\,, \qquad M_s(\eta) = (\boldsymbol{k}+\eta)^2 - H^s\,, \qquad \boldsymbol{k} = -i\partial_t\,. \tag{4.14}$$

Here $H^s w$ is defined pointwise by the equation $(H^s w)(t) = H^s(t)w(t)$. If η is real, then $M_s(\eta)$ is self-adjoint as a linear operator on the Hilbert space $\mathfrak{H} = \mathrm{L}^2([0,2\pi],\mathcal{H})$ with the inner product

$$\langle w, w'\rangle = \frac{1}{\pi}\int_0^{2\pi} \langle w(t), w'(t)\rangle\, dt\,. \tag{4.15}$$

To be more precise, the domain of $M_s(\eta)$ is the set of all functions $w \in \mathfrak{H}$ with the property that $\boldsymbol{k}w$ belongs to $\mathfrak{H}$. Then $M_s(\eta)$ is a Fredholm operator on $\mathfrak{H}$, and in particular, the spectrum of $M_s(\eta)$ consists of eigenvalues with finite multiplicity. It is straightforward to show that the eigenfunctions of $M_s(\eta)$ are continuous, and that any nonzero vector w in the null space of $M_s(\eta)$ yields an eigenvalue $\mathfrak{v}$ of $\Phi_s(2\pi)$ with eigenvalue $\lambda = e^{2\pi i\eta}$. For details we refer to [13].

We need the above only for operators H^s in an affine family (4.13), where $(Dw)_j = d_j w_j$ for a bounded sequence $j \mapsto d_j$ of positive real numbers. Consider the affine family $H^s = H^0 + sD$ for $s \in [-1,1]$. With $M_s(\eta)$ as defined in (4.14), our goal is to show that, for some given $\eta \in \mathbb{R}$, none of the operators $M_s(\eta)$ with $s \in [-1,1]$ have an eigenvalue zero.

It is convenient to replace $M_s(\eta)$ by a bounded linear operator $\hat{M}_s(\eta)$ as follows.

Definition 4.7. *For every integer k, define $\theta_k = \max(1,|k|)^{-1}$. Denote by θ the (unique) continuous linear operator on $\mathfrak{H}$ with the property that for each j, if $w_j(t) = e^{ikt}$ for all t, then $(\theta w)_j(t) = \theta_k e^{ikt}$ for all t. If M is any linear operator on $\mathfrak{H}$, then we define $\hat{M}w = \theta M\theta w$, whenever $w \in \mathfrak{H}$ and θw belongs to the domain of M.*

Clearly, $M_s(\eta)$ has an eigenvalue zero if and only if $\hat{M}_s(\eta)$ has an eigenvalue zero. The operators $\theta(\boldsymbol{k}+\eta)^2\theta$ and $\hat{D}$ are trivial to represent. The operator $\hat{H}^0$ is less easy to handle. But it is compact, so we approximate it by a simpler (finite rank) operator $\check{H}^0$.

In order to show that $e^{2\pi i\eta}$ is not an eigenvalue of $\Phi_s(2\pi)$, it suffices now to verify the hypotheses of the following lemma.

Proposition 4.8. [13] *Consider parameters values $-1 = s_0 < s_1 < \ldots < s_m = 1$. Let $C > \|\hat{H}^0 - \check{H}^0\|$. Assume that the operator $\check{M}_{s_j}(\eta) = \theta(\boldsymbol{k}+\eta)^2\theta - \check{H}^0 - s\hat{D}$ has no eigenvalues in $[-C,C]$, and that*

$$(s_j - s_{j-1})\|\hat{D}\| < 2C\,, \tag{4.16}$$

for $j=1,2,\ldots,m$. Then none of the operators $\hat{M}_s(\eta)$ has an eigenvalue zero.

4.4. Proof of Theorem 1.2

In view of Proposition 4.6, we need to find useful upper and lower bounds of the form $H^0-D\leqslant H_n\leqslant H^0+D$ on the operators H_n defined by the equation (2.43). To be more precise, such bounds are needed only for sufficiently large n. Thus, let us start by determining upper and lower bounds on the non-truncated operator H given by (2.6). Instead of $H(q(t))$, we write here $H(t)$ or just H. We assume that $\beta_j\geq 0$ for all j.

The space considered here is $\mathcal{H}=\ell^2(\mathbb{Z})$. We start by defining a self-adjoint truncation H^o of the operator H. After fixing a cutoff $j_*>0$, H^o is defined by the quadratic form

$$\langle v,H^o v\rangle=\sum_{\sigma-j_*\leq j<j_*}\alpha_{j-\sigma+1}|v_{j+1}-v_j|^2+\sum_{\sigma-j_*\leq j\leq j_*}\bigl(\beta_j-\bar\beta\bigr)|v_j|^2+\bar\beta\langle v,v\rangle\,, \tag{4.17}$$

where α and β are the functions defined in (2.4), and where $\bar\beta=\omega^{-2}\psi_2$. The truncation error $\mathcal{E}=H-H^o$ is then given by

$$\langle v,\mathcal{E}v\rangle=\sum_{j\geq j_*\ \text{or}\ j<\sigma-j_*}\alpha_{j-\sigma+1}|v_{j+1}-v_j|^2+\sum_{j>j_*\ \text{or}\ j<\sigma-j_*}\bigl(\beta_j-\bar\beta\bigr)|v_j|^2\,. \tag{4.18}$$

In order to estimate $\mathcal{E}$, we determine for $j=j_*$ and for $j=\sigma-j_*$ an interval $[\alpha_j^-,\alpha_j^+]$ that includes $\{0,\alpha_j\}$. In addition, we determine an interval $[\alpha_\infty^-,\alpha_\infty^+]$ that includes $\{0,\alpha_j\}$ whenever $j<1-j_*$ or $j>j_*-\sigma$. And we choose constants $\gamma^-\leq 0\leq\gamma^+$ such that

$$\gamma^-\leq 4\alpha_\infty^\pm+\beta_j-\bar\beta\leq\gamma_j^+\,,\qquad \text{if } j<\sigma-j_* \text{ or } j>j_*\,. \tag{4.19}$$

Now define

$$\pm d_j^\pm=\begin{cases}2\alpha_{j-s+1}^\pm & \text{if } j=j_*\,,\\ 2\alpha_{j-s}^\pm & \text{if } j=\sigma-j_*\,,\\ \gamma^\pm & \text{if } j<\sigma-j_* \text{ or } j>j_*\,,\\ 0 & \text{if } \sigma-j_*<j<j_*\,.\end{cases} \tag{4.20}$$

Using the trivial inequality $|x-y|^2\leq 2|x|^2+2|y|^2$, we find that

$$-D^-\leqslant\mathcal{E}\leqslant D^+\,,\qquad D^\pm=\operatorname{diag}\bigl(d^\pm+\epsilon\bigr)\,, \tag{4.21}$$

for any $\epsilon>0$. Finally, define

$$H^s=H^0+sD\,,\qquad H^0=H^o+\tfrac12\bigl(D^+-D^-\bigr)\,,\qquad D=\tfrac12\bigl(D^++D^-\bigr)\,. \tag{4.22}$$

Then $D\geqslant 0$ and

$$H^{-1}=H^o-D^-\leqslant H\leqslant H^o+D^+=H^1\,. \tag{4.23}$$

It is straightforward to check that the same holds if H is replaced by any of the operators H_n with n sufficiently large.

Remark 5. The function H^o that is used in our computer-assisted proof differs from (4.17) in the sense that α_j and β_j are replaced by function α_j^o and β_j^o that are very close to α_j and β_j, respectively. The value of $\epsilon > 0$ in the definition (4.21) is chosen to (over)compensate for the resulting error.

Notice that a chain v that is supported at a single point $j < \sigma - j_*$ or $j > j_*$ is an eigenvector of H^s, with eigenvalue

$$\mu_s = \bar{\beta} + \tfrac{1-s}{2}(\gamma^- - \epsilon) + \tfrac{1+s}{2}(\gamma^+ + \epsilon)\,. \tag{4.24}$$

The corresponding eigenvalues of $\Phi_s(2\pi)$ need to be considered as well in our application of Proposition 4.6. In $\mathcal{H}$ they have infinite multiplicity, but when considering H_n in place of H, only the eigenvalues with eigenvectors in $P_n\mathcal{H}$ are relevant. In order to compute their Krein signature, write $\mu_s = -\eta_s^2$. Then the corresponding eigenvalues for X_s are $\pm i\eta_s$. So we need $\mu_s \le 0$ for $\Phi_s(2\pi)$ to be spectrally stable. Assume that $\eta_s > 0$. Using (4.11), one easily finds that the eigenvector for the eigenvalue $e^{2\pi i\eta_s}$ of $\Phi_s(2\pi)$ has a negative Krein signature, while $e^{-2\pi i\eta_s}$ has a positive Krein signature.

The following two lemmas are proved with the assistance of a computer, as described in Section 5.

Lemma 4.9. *For each of the solutions* 2, 3, 4, 8, *and* 13, *there exists a family of operators* $s \mapsto H^s$ *as described above, satisfying* $H^{-1} \lessdot H_n \lessdot H^1$ *for large* n. *In addition, there exists a minimal separating set* Z *for* $\Phi_{-1}(2\pi)$, *a finite rank operator* $\check{H}^0$, *a constant* $C > 0$, *and parameter values* $-1 = s_0 < s_1 < \ldots < s_m = 1$, *such that the hypotheses of Proposition 4.8 are satisfied, for every primary point* $e^{2\pi i\eta}$ *in* Z.

For the definition of a minimal separating set Z and of a primary point in Z, see Subsection 4.3.

Lemma 4.10. *For each of the solutions* 1 *and* 11, *there exist real numbers* $\alpha_*^{\pm}$, $\beta_*^{\pm}$, *and* $\mu > 1$ *such that the hypotheses of Proposition 4.2 are satisfied, with* H^o *as described above.*

We note that the set Z described in Lemma 4.9 is determined by first computing the eigenvalues for $\Phi_1(2\pi)$. Then we check that Z is a separating set for $\Phi_{-1}(2\pi)$, by computing accurate bounds on the eigenvalues of $\Phi_{-1}(2\pi)$. Here, $\Phi_{\pm1}(2\pi)$ denotes the time-2π map associated with the operator $H^{\pm1}$. The nontrivial part of $\Phi_{\pm1}(2\pi)$ is just a $2k_* \times 2k_*$ matrix, where $k_* = 2j_* + 1 - \sigma$. It is obtained by integrating the flow $\dot{\mathfrak{v}} = X_{\pm1}\mathfrak{v}$ associated with the second order equation $\ddot{v} = -H^{\pm1}v$.

In order to make this part of our programs [14] more transparent, let us write down the equations that are being integrated. To simplify notation, consider the operator H^o in place of $H^{\pm1}$. After a change of variables (indices) $g_k = v_{k-j_*-1+\sigma}$, the equation $\ddot{v} = -H^o v$ becomes

$$\ddot{g}_k = \alpha_{k-j_*}g_{k+1} - (\alpha_{k-j_*-1} + \alpha_{k-j_*} + \beta_{k-j_*-1+s})g_k + \alpha_{k-j_*-1}g_{k-1}\,, \tag{4.25}$$

for $1 < k < k_*$, and

$$\begin{aligned} \ddot{g}_1 &= \alpha_{1-j_*}g_2 - (\alpha_{1-j_*} + \beta_{-j_*+s})g_1\,,\\ \ddot{g}_{k_*} &= -(\alpha_{j_*-s} + \beta_{j_*})g_{k_*} + \alpha_{j_*-s}g_{k_*-1}\,. \end{aligned} \tag{4.26}$$

Based on Lemmas 4.9 and 4.10, we can now give a

Proof of Theorem 1.2. . Consider first one of solutions 2, 3, 4, 8, and 13, that we claim to be spectrally stable. By Lemma 4.9, we can apply Proposition 4.8 to conclude that none of the points in Z is an eigenvalue of any of the operators $\Phi_s(2\pi)$. Here, we have used also that an eigenvalue of $\Phi_s(2\pi)$ cannot hit a non-primary point without first crossing a primary point, as s is increased. Now we can apply Proposition 4.6, with H_n in place of H, and n sufficiently large. It shows that Z is a separating set for $\Phi_n(2\pi)$, and that all eigenvalues of $\Phi_n(2\pi)$ lie on Λ. Taking $n \to \infty$ along a suitable subsequence, we conclude from Theorem 2.13 that all eigenvalues of $\Phi(2\pi)$ lie on the unit circle and are bounded away from 1, with the possible exception of two eigenvalues on the closure of the cross determined by Z. But we already know that $\Phi(2\pi)$ has an eigenvalue 1, as described in Remark 2, and this eigenvalue must have an even multiplicity by symplecticity. This implies that all eigenvalues of $\Phi(2\pi)$ lie on the unit circle.

Next, consider one of solutions that we claim to be spectrally unstable. As mentioned at the beginning of Subsection 4.1, it suffices to consider the solutions 1 and 11. In these cases, Lemma 4.10 and Proposition 4.2 imply that $\Phi(2\pi)$ has at least one real eigenvalue larger than 1. This concludes the proof of Theorem 1.2. **QED**

5. Computer estimates

In order to complete our proof of Theorems 1.1, 1.2, and 1.3, we need to verify the assumptions of the Lemmas 3.1, 3.4, 4.9, and 4.10. The strategy is to reduce each of these lemmas to successively simpler propositions, until the claims are trivial numerical statements that can be (and have been) verified by a computer. This part of the proof is written in the programming language Ada [15] and can be found in [14].

The following is meant to be a rough guide for the reader who wishes to check the correctness of our programs. The first part of the above-mentioned reduction is organized by the main program `Run_All`. It divides up the given task among five standalone procedures. The first is `Approx_Fixpt`, which is purely numerical and computes the finite-rank part $A' = A - \mathrm{I}$ of the operator A that appears in (3.6). The approximate solution $\bar{q}$ is read from the `data` directory, and the necessary parameters are specified in the Ada package `Params`. (If desired, `Approx_Fixpt` can be used also to improve the quality of the approximate solution.) Now that the map $\mathcal{F}$ is well-defined, the procedure `Check_Fixpt` is called to verify the assumptions of Lemma 3.1 and Lemma 3.4. At this point, we have an enclosure for the fixed point q of G. Enclosures for chains in $\mathcal{B}^{\sigma;\tau}_{r,\rho}$ are represented by the data type `FChain`, using enclosures of type `CosSin1` for functions in $\mathcal{A}^{\tau}_{r}$. Data associated with q that are needed later, such as the functions α, β, and upper bounds on the numbers $d^{\pm}_j$ defined in (4.20), are computed and saved by the procedure `Save_Data`. This procedure also determines a bound `NPD` on the operator norm $\|H - H^o\|$ that appears in (4.5). Bounds on the maps $\Phi_{\pm 1}(2\pi)$ and on its eigenvalues are determined by the procedures `Phi2Pi` and `Eigen`. For the solutions that are expected to be unstable, `Eigen` also calls the procedure `ScalVectors.Phi.Check_Unstable` to verify the the assumptions of Lemma 4.10. For the solutions that are expected to be stable, `Run_All` calls the procedure `Separation` to verify the assumptions of Lemma 4.9.

The next steps in the reduction process require specialized knowledge and tools, so each of the above-mentioned procedures first instantiates a few specialized Ada packages and then hands the task to some procedure(s) that are defined in those packages. An Ada package is simply a collection of definitions and procedures, centered around a few specific data types. In particular, the package `CosSins1` and its child `CosSins1.Chain` implement basic bounds involving the data types `CosSin1` and `FChain`, respectively. The type `CosSin1` is equivalent to the type `Fourier` that is used and documented in [4]. Our type `FChain` is in essence an array of `CosSin1`, one `CosSin1` for each site in $\mathcal{J} = \{j \in \mathbb{Z} : \sigma - \ell \leq j \leq \ell\}$. If `Q` is an `FChain` specifying an enclosure for a chain $q \in \mathcal{B}^{\sigma;\tau}_{r,\rho}$, then the components $\mathtt{Q}_{\sigma-\ell}$ and $\mathtt{Q}_\ell$ consist of error bounds on the tails $(\ldots, q_{\sigma-\ell-1}, q_{\sigma-\ell})$ and $(q_\ell, q_{\ell+1}, \ldots)$, respectively. The remaining components $\mathtt{Q}_j$ define enclosures for the functions $q_j \in \mathcal{A}^\tau_r$, with $\sigma - \ell < j < \ell$. In our programs, ℓ is named `JEMax`. And the cutoff $j_* < \ell$ used in (4.17) is named `JAst`.

As can be seen in `Check_Fixpt`, the specialized bounds that are needed in the proofs of Lemma 3.1 and Lemma 3.4 are implemented in the child package `CosSins1.Chain.Fix`. This includes bounds on the maps G, $\mathcal{F}$, DG, and $D\mathcal{F}$. Similarly, the proof of Lemma 4.9 is organized by the procedure `CheckEta` in the package `CosSins1.Chain.Pairs.FlokM`. As the package structure indicates, bounds in `CosSins1.Chain.Pairs.FlokM` are reduced in stages to bounds defined in `CosSins1`, and those reduce further to bounds on data of type `Scalar`, etc. Following these instructions, a computer ends up with a finite number of basic numerical operations, which are carried out with rigorous upper and (if necessary) lower bounds.

All this is described in full detail by the source code of our programs [14]. But some remarks may be in order concerning the choice of algorithms. Whenever implicit equations need to be solved, our approach is the same as for the equation $G(q) = q$. After determining an approximate solution $\bar{q}$, we use the contraction mapping theorem for a Newton-type map $\mathcal{F}$ to obtain a rigorous bound on the error $q - \bar{q}$. This approach is used e.g. to obtain bounds on the eigenvalues λ_k of the symplectic matrix for the nontrivial part of $\Phi_{\pm 1}(2\pi)$ or $\Phi_o(2\pi)$, after determining a polynomial whose roots are the numbers $\frac{1}{2}\lambda_k + \frac{1}{2}\lambda_k^{-1}$. The computation of the matrix itself is entirely explicit: here we use a Taylor method to integrate the nontrivial part of the vector field $X_{\pm 1}$ (for Lemma 4.9) or X_o (for Lemma 4.10) associated with the operators $H^o \pm D_{\pm 1}$ or H^o, respectively, described in Subsection 4.4. See also the comments after Lemma 4.10. (In the spectrally stable case, computing both $\Phi_{\pm 1}(2\pi)$ has the added advantage of yielding accurate bounds on $2k_*$ eigenvalues of $\Phi(2\pi)$ via Proposition 4.5.) Verifying the assumptions of Lemma 4.9 is an explicit computation as well. Here, "computing" an object means finding a rigorous enclosure (specified by finitely many representable numbers) for that object. To prove that the operator $L = \check{M}_{s_j}(\eta)$ described in Proposition 4.8 has no eigenvalue in $[-C, C]$, we simply compute the inverse of L and check that $\|L^{-n}\| < C^n$ for some positive integer n (a power of 2). A more detailed description of the algorithms used to integrate a vector field and to compute eigenvalues can be found in [13], where we considered a similar spectral problem.

We will not explain here the more basic ideas and techniques underlying computer-assisted proofs in analysis. This has been done to various degrees in many other papers, including [13]. As far as our proof of the Lemmas 3.1, 3.4, 4.9, and 4.10 is concerned, the ultimate reference is the source code of our programs [14]. For the set of representable numbers (`Rep`) we choose either standard [17] extended floating-point numbers (type `LLFloat`) or high precision [18] floating-point numbers (type `MPFloat`), depending on the precision needed. Both types support controlled rounding. Our programs were run successfully on a standard desktop machine, using a public version of the gcc/gnat compiler [16]. Instructions on how to compile and run these programs can be found in the file README that is included with the source code [14].

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