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Segal-Bargmann-Fock modules of monogenic functions

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In this paper, we introduce the classical Segal-Bargmann transform starting from the basis of Hermite polynomials and extending it to Clifford algebra-valued functions. Then we apply the results to monogenic functions and prove that the Segal-Bargmann kernel corresponds to the kernel of the Fourier-Borel transform for monogenic functionals. This kernel is also the reproducing kernel for the monogenic Bargmann module. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5008651>

I. INTRODUCTION

The Segal-Bargmann transform can be seen as a unitary map from spaces of square-integrable functions to spaces of holomorphic functions square-integrable with respect to a Gaussian density (see Refs. 1, 12, and 13). The latter class of functions is known in the literature under different names: Bargmann or Segal-Bargmann or Fock spaces either in one or several complex variables (see Refs. 6, 10, and 17). In this paper, we do not enter historical discussion on the nomenclature, and we will call them Segal-Bargmann-Fock spaces. These spaces are important in several different settings: in quantum mechanics, where they are used for the description of the spaces via tensor products; in infinite dimensional analysis and in free analysis, since these spaces are related to the white noise space (a probability space) and to the theory of stochastic distributions, see Ref. 16.

It is natural to consider another higher dimensional extension, namely, the one based on monogenic functions with values in a Clifford algebra. Recently, in Refs. 8 and 9, the authors have pointed out that this approach can be useful in studying quantum systems with internal, discrete degrees of freedom corresponding to nonzero spins.

Square-integrable holomorphic functions are well known in the literature, and they may belong to Hardy or Bergman spaces, whose reproducing kernels are rational functions. In the case of the Segal-Bargmann-Fock spaces, the kernel is an exponential function. This reproducing kernel is not bounded, and this fact makes some computations in this framework more complicated. On the other hand, the fact that Segal-Bargmann-Fock spaces are defined on the whole space (either \mathbb{C} or \mathbb{C}^m) makes other techniques from Fourier analysis available. From the physics point of view, the (normalized) reproducing kernels of the Segal-Bargmann-Fock spaces are the so-called coherent states in quantum mechanics.

In this paper, we consider the higher dimensional framework based on monogenic functions. We can introduce a notion of Segal-Bargmann module (over the Clifford algebra) and of Segal-Bargmann-Fock transform. The fact that we have a Fischer decomposition allows proving a relation between the projection of the transform onto its monogenic part and the Fourier-Borel kernel. Equipping the Segal-Bargmann-Fock module with the Fischer inner product, we can also show that the Segal-Bargmann transform is still an isometry on harmonic polynomials.

The plan of this paper is the following. In Sec. II, we revise the classical Segal-Bargmann transform. Then in Sec. III, we apply the results of Sec. II to the case of monogenic functions. In Sec. IV, we study monogenic Bargmann modules. Finally in Sec. V, we make summarizing remarks.

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II. THE CLASSICAL SEGAL-BARGMANN TRANSFORM

The Segal-Bargmann transform is a very well-known operator acting from the space $L^2(\mathbb{R})$ and the Segal-Bargmann-Fock space which is unitary and allows identifying these two spaces (see Refs. 1, 12, and 13 and the recent monographs^{10,17}).

The Segal-Bargmann-Fock space $\mathcal{B}(\mathbb{C})$ is the Hilbert space of entire functions which are square-integrable with respect to the Gaussian density, i.e.,

$$\frac{1}{\pi} \int_{\mathbb{C}} \exp(-|z|^2) |f(z)|^2 dx dy < \infty.$$

For any $f \in L^2(\mathbb{R})$, we define the Segal-Bargmann transform $\mathcal{B}: L^2(\mathbb{R}) \rightarrow \mathcal{B}(\mathbb{C})$ (sometimes called the Segal-Bergmann-Fock transform) as

$$\mathcal{B}[f](z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{z^2}{2} + xz - \frac{x^2}{4}\right) f(x) dx,$$

for any $f \in L^2(\mathbb{R})$.

Let us consider in $L^2(\mathbb{R})$, the basis $\{\psi_k(x)\}$, where

$$\psi_k(x) = H_k(x)e^{-x^2/4}, \quad H_k(x) = (-1)^k e^{x^2/2} \partial_x^k e^{-x^2/2}.$$

Note that $H_k(x)$ are the well-known Hermite polynomials. This basis is orthogonal, in fact,

$$\langle \psi_\ell, \psi_k \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{\psi_\ell(x)} \psi_k(x) dx = \delta_{\ell k} k!,$$

where $\delta_{\ell k}$ is the Kronecker delta.

Using integration by parts, we obtain

$$\begin{aligned} \mathcal{B}[\psi_k](z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{(z-x)^2}{2}\right) H_k(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \partial_x^k \left(\exp\left(-\frac{z^2}{2} + xz\right) \right) \exp(-x^2/2) dx \\ &= \frac{1}{\sqrt{2\pi}} z^k \int_{\mathbb{R}} \exp\left(-\frac{(z-x)^2}{2}\right) dx \\ &= z^k. \end{aligned}$$

Let us consider the so-called Fischer space, namely, the space of real analytic functions equipped with the inner product

$$[R, S] = \overline{R(\partial_x)} S(x)|_{x=0}.$$

To consider functions defined over the real numbers, it is necessary to obtain a match between the Fischer and the Bargmann inner products.

If we consider instead of the complex variable z a real variable u , we can use the Fischer inner product to get the equality $[u^\ell, u^k] = \delta_{\ell k} k!$.

The above computations show that the Segal-Bargmann-Fock transform $f(x) \mapsto \mathcal{B}[f](u)$ is an isometry between $L^2(\mathbb{R})$ and the Fischer space.

On the other hand, we can consider the space $\mathcal{O}(\mathbb{C})$ of entire holomorphic functions equipped with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} \exp(-|z|^2) \overline{f(z)} g(z) dx dy, \quad z = x + iy.$$

We note that the basis $\{z^k\}$ is orthogonal, in fact,

$$\langle z^\ell, z^k \rangle = \frac{1}{\pi} \int_{\mathbb{C}} \exp(-|z|^2) \overline{z}^k z^\ell dx dy = 0 \quad \text{for } k \neq \ell.$$

Moreover

$$\langle z^k, z^k \rangle = \frac{1}{\pi} \int_0^{2\pi} \int_0^\infty \exp(-r^2) r^{2k} r dr d\theta = k!, \quad \text{where } z = re^{i\theta}.$$

Then we have that $\mathcal{B}: L^2(\mathbb{R}) \rightarrow \mathcal{B}(\mathbb{C})$ is an isometry. Indeed, the Fischer inner product for functions defined in \mathbb{R} is equal to the Bargmann inner product of the holomorphic extensions.

This discussion can be extended from one to several complex variables.

Definition 2.1. The Segal-Bargmann-Fock space $\mathcal{B}(\mathbb{C}^m)$ is defined as the Hilbert space of entire functions f in \mathbb{C}^m which are square-integrable with respect to the $2m$ -dimensional Gaussian density, i.e.,

$$\frac{1}{\pi^m} \int_{\mathbb{C}^m} \exp(-|\underline{z}|^2) |f(\underline{z})|^2 d\underline{x}d\underline{y} < \infty, \quad \underline{z} = \underline{x} + i\underline{y}$$

and equipped with the inner product

$$\langle f, g \rangle = \frac{1}{\pi^m} \int_{\mathbb{C}^m} \exp(-|\underline{z}|^2) \overline{f(\underline{z})} g(\underline{z}) d\underline{x}d\underline{y}.$$

Let us denote by \underline{x} the m -tuple $(x_1, \dots, x_m) \in \mathbb{R}^m$. We consider the space $L^2(\mathbb{R}^m)$ with the basis $\{\psi_{k_1 \dots k_m}(\underline{x})\}$, where

$$\psi_{k_1 \dots k_m}(\underline{x}) = H_{k_1 \dots k_m}(\underline{x}) e^{-|\underline{x}|^2/4}$$

and $H_{k_1 \dots k_m}$ are the Hermite polynomials in \mathbb{R}^m given by

$$H_{k_1 \dots k_m}(\underline{x}) e^{-|\underline{x}|^2/2} = (-1)^{k_1 + \dots + k_m} \partial_{x_1}^{k_1} \dots \partial_{x_m}^{k_m} e^{-|\underline{x}|^2/2}.$$

Definition 2.2. The Segal-Bargmann-Fock transform $\mathcal{B}: L^2(\mathbb{R}^m) \rightarrow \mathcal{B}(\mathbb{C}^m)$ is defined by

$$\mathcal{B}[f](\underline{z}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{\underline{z} \cdot \underline{z}}{2} + \underline{x} \cdot \underline{z} - \frac{\underline{x} \cdot \underline{x}}{4}\right) f(\underline{x}) d\underline{x}, \quad \underline{x} \cdot \underline{z} = \sum_{j=1}^m x_j z_j,$$

for any $f \in L^2(\mathbb{R}^m)$.

Remark 2.3. Similar to the one variable case, we have that

$$\mathcal{B}[\psi_{k_1 \dots k_m}](\underline{z}) = z_1^{k_1} \dots z_m^{k_m}.$$

Finally note that $\|\psi_{k_1 \dots k_m}\|^2 = k_1! \dots k_m!$. It follows from this that $\mathcal{B}: L^2(\mathbb{R}^m) \rightarrow \mathcal{B}(\mathbb{C}^m)$ is an isometry.

III. THE MONOGENIC CASE

Let us denote by \mathbb{R}_m the real Clifford algebra generated by m imaginary units e_1, \dots, e_m . The multiplication in this associative algebra is determined by the relations

$$e_j e_k + e_k e_j = -2\delta_{ij}.$$

An element $a \in \mathbb{R}_m$ can be written as

$$a = \sum_A a_A e_A, \quad x_A \in \mathbb{R},$$

where the basis elements $e_A = e_{j_1} \dots e_{j_k}$ are defined for every subset $A = \{j_1, \dots, j_k\}$ of $\{1, \dots, m\}$ with $j_1 < \dots < j_k$. For the empty set, one puts $e_\emptyset = 1$, the latter being the identity element.

Observe that the dimension of \mathbb{R}_m as a real linear space is 2^m . Furthermore, conjugation in \mathbb{R}_m is given by $\bar{a} = \sum_A a_A \bar{e}_A$, where $\bar{e}_A = \bar{e}_{j_1} \dots \bar{e}_{j_k}$ with $\bar{e}_j = -e_j, j = 1, \dots, m$.

In the Clifford algebra \mathbb{R}_m , we can identify the so-called 1-vectors, namely, the linear combinations with real coefficients of the elements $e_j, j = 1, \dots, m$, with the vectors in the Euclidean space \mathbb{R}^m . The correspondence is given by the map $(x_1, \dots, x_m) \mapsto \underline{x} = x_1 e_1 + \dots + x_m e_m$ and it is obviously one-to-one.

The norm of a 1-vector \underline{x} is defined as $|\underline{x}| = \sqrt{x_1^2 + \dots + x_m^2}$ and clearly $\underline{x}^2 = -|\underline{x}|^2$. The product of two 1-vectors $\underline{x} = \sum_{j=1}^m x_j e_j$ and $\underline{y} = \sum_{j=1}^m y_j e_j$ splits into a scalar part and a 2-vector or the so-called bivector part

$$\underline{x} \underline{y} = -\langle \underline{x}, \underline{y} \rangle + \underline{x} \wedge \underline{y},$$

where

$$\begin{aligned} \langle \underline{x}, \underline{y} \rangle &= \underline{x} \cdot \underline{y} = -\frac{1}{2}(\underline{x}\underline{y} + \underline{y}\underline{x}) = \sum_{j=1}^m x_j y_j, \\ \underline{x} \wedge \underline{y} &= \frac{1}{2}(\underline{x}\underline{y} - \underline{y}\underline{x}) = \sum_{j=1}^m \sum_{k=j+1}^m (x_j y_k - x_k y_j) e_j e_k. \end{aligned}$$

The complex Clifford algebra \mathbb{C}_m can be seen as the complexification of the real Clifford algebra \mathbb{R}_m , i.e., $\mathbb{C}_m = \mathbb{R}_m \oplus i\mathbb{R}_m$. Any complex Clifford number $c \in \mathbb{C}_m$ may be written as $c = a + ib$, $a, b \in \mathbb{R}_m$, leading to the definition of the Hermitian conjugation: $c^\dagger = \bar{a} - ib$.

The first order differential operator

$$\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$$

is called the Dirac operator in \mathbb{R}^m . Functions in the kernel of this operator are known as monogenic functions (see, e.g., Refs. 2–4 and 7).

Definition 3.1. A function $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{C}_m$ defined and continuously differentiable in the open set Ω is said to be (left) monogenic if $\partial_{\underline{x}} f(\underline{x}) = 0$ in Ω . We denote by $\mathcal{M}(\mathbb{R}^m)$ the right \mathbb{C}_m -module of monogenic functions in \mathbb{R}^m .

A basic result in Clifford analysis is the so-called Fischer decomposition. Every homogeneous polynomial R_k of degree k can be uniquely decomposed as

$$R_k(\underline{x}) = M_k(\underline{x}) + \underline{x}R_{k-1}(\underline{x}),$$

where M_k and R_{k-1} are homogeneous polynomials and $M_k \in \mathcal{M}(\mathbb{R}^m)$. The monogenic polynomial M_k is called the monogenic part of R_k denoted by $\mathcal{M}(R_k)$.

If f is real analytic function near the origin, then it admits a decomposition of the form $f(\underline{x}) = \sum_{k=0}^{\infty} R_k(\underline{x})$ in an open ball centred at the origin. The monogenic part of f is thus defined by

$$\mathcal{M}(f(\underline{x})) = \sum_{k=0}^{\infty} \mathcal{M}(R_k(\underline{x})).$$

Let us recall the definition of the Clifford-Hermite polynomials $H_{s,k}(\underline{x})$. They are polynomials with real coefficients in \underline{x} of degree s satisfying

$$H_{s,k}(\underline{x})e^{-|\underline{x}|^2/2}P_k(\underline{x}) = (-1)^s \partial_{\underline{x}}^s (P_k(\underline{x})e^{-|\underline{x}|^2/2}),$$

where $P_k(\underline{x})$ denotes a homogeneous polynomial of degree k in $\mathcal{M}(\mathbb{R}^m)$.

Put $\psi_{s,k}(\underline{x}) = H_{s,k}(\underline{x})e^{-|\underline{x}|^2/4}$. The set of functions $\{\psi_{s,k}(\underline{x})P_k(\underline{x}) : s, k \in \mathbb{N}\}$ is an orthogonal basis for $L^2(\mathbb{R}^m)$ (see Ref. 14). Recalling Definition 2.2, we have the following:

Theorem 3.2. Assume that $P_k(\underline{x})$ is a homogeneous polynomial of degree k in $\mathcal{M}(\mathbb{R}^m)$. Then the following formula holds:

$$\mathcal{B}[\psi_{s,k}(\underline{x})P_k(\underline{x})](\underline{z}) = \underline{z}^s \mathcal{B}[P_k(\underline{x})e^{-|\underline{x}|^2/4}](\underline{z}).$$

Proof. The formula follows using integration by parts in higher dimensions,

$$\begin{aligned} \mathcal{B}[\psi_{s,k}(\underline{x})P_k(\underline{x})](\underline{z}) &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{\underline{z} \cdot \underline{z}}{2} + \underline{x} \cdot \underline{z} - \frac{\underline{x} \cdot \underline{x}}{4}\right) \psi_{s,k}(\underline{x})P_k(\underline{x}) d\underline{x} \\ &= \frac{(-1)^s}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{\underline{z} \cdot \underline{z}}{2} + \underline{x} \cdot \underline{z}\right) \partial_{\underline{x}}^s (P_k(\underline{x})e^{-|\underline{x}|^2/2}) d\underline{x} \\ &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \partial_{\underline{x}}^s \left(\exp\left(-\frac{\underline{z} \cdot \underline{z}}{2} + \underline{x} \cdot \underline{z}\right) \right) P_k(\underline{x})e^{-|\underline{x}|^2/2} d\underline{x} \\ &= \underline{z}^s \mathcal{B}[P_k(\underline{x})e^{-|\underline{x}|^2/4}](\underline{z}). \end{aligned}$$

□

The next problem is to compute $\mathcal{B}[P_k(\underline{x})e^{-|\underline{x}|^2/4}]$. To this end, we consider an example from which the general result will follow.

Lemma 3.3. Let $P_k(\underline{x}) = (x_1 - e_1 e_2 x_2)^k$. Then

$$\mathcal{B}[P_k(\underline{x})e^{-|\underline{x}|^2/4}](\underline{z}) = (z_1 - e_1 e_2 z_2)^k = P_k(\underline{z}).$$

Proof. Let $Z = x_1 - e_1 e_2 x_2$ and $2\partial_{\underline{z}} = \partial_{x_1} - e_1 e_2 \partial_{x_2}$. Note that

$$e^{-|\underline{x}|^2/2} = \exp\left(-\frac{1}{2}Z\bar{Z}\right) \exp\left(\sum_{j=3}^m -\frac{1}{2}x_j^2\right).$$

Thus we have

$$\begin{aligned} Z^k e^{-|\underline{x}|^2/2} &= \left((-2\partial_{\underline{z}})^k \exp\left(-\frac{1}{2}Z\bar{Z}\right)\right) \exp\left(\sum_{j=3}^m -\frac{1}{2}x_j^2\right) \\ &= (-\partial_{x_1} - e_1 e_2 \partial_{x_2})^k e^{-|\underline{x}|^2/2}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{B}[P_k(\underline{x})e^{-|\underline{x}|^2/4}](\underline{z}) &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \left((\partial_{x_1} - e_1 e_2 \partial_{x_2})^k \exp\left(-\frac{\underline{z} \cdot \underline{z}}{2} + \underline{x} \cdot \underline{z}\right)\right) e^{-|\underline{x}|^2/2} d\underline{x} \\ &= \frac{(z_1 - e_1 e_2 z_2)^k}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{(\underline{z} - \underline{x}) \cdot (\underline{z} - \underline{x})}{2}\right) d\underline{x} \\ &= \frac{(z_1 - e_1 e_2 z_2)^k}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x}|^2}{2}\right) d\underline{x} \\ &= (z_1 - e_1 e_2 z_2)^k. \end{aligned}$$

□

We can now prove the following simple but important result:

Theorem 3.4. Suppose that $P_k(\underline{x})$ is a homogeneous polynomial of degree k in $\mathcal{M}(\mathbb{R}^m)$. Then the following formula holds:

$$\mathcal{B}[H_{s,k}(\underline{x})e^{-|\underline{x}|^2/4}P_k(\underline{x})](\underline{z}) = \underline{z}^s P_k(\underline{z}).$$

Proof. We show that the formula that we have established for $P_k(\underline{x}) = (x_1 - e_1 e_2 x_2)^k$ in Lemma 3.3 holds for a general $P_k(\underline{x})$. First of all, due to the spin(m)-invariance, the result also holds for monogenic plane wave functions of the form

$$\langle \underline{x}, \underline{t} \rangle - \underline{t} \underline{s} \langle \underline{x}, \underline{s} \rangle^k,$$

where \underline{t} and \underline{s} are orthogonal unit vectors. Then we notice that the space of homogeneous monogenic polynomials of degree k is spanned by finitely many monogenic plane waves $\langle \underline{x}, \underline{t} \rangle - \underline{t} \underline{s} \langle \underline{x}, \underline{s} \rangle^k$ for some choices of the parameters $(\underline{t}, \underline{s})$. This, combined with Theorem 3.2, proves the result. □

IV. MONOGENIC BARGMANN MODULES

We will use the following notations.

Definition 4.1. Let $s \in \mathbb{N}$. We denote by $\mathcal{M}^s(\mathbb{R}^m)$ the right \mathbb{C}_m -module of s -monogenic functions on \mathbb{R}^m , namely, the set of smooth functions in the kernel of $\partial_{\underline{x}}^s$. Next, by $\mathcal{M}^s(\mathbb{C}^m)$, we denote the right \mathbb{C}_m -module of s -monogenic functions on \mathbb{C}^m , namely, the set of entire holomorphic functions in the kernel of $\partial_{\underline{z}}^s$, where $\partial_{\underline{z}} = \sum_{j=1}^m e_j \partial_{z_j}$ is the complexified Dirac operator.

Note that $\mathcal{M}^s(\mathbb{C}^m)$ is spanned by the set of polynomials of the form $\underline{z}^j P_k(\underline{z}), j = 0, \dots, s-1$, where $P_k(\underline{z})$ is the complex spherical monogenics of degree $k \in \mathbb{N}$.

Let us now define the s -monogenic Bargmann modules.

Definition 4.2. For any $s \in \mathbb{N}$, the s -monogenic Bargmann module $\mathcal{MB}^s(\mathbb{C}^m)$ is defined as

$$\mathcal{MB}^s(\mathbb{C}^m) = \mathcal{M}^s(\mathbb{C}^m) \cap \mathcal{B}(\mathbb{C}^m),$$

and it is equipped with the inner product defined in $\mathcal{B}(\mathbb{C}^m)$.

The Segal-Bargmann transform may act on s -monogenic functions as described in the following:

Proposition 4.3. The map

$$\mathcal{B}: \mathcal{M}^s(\mathbb{R}^m)e^{-|\underline{x}|^2/4} \cap L^2(\mathbb{R}^m) \rightarrow \mathcal{MB}^s(\mathbb{C}^m)$$

is an isometry (and it is even unitary).

Proof. The proof follows the standard arguments in the complex case and since $H_{s,k}(\underline{x})P_k(\underline{x})$, $j = 0, \dots, s - 1$, span $\mathcal{M}^s(\mathbb{R}^m)$, it is a consequence of the previous Theorem 3.4. \square

Remark 4.4. From the Fischer decomposition we obtain

$$\frac{\langle \underline{x}, \underline{u} \rangle^k}{k!} = \sum_{s=0}^k \underline{x}^s Z_{k,s}(\underline{x}, \underline{u}) \underline{u}^s,$$

where $Z_{k,s}(\underline{x}, \underline{u})$ are the so-called zonal spherical monogenics (see Ref. 15). The latter are homogeneous polynomials of degree $k - s$ in \underline{x} and \underline{u} and satisfy the two-sided biregular system $\partial_{\underline{x}} Z_{k,s}(\underline{x}, \underline{u}) = Z_{k,s}(\underline{x}, \underline{u}) \partial_{\underline{u}} = 0$.

One can also check that

$$Z_{k,s}(\underline{x}, \underline{u}) = \frac{Z_{k-s,0}(\underline{x}, \underline{u})}{\beta_{s,k-s} \dots \beta_{1,k-s}}, \quad k \geq s$$

with $\beta_{2s,k} = -2s$, $\beta_{2s+1,k} = -(2s + 2k + m)$. Moreover,

$$\begin{aligned} Z_k(\underline{x}, \underline{u}) &= Z_{k,0}(\underline{x}, \underline{u}) \\ &= \frac{\Gamma\left(\frac{m}{2} - 1\right)}{2^{k+1} \Gamma\left(k + \frac{m}{2}\right)} (|\underline{x}||\underline{u}|)^k \left[(k + m - 2) C_k^{\frac{m}{2}-1}(t) + (m - 2) \frac{\underline{x} \wedge \underline{u}}{|\underline{x}||\underline{u}|} C_{k-1}^{\frac{m}{2}}(t) \right], \end{aligned}$$

where $C_k^\alpha(t)$ denotes the classical Gegenbauer polynomial and $t = \frac{\langle \underline{x}, \underline{u} \rangle}{|\underline{x}||\underline{u}|}$.

Remark 4.5. Therefore, the Fischer decomposition of $e^{\langle \underline{x}, \underline{u} \rangle}$ has the form

$$e^{\langle \underline{x}, \underline{u} \rangle} = \sum_{k=0}^{\infty} \frac{\langle \underline{x}, \underline{u} \rangle^k}{k!} = \sum_{k=0}^{\infty} \sum_{s=0}^k \underline{x}^s Z_{k,s}(\underline{x}, \underline{u}) \underline{u}^s = \sum_{s=0}^{\infty} \underline{x}^s E_s(\underline{x}, \underline{u}) \underline{u}^s,$$

with $E_s(\underline{x}, \underline{u}) = \sum_{k=s}^{\infty} Z_{k,s}(\underline{x}, \underline{u})$.

We note that $E(\underline{x}, \underline{u}) = E_0(\underline{x}, \underline{u})$ is the monogenic part of $e^{\langle \underline{x}, \underline{u} \rangle}$, which is known as the Fourier-Borel kernel. This function has been computed in a closed form using Bessel functions in Ref. 5 (see also Ref. 11).

Let us again denote by \mathcal{M} the projection of a function onto its monogenic part in the complex monogenic Fischer decomposition.

Proposition 4.6. Let $f(\underline{x}) = g(\underline{x})e^{-|\underline{x}|^2/4} \in L^2(\mathbb{R}^m)$. Then we have

$$\mathcal{M}(\mathcal{B}[f](\underline{z})) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} E(\underline{z}, \underline{x}) g(\underline{x}) e^{-|\underline{x}|^2/2} d\underline{x} = [E(\underline{z}, \underline{x})^\dagger, g(\underline{x})],$$

where $[\cdot, \cdot]$ denotes the Fischer inner product. In particular when g is monogenic, we obtain $\mathcal{M}(\mathcal{B}[f](\underline{z})) = g(\underline{z})$.

Proof. From the above remarks, we obtain

$$\mathcal{M}\left(\exp\left(-\frac{\underline{z}\cdot\underline{z}}{2} + \underline{x}\cdot\underline{z} - \frac{\underline{x}\cdot\underline{x}}{4}\right)\right) = \mathcal{M}\left(\exp(\underline{x}\cdot\underline{z})\right)e^{-|\underline{x}|^2/4} = E(\underline{z}, \underline{x})e^{-|\underline{x}|^2/4}.$$

The statement follows using

$$\mathcal{M}(\mathcal{B}[f](\underline{z})) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \mathcal{M}\left(\exp\left(-\frac{\underline{z}\cdot\underline{z}}{2} + \underline{x}\cdot\underline{z} - \frac{\underline{x}\cdot\underline{x}}{4}\right)\right) f(\underline{x}) d\underline{x}.$$

□

V. REAL *s*-MONOGENIC BARGMANN MODULES

Let us recall that for harmonic polynomials $R(\underline{x}), S(\underline{x})$, we have (see also Ref. 7)

$$[R(\underline{x}), S(\underline{x})] = R(\partial_{\underline{x}})^\dagger S(\underline{x})|_{\underline{x}=0} = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} R(\underline{x})^\dagger S(\underline{x}) e^{-|\underline{x}|^2/2} d\underline{x}.$$

For $P_k(\underline{x}) \in \mathcal{M}(\mathbb{R}^m)$, we have

$$P_k(\underline{u}) = [Z_k(\underline{u}, \underline{x})^\dagger, P_k(\underline{x})] = [E(\underline{u}, \underline{x})^\dagger, P_k(\underline{x})].$$

From this formula, one also obtains in another way that for every monogenic $g(\underline{x})$ such that $g(\underline{x})e^{-|\underline{x}|^2/4} \in L^2(\mathbb{R}^m)$, the following formula holds:

$$g(\underline{u}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} E(\underline{u}, \underline{x}) g(\underline{x}) e^{-|\underline{x}|^2/2} d\underline{x}.$$

For general polynomials $R(\underline{x}), S(\underline{x})$, this link with the Fischer inner product no longer holds. However, we have that

$$[R(\underline{x}), S(\underline{x})] = \frac{1}{\pi^m} \int_{\mathbb{C}^m} R(\underline{z})^\dagger S(\underline{z}) e^{-|\underline{z}|^2} d\underline{z}.$$

For instance, take R, S of the form $z_1^{k_1} \dots z_m^{k_m}$. Moreover, \mathcal{B} is an isometry between $L^2(\mathbb{R}^m)$ and the Segal-Bargmann-Fock module equipped with the above inner product. In particular, if $g \in \mathcal{M}^s(\mathbb{R}^m)$, we have that the map

$$\mathcal{B}: g(\underline{x})e^{-|\underline{x}|^2/4} \mapsto h(\underline{z}) = \mathcal{B}(g(\underline{x})e^{-|\underline{x}|^2/4})$$

is an isometry. Note also that h is s -monogenic, but it is no longer true that $h(\underline{u}) = g(\underline{u})$ for $\underline{u} \in \mathbb{R}^m$.

Let us consider $f \in \mathcal{B}(\mathbb{C}^m)$ and let \mathcal{M}_s be the orthogonal projection onto the submodule of s -monogenic functions $\mathcal{M}\mathcal{B}^s(\mathbb{C}^m)$. To be more precise, if

$$f(\underline{z}) = \sum_{j=0}^{\infty} \underline{z}^j f_j(\underline{z}), \quad \partial_{\underline{z}} f_j(\underline{z}) = 0,$$

is the Fischer decomposition of f , then $\mathcal{M}_s(f(\underline{z})) = \sum_{j=0}^{s-1} \underline{z}^j f_j(\underline{z})$.

Define the function

$$B_s(\underline{z}, \underline{x}) = \mathcal{M}_s\left(\exp\left(-\frac{\underline{z}\cdot\underline{z}}{2} + \underline{x}\cdot\underline{z}\right)\right).$$

Theorem 5.1. *The following formula holds:*

$$B_s(\underline{z}, \underline{x}) = \sum_{j=0}^{s-1} \underline{z}^j \sum_{\ell=0}^{[j/2]} \frac{1}{2^\ell \ell!} E_{j-2\ell}(\underline{z}, \underline{x}) \underline{x}^{j-2\ell}.$$

Proof. We have that

$$\exp\left(-\frac{\underline{z}\cdot\underline{z}}{2}\right) = \sum_{\ell=0}^{\infty} \frac{\underline{z}^{2\ell}}{2^\ell \ell!}$$

and

$$\exp(\underline{x}\cdot\underline{z}) = \sum_{k=0}^{\infty} \underline{z}^k E_k(\underline{z}, \underline{x}) \underline{x}^k.$$

Therefore

$$\begin{aligned} \exp\left(-\frac{\underline{z} \cdot \underline{z}}{2} + \underline{x} \cdot \underline{z}\right) &= \sum_{k,\ell=0}^{\infty} \frac{1}{2^\ell \ell!} \underline{z}^{k+2\ell} E_k(\underline{z}, \underline{x}) \underline{x}^k \\ &= \sum_{j=0}^{\infty} \underline{z}^j \sum_{\ell=0}^{\lfloor j/2 \rfloor} \frac{1}{2^\ell \ell!} E_{j-2\ell}(\underline{z}, \underline{x}) \underline{x}^{j-2\ell} \end{aligned}$$

from which the desired result easily follows. \square

Remark 5.2. Note also the formula

$$\mathcal{M}_s \left(\mathcal{B}[g(\underline{x})e^{-|\underline{x}|^2/4}](\underline{z}) \right) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \mathcal{M}_s \left(\exp\left(-\frac{\underline{z} \cdot \underline{z}}{2} + \underline{x} \cdot \underline{z}\right) \right) g(\underline{x}) e^{-|\underline{x}|^2/2} d\underline{x}.$$

Next, one may introduce the real s -monogenic Bargmann module, which is the set of $g \in \mathcal{M}^s(\mathbb{R}^m)$ such that $g(\underline{x})e^{-|\underline{x}|^2/4}$ is square-integrable.

Definition 5.3. For any $s \in \mathbb{N}$, the real s -monogenic Bargmann module $\mathcal{M}^s(\mathbb{R}^m)$ is defined by

$$\mathcal{M}^s(\mathbb{R}^m) = \mathcal{M}^s(\mathbb{R}^m) \cap L^2(\mathbb{R}^m, e^{-|\underline{x}|^2/2}).$$

It is equipped with the inner product in $L^2(\mathbb{R}^m, e^{-|\underline{x}|^2/2})$,

$$\langle f, g \rangle = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f(\underline{x})^\dagger g(\underline{x}) e^{-|\underline{x}|^2/2} d\underline{x}.$$

As an additional remark, we note that for any $f \in \mathcal{M}^s(\mathbb{R}^m)$, we can define its Weierstrass transform $\mathcal{W}: \mathcal{M}^s(\mathbb{R}^m) \rightarrow \mathcal{M}^s(\mathbb{C}^m)$ by

$$\mathcal{W}[f](\underline{z}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{(\underline{z} - \underline{x}) \cdot (\underline{z} - \underline{x})}{2}\right) f(\underline{x}) d\underline{x} = \mathcal{B}[f(\underline{x})e^{-|\underline{x}|^2/4}](\underline{z}).$$

With standard techniques, one can show that the map $\mathcal{W}: \mathcal{M}^s(\mathbb{R}^m) \rightarrow \mathcal{M}^s(\mathbb{C}^m)$ is an isometry, moreover

$$\mathcal{W}[f](\underline{z}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} B_s(\underline{z}, \underline{x}) f(\underline{x}) e^{-|\underline{x}|^2/2} d\underline{x}.$$

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