Diametrically complete sets and normal structure ☆

Elisabetta Maluta [a,*], Pier Luigi Papini [b]

[a] *Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci, 32, 20133 Milano MI, Italy*

[b] *Via Martucci, 19, 40136 Bologna BO, Italy*

Received 11 August 2014
Available online 5 December 2014
Submitted by T. Domínguez Benavides

0. Introduction

The concept of *diametrically complete set*, i.e. a set such that addition of any point produces a set of larger diameter, was introduced by Meissner at the beginning of the last century as a property enjoyed by all *sets of constant width*. Diametrically complete sets have also been called *diametrically maximal* or simply *complete* in the literature. Although the notions of set of constant width and of diametrically complete set are equivalent in 2-dimensional and in Euclidean spaces, a fundamental result by Eggleston [6] shows that, even in 3-dimensional spaces, diametrically complete sets need not have constant width. Besides, results by Yost [23] prove that, already in finite dimensional spaces, the class of constant width sets is poor, reducing to balls and singletons any time the unit ball of the space is irreducible, and suggest to consider the wider class of diametrically complete sets for further exploration.

Existence of constant width sets which are not balls (the most famous the Reuleaux triangle) has been known for a long time in Mathematics, and a thorough study of such sets, as well as of diametrically complete sets, has been conducted in finite dimensional spaces. Interest about these notions in the infinite

☆ Research of both authors was partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) of Italy.

* Corresponding author.
E-mail addresses: elisabetta.maluta@polimi.it (E. Maluta), plpapini@libero.it (P.L. Papini).

dimensional setting started around 1985 with papers by Behrends, Harmand, Payá, Rodriguez-Palacios and Yost, mostly connected with the study of M-ideals, and several scattered results appeared in the following twenty years.

Then, in 2006, Moreno, Phelps and the second author published two joint papers [13,14] totally devoted to the study of these properties in infinite dimensional spaces. An exhaustive report of the results obtained till that time is contained in [13] and we refer to it for a more detailed history of the subject.

In [13] Moreno, Papini and Phelps determine the class of constant width sets in $\mathcal{C}(K)$ spaces and in c_o, and they provide both an example, in c_0, of a set of constant width with empty interior and an example of a subset of $C([0,1])$ which is diametrically complete and is contained in a hyperplane. In [14] they consider, beside the two classes of sets already mentioned, two other related families of sets; in particular the class of sets with *constant radius from the boundary*, a property introduced by Eggleston and possessed by all diametrically complete sets (hence by all constant width sets).

Papers [13] and [14] focus mainly on $\mathcal{C}(K)$ spaces and their subspaces. Few examples and results exist in reflexive spaces, except for finite dimensional and Hilbert spaces.

Therefore one of the aims of this paper is to examine properties of the aforementioned classes of sets as well as their mutual relationships in generic reflexive spaces.

In doing that, the notion of normal structure, that has been sporadically used in [13,14] and in some subsequent work by Moreno [12] and Moreno and Schneider [15] in connection with the properties we are interested in, turns out to be fundamental. A detailed study regarding how properties like normal structure can help to shed new light on diametrically complete sets or even constant width sets has not yet appeared in the literature and it is our intention to present it, in generic normed spaces, in Section 2. Our main result of the section is Proposition 2.5 stating that the class of diametrically complete sets and the class of sets of constant radius from the boundary coincide if the space has normal structure. Besides, also Theorem 3.4 (which is in Section 3 because its usefulness is in reflexive spaces) enlightens relationships between properties of normal type and existence of diametrically complete sets which are diametral.

The main result of the paper is Theorem 3.5 in Section 3 where we show that in reflexive spaces satisfying the non-strict Opial's condition diametral sets (if any) can be enlarged to sets which are both diametral and diametrically complete.

The first consequence of this result is that, in those spaces, normal structure is equivalent to the weaker property that the absolute Chebyshev radius is strictly smaller than the diameter for every bounded set.

Our main result is of course strictly connected also with the question whether diametrically complete (or even constant width) sets exist whose interior is empty. In regard to this problem, the situation seems to be quite different in nonreflexive and in reflexive spaces. In nonreflexive spaces the results we quoted from [13] prove that it is actually possible to construct explicit examples. In reflexive spaces, our result guarantees existence of diametrically complete sets with empty interior. Moreover, starting from a set of that kind, we construct a diametrically complete set which is contained in a hyperplane. On the other side, we are not aware of any example, in reflexive spaces, of a set of constant width with empty interior.

The last Section 4 presents some known results and open problems about the structures of normal type considered in the paper and their uniform versions.

1. Notation and relevant definitions

Throughout this paper, X denotes an infinite dimensional real Banach space, X^* its topological dual space and B_X its unit ball.

We denote by $\delta(A)$ the diameter of a bounded set A, by A^o, ∂A and $\overline{A}$ its interior, its boundary and its closure, respectively, and by $\overline{co}\,A$ the closed convex hull of A.

C denotes a nonempty, nonsingleton, bounded, closed and convex set.

We recall the relevant definitions.

We say that A is a *diametrically complete set* in X, DC for short, if

$$\delta\big(A \cup \{x\}\big) > \delta(A) \quad \text{for every } x \in X \setminus A.$$

Obviously a diametrically complete set must be bounded, closed and convex.

Eggleston proved that diametrically complete sets are characterized by the *spherical intersection property*, i.e. that *A is diametrically complete if and only if*

$$A = \bigcap_{x \in A} \big(x + \delta(A) \cdot B_X\big).$$

Given a set A, a diametrically complete set C such that $A \subseteq C$ and $\delta(A) = \delta(C)$ is called *diametric completion* of C.

It is simple to see (via Zorn's lemma) that *every bounded set admits diametric completions.*

We say that C is a *constant width set*, CW for short, *of width* λ, if for every $f \in X^*$, $\|f\| = 1$, we have

$$\sup f(C) - \inf f(C) = \lambda.$$

A simple application of the separation theorem shows that *in every Banach space X, sets of constant width are diametrically complete.*

The converse is valid when X is Hilbert (for the infinite dimensional case this follows easily from Theorem 1 in [20]; see also [2] for a direct proof) or 2-dimensional but fails already for certain 3-dimensional spaces (see [6]).

For sets $A, B \subset X$, A, B bounded and nonempty, and for $x \in X$ we set

$$r(A, x) = \sup\big\{\|y - x\| \,:\, y \in A\big\} \quad \text{and} \quad r(A, B) = \inf\big\{r(A, x) \,:\, x \in B\big\}.$$

$r(A, X)$, usually denoted by $r(A)$, is called *(absolute) Chebyshev radius* of A; $r(A, \overline{co}\, A)$ is called *Chebyshev self-radius* of A.

The sets

$$\mathcal{C}(A) = \big\{x \in X : r(A, x) = r(A)\big\} \quad \text{and} \quad \mathcal{C}_S(A) = \big\{x \in \overline{co}\, A : r(A, x) = r(A, \overline{co}\, A)\big\}$$

are called respectively *Chebyshev center* and *Chebyshev self-center* of A.

We say that C is a *set of constant radius (from its boundary)*, CR for short, if

$$r(C, x) = \delta(C) \text{ for every } x \in \partial C.$$

CR sets, together with CW and DC sets, were considered in [14] where it was proved that in any normed space *diametrically complete sets have constant radius.*

We recall now some definitions connected with the notion of normal structure.

A set C is called *diametral* if $r(C, C) = \delta(C)$. Analogously a sequence $\{x_n\} \subset X$ is called *diametral* if $\lim_{n \to +\infty} \operatorname{dist}(x_n, co(x_1, x_2, ..., x_{n-1})) = \delta(\{x_n\})$. For $\{x_n\}$ diametral, $\overline{co}\{x_n\}$ is a diametral set.

A space X has *normal structure* if it does not contain any diametral set, i.e. if $r(C, C) < \delta(C)$ for every $C \subset X$.

If we require only the absolute radius to be strictly smaller than the diameter for every bounded set, we obtain a property strictly weaker than normal structure.

We say that a space X has *property (G)* if $r(C) < \delta(C)$ for every $C \subset X$.

Both concepts have a uniform version; precisely we define two constants of the space, $J(X)$, called *Jung constant*, and $J_s(X)$, called *self-Jung constant*, as

$$J(X) = \sup\{2r(C) : C \subset X;\, \delta(C) = 1\}$$

and

$$J_s(X) = \sup\{2r(C,C) : C \subset X;\, \delta(C) = 1\}.$$

When $J_s(X) < 2$ we say that X has *uniform normal structure.* The following definition, which is new, will be essential in Section 3.

Definition 1.1. A diametral set is called *maximal diametral* if it is not properly contained in any diametral set with the same diameter.

It can be proved easily, again via Zorn's lemma, that any diametral set C is contained in a maximal diametral set $\tilde{C}$ with the same radius. Any such $\tilde{C}$ is said to be *generated* by C.

It is easy to see that:

Remark 1.2. A diametral set has empty interior and constant radius. A diametral set can be diametrically complete only if it is a maximal diametral set.

2. CW, DC, CR sets and normal structure

The strict relationships between the group of properties that we are examining and normal structure can be inferred from the next result from [13].

Proposition 2.1. *(See [13, Theorem 3.2].) Let C be diametrically complete: if $C^o = \emptyset$, then C is diametral.*

The result can be made slightly more detailed:

Proposition 2.2. *Let C be diametrically complete: $C^o = \emptyset$ if and only if C is a maximal diametral set.*
Let C be a set of constant radius: $C^o = \emptyset$ if and only if C is diametral.

Proof. The *if* implication is trivial because any diametral set has empty interior.
For the reverse implication we prove the second claim first. For a set of constant radius C, $C^o = \emptyset$ implies $r(C, x) = \delta(C)$ for every $x \in C$, i.e. $r(C, C) = \delta(C)$.
Now a diametrically complete set C has constant radius, and is therefore diametral whenever $C^o = \emptyset$; were C not maximal, a diametral set $\tilde{C}$ would exist, with $\delta(\tilde{C}) = \delta(C)$, which contains C properly. Then, adding to C any point from $\tilde{C} \setminus C$ would not enlarge the diameter of C, contradicting the assumption that C is diametrically complete. □

Results in the next proposition are almost immediate consequences of Proposition 2.2; the second claim was already remarked by Moreno [12, p. 176].

Proposition 2.3.

(a) *A space has normal structure if and only if every set of constant radius has nonempty interior.*
(b) *A space has property (G) if and only if every diametrically complete set has nonempty interior; in particular, in spaces that enjoy property (G) every set of constant width has nonempty interior.*

Proof.

(a) is equivalent to the second statement of Proposition 2.2.

(b) We recall that in [13, Theorem 3.2], it was proved that $r(C) = r(C, C)$ for every diametrically complete set C.
If (G) holds, for any diametrically complete C we have

$$r(C) = r(C, C) < \delta(C)$$

so C is not diametral and its interior cannot be empty.
If (G) does not hold, let C be such that $r(C) = \delta(C)$ and $\tilde{C}$ be a diametrical completion of C. Then

$$r(C) \leq r(\tilde{C}) \leq \delta(\tilde{C}) = \delta(C) \quad \text{and} \quad r(\tilde{C}) = r(\tilde{C}, \tilde{C})$$

imply that $\tilde{C}$ is diametral, hence has empty interior. $\square$

We have just proved that constant width, diametrically complete and constant radius sets with empty interior can exist only in spaces which lack normal structure. As about constant radius sets, we have remarked in Section 1 that every diametral set is of constant radius, hence we have plenty of constant radius sets whose interior is empty. But do constant width sets or diametrically complete sets with empty interior exist?

A positive answer to this question was given in [13] in nonreflexive spaces. We'll deal with the problem in reflexive spaces in Section 3.

We have recalled in Section 1 that each of the classes of constant width, diametrically complete and constant radius sets is contained in the next one, and in general the inclusion is strict.

Concerning equivalence of the classes of diametrically complete and of constant radius sets, Moreno, Papini and Phelps proved the following

Proposition 2.4. *(See [14, Prop. 2.2].) Every diametrically complete set has constant radius. The converse, which is false in general, holds for sets with nonempty interior. In particular, the two notions coincide in finite dimensional spaces. They also coincide in Hilbert spaces.*

We extend their result to spaces with normal structure. We remark that the problem whether the coincidence of the two classes characterizes spaces with normal structure is still open.

Proposition 2.5. *The class of diametrically complete sets and that of sets of constant radius coincide if the space X has normal structure.*

Proof. Since diametrically complete sets do have constant radius, we need only prove that in spaces with normal structure every set of constant radius is a diametrically complete set.

Assume that X has normal structure and let C be any set of constant radius in X. By Proposition 2.3(a), $C^o \neq \emptyset$, and the result just quoted from [14] implies that it is diametrically complete.

Propositions 2.5 and 2.2 show that the class of sets of constant radius is the union of the class of diametrically complete sets with the family of all diametral sets.

Proposition 2.5 together with Proposition 4.1 in [14] allows us to extend immediately Corollary 4.2 there to

Corollary 2.6. *The class of diametrically complete sets is closed with respect to the Hausdorff metric in any space with normal structure.*

The next two results connect uniqueness of the diametric completion of a set C to the diameter of its Chebyshev center $\mathcal{C}(C)$. The first one was presented by the second author in [16]: since that paper is not easily accessible, we give again its proof here.

Proposition 2.7. *(See [16, Prop. 3.5].) If the set C has a unique diametric completion in X, then $\delta(\mathcal{C}(C)) \leq 2r(C) - \delta(C)$.*

Proof. Set

$$C' = \bigcup\{\tilde{C} : \tilde{C} \text{ a diametric completion of } C\}$$

and, for $\varepsilon \geq 0$,

$$\mathcal{C}_\varepsilon(C) = \big\{x \in X : r(C,x) \leq r(C) + \varepsilon\big\}.$$

We may assume $\mathcal{C}(C)$ nonempty, and, by contradiction, we suppose $\delta(\mathcal{C}(C)) > 2r(C) - \delta(C)$.

It is easy to see that, for any ε,

$$\mathcal{C}(C) + \varepsilon B_X \subset \mathcal{C}_\varepsilon(C),$$

hence

$$\delta\big(\mathcal{C}_\varepsilon(C)\big) \geq \delta\big(\mathcal{C}(C)\big) + 2\varepsilon.$$

Therefore, if we choose $\varepsilon = \delta(C) - r(C)$, we have that

$$\mathcal{C}_\varepsilon(C) = \big\{x \in X : r(C,x) \leq \delta(C)\big\} = C' \quad \text{and} \quad \delta\big(C'\big) \geq \delta\big(\mathcal{C}(C)\big) + 2\varepsilon > \delta(C)$$

i.e. there exist more than one diametric completion of C in X. $\square$

Corollary 2.8. *Any bounded set C such that $\frac{\delta(\mathcal{C}(C))}{\delta(C)} > J(X) - 1$ has more than one diametric completion. In particular, $\frac{\delta(\mathcal{C}(C))}{\delta(C)} \leq J(X) - 1$ for every diametrically complete set C.*

Proof. We may assume $\delta(C) = 1$. Then our assumption becomes $\delta(\mathcal{C}(C)) > J(X) - 1 \geq 2r(C) - \delta(C)$ and the thesis follows from Proposition 2.7.

The estimates given in Corollary 2.8 are sharp, as shown by the next

Example 2.9. Let X be c_0, and consider its subset $C = \{(x_i) : 0 \leq x_i \leq 1,\ i = 1, 2, ...\}$. It is easy to see that C is a diametral and diametrically complete set with $\delta(C) = 1$, hence it coincides with $\mathcal{C}(C)$. It is well known that $J(c_0) = 2$, therefore $1 = \frac{\delta(\mathcal{C}(C))}{\delta(C)} = J(X) - 1$ which shows that the bound given in Corollary 2.8 is sharp.

Moreover, for $\varepsilon > 0$, set $C_\varepsilon = \{(x_i) : 0 \leq x_1 \leq 1 - \varepsilon \wedge 0 \leq x_i \leq 1,\ i = 2, 3...\}$; still $\delta(C_\varepsilon) = 1$. Now $\mathcal{C}(C_\varepsilon) = \{(c_i) \in c_0 : -\varepsilon \leq c_1 \leq 1 \wedge 0 \leq c_i \leq 1,\ i = 2, 3...\}$ and $\delta(\mathcal{C}(C_\varepsilon)) = J(X) - 1 + \varepsilon$ which implies that C_ε has more than one diametric completion.

3. Diametral and diametrically complete sets in reflexive spaces

We begin this section with a classical example of a diametral set which we immediately see not to be diametrically complete.

Here X is the space $E_{\sqrt{2}}$, a renorming of l^2 introduced by R.C. James to show that not all reflexive spaces possess normal structure. We recall the general definition of such spaces, since they will be used more than once in the following.

For $\beta > 1$ let $E_\beta = (l^2, |\cdot|_\beta)$ be the space l^2 renormed according to

$$|x|_\beta = \max\{\|x\|_2, \beta\|x\|_\infty\}$$

where $\|x\|_2$, $\|x\|_\infty$ denote respectively the l^2 and l^∞ norms of x.

Example 3.1. For $X = E_{\sqrt{2}}$ set $C = \overline{co}\{e_i\}$, where $\{e_i\}$ is the canonical basis of l^2. It is known, and easy to see, that C is a diametral set in $E_{\sqrt{2}}$, with $r(C,C) = \delta(C) = \sqrt{2}$. We claim that $r(C) = \sqrt{2}$. In fact, take any $x \in E_{\sqrt{2}}$: since $x \in l^2$, $\forall \varepsilon > 0$ $\exists i_o(\varepsilon)$ such that $|x_i| < \varepsilon$ $\forall i \geq i_o$. Then

$$|x - e_{i_o}|_{\sqrt{2}} \geq \sqrt{2}\|x - e_{i_o}\|_\infty \geq \sqrt{2}(1-\varepsilon)$$

which implies $r(C,x) \geq \sqrt{2}$ for every $x \in E_{\sqrt{2}}$. So C is a diametral set such that $\sqrt{2} = r(C) = r(C,C)$; nevertheless it is neither diametrically complete, since $y = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ..., 0, ...)$ does not belong to C but $r(C,y) = \sqrt{2}$, nor maximal diametral because $\sqrt{2} = r(C) \leq r(C,z) \leq \sqrt{2}$ for each $z \in co(C \cup \{y\})$ implies that also $\overline{co}(C \cup \{y\})$ is diametral.

The equality $r(C) = \delta(C)$ in Example 3.1 shows that $E_{\sqrt{2}}$ lacks also property (G); besides, we could prove directly that a maximal diametral set generated by the set C is diametrically complete, but we'll obtain these results as particular cases of the next Theorem 3.5 and Corollary 3.6.

In Section 2 we asked about existence, in reflexive spaces, of constant width or at least diametrically complete sets whose interior is empty. Taking into account Proposition 2.2 we know that they have to be maximal diametral sets. This leads us to formulate the main question of this section.

Main Question 3.2. *In reflexive spaces, must every maximal diametral set be diametrically complete?*

A weaker version of it is

Question 3.3. *If a reflexive space contains a diametral set, does it contain also a diametrically complete set which is still diametral?*

Interest in finding classes of spaces where the problem has an affirmative answer lies also in the fact that those spaces lack normal structure (if and) only if they lack property (G).

All this is made precise in

Theorem 3.4. *Let X be a normed space and consider the following conditions:*

(a) *Any maximal diametral set is diametrically complete;*
(b) *for any maximal diametral set $C \subset X$, $r(C) = r(C,C)$;*
(c) *if a diametral set exists in X, then there exists also a diametrically complete set which is diametral;*
(d) *if X has property (G) then X has normal structure.*

(a) *and* (b) *are equivalent; they imply* (c) *which is equivalent to* (d).

Proof. (a) ⇒ (b) Proved in [13, Theorem 3.2].

(b) ⇒ (a) If C were not diametrically complete, it would exist $y \notin C$, such that $\delta(C \cup y) = \delta(C)$, hence $r(C, y) \leq \delta(C)$ and consequently (being $r(C, \cdot)$ a convex function) $r(C, z) \leq \delta(C)$ for any $z \in \overline{co}(C \cup \{y\})$.

Now our assumption that $r(C) = r(C, C) = \delta(C)$ implies that $r(C, z) = \delta(C)$ for every $z \in \overline{co}(C \cup \{y\})$, i.e. that $\overline{co}(C \cup \{y\})$ is diametral, contradicting maximality of C.

(a) ⇒ (c) Let C be a diametral set. Consider the class $\mathcal{A}$ of all diametral sets containing C and with diameter $\delta(C)$: by Zorn's lemma, there is a diametral set $\tilde{C} \in \mathcal{A}$, which is maximal with respect to the (partial) order relation defined by inclusion; $\tilde{C}$ must be diametrically complete because of (a).

(c) ⇒ (d) Suppose X lacks normal structure: in X there exists a diametral set, hence by (c) there exists also a set C which is both diametrically complete and diametral. So $r(C) = r(C, C) = \delta(C)$ and X lacks property (G) as well.

(d) ⇒ (c) If a diametral set exists, by hypothesis X lacks also property (G). Then, Proposition 2.3 guarantees the existence of a diametrically complete set with empty interior, which is diametral by Proposition 2.2.

We point out that, though Theorem 3.4 holds in any normed space, nonreflexive spaces do exist where none of the conditions listed above is fulfilled: for instance, l^∞ enjoys property (G) but lacks normal structure, hence it contains a maximal diametral set which is not diametrically complete and is contained in a ball of radius strictly smaller than its diameter.

We proceed now to prove that the answer to our main question is affirmative in the class of reflexive spaces satisfying the non-strict Opial's property.

We recall that a space X satisfies the *non-strict Opial's property* if, for any sequence $\{x_n\} \subset X$, if $w\text{-}\lim_{n\to+\infty} x_n = x$ then, for every $y \in X$,

$$\liminf \|x_n - x\| \leq \liminf \|x_n - y\|. \tag{1}$$

It is known that the E_β's as well as the spaces $l_{p,\infty}$, which are renormings of the l^p spaces, do have the non-strict Opial's property (see [5]), and also that none of them enjoys normal structure. On the other side, the strict Opial's property (defined like (1) but with a strict inequality) is known to imply normal structure.

Theorem 3.5. *Let X be a reflexive Banach space which satisfies the non-strict Opial's property. If C is a maximal diametral set in X, then C is diametrically complete in X.*

Proof. Assume C is not diametrically complete: as a first step, we claim that a point $y \in X \setminus C$ exists such that $r(C, y) < \delta(C)$. In fact, we have $r(C, y) \leq \delta(C)$ for some points $y \in X \setminus C$. If $r(C, y) = \delta(C)$ for all such y's, we would have $r(C, z) = \delta(C)$ for any y and any $z \in \overline{co}(C \cup \{y\})$, contradicting maximality of C.

Secondly, it is easy to prove, and known in literature, that from any diametral set C it is possible to extract a diametral sequence $\{x_n\} \subset C$ with the same diameter as C, i.e. a sequence such that $\lim_{n\to+\infty} \operatorname{dist}(x_n, co(x_i)_1^{n-1}) = \delta(\{x_n\}) = \delta(C)$. Notice that any subsequence of a diametral sequence is still diametral, with the same diameter. So weak compactness of C allows us to assume, passing to a subsequence if necessary, that $\{x_n\}$ converges weakly to some point $z \in \overline{co}\{x_n\}$. Now, for $k = 1, 2, \ldots$, choose $z_k \in co(x_i)_1^{n_k-1}$ such that $\|z - z_k\| < \frac{1}{k}$; we may assume that $\{n_k\}$ is increasing. Considering the sequence $\{x_{n_k}\}$, which is still diametral with $\delta(\{x_{n_k}\}) = \delta(C)$, we have

$$\begin{aligned}\liminf \|x_{n_k} - z\| &\geq \liminf\bigl(\|x_{n_k} - z_k\| - \|z_k - z\|\bigr)\\ &\geq \lim\Bigl(\operatorname{dist}\bigl(x_{n_k}, co(x_i)_1^{n_k-1}\bigr) - \frac{1}{k}\end{aligned}$$

$$= \delta(\{x_{n_k}\}) = \delta(C) > r(C, y)$$
$$\geq \liminf \|x_{n_k} - y\|$$

contradicting the non-strict Opial's condition, which proves the thesis.

Taking into account Theorem 3.4 we can state the next

Corollary 3.6. *Let X be a reflexive Banach space which satisfies the non-strict Opial's property: if X lacks normal structure, X lacks also property (G).*

While l^∞ is an example of a space with property (G) (see Section 4) and without normal structure, we do not know of any such example in reflexive space. A famous result obtained independently by Klee [10] and Garkavi [8], assures that in every normed space X which is not an inner product space and has $dim(X) \geq 3$ we can find bounded convex sets C such that $r(C) < r(C, C)$.

Our main question is equivalent to ask if in reflexive spaces we may have maximal diametral sets satisfying this inequality.

We turn now to the problem of the existence of diametrically complete sets (or even constant width sets) with empty interior. We have already remarked in the Introduction that such examples were provided in [13] in nonreflexive spaces, therefore we restrict our investigation to reflexive spaces.

Theorem 2.3 shows that such sets can exist only in spaces without normal structure, and Theorem 3.5 shows that in every reflexive space which satisfies the non-strict Opial's condition and lacks normal structure (equivalently, lacks property (G)) we must have a diametrically complete one which is diametral, hence a diametrically complete set with empty interior. Therefore we can state

Corollary 3.7. *Every reflexive space which satisfies the non-strict Opial's condition and lacks normal structure contains diametrically complete sets whose interior is empty.*

We remark that in [3, Theorem 3.2], it was claimed that no diametrically complete set other than single points could admit empty interior, which is contradicted both by examples in [13] and by our results.

In particular, the set C in Example 3.1 has a diametric completion which is still diametral and has empty interior. Obviously, a completion of that C cannot be contained in a hyperplane, but we can produce also diametrically complete sets with that property, as shown by the following result.

Theorem 3.8. *Let $X = E_{\sqrt{2}} \oplus_2 \mathbf{R}$. There exists a diametrically complete set C contained in the hyperplane $E_{\sqrt{2}}$ of X.*

Proof. Let $\mathcal{A}$ be the collection of all diametral subsets of $E_{\sqrt{2}}$ which contain $\overline{co}\{e_i\}$ and have diameter $\sqrt{2}$. With an easy application of Zorn's lemma as in Theorem 3.4, we produce a maximal set C in $\mathcal{A}$. We claim that

(a) C is diametrically complete in $E_{\sqrt{2}}$;
(b) C is diametrically complete also in $X = E_{\sqrt{2}} \oplus_2 \mathbf{R}$.

In fact

(a) C, as a maximal diametral set in the space $E_{\sqrt{2}}$ which satisfies the non-strict Opial's property, is diametrically complete: in particular

$$r(C, x) = \sqrt{2} \quad \forall x \in C \quad \text{and} \quad r(C, x) > \sqrt{2} \quad \forall x \in E_{\sqrt{2}} \setminus C;$$

(b) let $z \in E_{\sqrt{2}} \oplus_2 \mathbf{R}$, $z = (\tilde{z}, \zeta)$ with $\tilde{z} \in E_{\sqrt{2}}$ and $\zeta \in \mathbf{R} \setminus \{0\}$. For any $x \in C$,

$$\|z - x\| = \left(|\tilde{z} - x|^2_{\sqrt{2}} + \zeta^2\right)^{\frac{1}{2}} > |\tilde{z} - x|_{\sqrt{2}}.$$

As a consequence, for every $z \in (E_{\sqrt{2}} \oplus_2 \mathbf{R}) \setminus E_{\sqrt{2}}$, we have

$$r(C, z) = \left(r(C, \tilde{z})^2 + \zeta^2\right)^{\frac{1}{2}} > r(C, \tilde{z}) \geq \sqrt{2}$$

which, together with (a), proves that C is diametrically complete in $E_{\sqrt{2}} \oplus_2 \mathbf{R}$.

Therefore C is a diametrically complete set contained in a hyperplane of a reflexive space.

Having proved that diametrically complete sets may have empty interior also in reflexive spaces, we are led to ask what can be said about constant width sets with empty interior. As recalled in Section 2, such an example was given in c_0; the set in that example is a proper pseudo-ball (a set whose w^*-closure in X^{**} is a ball and which is not a ball itself), and of course the example cannot be extended to reflexive spaces.

Existence of proper pseudo-balls characterizes M-ideals among Banach spaces (see [17]).

A preliminary question is if proper constant width sets (i.e. neither singleton nor balls) may exist in infinite dimensional reflexive spaces. An affirmative answer is provided by results in [22] saying that any Banach space has an isomorphic copy containing proper constant width sets (i.e. not balls and not singletons), so we can say that reflexive spaces containing proper constant width sets do exist.

Therefore it is not meaningless to look for the "worst", from a topological point of view, constant width sets, so bad to have empty interior.

Unfortunately, we must leave the problem open: our last example, being contained in a hyperplane, certainly cannot have constant width and we'll show that, at least in the case of $X = E_{\sqrt{2}}$, also the construction in Theorem 3.5 does not lead to a constant width set.

With this aim, we start from the set $\overline{co}\{e_i\}$ in $E_{\sqrt{2}}$ and give a clear geometric characterization of the maximal diametral set C that it generates. Theorem 3.5 states that C is a diametric completion of $\overline{co}\{e_i\}$.

Theorem 3.9. *If C is a maximal diametral set generated by $\overline{co}\{e_i\}$ in $E_{\sqrt{2}}$, C is the intersection of the closed positive cone of l^2 with the unit ball in the l^2-norm. In particular, C is the unique diametric completion of $\overline{co}\{e_i\}$.*

Proof. Let $c = (c_i)$ be any point of C: observe first that $c_{\bar{\imath}} < 0$ for some $\bar{\imath}$ would imply $|c - e_{\bar{\imath}}|_{\sqrt{2}} \geq \sqrt{2}|c_{\bar{\imath}} - 1| > \sqrt{2}$ contradicting $\delta(C) = \sqrt{2}$.

Now suppose that $\|c\|_2 > 1$: since $c \in l^2$ we have that $\forall \varepsilon > 0$ $\exists i_o(\varepsilon)$ such that $|x_i| < \varepsilon$ $\forall i \geq i_o$. Then, for a sufficiently small ε

$$|c - e_{i_0}|^2_{\sqrt{2}} \geq \|c - e_{i_o}\|^2_2 = \sum_{i \neq i_0} c_i^2 + (1 - c_{i_0})^2 = \|c\|^2_2 + 1 - 2c_{i_0} > 2$$

contradicting again $\delta(C) = \sqrt{2}$.

To prove the reverse implication, take $x = (x_i)$ with $x_i \geq 0$ $\forall i$ and $\|x\|_2 \leq 1$ and suppose that x does not belong to C. Since C is diametrically complete, this is equivalent to say that $r(C, x) > \sqrt{2}$, i.e. $\exists c \in C$ such that $|x - c|_{\sqrt{2}} > \sqrt{2}$. Then one of the following two inequalities must be satisfied:

(a) $\exists \bar{\imath}$ such that $|x_{\bar{\imath}} - c_{\bar{\imath}}| > 1$
(b) $\|x - c\|_2 > \sqrt{2}$.

If (a) held, since $x_{\bar{\imath}} \leq 1$ and both $x_{\bar{\imath}}$ and $c_{\bar{\imath}}$ are non-negative, we would have

$$|x_{\bar{\imath}} - c_{\bar{\imath}}| = c_{\bar{\imath}} - x_{\bar{\imath}} > 1$$

hence

$$c_{\bar{\imath}} > 1 \quad \text{which implies} \quad |c|_{\sqrt{2}} \geq \sqrt{2}|c_{\bar{\imath}}| > \sqrt{2}$$

contradicting $\delta(C) = \sqrt{2}$ because $0 = w\text{-}\lim e_i \in C$.

In case (b), remember that we have just proved that for each $c \in C$ we have $\|c\|_2 \leq 1$. Therefore

$$\|x - c\|_2^2 = \sum_{i=1}^{+\infty} (x_i - c_i)^2 = \sum_{i=1}^{+\infty} \big(x_i^2 + c_i^2 - 2x_i c_i\big) \leq \|x\|_2^2 + \|c\|_2^2 \leq 2$$

again a contradiction. □

Corollary 3.10. *In $E_{\sqrt{2}}$, the set C, diametric completion of $\overline{co}\{e_i\}$, cannot be of constant width.*

Proof. It was proved in [17] that a set C is of constant width in X if and only if $\overline{C - C} = \delta(C)B_X$; by reflexivity of $E_{\sqrt{2}}$ we need only show that $C - C \neq \sqrt{2}B_X$ and it is sufficient to notice that the point $y = (1, 1, 0, \ldots)$ belongs to $\sqrt{2}B_X$, but not to $C - C$.

More generally, any point y of $\sqrt{2}B_X$ such that $1 < \|y\|_2 \leq \sqrt{2}$ and $0 \leq y_i \leq 1$ for all i's cannot be in $C - C$. In fact, suppose $y = z - w$; $z, w \in C$. Since, by Theorem 3.9, $w_i \geq 0$ $\forall i$, we must have $z_i \geq y_i$ $\forall i$, hence $\|z\|_2 \geq \|y\|_2 > 1$ contradicting, again because of Theorem 3.9, the assumption $z \in C$. □

4. Jung and self-Jung constants

In this section we present a few known results about Jung and self-Jung constants J and J_S, to describe the environmental background of the aspects related to normal structure in our main question of Section 3.

The fact that the two constants may have a strongly different behavior in nonreflexive spaces can be deduced from a few classical results:

- $1 \leq J(X) \leq 2$;
- $J(X) = 1$ if $X = C(\Omega)$, Ω an extremally disconnected compact set (see [9, p. 193]); $J(C(\Omega)) = 2$ otherwise (Amir [1], Franchetti [7]); as a consequence

$$J(l^\infty) = J\big(L^\infty([0, 1])\big) = 1; \qquad J(c) = J\big(C[0, 1]\big) = 2;$$

- $J(l^1) = J(L^1([0, 1])) = 2$ (Pichugov [18]);
- $\sqrt{2} \leq J_s(X) \leq 2$ (Maluta [11]);
- $J_s(X) = 2$ if X is nonreflexive (Maluta [11]).

Actually, the anomalous case is $X = C(\Omega)$, Ω an extremally disconnected compact set, which is the unique case where $J(X) = 1$, i.e. $r(C) = \frac{1}{2}\delta(C)$ for every bounded C (Davis [4]; see also [9]). All other classic nonreflexive spaces have $J(X) = 2$.

On the other side, it is very difficult to evaluate the two constants in cases which are not extreme, and they are known in very few spaces: in "nice" spaces, for instance in the reflexive L^p's, they coincide:

- $J(X) = J_s(X) = \sqrt{2}$ if X is Hilbert (Routledge [21]);
- $J(l^p) = J(L^p([0,1])) = J_s(l^p) = J_s(L^p([0,1])) = 2^{\max\{\frac{1}{p}, \frac{p-1}{p}\}}$ for $1 < p < +\infty$ (Pichugov [18,19]).

Regarding their mutual implications, first we remark that like normal structure implies property (G), analogously $J_s(X) < 2$ implies $J(X) < 2$. As about the reverse implications, the just mentioned results solve easily the problem for nonreflexive X's. In fact

- $J(l^\infty) = 1 < 2 = J_s(l^\infty)$;
- l^∞ has property (G), though it is well known that it lacks normal structure.

When we pass to reflexive spaces, we come again to our main question of Section 3 that, in this environment, we formulate in the weaker form as

Question 4.1. *In reflexive spaces, does property (G) imply normal structure?*

We point out that the analogous question for the corresponding uniform properties was given a negative answer in [1]. Unfortunately, the related claim is incorrect.

The relevant space is $X = (\sum_1^{+\infty} l_n^\infty)_2$; X is reflexive (but not superreflexive), it has normal structure but $J_s(X) = 2$ (Baillon, 1981).

In [1] it was claimed that $J(X) = 1$ (which would also contradict [4]), but it is enough to consider the sequence $\{x_n\} \subset X$, where, for each x_n, the only non-zero coordinate is the nth, and this one is the vector $(1, 0, ..., 0)$ of $\mathbb{R}^n$, to see that $J(X)$ must be at least $\sqrt{2}$.

In fact, $\delta(\{x_n\}) = \sqrt{2}$, and, for any $x \in X$ and for any $\varepsilon > 0$, for a sufficiently large n we have $\|x - x_n\| > 1 - \varepsilon$, proving that $r(\{x_n\}, x) \geq 1$, so that $r(\{x_n\}) \geq 1$.

We do not know if for this space $J(X)$ is actually less than two: $\sqrt{2}$ seems to be a reasonable candidate for its value. In any case, since X does possess normal structure, it would not provide a negative answer to our question.

Acknowledgment

The authors thank the referee for the careful reading of the paper and for his valuable comments.

References

[1] D. Amir, On Jung's constant and related constants in normed linear spaces, Pacific J. Math. 118 (1985) 1–15.
[2] L. Caspani, P.L. Papini, On constant width sets in Hilbert spaces and around, J. Convex Anal. (2014), in press.
[3] L. Dalla, N.K. Tamvakis, Sets of constant width and diametrically complete sets in normed spaces, Bull. Soc. Math. Grèce (N.S.) 26 (1985) 27–39.
[4] W.J. Davis, A characterization of $\mathcal{P}_1$ spaces, J. Approx. Theory 21 (1977) 315–318.
[5] T. Dominguez Benavides, G. López Acedo, H.K. Xu, Qualitative and quantitative properties for the space $l_{p,q}$, Houston J. Math. 22 (1996) 89–100.
[6] G. Eggleston, Sets of constant width in finite dimensional spaces, Israel J. Math. 3 (1965) 163–172.
[7] C. Franchetti, Projections onto hyperplanes in Banach spaces, J. Approx. Theory 38 (1983) 319–323.
[8] A. Garkavi, The best possible net and the best possible cross-section of a set in a normed space, Izv. Akad. Nauk SSSR 26 (1962) 87–106 (in Russian).
[9] R.B. Holmes, A Course on Optimization and Best Approximation, Lecture Notes in Math., vol. 257, Springer-Verlag, Berlin, 1972.
[10] V. Klee, Circumspheres and inner products, Math. Scand. 8 (1960) 363–370.
[11] E. Maluta, Uniformly normal structure and related coefficients, Pacific J. Math. 111 (1984) 357–369.
[12] J.P. Moreno, Porosity and unique completion in strictly convex spaces, Math. Z. 267 (2011) 173–184.
[13] J.P. Moreno, P.L. Papini, R.R. Phelps, Diametrically maximal and constant width sets in Banach spaces, Canad. J. Math. 58 (2006) 820–842.
[14] J.P. Moreno, P.L. Papini, R.R. Phelps, New families of convex sets related to diametrical maximality, J. Convex Anal. 13 (2006) 823–837.

[15] J.P. Moreno, R. Schneider, Local Lipschitz continuity of the diametric completion mapping, Houston J. Math. 38 (2012) 1207–1223.
[16] P.L. Papini, Complete sets and surroundings, in: Proc. Int. Symposium on Banach and Function Spaces IV, Kitakyushu, 2012, pp. 149–163.
[17] R. Payà, A. Rodriguez-Palacios, Banach spaces which are semi-L-summands in their biduals, Math. Ann. 289 (1991) 529–542.
[18] S.A. Pichugov, Jung's constants of L^p space, Mat. Zametki 43 (1988) 604–614 (in Russian); translation in Math. Notes 43 (1988) 348–354.
[19] S.A. Pichugov, Relative Jung's constants of L^p space, Ukrain. Mat. Zh. 42 (1990) 122–125 (in Russian); translation in Ukrainian Math. J. 42 (1990) 111–113.
[20] E.S. Polovinkin, Convex bodies of constant width, Dokl. Math. 70 (2004) 560–562.
[21] N. Routledge, A result in Hilbert spaces, Quart. J. Math. 3 (1952) 12–18.
[22] D. Yost, Banach spaces isomorphic to proper M-ideals, Colloq. Math. 56 (1988) 99–106.
[23] D. Yost, Irreducible convex sets, Mathematika 38 (1991) 134–155.