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Low-thrust Trajectory Optimisation through Differential Dynamic Programming Method based on Keplerian Orbital Elements

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Abstract

The optimisation of low-thrust orbital trajectories represents one of the classic non-linear constrained optimal control problems in space applications. Low-thrust systems are getting more involved in the design of new missions, such as for example the all-electric spacecraft, since they grant a greater final operational mass thanks to their high specific impulse.

One of the methods for solving such difficult problem is Differential Dynamic Programming (DDP), which is based on the identification of optimal feedback control laws by the discretisation of the dynamics and the application of Bellman's principle of optimality. Despite the recent development of many advanced techniques in the field of DDP, the formulation of the dynamics in all these works has always been in Cartesian coordinates and no attempt was made to couple DDP with Keplerian orbital elements as state variables.

In this paper a low-thrust trajectory optimisation through a DDP approach based on Keplerian orbital elements is derived. Lagrange's and Gauss' planetary equations are used to model the dynamics of the spacecraft in such a way that orbital perturbations can be included if their disturbing potential is expressed in terms of orbital elements or, in case of aerodynamic drag if the disturbing acceleration is properly modelled. The adoption of orbital elements as state variables presents all the advantages coming from the variational equations for the propagation of the dynamics and the state transition matrix can be easily computed.

An interplanetary transfer to a near-Earth asteroid is used as example to test and assess the proposed approach since its dynamics is well described by using the variational equations.
Keywords: low-thrust, optimal control, orbital elements, differential dynamic programming.

Nomenclature

α	difference in cost prediction
β_1	optimal control law coefficient with respect to the state variation
β_2	optimal control law coefficient with respect to the Lagrange multipliers variation
$\delta\boldsymbol{b}$	Lagrange multipliers variation
$\delta\boldsymbol{u}$	control variation
$\delta\boldsymbol{x}$	state variation
ε	tuning coefficient for the Lagrange multipliers variation
μ	gravitational parameter $[\mathrm{km}^3/s^2]$
ν	true anomaly [rad]
$\boldsymbol{\varphi}$	endpoint equality constraints
ω	pericentre anomaly [rad]
Ω	right ascension of the ascending node [rad]
a	semi-major axis [km]
$\tilde{a}$	adimensional semi-major axis
a_0	initial semi-major axis [km]
$\boldsymbol{b}$	Lagrange multipliers
$\overline{\boldsymbol{b}}$	nominal set of Lagrange multipliers
e	eccentricity
f	dynamics of the system
g_0	sea-level Earth gravity acceleration [km/s^2]
h	specific angular momentum $[\mathrm{km}^2/s]$
H	Hamiltonian function
i	orbit inclination [rad]
Isp	specific impulse [s]
J	cost function
L_{ref}	reference length [km]
m	satellite mass [kg]
$\tilde{m}$	adimensional mass
m_0	initial satellite mass [kg]
m_{ref}	reference mass [kg]
r	satellite distance [km]
t	time variable [s]
$\tilde{t}$	adimensional time
t_{ref}	reference time [s]
$\boldsymbol{u}$	control thrust [N]
$\overline{\boldsymbol{u}}$	nominal control
$\tilde{\boldsymbol{u}}$	adimensional control thrust
$\boldsymbol{u}^*$	control minimising the Hamiltonian function
u_0	maximum control thrust magnitude [N]

u_h	out-of-plane control thrust [N]
$\tilde{u}_h$	adimensional out-of-plane control thrust
u_n	normal control thrust [N]
$\tilde{u}_n$	adimensional normal control thrust
u_t	tangential control thrust [N]
$\tilde{u}_t$	adimensional tangential control thrust
u_{ref}	reference thrust [N]
v	satellite velocity magnitude [km/s]
V	value function
$\boldsymbol{x}$	satellite state vector
$\overline{\boldsymbol{x}}$	nominal state vector
$\boldsymbol{x}_f$	final state vector

Acronyms/Abbreviations

ESA	European Space Agency
GNSS	Global Navigation Satellite Systems
HDPP	Hybrid Differential Dynamic Programming
SDDP	Stochastic Differential Dynamic Programming
DDP	Differential Dynamic Programming
PDE	Partial Differential Equation
HJB	Hamilton-Jacobi-Bellman
LQE	Linear Quadratic Expansion
RAAN	Right Ascension of the Ascending Node
TOF	Time Of Flight

1. Introduction

Electric spacecraft are the new frontier of next space missions, not only for planetary missions, but also for interplanetary missions. This can be inferred by looking at the latest space missions like the ESA mission BepiColombo [1] towards Mercury, or the Galileo [2] satellites belonging to the GNSS services.

The low-thrust systems onboard of the electric spacecraft present the great advantage in maximising the final operational mass of the spacecraft but the design of the trajectories involving these systems are more involved because their dynamics cannot be represented by ballistic motion but it is a continuous dynamics where the thruster is always providing an acceleration.

There are a lot of existing techniques dealing with the problem of low-thrust trajectory optimisation in literature. One of the most interesting techniques for solving non-linear optimal control problems is DDP. This method is based on Bellman's principle of optimality [3] which states that an optimal policy has the property to be the same even if the optimal control is found starting from an intermediate state, and so it is independent on the initial guess used for the trajectory of the dynamics. This principle is mathematically expressed by a PDE which is the HJB equation. Unfortunately, this PDE has no analytical solution and the numerical solution cannot be provided since the dimension of the searching space is not finite. The DDP proposes to apply the dynamic programming in a neighbourhood of a nominal non-optimal trajectory. This method is more effective the closer the non-optimal trajectory is to the optimal solution. Colombo et al. [4] presented a modified DDP algorithm for the optimisation of low-thrust trajectories where the problem is discretised in several decision steps, so that the optimisation process requires the solution of a great number of small problems. Lantoine and Russell [5] developed a new second-order algorithm based on DDP, called HDDP, which maps the required derivatives recursively through first-order and second-order state transition matrices. Ozaki et al. [6] proposed a SDDP where random perturbations enter the dynamics of the problem and their expected values are computed by the unscented transform. However, this kind of technique has not been further explored and it has been used within the Cartesian framework.

This paper presents a low-thrust trajectory optimisation using a DDP algorithm which is based on Keplerian orbital elements as state representation to prove that the methodology can work also in a different framework like the one proposed by the orbital elements. The dynamics of the system will be expressed by Gauss' planetary equations in the $[\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{h}}]$ (tangential, normal, out-of-plane) reference frame because the low-thrust acceleration cannot be modelled as a conservative force.

The paper is structured as follows: Section 2 presents the general DDP, while in Section 3 the modification due to the new representation in terms of the orbital elements will be presented. The results of the optimisation will be shown in Section 4, whereas Section 5 is devoted to the discussion of the results and of the methodology. Finally, Section 6 concludes the paper.

2. Methodology and mathematical theory

In this section the main problem of finding an optimal control law for trajectory design will be presented together with the fundamental theory and methodology of the DDP algorithm.

2.1 Trajectory optimisation problem

The main problem to be solved is to find the optimal control law, $\boldsymbol{u}(t)$, that inserted in the dynamics of the system provides a trajectory resulting in the minimisation of a functional cost subject to final equality constraints.

$$J = \int_{t_0}^{t_f} [u(t)]^2 dt \quad \text{subject to} \quad \boldsymbol{\varphi} = \boldsymbol{x}(t_f) - \boldsymbol{x}_f \qquad (1)$$

In the Keplerian orbital elements framework, the equations of motions of the system are represented by Gauss' planetary equations in $[\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{h}}]$ reference frame since the low-thrust force is not a conservative force. The dynamics of the satellite consists of the equations of motions taken from Battin [7] and the equation for the mass rate variation:

$$\frac{da}{dt} = 2\frac{a^2 \mathrm{v}}{\mu}\frac{u_t}{m}$$

$$
\begin{aligned}
\frac{de}{dt} &= \frac{1}{\mathrm{v}}\left[2(e+\cos\nu)\frac{u_t}{m} - \frac{r}{a}\sin\nu\frac{u_n}{m}\right] \\
\frac{di}{dt} &= \frac{r}{h}\cos(\omega+\nu)\frac{u_h}{m} \\
\frac{d\Omega}{dt} &= \frac{r}{h}\frac{\sin(\omega+\nu)}{\sin i}\frac{u_h}{m} \\
\frac{d\omega}{dt} &= \frac{1}{e\mathrm{v}}\left[2\sin\nu\frac{u_t}{m} + \left(2e+\frac{r}{a}\cos\nu\right)\frac{u_n}{m}\right] - \frac{r}{h}\frac{\sin(\omega+f)\cos i}{\sin i}\frac{u_h}{m} \\
\frac{d\nu}{dt} &= \frac{h}{r^2} - \frac{1}{e\mathrm{v}}\left[2\sin\nu\frac{u_t}{m} + \left(2e+\frac{r}{a}\cos\nu\right)\frac{u_n}{m}\right] \\
\frac{dm}{dt} &= -\frac{(u_t^2+u_n^2+u_h^2)^{\frac{1}{2}}}{Isp\, g_0}
\end{aligned} \tag{2}
$$

where $a, e, i, \Omega, \omega, \nu$ are the Keplerian elements defining respectively the semi-major axis, the orbit eccentricity, inclination, RAAN, pericentre anomaly and true anomaly, while $m, \mu, u_t, u_n, u_h, Isp, g_0$ represent the satellite mass, the gravitational parameter, tangential, normal and out-of-plane components of the control thrust, the specific impulse and the mean surface Earth gravitational acceleration respectively. The other variables v, r, h appearing in Eq. (2) are the velocity magnitude, satellite distance and specific angular momentum which are related to the Keplerian orbital elements using the following relations:

$$
\begin{aligned}
\mathrm{v} &= \sqrt{\frac{\mu}{a(1-e^2)}\left[1+2e\cos\nu+e^2\right]} \\
r &= \frac{a(1-e^2)}{(1+e\cos\nu)} \\
h &= \sqrt{\mu a(1-e^2)}
\end{aligned} \tag{3}
$$

The state of the system is represented by the Keplerian orbital elements set and the spacecraft mass:

$$\boldsymbol{x} = [a, e, i, \Omega, \omega, \nu, m]^{\boldsymbol{T}}$$

2.2 DDP theory and fundamental algorithm

Differential dynamic programming is a numerical technique for the resolution of non-linear optimal control problems, and it is a simplification of the most general concept of dynamic programming. It is based on Bellman's principle of optimality [3] which states that an optimal policy has the property to be always the same no matters of the initial guess that is considered for the optimisation. This principle results in the derivation of the HJB equation:

$$-\frac{\partial V(\boldsymbol{x},\boldsymbol{u};t)}{\partial t} = \min_{\forall \boldsymbol{u}\in U}[J(\boldsymbol{x},\boldsymbol{u};t) + \langle V_x(\boldsymbol{x};t), f(\boldsymbol{x},\boldsymbol{u};t)\rangle] \tag{4}$$

where V represents the value function, J is the functional cost, V_x are the costates and f represents the dynamics of the system. The principle of dynamic programming suggests that it is possible to obtain the optimal solution thanks to a backward integration followed by a forward integration of the dynamics. Starting from the final conditions, all the possible paths should be investigated and the one resulting in the optimal choice corresponds also to the real optimal solution according to the principle of optimality. However, this procedure cannot be applied in a rigorous way since from the mathematical point of view no analytical solution exists for the PDE, and from the numerical point of view the dimension of the problem is infinite because the number of paths belongs to a non-Euclidean space. This makes the dynamic programming useless for the optimisation problems. The DDP tries to overcome the "curse of dimensionality" associated to the dynamic programming considering a LQE of the HJB equation in the neighbourhood of a nominal, non-optimal solution [8].

2.2.1 Unconstrained optimisation

The starting point of DDP is the HJB equation. The equation can be written in terms of a nominal trajectory by setting:

$$
\begin{aligned}
\boldsymbol{x} &= \bar{\boldsymbol{x}} + \delta\boldsymbol{x} \\
\boldsymbol{u} &= \bar{\boldsymbol{u}} + \delta\boldsymbol{u}
\end{aligned} \tag{5}
$$

where $\delta\boldsymbol{x}$ and $\delta\boldsymbol{u}$ are the state and the control variations, respectively, measured with respect to the nominal quantities $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{u}}$, and they are not necessarily small variables. The HJB equation takes the following form:

$$-\frac{\partial V(\bar{x}+\delta x;t)}{\partial t} = \min_{\delta\boldsymbol{u}}[J(\bar{\boldsymbol{x}}+\delta\boldsymbol{x}, \bar{\boldsymbol{u}}+\delta\boldsymbol{u};t) + \langle V_x(\bar{\boldsymbol{x}}+\delta\boldsymbol{x};t), f(\bar{\boldsymbol{x}}+\delta\boldsymbol{x}, \bar{\boldsymbol{u}}+\delta\boldsymbol{u};t)\rangle] \tag{6}$$

Assuming that the value function is sufficiently well-behaved it is possible to perform a Taylor series expansion of both sides of Eq. (6) stopping at the second order.

$$V(\bar{\boldsymbol{x}}+\delta\boldsymbol{x};t) = V(\bar{\boldsymbol{x}};t) + \langle V_x, \delta\boldsymbol{x}\rangle + \frac{1}{2}\langle \delta\boldsymbol{x}, V_{xx}\delta\boldsymbol{x}\rangle \tag{7}$$

The value function, which is equivalent to the optimal cost function, at the nominal state, $\bar{\boldsymbol{x}}$, is put equal to:

$$V(\bar{\boldsymbol{x}};t) = \bar{V}(\bar{\boldsymbol{x}};t) + \alpha(\bar{\boldsymbol{x}};t) \tag{8}$$

where α is defined as the difference between the optimal cost $V(\bar{\boldsymbol{x}};t)$, obtained by using the optimal control, and the nominal cost $\bar{V}(\bar{\boldsymbol{x}};t)$, obtained using the nominal control. Replacing Eq. (8) in Eq. (7), the following result is obtained:

$$V(\bar{\boldsymbol{x}} + \delta\boldsymbol{x}; t) = \bar{V}(\bar{\boldsymbol{x}}; t) + \alpha(\bar{\boldsymbol{x}}; t) + \langle V_x, \delta\boldsymbol{x}\rangle + \frac{1}{2}\langle \delta\boldsymbol{x}, V_{xx}\delta\boldsymbol{x}\rangle \tag{9}$$

Similarly for the value function, also the costate of the dynamics can be expanded in Taylor series up to the second order in $\delta\boldsymbol{x}$ resulting in:

$$V_x(\bar{\boldsymbol{x}} + \delta\boldsymbol{x}; t) = V_x(\bar{\boldsymbol{x}}; t) + \langle V_{xx}, \delta\boldsymbol{x}\rangle + \frac{1}{2}\langle \delta\boldsymbol{x}, V_{xxx}\delta\boldsymbol{x}\rangle \tag{10}$$

Substituting Eq. (9) and Eq. (10) in the HJB equation it follows:

$$-\frac{\partial\bar{V}}{\partial t} - \frac{\partial\alpha}{\partial t} - \langle\frac{\partial V_x}{\partial t}, \delta\boldsymbol{x}\rangle - \frac{1}{2}\langle\delta\boldsymbol{x}, \frac{\partial V_{xx}}{\partial t}\delta\boldsymbol{x}\rangle = \min_{\delta\boldsymbol{u}}[H(\bar{\boldsymbol{x}} + \delta\boldsymbol{x}, \bar{\boldsymbol{u}} + \delta\boldsymbol{u}, V_x; t) + \langle V_{xx}\delta\boldsymbol{x}, f(\bar{\boldsymbol{x}} + \delta\boldsymbol{x}, \bar{\boldsymbol{u}} + \delta\boldsymbol{u}; t)\rangle] \tag{11}$$

where the term including V_{xxx} has been discarded since it is associated to a higher order infinitesimal and the Hamiltonian function, H, has been defined as:

$$H(\boldsymbol{x}, \boldsymbol{u}, V_x; t) = J(\boldsymbol{x}, \boldsymbol{u}; t) + \langle V_x, f(\boldsymbol{x}, \boldsymbol{u}; t)\rangle \tag{12}$$

At this step Eq. (11) is first considered at the nominal state $\boldsymbol{x} = \bar{\boldsymbol{x}}$**,** and it becomes:

$$-\frac{\partial\bar{V}}{\partial t} - \frac{\partial\alpha}{\partial t} = \min_{\delta\boldsymbol{u}} H(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}} + \delta\boldsymbol{u}, V_x; t) \tag{13}$$

The Hamiltonian is completely minimised either analytically or, if necessary, numerically. In this paper the minimisation is performed numerically using the MATLAB solver *fmincon*. Assuming that the minimising control, $\boldsymbol{u}$, is given by:

$$\boldsymbol{u}^* = \bar{\boldsymbol{u}} + \delta\boldsymbol{u}^* \tag{14}$$

the variation in the state, $\delta\boldsymbol{x}$, is reintroduced in Eq. (13), but now the minimising control for $\boldsymbol{x} = \bar{\boldsymbol{x}} + \delta\boldsymbol{x}$ will be given by:

$$\boldsymbol{u} = \boldsymbol{u}^* + \delta\boldsymbol{u} \tag{15}$$

where the variation $\delta\boldsymbol{u}$ is still to be determined and this time is measured with respect to the control $\boldsymbol{u}^*$ and no longer to the nominal control. Because $\boldsymbol{u}^*$ minimises the Hamiltonian function, the necessary condition

$$H_u(\bar{\boldsymbol{x}}, \boldsymbol{u}^*, V_x; t) = 0 \tag{16}$$

holds and the only terms involving $\delta\boldsymbol{u}$ in the equation obtained expanding Eq. (11) around the new control are:

$$\langle\delta\boldsymbol{u}, (H_{ux} + f_u^T V_{xx})\delta\boldsymbol{x}\rangle + \frac{1}{2}\langle\delta\boldsymbol{u}, H_{uu}\delta\boldsymbol{u}\rangle \tag{17}$$

If $\delta\boldsymbol{u}$ is of the same order of $\delta\boldsymbol{x}$, then these terms will be quadratic in $\delta\boldsymbol{x}$ plus higher-order in $\delta\boldsymbol{x}$**.** There is, therefore, no point in finding a relationship between $\delta\boldsymbol{u}$ and $\delta\boldsymbol{x}$ that is of higher order than linear, because terms higher than second-order in $\delta\boldsymbol{x}$ are neglected. A relationship of the following form is, therefore, required:

$$\delta\boldsymbol{u} = \beta_1\delta\boldsymbol{x} \tag{18}$$

where β_1 is the coefficient matrix used for the minimisation of the right-hand side of the expansion of the HJB equation keeping the necessary condition of optimality, and its value is:

$$\beta_1 = -H_{uu}^{-1}(H_{ux} + f_u^T V_{xx}) \tag{19}$$

The new control can be replaced in the Eq. (11) and the only unknown variation is the one associated to $\delta\boldsymbol{x}$. Since equality holds for all $\delta\boldsymbol{x}$ sufficiently small, the coefficients of like powers of $\delta\boldsymbol{x}$ should be equated to obtain a system of differential equations. After several manipulations the result is:

$$\begin{aligned} -\dot{\alpha} &= H - H(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}, V_x; t) \\ -\dot{V}_x &= H_x + V_{xx}\big(f - f(\bar{x}, \bar{u}; t)\big) \\ -\dot{V}_{xx} &= H_{xx} + f_x^T V_{xx} + V_{xx} f_x - \beta_1^T H_{uu}\beta_1 \end{aligned} \tag{20}$$

The previous system of differential equations is propagated backwards starting from the final conditions up to the initial time, storing at each time instant the coefficient β_1 which gives the control law. The final conditions are defined considering the endpoint constraints:

$$\begin{aligned} &\alpha(t_f) = 0 \\ &V_x(t_f) = F_x(\bar{\boldsymbol{x}}(t_f); t_f) + \boldsymbol{\varphi}^T(\bar{\boldsymbol{x}}(t_f); t_f)\bar{\boldsymbol{b}} \\ &V_{xx}(t_f) = F_{xx}(\bar{\boldsymbol{x}}(t_f); t_f) + \bar{\boldsymbol{b}}\boldsymbol{\varphi}_{xx}(\bar{\boldsymbol{x}}(t_f); t_f) \end{aligned} \tag{21}$$

Once the backward propagation completed, a forward propagation is performed considering the new control:

$$\boldsymbol{u} = \boldsymbol{u}^* + \beta_1\delta\boldsymbol{x} \tag{22}$$

A check on the value function is done for verifying whether it is lower than the one computed with the nominal values. If this is the case, the nominal control is replaced with the new control and a new backward integration is carried out until the optimal control function has been found. If the value function is worse than the nominal one, the step-size adjustment method is applied for which the reader is pointed to Jacobson and Mayne [8].

At the end of the convergence, the unconstrained part of the problem is over, and the second part associated to the fulfilment of the final equality constraints begins.

2.2.2 Constrained optimisation

The constrained optimisation loop starts as soon as the optimal control minimising the value function considering a nominal value of the Lagrange multipliers has been obtained. The procedure does not differ so much from the unconstrained part for the derivation of the differential equations to be solved backwards. Indeed, starting from the definition of the control $\boldsymbol{u}^*$ minimising the Hamiltonian function, this time both the variations in state, $\delta\boldsymbol{x}$, and in the Lagrange multipliers, $\delta\boldsymbol{b}$, are reintroduced in the Taylor expansion of the HJB equation in Eq. (11). Again, a linear relationship between the control variation with respect to the variations in state and in the Lagrange multipliers is introduced:

$$\delta\boldsymbol{u} = \beta_1\delta\boldsymbol{x} + \beta_2\delta\boldsymbol{b} \tag{23}$$

where the coefficient β_2 is found similarly to coefficient β_1, and its expression is given by:

$$\beta_2 = -H_{uu}^{-1} f_u^T V_{xb} \tag{24}$$

It is possible to introduce the new control variation in the LQE of the HJB equation where only the terms $\delta\boldsymbol{x}$ and $\delta\boldsymbol{b}$ appear. Again, the equality of the two sides of Eq. (11) should hold if the variations are small and this leads to the derivation of three new differential equations that should be solved together with the previous ones in Eq. (20):

$$\begin{aligned} -\dot{V}_b &= V_{xb}^T\big(f - f(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}};t)\big) \\ -\dot{V}_{xb} &= \left(f_x^T + \beta_1^T f_u^T\right)V_{xb} \\ -\dot{V}_{bb} &= -V_{xb}^T f_u H_{uu}^{-1} f_u^T V_{xb} \end{aligned} \tag{25}$$

Also, in this case the terminal conditions are derived from the endpoint equality constraints:

$$\begin{aligned} V_b(t_f) &= \boldsymbol{\varphi}\big(\bar{\boldsymbol{x}}(t_f);t_f\big) \\ V_{xb}(t_f) &= \boldsymbol{\varphi}_x^T\big(\bar{\boldsymbol{x}}(t_f);t_f\big) \\ V_{bb}(t_f) &= 0 \end{aligned} \tag{26}$$

Since the constrained problem is solved after the unconstrained one, the values of the nominal trajectory and of the one minimising the Hamiltonian function are practically the same. This implies that:

$$\dot{V}_b = 0 \tag{27}$$

and the value of V_b is equal to the final condition. In order to reduce the value of the endpoint equality constraints to 0, it is necessary that:

$$V_b + V_{bb}\delta\boldsymbol{b} = 0 \tag{28}$$

From Eq. (28) it is possible to derive the variation of the Lagrange multipliers to be provided to enforce the endpoint constraints:

$$\delta\boldsymbol{b} = -\varepsilon V_{bb}^{-1}(t_0)\boldsymbol{\varphi}\big(\bar{\boldsymbol{x}}(t_f);t_f\big) \tag{29}$$

In the previous expression, the tuning parameter ε is introduced to control the variation of the Lagrange multipliers. This coefficient varies in the interval [0,1] and it is necessary to avoid that the variation becomes too large so that the Taylor expansion is not valid anymore. One method to check if the variation of the Lagrange multipliers obtained is consistent or not with the accuracy of the Taylor expansion is to apply the following condition taken from Gershwin and Jacobson [9]:

$$\begin{aligned} &\bar{V}\big(\bar{\boldsymbol{x}}_0, \bar{\boldsymbol{b}} + \delta\boldsymbol{b}; t_0\big) - \bar{V}\big(\bar{\boldsymbol{x}}_0, \bar{\boldsymbol{b}}; t_0\big) = \alpha - \Big(\varepsilon + \\ &-\tfrac{1}{2}\varepsilon^2\Big)\boldsymbol{\varphi}^T\big(\bar{\boldsymbol{x}}(t_f);t_f\big)V_{bb}^{-1}(t_0)\boldsymbol{\varphi}\big(\bar{\boldsymbol{x}}(t_f);t_f\big) \end{aligned} \tag{30}$$

If the previous condition is respected within a prescribed tolerance, then the variation of the Lagrange multipliers provided is consistent with the Taylor expansion. If this is not the case, the value of ε is halved until Eq. (30) is satisfied.

At this stage the new control, trajectory and cost replace the nominal ones, and a new iteration from the first loop of the unconstrained case is performed until the cost is minimised and the endpoint constraints hold.
Fig. 1 is summing up the algorithm steps to get the optimal solution.

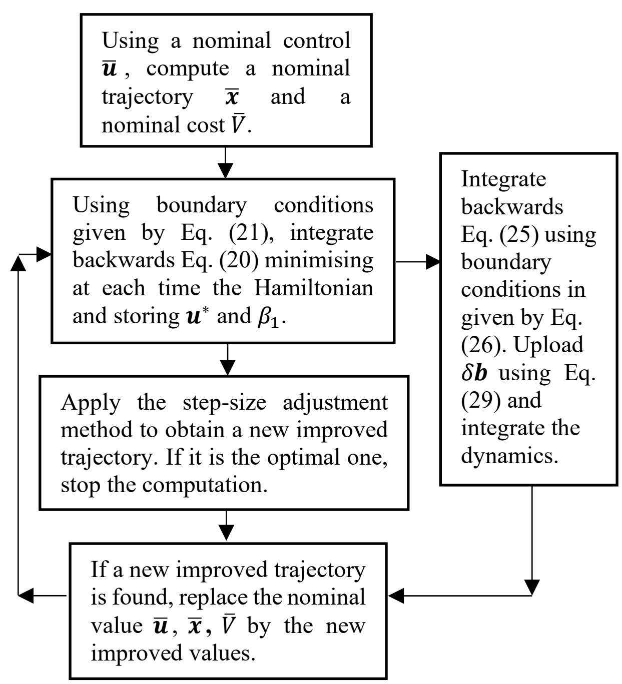

Fig. 1. DDP overall procedure

3. DDP based on Keplerian orbital elements

The DDP algorithm described in the previous section has always been used in the past considering Cartesian coordinates as state representation. In this paper, the assessment of a DDP algorithm based on Keplerian orbital elements as state representation is investigated.

Even if the structure of the algorithm has been kept the same, some modifications are required to let the DDP work in the new framework. The largest part of the problems occurs in the constrained part of the optimisation. Several reasons can be identified for such difficult behaviour:

- different order of magnitude in the state vector
- selection of the tuning parameters
- selection of the initial guess

The first problem associated to the different order of magnitude in the state vector can be solved considering an adimensionalisation of the Gauss' planetary equations. This operation should be done considering reference quantities such that all the elements of the state vector have the same order of magnitude and are quite close. The following reference quantities are proposed:

- $L_{ref} = a_0$, reference length equal to the initial semi-major axis.
- $u_{ref} = u_{max}$, reference thrust equal to the maximum thrust magnitude.
- $m_{ref} = m_0$, reference mass equal to the initial satellite wet mass.
- $t_{ref} = \sqrt{\frac{\mu}{L_{ref}}}\frac{m_{ref}}{u_{ref}}$, reference time chosen to make some coefficients in the equations equal to 1.

The dynamics of the system thus assumes the following form:

$$\frac{d\tilde{a}}{d\tilde{t}} = 2\sqrt{\frac{\tilde{a}^3(1+2e\cos\nu+e^2)}{\mu(1-e^2)}}\frac{\tilde{u}_t}{\tilde{m}}$$

$$\frac{de}{d\tilde{t}} = \sqrt{\frac{\tilde{a}(1-e^2)}{1+2e\cos\nu+e^2}}\left[2(e+\cos\nu)\frac{\tilde{u}_t}{\tilde{m}} - \frac{(1-e^2)\sin\nu}{1+e\cos\nu}\frac{\tilde{u}_n}{\tilde{m}}\right]$$

$$\frac{di}{d\tilde{t}} = \sqrt{\tilde{a}(1-e^2)}\frac{\cos(\omega+\nu)}{1+e\cos\nu}\frac{\tilde{u}_h}{\tilde{m}}$$

$$\frac{d\Omega}{d\tilde{t}} = \sqrt{\tilde{a}(1-e^2)}\frac{\sin(\omega+\nu)}{(1+e\cos\nu)\sin i}\frac{\tilde{u}_h}{\tilde{m}} \quad (31)$$

$$\frac{d\omega}{d\tilde{t}} = \frac{1}{e}\sqrt{\frac{\tilde{a}(1-e^2)}{1+2e\cos\nu+e^2}}\left[2\sin\nu\frac{\tilde{u}_t}{\tilde{m}} + \left(2e + \frac{1-e^2}{1+e\cos\nu}\cos\nu\right)\frac{\tilde{u}_n}{\tilde{m}}\right] - \sqrt{\tilde{a}(1-e^2)}\frac{\sin(\omega+\nu)\sin i}{(1+e\cos\nu)\cos i}\frac{\tilde{u}_h}{\tilde{m}}$$

$$\frac{d\nu}{d\tilde{t}} = \frac{(1+e\cos\nu)^2}{\sqrt{\tilde{a}^3(1-e^2)^3}}\frac{\mu m_{ref}}{u_{ref}L_{ref}^2} - \frac{1}{e}\sqrt{\frac{\tilde{a}(1-e^2)}{1+2e\cos\nu+e^2}}\left[2\sin\nu\frac{\tilde{u}_t}{\tilde{m}} + \left(2e + \frac{1-e^2}{1+e\cos\nu}\cos\nu\right)\frac{\tilde{u}_n}{\tilde{m}}\right]$$

$$\frac{d\tilde{m}}{d\tilde{t}} = -\sqrt{\frac{\mu}{L_{ref}}}\frac{(\tilde{u}_t^2+\tilde{u}_n^2+\tilde{u}_h^2)^{\frac{1}{2}}}{Isp\, g_0}$$

It is possible to put constraints on the value of the other reference quantities to make the coefficients appearing in the true anomaly equation and in the mass variation equation equal to 1. However, this choice leads to a state vector which is still not uniform in terms of order of magnitude. For this reason, this kind of form has been decided for the problem. In this way the first difficulty has been solved such that the matrices are not ill-conditioned.

The second problem is related to the selection of the tuning parameters which are used during the step-size adjustment method and in the constrained part to establish when the variation of the Lagrange multipliers is consistent with the Taylor expansion. Unfortunately, this kind of problem has not a unique solution, but according to the problem that must be solved the parameters are tuned and chosen empirically until the algorithm works.

The selection of the initial guess is another aspect to be discussed that is not only a problem of the current methodology, but in general it affects all the optimisation techniques. DDP is a technique based on the LQE of the HJB equation around a nominal control and a nominal trajectory, and so if the nominal guess is far from the optimal solution the algorithm will either not work or spend a lot of iterations to get the convergence.

All these aspects make understand how difficult is DDP as optimisation technique. In this paper the following procedure has been adopted:

- definition of a good initial guess.
- pseudo-optimisation of the in-plane problem.
- pseudo-optimisation of the out-of-plane problem.
- complete optimisation.

The definition of a good initial guess has been done analysing the final conditions and using a constant thrust such that the final values are not so far from the target conditions. This operation must be done manually considering different cases until the proper values of the initial guesses for the control in both the three directions are found.

At this stage, the DDP algorithm is used to solve the in-plane problem, that is the optimal control law to minimise the cost function and enforce the endpoint in-plane constraints is found. For this purpose, the nominal control guess will be planar and equal to the in-plane components of the initial guess that has been found before. Since, no out-of-plane component will be present in this part, the out-of-plane elements will not change.

After the optimisation of the in-plane problem, the current pseudo-optimal solution is used for solving the out-of-plane problem considering this time as endpoint constraints only the one related to the RAAN and the orbital inclination. Just the out-of-plane component of the control will be derived at this stage. At the end of this step, there will be a pseudo-optimal control that has been formed combining the in-plane optimisation and the out-of-plane optimisation and the endpoint constraints will be all satisfied apart from the one involving the pericentre anomaly ω. Indeed, the in-plane optimisation involves only the semi-major axis and the eccentricity as variables to be optimised. This strategy has been deduced looking at the dependence of the orbital elements' variations on the components of the low-thrust control. Indeed, the tangential thrust affects only the in-plane components while keeping untouched the out-of-plane components while the out-of-plane control thrust affects only the RAAN and the inclination. The pericentre anomaly is affected both by in-plane and out-of-plane variations and for this reason has been left at the end.

As last passage, a new optimisation using the DDP can be performed considering as nominal initial guess the pseudo-optimal control derived from the previous steps. Indeed, the previous passages do not lead to the optimal solution, but they are used only for the derivation of a new nominal guess for the complete DDP algorithm to increase the probability to converge to an optimal solution.

4. Results

The new DDP algorithm has been applied to an interplanetary transfer from the Earth's orbit to the near-Earth asteroid Apophis. The parameters of the problem are summarised in Table 1.

The problem is to find the optimal control law to perform the interplanetary transfer in a given TOF which minimises the propellant consumption to reach the Apophis' orbit.

Table 1. Initial data for the interplanetary transfer to Apophis

Data	Value
Initial mass	500 kg
Specific impulse	3250 s
Departure date	8630.95 MJD2000
Time of flight	990.4 days
Time steps	10000

In Fig. 2 it is shown the nominal control guess that is used for the initialisation of the DDP algorithm.

$$\boldsymbol{u} = [20, 10, -100]\, mN \tag{32}$$

Such a choice comes from the fact that both in-plane and out-of-plane variations occur in this problem and this control allows to make the trajectory close to the final target endpoint constraints.

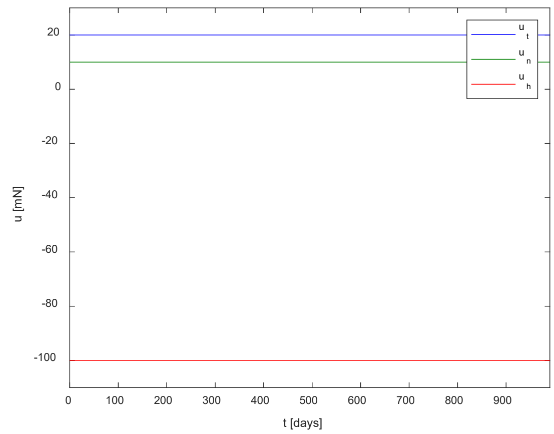

Fig. 2. Initial nominal control guess for the DDP

The output of the DDP algorithm is directly represented by the three components of the low-thrust control varying with time and it is shown in Fig. 3.

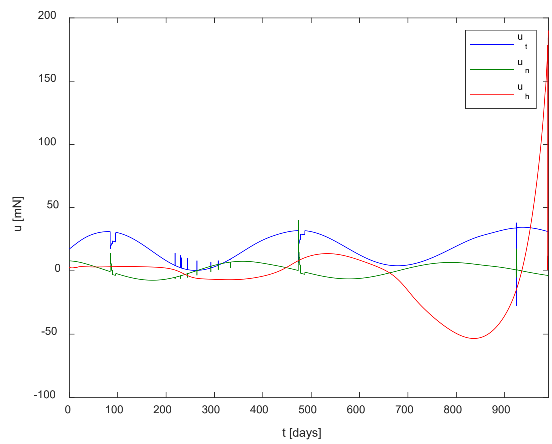

Fig. 3. Optimal control law after the DDP optimisation

The discontinuities found in the optimal control law are related to numerical errors and to the step-size adjustment theory, which basically patches two different control laws together to verify if a new suboptimal policy has been found. By using this optimal control law, the Keplerian orbital elements' variation in time can be derived integrating the dynamics. The results are shown in Fig. 4 and they are compared to the variation of the Keplerian orbital elements obtained by using the initial control guess.

As it can be seen from Fig. 4, all the Keplerian orbital elements are converging to the endpoint conditions and the mass consumption has been considerably reduced with respect to the initial guess. One of the disadvantages associated to the choice of Keplerian orbital elements as state representation is to handle angles which vary in a fixed interval and not in a linear way.

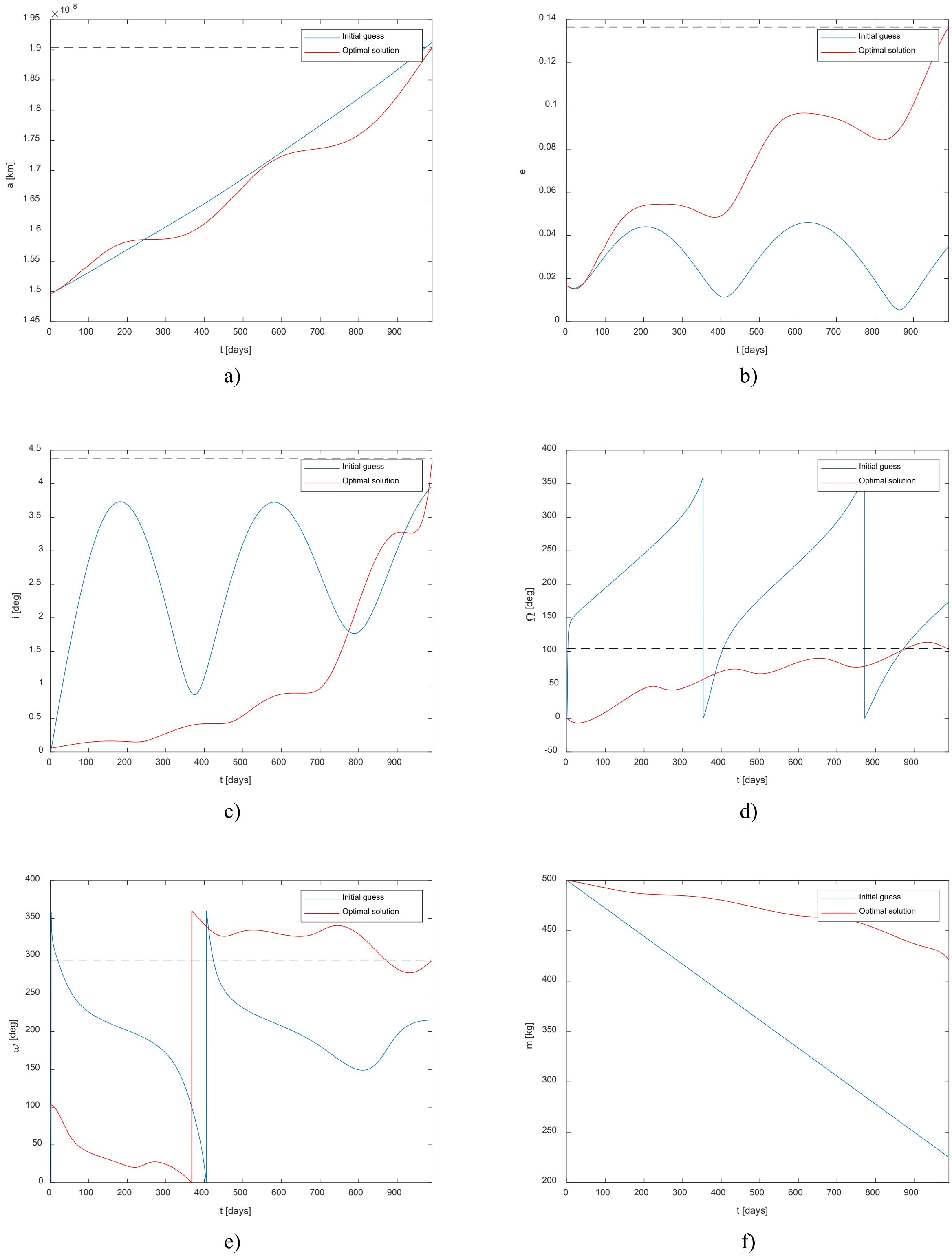

Fig. 4. Variation of the Keplerian orbital parameters considering the nominal and optimal control law

5. Discussion

The decision of using Keplerian orbital elements as state representation is that they represent the natural orbital dynamics. Indeed, while the Cartesian state has an oscillatory variation in time even if an unperturbed scenario is considered, the Keplerian elements have the property to be constant and the only variations in the state are deputed to the effects of external forces that in this case are associated to the low-thrust equipped on board of the spacecraft.

The strategy described in the previous section can be modified considering different ways to perform the "staged optimisation" that will be explored in the future work. For example, one idea is to consider only the semi-major axis as variable to be optimised in the in-plane part because it is only depending on the tangential thrust and in this way the normal component of the thrust can be used to modify both the eccentricity and the pericentre anomaly. However, this kind of analysis is done to define a good initial guess and so it can depend on the problem to be solved.

Another aspect that will be investigated is the reformulation of the control thrust inside the Gauss' planetary equations. Indeed, the actual form of Gauss' planetary equations is linearly dependant on the thrust components, and this leads to second derivatives with respect to the control equal to 0. In the backward integration the inverse of the Hessian of the Hamiltonian with respect to the control is present. However, the Hamiltonian is composed by a piece given by the cost function and one coming from the dynamics that in this case is not present because of the previous consideration. This makes the unconstrained optimisation part very sensitive to the nominal Lagrange multipliers selection because the algorithm is not able to optimise the control considering the contributions of the Lagrange multipliers that are associated to the Hessian of the dynamics.

Finally, because the DDP is based on a LQE of the HJB equation, the only information needed for the construction of the algorithm are gradients, Jacobians and Hessians. This means that it is possible to include in the problem all the desired accelerations provided that the previous quantities are available. One of the future developments will be the addiction of the orbital perturbations inside the dynamics of the problem considering their formulation with the semi-analytical techniques. This choice has an advantage to be accurate for the modelling of the orbital perturbation effects in the long period and to give the analytical formulations of the disturbing accelerations.

6. Conclusions

In this paper a DDP algorithm based on Keplerian elements as state representation of the problem has been investigated. The structure of the algorithm has been kept unchanged from the traditional one, but a new formulation for the dynamics has been provided through Gauss' planetary equations and a new algorithm strategy has been proposed. The method has been assessed testing an interplanetary transfer to a near-Earth asteroid. A future work will include the effects of the perturbations expressed analytically thanks to the semi-analytical formulations inside the dynamics of the system to enhance the effects of the orbital perturbations.

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