Mixed L_-/l_1 Fault Detection Observer Design for Positive Switched Systems with Time-varying Delay via Delta Operator Approach

Shuo Li, Zhengrong Xiang*, and Hamid Reza Karimi

Abstract: This paper investigates the problem of fault detection observer design for positive switched systems with time-varying delay via delta operator approach. A new fault sensitivity measure, called L_- index, is proposed. The L_- fault detection observer design and multi-objective L_-/l_1 fault detection observer design problems are addressed. Based on the average dwell time approach and the piecewise co-positive type Lyapunov-Krasovskii functional method in delta domain, sufficient conditions for the existence of such two kinds of fault detection observers are firstly given, and then the design methods are presented. Finally, two examples are provided to show the effectiveness and the applicability of the proposed methods.

Keywords: Average dwell time, delta operator, fault detection, fault sensitivity, positive switched systems.

1. INTRODUCTION

Recently, positive switched systems have been highlighted and investigated by many researchers, due to the broad applications in communication systems [1], formation flying [2], the viral mutation dynamics under drug treatment [3] and systems theories [4-6]. A positive switched system is a type of switched system that consists of a family of positive subsystems and a switching signal defining a specific positive subsystem being activated during a certain interval. It should be pointed out that a linear co-positive Lyapunov functional is powerful for the analysis and synthesis of positive systems [7]. Some results on the stability and stabilization of such systems have been obtained in [8,9]. It has been shown that the l_1 performance index can characterize effectively the disturbance attenuation performance because of the peculiar nonnegative property of positive systems [10,11]. And the l_1-gain of positive switched systems has been investigated in [12,13].

In practice, time-delay phenomena widely exist in dynamic systems. The existence of time-delay may give rise to the deterioration of system performance and instability. Although many results have been reported for time-delay systems [14,15], only recently has the positive switched system with time delay become a topic of major interest [16-18], which is of significance to numerous applications.

On the other hand, the research of fault detection and isolation in dynamic systems has received considerable attention during the past decades, and some model-based fault detection approaches have been proposed in [19-21]. Among them, the basic idea is to use state observers or filters to construct a residual signal, and then to determine the residual evaluation function, which is to be compared with a predefined threshold. When the residual evaluation function has a value larger than the threshold, an alarm is generated. However, noises and disturbances may change the residual and result in false alarms. This means that the fault detection and isolation systems have to be sensitive to faults and simultaneously robust to the noise and disturbances. Several approaches using the H_∞ norm techniques have been largely developed for the design of robust fault detection observers or filters [22-24], where the H_∞ norm has been widely accepted as a good measure of robustness against unknown noise and disturbances. However, it should be pointed out that H_∞ norm does not consist with the main objective of fault detection, i.e., sensitivity to faults, because it measures the maximum effect of an input on an output. In general, high fault sensitivity (i.e., high sensitivity of the residual signal to faults) is preferred.

Recently, the study on the smallest singular value of a transfer function matrix has attracted considerable attention, which aims to maximize the minimum fault sensitivity to ensure the detection of the worst possible faults. Among various proposed measures, some H_- norms have been defined by using the minimum ‘nonzero” singular value, taken either at $w = 0$ [25], or over nonzero frequency ranges in [26,27]. This sensitive measure is closer, but not yet the worst-case sensitivity

---

Manuscript received October 15, 2013; revised December 25, 2013; accepted February 14, 2014. Recommended by Associate Editor Bin Jiang under the direction of Editor Zengqi Sun.

This work was supported by the National Natural Science Foundation of China under Grant No. 61273120 and by the Polish-Norwegian Research Programme operated by the National Centre for Research and 24 Development under the Norwegian Financial Mechanism 2009-2014 in the frame of Project Contract no. Pol-Nor/200957/47/2013.

Shuo Li and Zhengrong Xiang are with the School of Automation, Nanjing University of Science and Technology, Nanjing 210094, P. R. China (e-mails: lishuojx@126.com, xiangzr@mail.njust.edu.cn).

Hamid Reza Karimi is with the Department of Engineering, Faculty of Engineering and Science, University of Agder, N-4898 Grimstad, Norway (e-mail: hamid.r.karimi@uia.no).

* Corresponding author.

measure during to the exclusion of possible zero singular values in the detection. Moreover they were not used in the analysis directly because of the lack of an effective characterization of H_- norm performance. Instead, a true sensitivity measure called the H_- index, which is defined as the minimum singular value of the transfer function matrix over a given frequency range, was introduced to investigate the fault detection problem in [28,29]. The inclusion of possible zero singular values in the definition renders the H_- index of a true worst-case sensitivity measure. In addition, based on the H_- index, mixed H_-/H_∞ fault detection problem has attracted a great deal of attention and has been investigated in [30,31].

As a novel method with good finite word length performance under fast sampling rates, the delta operator has drawn considerable interest in the past three decades. As is known that the standard shift operator is often applied in the discrete systems control theories. When the sampling period tends to zero, namely data are taken at high sampling rates, however, the dynamic response of a discrete system does not converge smoothly to its continuous counterpart. The above problem is avoided until a delta operator method is proposed to take place of the traditional shift operator in [32]. It was shown that when implemented in fixed-point digital control processors, delta operator requires smaller word length than shift operator does [33]. Thus the delta operator model can be regarded as a useful approach to deal with discrete-time systems under high sampling rates through the analysis methods of continuous-time systems [34]. Based on some significant early investigations [35,36], the numerical properties and practical applications of delta operator formulated model have been extensively investigated [37,38]. However, to the best of our knowledge, the fault detection observer design problem of positive switched systems via delta operator approach has not been fully investigated to date, which motivates the present research.

In this paper, we focus our attention on investigating the fault detection observer design problem of positive switched systems with time-varying delay via delta operator approach. The main contributions of this paper are four-fold: 1) The fault detection problems of positive switched systems via delta operator are investigated for the first time; 2) In the framework of a linear Lyapunov function, an l_- index, as a new sensitivity measure of the residual signal to faults, is introduced for the design of positive fault detection observer; 3) Based on the proposed l_- index as well as the average dwell time, a fault sensitivity condition is developed and a positive l_- fault detection observer is designed; 4) The mixed l_-/l_1 fault detection observer design scheme is proposed such that the fault detection system is sensitive to faults and simultaneously robust to unknown disturbances.

The remainder of the paper is as follows. The problem formulation and some necessary lemmas are provided in Section 2. In Section 3, l_1 index robustness conditions and l_- index fault sensitivity conditions are provided, respectively. Then based on the obtained results, the problems of the l_- fault detection observer design and the multi-objective l_-/l_1 fault detection observer design are addressed and solved. Two numerical examples are presented to demonstrate the feasibility of the obtained results in Section 4. In Section 5, concluding remarks are given.

Notations: $A \succeq 0 (\preceq, \succ, \prec)$ means that all entries of matrix A are nonnegative (nonpositive, positive, negative); $A \succ B (A \succeq B)$ means $A - B \succ 0 (A - B \succeq 0)$; A^T means the transpose of matrix A; $R(R_+)$ is the set of all real (positive real) numbers; $R^n(R_+^n)$ is n-dimensional real (positive real) vector space; $R^{m\times n}$ is the set of all $m\times n$-dimensional real matrices; The vector 1-norm is denoted by $\|x\| = \sum_{k=1}^{n} |x_k|$ where x_k is the kth element of $x \in R^n$; Given $v : R \to R^n$, the l_1-norm is defined by $\|v\|_{L_1} = \sum_{k=k_0}^{\infty} \|v(k)\|$; We define $\underline{n} = \{1, 2, ..., n\}$ and $1_q = [1,1,...,1]_{1\times q}^T$; $l_1[k_0, \infty)$ is the space of absolute summable sequences on $[k_0, \infty)$, i.e., we say $z : [k_0, \infty) \to R^k$ is in $l_1[k_0, \infty)$ if $\sum_{k=k_0}^{\infty} \|z(k)\| < 0$.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following switched delta operator system with time-varying delay:

$$\begin{cases} \delta x(k) = A_{\sigma(k)} x(k) + A_{d\sigma(k)} x(k - d_k) \\ \qquad + E_{\sigma(k)} w(k) + G_{\sigma(k)} f(k), \\ y(k) = C_{\sigma(k)} x(k) + D_{\sigma(k)} w(k) + H_{\sigma(k)} f(k), \\ x(k_0 + \theta) = \varphi(\theta), \quad \theta = -\bar{d}, -\bar{d} + 1, ..., 0, \end{cases} \tag{1}$$

where $x(k) \in R^n$ denotes the state; $y(k) \in R^p$ is the measured output; $w(k) \in R^q$, $f(k) \in R^z$ are the disturbance input and the fault input, respectively, which belong to $l_1[k_0, \infty)$. k means the time $t = kT$ and $T > 0$ is the sampling period; k_0 is the initial time. $\sigma(k)$: $[k_0, \infty) \to \underline{M} = \{1, 2, ..., M\}$ is the switching signal, with M representing the number of subsystems; A_i, A_{di}, C_i, D_i, E_i, H_i and G_i, $i \in \underline{M}$, are constant matrices with appropriate dimensions. d_k denotes the time-varying discrete delay which satisfies $0 \le \underline{d} \le d_k \le \bar{d}$ for known constants $\underline{d}$ and $\bar{d}$; $\{\varphi(\theta), \theta = -\bar{d}, -\bar{d} + 1, ..., 0\}$ is a given discrete vector-valued initial condition. The switch is assumed to only occur at the sampling time in this paper. The delta operator is defined by

$$\delta x(t) = \begin{cases} dx(t)/dt, & T = 0 \\ (x(t + T) - x(t))/T, & T \neq 0, \end{cases} \tag{2}$$

where T is a sampling period. When $T \to 0$, the delta operator model will approach the continuous system before discretization and reflect a quasi-continuous performance.

Remark 1: Since a delta operator system can be regarded as a quasi-continuous system when $T \to 0$, the term $\delta x(k)$ can be applicable in normal continuous-time systems.

Definition 1: System (1) is said to be positive if, for any initial conditions $\varphi(\theta) \succeq 0$, $\theta = -\bar{d}, -\bar{d}+1, \ldots, 0$, $w(k) \succeq 0$, $f(k) \succeq 0$ and any switching signals $\sigma(k)$, the corresponding trajectories $x(k) \succeq 0$ and $y(k) \succeq 0$ hold for all $k \ge k_0$.

Lemma 1: System (1) is positive if and only if $(I + TA_i) \succeq 0$, $A_{di} \succeq 0$, $E_i \succeq 0$, $G_i \succeq 0$, $C_i \succeq 0$, $D_i \succeq 0$ and $H_i \succeq 0$, $\forall i \in \underline{M}$.

Proof: From the definition of delta operator δ, the discrete form of system (1) can be obtained as follows:

$$\begin{cases} x(k+1) = (I + TA_{\sigma(k)})x(k) + TA_{d\sigma(k)}x(k-d_k) \\ \qquad\qquad + TE_{\sigma(k)}w(k) + TG_{\sigma(k)}f(k), \\ y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}w(k) + H_{\sigma(k)}f(k), \\ x(k_0+\theta) = \varphi(\theta), \quad \theta = -\bar{d}, -\bar{d}+1, \ldots, 0. \end{cases} \tag{3}$$

Combining Lemma 2 in [39] and Lemma 1 in [16], one can obtain the remaining proof easily.

Remark 2: When $T \to 0$, system (1) degenerates to a general continuous-time positive switched system as follows:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d(t)) \\ \qquad\quad + E_{\sigma(t)}w(t) + G_{\sigma(t)}f(t), \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}w(t) + H_{\sigma(t)}f(t), \\ x(t_0+\theta) = \varphi(\theta), \quad \theta \in [-\bar{d}, 0], \end{cases} \tag{4}$$

where $d(t)$ denotes the time-varying delay which is everywhere time-differentiable and satisfies $0 \le \underline{d} \le d(t) \le \bar{d}$ for known constants $\underline{d}$ and $\bar{d}$. Then according to [12], system (4) is positive if and only if A_i are Metzler matrices, and $A_{di} \succeq 0$, $C_i \succeq 0$, $D_i \succeq 0$, $E_i \succeq 0$, $G_i \succeq 0$ and $H_i \succeq 0$, $\forall i \in \underline{M}$.

The fault detection observer has the form

$$\begin{cases} \delta\hat{x}(k) = A_{\sigma(k)}\hat{x}(k) + A_{d\sigma(k)}\hat{x}(k-d_k) \\ \qquad\qquad + K_{\sigma(k)}(y(k) - \hat{y}(k)), \\ \hat{y}(k) = C_{\sigma(k)}\hat{x}(k), \\ r(k) = y(k) - \hat{y}(k), \\ \hat{x}(k_0+\theta) = \hat{\varphi}(\theta), \quad \theta = -\bar{d}, -\bar{d}+1, \ldots, 0, \end{cases} \tag{5}$$

where $\hat{x}(k) \in R^n$ and $r(k) \in R^p$ are the estimated state and the residual, respectively. K_i, $i \in \underline{M}$, are the observer gain matrices to be determined.

Remark 3: For a non-positive system, the state of designed observer is only required to tract asymptotically that of the considered system. However, as stated in [40], this requirement is not enough for positive system (1), the designed observer should also ensure that the estimated state $\hat{x}(k)$ is positive. That's to say, the observer gain matrices K_i should guarantee that $I + T(A_i - K_iC_i) \succeq 0$ and $K_iC_i \succeq 0$.

Letting $e(k) = x(k) - \hat{x}(k)$ be the state estimation error, the residual error dynamic equations can be obtained from (1) and (5) as follows:

$$\begin{cases} \delta e(k) = (A_{\sigma(k)} - K_{\sigma(k)}C_{\sigma(k)})e(k) + A_{d\sigma(k)}e(k-d_k) \\ \qquad\qquad + (E_{\sigma(k)} - K_{\sigma(k)}D_{\sigma(k)})w(k) \\ \qquad\qquad + (G_{\sigma(k)} - K_{\sigma(k)}H_{\sigma(k)})f(k), \\ r(k) = C_{\sigma(k)}e(k) + D_{\sigma(k)}w(k) + H_{\sigma(k)}f(k), \\ e(k_0+\theta) = \varphi(\theta) - \hat{\varphi}(\theta), \quad \theta = -\bar{d}, -\bar{d}+1, \ldots, 0. \end{cases} \tag{6}$$

Set $\tilde{A}_i = A_i - K_iC_i$, $\tilde{E}_i = E_i - K_iD_i$ and $\tilde{G}_i = G_i - K_iH_i$, then according to Lemma 1, the above residual error delta operator system (6) is positive if $I + T\tilde{A}_i \succeq 0$, $\tilde{E}_i \succeq 0$ and $\tilde{G}_i \succeq 0$. Also the fault detection observer will be designed to maximize both the robustness against disturbance input $w(k)$ and the sensitivity to fault input $f(k)$.

Remark 4: As stated in [40], the positivity requirement on the estimated error $e(k)$ is introduced not only to be consistent with the observer case, but also to facilitate the synthesis of the desired positive observer. It should be pointed out that although this requirement may cause some certain conservatism, the positivity of $e(k)$ will not affect that of the estimated state $\hat{x}(k)$. If the initial condition does not satisfy $x(k) \succeq \hat{x}(k)$, the estimated error $e(k) \succeq 0$ may not hold for all $k \ge k_0$, but $\hat{x}(k)$ will still remain positive.

Definition 2 [41]**:** System (6) with $w(k) = 0$ and $f(k) = 0$ is said to be exponentially stable under $\sigma(k)$ if, for constants $\alpha > 0$ and $\kappa > 0$, the solution $e(k)$ satisfies

$$\|e(k)\| \le \alpha \|e(k_0)\|_c \, e^{-\kappa(k-k_0)}, \quad \forall k \ge k_0, \tag{7}$$

where $\|e(k_0)\|_c = \sup_{-\bar{d} \le \theta \le 0} \|e(k_0+\theta)\|$.

Definition 3 [42]**:** For any switching signal $\sigma(k)$ and any $k_2 > k_1 \ge 0$, let $N_\sigma(k_1, k_2)$ denote the number of switches of $\sigma(k)$ over the interval $[k_1, k_2)$. For given $\tau_a > 0$ and $N_0 \ge 0$, if the inequality

$$N_\sigma(k_1, k_2) \le N_0 + \frac{k_2 - k_1}{\tau_a} \tag{8}$$

holds, then the positive constant τ_a is called the average dwell time and N_0 is called the chattering bound.

Without loss of generality, we choose $N_0 = 0$ in this paper.

Definition 4 [12]**:** Given positive scalars $0 < \alpha < \frac{1}{T}$ and γ, system (6) is said to have a weighted l_1 performance index γ, if under zero initial condition, i.e., $e(k_0+\theta) = 0$, $\theta = -\bar{d}, -\bar{d}+1, \ldots, 0$, it holds that

$$\sup_{w(k) \ne 0, f(k)=0} \frac{\sum_{k=k_0}^{\infty} (1 - T\alpha)^{(k-k_0)} \|r(k)\|}{\sum_{k=k_0}^{\infty} \|w(k)\|} < \gamma,$$

$$w(k) \in l_1[k_0, \infty) \tag{9}$$

Remark 5: As stated in [10], in spite of being computed with the assumption of nonnegative state values and nonnegative input signals which belong to $l_1[k_0, \infty)$, the l_1 index is valid for any nonnegative initial state and any input signal in $l_1[k_0, \infty)$.

Remark 6: In Definition 4, as shown in [12,13], the l_1 index characterizes system's disturbance attenuation performance as the frequently-used H_∞ index. The smaller the value of γ is, the better the performance of the system is.

Definition 5: Given positive scalars $0 < \alpha < \frac{1}{T}$ and β, system (6) is said to have a weighted l_- performance index β, if under zero initial condition, i.e., $e(k_0 + \theta) = 0$, $\theta = -\bar{d}, -\bar{d}+1, \ldots, 0$, it holds that

$$\inf_{w(k)=0, f(k)\neq 0} \frac{\sum_{k=k_0}^{\infty} \|r(k)\|}{\sum_{k=k_0}^{\infty} (1-T\alpha)^{(k-k_0)} \|f(k)\|} > \beta,$$

$$f(k) \in l_1[k_0, \infty). \tag{10}$$

Remark 7: The l_- index is defined based on the l_1 signal spaces, which is different from the H_- index proposed in the literature [28,43]. Moreover, (10) means that the lower bound of the weighted l_1 gain from faults to residuals for any fault signals in $l_1[k_0, \infty)$ is not less than β, which is contrary to the aim of the l_1 index. Therefore the proposed l_- index can be regarded as a measure of the fault sensitivity.

Definition 6: Given positive switched delta operator system (1), for two positive scalars γ and β, the observer (5) is said to be an l_-/l_1 fault detection observer if

1) Error system (6) is exponentially stable when $w(k) = 0$ and $f(k) = 0$;
2) Under zero initial conditions, (9) and (10) hold.

The objective considered in this paper is to develop an admissible fault detection observer (5) for positive switched delta operator system (1) to minimize γ and to maximize β simultaneously.

l_- fault detection observer design: Given switched positive system (1) and a performance bound $\beta > 0$, find a fault detection observer (5), if exists, such that the residual error system (6) is exponentially stable under the switching signals with average dwell time when $w(k) = 0$, and (9) holds under zero initial conditions. Then the observer (5) is called an l_- fault detection observer.

Mixed l_-/l_1 fault detection observer design: Given switched positive system (1), find an l_-/l_1 fault detection observer, if exists, such that the residual error system (6) is exponentially stable under the switching signals with average dwell time when $w(k) = 0$ and $f(k) = 0$, as well as (9) and (10) hold with $\gamma - \beta$ being minimized.

Remark 8: Various mixed H_- / H_∞ performance $(\gamma^2 - \beta^2, \gamma/\beta, etc.)$ criteria were proposed in [30,31] using the H_- index. In this paper, we adopt the $\gamma - \beta$ criterion, using the l_1 index for easier comparison.

After designing the residual generator, the remaining important task is to evaluate the generated residual. One of the widely adopted approaches is to select a threshold and a residual evaluation function. In this paper, the residual evaluation function is chosen as

$$J_r(T_\pi) = \sum_{k=k_0}^{T_\pi} \|r(k)\|, \tag{11}$$

where T_π is the evaluation time window.

Once the evaluation function has been selected, we are able to determine the threshold. It is reasonable to choose the threshold as

$$J_{th} = \sup_{w \in l_1[k_0,\infty), f=0} J_r(T_\pi). \tag{12}$$

Based on this, the faults can be detected by using the following logical relationships

$$J_r(T_\pi) > J_{th} \Rightarrow \text{With faults} \Rightarrow \text{Alarm} \tag{13}$$

$$J_r(T_\pi) \le J_{th} \Rightarrow \text{No Faults.} \tag{14}$$

3. MAIN RESULTS

3.1. Stability analysis

First, we consider the following error delta operator system with time-varying delay:

$$\begin{cases} \delta e(k) = \tilde{A}_{\sigma(k)} e(k) + A_{d\sigma(k)} e(k - d_k) \\ e(k_0 + \theta) = \varphi(\theta) - \hat{\varphi}(\theta), \quad \theta = -\bar{d}, -\bar{d}+1, \ldots, 0, \end{cases} \tag{15}$$

where $I + T\tilde{A}_i \succeq 0$ for $i \in \underline{M}$. d_k is defined as the same as system (1).

Sufficient conditions of exponential stability for system (15) are provided in the following theorem.

Theorem 1: Given a positive constant $0 < \alpha < \frac{1}{T}$, if there exist ν_i, υ_i, $\vartheta_i \in R_+^n$, such that, $\forall i \in \underline{M}$,

$$\tilde{A}_i^T \nu_i + \alpha \nu_i + (1-T\alpha)(\bar{d} - \underline{d} + 1)\upsilon_i + (1-T\alpha)\vartheta_i \preceq 0, \tag{16}$$

$$A_{di}^T \nu_i - (1-T\alpha)^{\bar{d}+1} \upsilon_i \preceq 0, \tag{17}$$

where $\nu_i = [\nu_{i1}, \nu_{i2}, \ldots, \nu_{in}]^T$, $\upsilon_i = [\upsilon_{i1}, \upsilon_{i2}, \ldots, \upsilon_{in}]^T$, $\vartheta_i = [\vartheta_{i1}, \vartheta_{i2}, \ldots, \vartheta_{in}]^T$, then system (15) is exponentially stable for any switching signals $\sigma(k)$ with average dwell time satisfying

$$\tau_a > \tau_a^* = -\frac{\ln \mu}{\ln(1-T\alpha)}. \tag{18}$$

Furthermore, the state decay of system (15) is given by

$$\|e(k)\| \le a b^{(k-k_0)} \|e(k_0)\|_c, \tag{19}$$

where

$$a = \frac{\varepsilon_2}{\varepsilon_1} + \frac{T\varepsilon_3 \bar{d}}{\varepsilon_1} + \frac{0.5T\varepsilon_3(\bar{d} - \underline{d})(\bar{d} + \underline{d} - 1)}{\varepsilon_1} + \frac{T\varepsilon_4 \bar{d}}{\varepsilon_1},$$

$$b = \mu^{\frac{1}{\tau_a}}(1-T\alpha), \quad \varepsilon_1 = \min_{(r,i)\in \underline{n}\times\underline{M}} \{\nu_{ir}\},$$

$\varepsilon_2 = \max_{(r,i)\in \underline{n}\times \underline{M}} \{\nu_{ir}\}, \quad \varepsilon_3 = \max_{(r,i)\in \underline{n}\times \underline{M}} \{\upsilon_{ir}\},$
$\varepsilon_4 = \max_{(r,i)\in \underline{n}\times \underline{M}} \{\vartheta_{ir}\}, \quad \|e(k_0)\|_c = \sup_{-\bar{d}\le\theta\le 0} \|e(k_0+\theta)\|,$

and $\mu \ge 1$ satisfies

$$\nu_i \preceq \mu\nu_j, \quad \upsilon_i \preceq \mu\upsilon_j, \quad \vartheta_i \preceq \mu\vartheta_j, \quad \forall i, j \in \underline{M}. \tag{20}$$

Proof: Choose the following piecewise co-positive type Lyapunov functional for the i th subsystem in system (15)

$$\begin{aligned} V_i(k,e(k)) = {} & V_{i1}(k,e(k)) + V_{i2}(k,e(k)) \\ & + V_{i3}(k,e(k)) + V_{i4}(k,e(k)), \end{aligned} \tag{21}$$

where

$$V_{i1}(k,e(k)) = e^T(k)\nu_i,$$
$$V_{i2}(k,e(k)) = T\sum_{s=k-d_k}^{k-1}(1-T\alpha)^{k-s}e^T(s)\upsilon_i,$$
$$V_{i3}(k,e(k)) = T\sum_{l=-\bar{d}+1}^{-\underline{d}}\sum_{s=k+l}^{k-1}(1-T\alpha)^{k-s}e^T(s)\upsilon_i,$$
$$V_{i4}(k,e(k)) = T\sum_{s=k-\bar{d}}^{k-1}(1-T\alpha)^{k-s}e^T(s)\vartheta_i, \quad \forall i\in \underline{M}.$$

For simplicity, $V_i(k,e(k))$ is written as $V_i(k)$ (correspondingly, $V(k,e(k))$ is written as $V(k)$) in the later section of the paper.

According to the definition $\delta e(k) = \frac{e(k+1)-e(k)}{T}$, the Lyapunov function in delta domain has the following form:

$$\begin{aligned} \delta V_{i1}(k,e(k)) &= \delta(e^T(k)\nu_i) = (\delta e^T(k))\nu_i \\ &= e^T(k)\tilde{A}_i^T\nu_i + e^T(k-d_k)A_{di}^T\nu_i, \end{aligned} \tag{22}$$

$$\begin{aligned} \delta V_{i2}(k,e(k)) &= \frac{1}{T}[V_{i2}(k+1) - V_{i2}(k)] \\ &\le -T\alpha\sum_{s=k-d_k}^{k-1}(1-T\alpha)^{k-s}e^T(s)\upsilon_i \\ &\quad + (1-T\alpha)e^T(k)\upsilon_i \\ &\quad - (1-T\alpha)^{\bar{d}+1}e^T(k-d_k)\upsilon_i \\ &\quad + \sum_{s=k+1-\bar{d}}^{k-\underline{d}}(1-T\alpha)^{k+1-s}e^T(s)\upsilon_i, \end{aligned} \tag{23}$$

$$\begin{aligned} \delta V_{i3}(k,e(k)) &= \frac{1}{T}[V_{i3}(k+1) - V_{i3}(k)] \\ &= -T\alpha\sum_{l=-\bar{d}+1}^{-\underline{d}}\sum_{s=k+l}^{k-1}(1-T\alpha)^{k-s}e^T(s)\upsilon_i \\ &\quad + (1-T\alpha)(\bar{d}-\underline{d})e^T(k)\upsilon_i \\ &\quad - \sum_{s=k+1-\bar{d}}^{k-\underline{d}}(1-T\alpha)^{k+1-s}e^T(s)\upsilon_i, \end{aligned} \tag{24}$$

$$\begin{aligned} \delta V_{i4}(k,e(k)) &= \frac{1}{T}[V_{i4}(k+1) - V_{i4}(k)] \\ &= -T\alpha\sum_{s=k-\bar{d}}^{k-1}(1-T\alpha)^{k-s}e^T(s)\vartheta_i \\ &\quad + (1-T\alpha)e^T(k)\vartheta_i \\ &\quad - (1-T\alpha)^{\bar{d}+1}e^T(k-\bar{d})\vartheta_i. \end{aligned} \tag{25}$$

According to (22)-(25), we have

$$\begin{aligned} & \delta V_i(k,e(k)) + \alpha V_i(k,e(k)) \\ & \le e^T(k)[\tilde{A}_i^T\nu_i + \alpha\nu_i + (1-T\alpha)(\bar{d}-\underline{d}+1)\upsilon_i \\ & \quad + (1-T\alpha)\vartheta_i] + e^T(k-d_k)[A_{di}^T\nu_i - (1-T\alpha)^{\bar{d}+1}\upsilon_i]. \end{aligned} \tag{26}$$

From (16) and (17), we have

$$\begin{aligned} & \delta V_i(k) + \alpha V_i(k) \le 0 \\ & \Rightarrow \quad \delta V_i(k) = \frac{V_i(k+1)-V_i(k)}{T} \le -\alpha V_i(k) \\ & \Rightarrow \quad V_i(k+1) \le (1-T\alpha)V_i(k). \end{aligned} \tag{27}$$

Then, along the trajectory of system (15), we have

$$V_i(k) \le (1-T\alpha)^{k-k_0^{(i)}}V_i(k_0^{(i)}), \quad k \ge k_0^{(i)}, \tag{28}$$

$$0 < 1-T\alpha < 1 \Rightarrow 0 < \alpha < \frac{1}{T}, \tag{29}$$

where $k_0^{(i)}$ denotes the initial instant of the i th activated subsystem.

Let $k_0 < k_1 < \cdots < k_g$, $g = 0,1,\cdots$, denote the switching instants of $\sigma(k)$ over the interval $[k_0, k)$. When $k \in [k_g, k_{g+1})$, consider the following piecewise Lyapunov functional candidate for system (15)

$$V(k) = V_{\sigma(k)}(k) = V_{\sigma(k_g)}(k), \quad \forall k\in[k_g, k_{g+1}). \tag{30}$$

From (20) and (21), we can obtain

$$V_{\sigma(k_g)}(k,e(k_g)) \le \mu V_{\sigma(k_g^-)}(k,e(k_g^-)). \tag{31}$$

Then, it follows from (18), (28), (30) and the relation $N_\sigma(k_0,k) \le \frac{k-k_0}{\tau_a}$ that

$$\begin{aligned} V_{\sigma(k)}(k) &= V_{\sigma(k_g)}(k) < (1-T\alpha)^{(k-k_g)}V_{\sigma(k_g)}(k_g) \\ &\le \mu(1-T\alpha)^{(k-k_g)}V_{\sigma(k_g^-)}(k_g^-) \\ &= \mu(1-T\alpha)^{(k-k_{g-1})}V_{\sigma(k_{g-1})}(k_{g-1}) \\ &= \mu^{N_\sigma(k_{g-1},k)}(1-T\alpha)^{(k-k_{g-1})}V_{\sigma(k_{g-1})}(k_{g-1}) \\ &\le \cdots \le \mu^{N_\sigma(k_0,k)}(1-T\alpha)^{(k-k_0)}V_{\sigma(k_0)}(k_0) \\ &\le [\mu^{\frac{1}{\tau_a}}(1-T\alpha)]^{(k-k_0)}V_{\sigma(k_0)}(k_0). \end{aligned} \tag{32}$$

Considering the definition of $V_{\sigma(k)}(k)$, ε_1, ε_2, ε_3 and ε_4 in the condition of Theorem 1, it yields that

$$V_{\sigma(k)}(k) \ge \varepsilon_1\|e(k)\|, \tag{33}$$

$$
\begin{aligned}
&V_{\sigma(k_0)}(k_0) \\
&\le \varepsilon_2 \left\| e(k_0) \right\| + T\bar{d}\varepsilon_3 \sup_{-d_{k_0} \le \theta \le -1} \left\| e(k_0+\theta) \right\| \\
&\quad + 0.5T\varepsilon_3(\bar{d}-\underline{d})(\bar{d}+\underline{d}-1) \sup_{-\bar{d}+1 \le \theta \le -1} \left\| e(k_0+\theta) \right\| \\
&\quad + T\bar{d}\varepsilon_4 \sup_{-\bar{d} \le \theta \le -1} \left\| e(k_0+\theta) \right\| \\
&\le (\varepsilon_2 + T\varepsilon_3\bar{d} + 0.5T\varepsilon_3(\bar{d}-\underline{d})(\bar{d}+\underline{d}-1) \\
&\quad + T\varepsilon_4\bar{d}) \sup_{-\bar{d} \le \theta \le 0} \left\| e(k_0+\theta) \right\|.
\end{aligned} \tag{34}
$$

Combining (32)-(34), we obtain

$$
\begin{aligned}
\left\| e(k) \right\| \le \frac{1}{\varepsilon_1} &[(\varepsilon_2 + T\varepsilon_3\bar{d} + 0.5T\varepsilon_3(\bar{d}-\underline{d})(\bar{d}+\underline{d}-1) \\
&+ T\varepsilon_4\bar{d}][\mu^{\frac{1}{\tau_a}}(1-T\alpha)]^{(k-k_0)} \left\| e(k_0) \right\|_c,
\end{aligned} \tag{35}
$$

where $\left\| e(k_0) \right\|_c = \sup_{-\bar{d} \le \theta \le 0} \left\| e(k_0+\theta) \right\|$.

When the dwell time of the system satisfies

$$
0 < \mu^{\frac{1}{\tau_a}}(1-T\alpha) < 1, \tag{36}
$$

$$
\frac{1}{\tau_a^*} \ln \mu + \ln(1-T\alpha) < 0, \tag{37}
$$

we have

$$
\tau_a > \tau_a^* = -\frac{\ln \mu}{\ln(1-T\alpha)}.
$$

Therefore, according to Definition 2, we can conclude that system (15) is exponentially stable.

This completes the proof.

3.2. l_1 Index robustness condition

In this subsection, the robustness requirement (9) is considered. Let $f(k)=0$ in (6), we have

$$
\begin{cases}
\delta e(k) = \tilde{A}_{\sigma(k)} e(k) + A_{d\sigma(k)} e(k-d_k) + \tilde{E}_{\sigma(k)} w(k), \\
r(k) = C_{\sigma(k)} e(k) + D_{\sigma(k)} w(k).
\end{cases} \tag{38}
$$

The following result establishes a sufficient condition for the existence of l_1 performance of system (38).

Theorem 2: For given positive constants $0 < \alpha < \frac{1}{T}$ and γ, if there exist ν_i, υ_i, $\vartheta_i \in R_+^n$, $\forall i \in \underline{M}$, such that

$$
I + T\tilde{A}_i \succeq 0, \quad \tilde{E}_i \succeq 0, \tag{39}
$$

$$
\begin{aligned}
&\tilde{A}_i^T \nu_i + \alpha \nu_i + (1-T\alpha)(\bar{d}-\underline{d}+1)\upsilon_i \\
&\quad + (1-T\alpha)\vartheta_i + \hat{c}_i \preceq 0,
\end{aligned} \tag{40}
$$

$$
A_{di}^T \nu_i - (1-T\alpha)^{\bar{d}+1} \upsilon_i \preceq 0, \tag{41}
$$

$$
\tilde{E}_i^T \nu_i + \hat{d}_i - \gamma 1_q \preceq 0, \tag{42}
$$

where $\hat{c}_i = [\| c_{i1} \|_1 \ \ \| c_{i2} \|_1 \ \ \cdots \ \ \| c_{in} \|_1]_{1\times n}^T$, $c_{i\varsigma}$ represents the ς th column of matrix C_i, $\varsigma \in \underline{n} = \{1,2,...,n\}$, $\hat{d}_i = [\| d_{i1} \|_1 \ \ \| d_{i2} \|_1 \ \ \cdots \ \ \| d_{iq} \|_1]_{1\times q}^T$, $d_{i\varsigma}$ represents the ς th column of matrix D_i, $\varsigma \in \underline{q} = \{1,2,...,q\}$, then system (38) is exponentially stable and has a weighted l_1 disturbance attenuation performance index γ for any switching signals $\sigma(k)$ with average dwell time (18), where $\mu \ge 1$ satisfies (20).

Proof: By Theorem 1, if (40)-(41) hold, the exponential stability of system (38) with $w(k)=0$ is ensured. To show the weighted l_1-gain performance, we choose the Lyapunov functional (21). From (20), we have

$$
V_{\sigma(k_g)}(k_g) \le \mu V_{\sigma(k_g^-)}(k_g^-), \quad g = 1,2,.... \tag{43}
$$

For any $k \in [k_g, k_{g+1})$, noticing (40)-(42), we have

$$
\begin{aligned}
V(k) \le &(1-T\alpha)^{(k-k_g)} V_{\sigma(k_g)}(k_g) \\
&- T \sum_{s=k_g}^{k\text{-}1} (1-T\alpha)^{(k-s-1)} \Lambda(s),
\end{aligned} \tag{44}
$$

where $\Lambda(s) = \left\| r(s) \right\| - \gamma \left\| w(s) \right\|$.

Combining (43) and (44) leads to

$$
\begin{aligned}
V(k) &\le \mu(1-T\alpha)^{(k-k_g)} V_{\sigma(k_g^-)}(k_g^-) \\
&\qquad - T \sum_{s=k_g}^{k-1} (1-T\alpha)^{(k-s-1)} \Lambda(s) \\
&\le \mu(1-T\alpha)^{(k-k_{g-1})} V_{\sigma(k_{g-1})}(k_{g-1}) \\
&\qquad - \mu T \sum_{s=k_{g-1}}^{k_g-1} (1-T\alpha)^{(k-s-1)} \Lambda(s) \\
&\qquad - T \sum_{s=k_g}^{k-1} (1-T\alpha)^{(k-s-1)} \Lambda(s) \\
&\le \cdots \le \mu^{N_\sigma(k_0,k)} (1-T\alpha)^{(k-k_0)} V_{\sigma(k_0)}(k_0) \\
&\qquad - \mu^{N_\sigma(k_0,k)} T \sum_{s=k_0}^{k_1-1} (1-T\alpha)^{(k-s-1)} \Lambda(s) \\
&\qquad - \cdots - T \sum_{s=k_g}^{k-1} (1-T\alpha)^{(k-s-1)} \Lambda(s) \\
&= \mu^{N_\sigma(k_0,k)} (1-T\alpha)^{(k-k_0)} V_{\sigma(k_0)}(k_0) \\
&\qquad - T \sum_{s=k_0}^{k-1} \mu^{N_\sigma(s,k)} (1-T\alpha)^{(k-s-1)} \Lambda(s).
\end{aligned} \tag{45}
$$

Under the zero initial condition, from (45), we have

$$
0 \le -T \sum_{s=k_0}^{k-1} \mu^{N_\sigma(s,k)} (1-T\alpha)^{(k-s-1)} \Lambda(s) \tag{46}
$$

namely,

$$
\begin{aligned}
&\sum_{s=k_0}^{k-1} \mu^{N_\sigma(s,k)} (1-T\alpha)^{(k-s-1)} \left\| r(s) \right\| \\
&\le \gamma \sum_{s=k_0}^{k-1} \mu^{N_\sigma(s,k)} (1-T\alpha)^{(k-s-1)} \left\| w(s) \right\|.
\end{aligned} \tag{47}
$$

Multiplying both sides of (47) by $\mu^{-N_\sigma(k_0,k)}$ yields

$$\begin{aligned}&\sum_{s=k_0}^{k-1}\mu^{-N_\sigma(k_0,s)}(1-T\alpha)^{(k-s-1)}\|r(s)\|\\&\quad\le\gamma\sum_{s=k_0}^{k-1}\mu^{-N_\sigma(k_0,s)}(1-T\alpha)^{(k-s-1)}\|w(s)\|.\end{aligned}\tag{48}$$

Noticing that $N_\sigma(k_0,s)\le(s-k_0)/\tau_a$ and $\tau_a>\tau_a^*=-\frac{\ln\mu}{\ln(1-T\alpha)}$, we have

$$\mu^{-N_\sigma(k_0,s)}\le(1-T\alpha)^{(s-k_0)}.\tag{49}$$

Combining (48) and (49) leads to

$$\begin{aligned}&\sum_{s=k_0}^{k-1}(1-T\alpha)^{(s-k_0)}(1-T\alpha)^{(k-s-1)}\|r(s)\|\\&\quad\le\gamma\sum_{s=k_0}^{k-1}(1-T\alpha)^{(k-s-1)}\|w(s)\|.\end{aligned}\tag{50}$$

Summing both sides of (50) from $k=k_0$ to ∞ leads to inequality:

$$\sum_{k=k_0}^{\infty}(1-T\alpha)^{(k-k_0)}\|r(k)\|\le\gamma\sum_{k=k_0}^{\infty}\|w(k)\|.$$

From Definition 4, it can be concluded that system (38) is exponentially stable with a prescribed l_1-gain performance level γ.

This completes the proof.

3.3. l_- index fault sensitivity condition

In this subsection, the fault sensitivity condition (10) is considered. Let $w(k)=0$ in (6), we have

$$\begin{cases}\delta e(k)=\tilde{A}_{\sigma(k)}e(k)+A_{d\sigma(k)}e(k-d_k)+\tilde{G}_{\sigma(k)}f(k),\\ r(k)=C_{\sigma(k)}e(k)+H_{\sigma(k)}f(k).\end{cases}\tag{51}$$

In the following theorem, a sufficient condition is provided for system (51) to have a weighted l_- fault sensitivity index β.

Theorem 3: For given positive constants $0<\alpha<\frac{1}{T}$ and β, if there exist ν_{oi}, υ_{oi}, $\vartheta_{oi}\in R_+^n$, $\forall i\in\underline{M}$, such that

$$I+T\tilde{A}_i\succeq 0,\quad \tilde{G}_i\succeq 0,\tag{52}$$

$$\begin{aligned}&\tilde{A}_i^T\nu_{oi}+\alpha\nu_{oi}+(1-T\alpha)(\bar{d}-\underline{d}+1)\upsilon_{oi}\\&\quad+(1-T\alpha)\vartheta_{oi}-\hat{c}_i\preceq 0,\end{aligned}\tag{53}$$

$$A_{di}^T\nu_{oi}-(1-T\alpha)^{\bar{d}+1}\upsilon_{oi}\preceq 0,\tag{54}$$

$$\tilde{G}_i^T\nu_{oi}-\hat{h}_i+\beta 1_z\preceq 0,\tag{55}$$

where $\hat{h}_i=[\|h_{i1}\|_1\ \ \|h_{i2}\|_1\ \ \cdots\ \ \|h_{iz}\|_1]_{1\times z}^T$, $h_{i\varsigma}$ represents the ςth column of matrix H_i, $\varsigma\in\underline{z}=\{1,2,\ldots,z\}$, then system (21) is positive and has a weighted l_- fault sensitivity index β for any switching signals $\sigma(k)$ with average dwell time (18), where $\mu\ge 1$ satisfies

$$\nu_{oi}\preceq\mu\nu_{oj},\quad \upsilon_{oi}\preceq\mu\upsilon_{oj},\quad \vartheta_{oi}\preceq\mu\vartheta_{oj},\quad \forall i,j\in\underline{M}\tag{56}$$

Proof: Choose the following piecewise co-positive type Lyapunov functional for the ith subsystem in system (51)

$$\begin{aligned}V_i(k,e(k))&=V_{i1}(k,e(k))+V_{i2}(k,e(k))\\&\quad+V_{i3}(k,e(k))+V_{i4}(k,e(k)),\end{aligned}\tag{57}$$

where

$$\begin{aligned}&V_{i1}(k,e(k))=e^T(k)\nu_{oi},\\&V_{i2}(k,e(k))=T\sum_{s=k-d_k}^{k-1}(1-T\alpha)^{k-s}e^T(s)\upsilon_{oi},\\&V_{i3}(k,e(k))=T\sum_{l=-\bar{d}+1}^{-\underline{d}}\sum_{s=k+l}^{k-1}(1-T\alpha)^{k-s}e^T(s)\upsilon_{oi},\\&V_{i4}(k,e(k))=T\sum_{s=k-\bar{d}}^{k-1}(1-T\alpha)^{k-s}e^T(s)\vartheta_{oi},\quad\forall i\in\underline{M},\end{aligned}$$

where ν_{oi}, υ_{oi} and $\vartheta_{oi}\in R_+^n$ are to be determined.

Denote $k_1,k_2,\ldots,k_g$ the switching instants on the interval $[k_0,k)$ and let $\sigma(k_g)=i$. Similar to the proof line of Theorem 1, for $k\in[k_g,k_{g+1})$, we can obtain

$$\begin{aligned}&\delta V_i(k)+\alpha V_i(k)+\beta\|f(k)\|-\|r(k)\|\\&\le e^T(k)[\tilde{A}_i^T\nu_{oi}+\alpha\nu_{oi}+(1-T\alpha)\vartheta_{oi}\\&\quad+(1-T\alpha)(\bar{d}-\underline{d}+1)\upsilon_{oi}-\hat{c}_i]\\&\quad+e^T(k-d_k)[A_{di}^T\nu_{oi}-(1-T\alpha)^{\bar{d}+1}\upsilon_{oi}]\\&\quad+f^T(k)(\tilde{G}_i^T\nu_{oi}-\hat{h}_i+\beta 1_z).\end{aligned}\tag{58}$$

It can be obtained from (53)-(55) that

$$\delta V_i(k)+\alpha V_i(k)+\beta\|f(k)\|-\|r(k)\|\le 0.$$

It follows that

$$\begin{aligned}V_{\sigma(k)}(k)&\le(1-T\alpha)^{(k-k_g)}V_{\sigma(k_g)}(k_g)\\&\quad-T\sum_{s=k_g}^{k-1}(1-T\alpha)^{(k-s-1)}\Delta(s),\end{aligned}\tag{59}$$

where $\Delta(s)=\beta\|f(k)\|-\|r(k)\|$.

Following the proof line of Theorem 2, we have

$$\sum_{k=k_0}^{\infty}(1-T\alpha)^{(k-k_0)}\beta\|f(k)\|\le\sum_{k=k_0}^{\infty}\|r(k)\|,$$

which means that (10) is satisfied.

This completes the proof.

Remark 9: In the derivation of Theorem 3, the co-positive type Lyapunov-Krasovskii functional is employed for the fault sensitivity analysis, which makes the results obtained be easier to analyze. However, unlike the l_1 performance analysis problem, conditions in (53)-(55) do not guarantee the negativity of the chosen Lyapunov-Krasovskii functional. Hence, these conditions given in Theorem 3 do not ensure the stability of system (51).

3.4. Design of l_- fault detection observer

First consider the l_- fault detection observer design problem. Note that the l_- index measure requires no stability, (53)-(55) do not always provide a stable solution.

Therefore, we should consider the stability of error system (6) in the design process. According to the stability condition given in Theorem 1, an additional condition for the stability of error system (6) is the existence of $\nu_{oi} = \nu_i \in R_+^n$, $\upsilon_{oi} = \upsilon_i \in R_+^n$ and $\upsilon_{oi} = \upsilon_i \in R_+^n$ such that

$$\tilde{A}_i^T \nu_{oi} + \alpha \nu_{oi} + (1 - T\alpha)(\bar{d} - \underline{d} + 1)\upsilon_{oi} + (1 - T\alpha)\vartheta_{oi} \preceq 0. \quad (60)$$

Note that the condition (60) implies (53). Then (54)-(55), and (60) provide a sufficient condition for the existence of a positive l_- fault detection observer. In what follows, a positive l_- fault detection observer design scheme is provided, and the designed observer gain matrices are obtained simultaneously.

Theorem 4: For given positive constants $0 < \alpha < \frac{1}{T}$ and β, if there exist vectors $\nu_{oi} = \nu_i \in R_+^n$, $\upsilon_{oi} = \upsilon_i \in R_+^n$, $\upsilon_{oi} = \upsilon_i \in R_+^n$ and $\rho_i \in R_+^q$, $\forall i \in \underline{M}$, such that

$$I + T(A_i - K_i C_i) \succeq 0, \quad G_i - K_i H_i \succeq 0, \quad (61)$$

$$A_i^T \nu_i - C_i^T \rho_i + \alpha \nu_i + (1 - T\alpha)(\bar{d} - \underline{d} + 1)\upsilon_i + (1 - T\alpha)\vartheta_i \preceq 0, \quad (62)$$

$$A_{di}^T \nu_i - (1 - T\alpha)^{\bar{d}+1} \upsilon_i \preceq 0, \quad (63)$$

$$G_i^T \nu_i - H_i^T \rho_i - \hat{h}_i + \beta 1_z \preceq 0, \quad (64)$$

then there exist some certain positive l_- fault detection observer satisfying (10) and guaranteeing the stability of error system (6) for any switching signals with average dwell time satisfying (18), where $\mu \ge 1$ satisfies (20).

Moreover, if the conditions above have a feasible solution, the observer gain matrices can be obtained by solving $\rho_i = K_i^T \nu_i$ and checking the condition (61).

Proof: Denoting $\rho_i = K_i^T \nu_i$ and substituting it into (54)-(55) and (60), we can obtain from Theorem 3 that the theorem is true.

It can be seen that (62)-(64) can be solved via the linear matrix inequality (LMI) technique due to the fact that (62)-(64) are linear constrains. Furthermore, for a fixed α, a suboptimal solution β can be obtained by solving the following optimization problem:

Problem 1:

$$\min_{\nu_i, \upsilon_i, \vartheta_i, \rho_i} -\beta$$

$$\text{s.t. (62)-(64)}, \quad i \in \underline{M},$$

then the corresponding suboptimal observer gain matrices can be obtained.

We are now in a position to give a procedure for constructing the designed l_- fault detection observer gain matrices.

Algorithm 1

Step 1: Given a parameter $0 < \alpha < \frac{1}{T}$, one can obtain the solution of ν_i, υ_i, ϑ_i and ρ_i by solving optimization Problem 1.

Step 2: Compute the observer gain matrices K_i by $\rho_i = K_i^T \nu_i$.

Step 3: Check the condition (61). If it holds, the desire observer gain matrices K_i are obtained. Otherwise, return to Step 1.

Step 4: Compute the values of μ and τ_a^* by (18) and (20).

3.5. Design of l_-/l_1 Fault Detection Observer

In this subsection, we focus on the mixed l_-/l_1 fault detection observer design. The following theorem gives a sufficient condition for the existence of a mixed l_-/l_1 fault detection observer.

Theorem 5: For given positive constants $0 < \alpha < \frac{1}{T}$, β and γ, if there exist vectors $\nu_{oi} = \nu_i \in R_+^n$, $\upsilon_{oi} = \upsilon_i \in R_+^n$, $\vartheta_{oi} = \vartheta_i \in R_+^n$ and $\rho_i \in R_+^q$, $\forall i \in \underline{M}$, such that

$$\begin{cases} I + T(A_i - K_i C_i) \succeq 0 \\ E_i - K_i D_i \succeq 0 \\ G_i - K_i H_i \succeq 0, \end{cases} \quad (65)$$

$$A_i^T \nu_i - C_i^T \rho_i + \alpha \nu_i + (1 - T\alpha)(\bar{d} - \underline{d} + 1)\upsilon_i + (1 - T\alpha)\vartheta_i + \hat{c}_i \preceq 0, \quad (66)$$

$$A_{di}^T \nu_i - (1 - T\alpha)^{\bar{d}+1} \upsilon_i \preceq 0, \quad (67)$$

$$G_i^T \nu_i - H_i^T \rho_i - \hat{h}_i + \beta 1_z \preceq 0, \quad (68)$$

$$E_i^T \nu_i - D_i^T \rho_i + \hat{d}_i - \gamma 1_q \preceq 0, \quad (69)$$

then there exist some certain positive l_-/l_1 fault detection observers satisfying (9) and (10) for any switching signals with average dwell time satisfying (18), where $\mu \ge 1$ satisfies (20).

Moreover, if the conditions above have a feasible solution, the observer gain matrices can be obtained by solving $\rho_i = K_i^T \nu_i$ and checking the condition (65).

Proof: Note that (60) can be directly obtained from (40) for $\nu_{oi} = \nu_i \in R_+^n$, $\upsilon_{oi} = \upsilon_i \in R_+^n$ and $\vartheta_{oi} = \vartheta_i \in R_+^n$. Thus (9)-(10) and the stability of error system (6) can be guaranteed under the conditions (40)-(42) and (55). Denoting $\rho_i = K_i^T \nu_i$ and substituting it into (40)-(42) and (55), we can obtain from Definition 6 that the theorem is true.

Remark 10: It should be noted that based on the widely accepted H_- index, mixed H_-/H_∞ fault detection problem has been investigated in the existing literatures [30,31]. However, because of the peculiar nonnegative property of positive systems, a straightforward application of available FDI observer designs for non-positive dynamical systems to positive dynamical systems may not be applicable. Thus an l_- index, as a new sensitivity measure of the residual signal to faults, is introduced for the design of positive fault detection observer. It is required that the designed l_-/l_1 observer not only ensures the robustness against disturbance input $w(k)$ and the sensitivity to fault input $f(k)$, but also guarantees the positivity of the residual error system (6).

Similarly, for a fixed α, a suboptimal solution to the mixed l_-/l_1 fault detection observer design problem can be obtained by solving the following optimization problem:

Problem 2:

$$\min_{\nu_i, \upsilon_i, \vartheta_i, \rho_i} \gamma - \beta$$
$$\text{s.t. (66)-(69)}, \quad i \in \underline{M},$$

then the corresponding suboptimal observer gain matrices can be obtained.

We are now in a position to give a procedure for constructing the designed l_-/l_1 fault detection observer gain matrices.

Algorithm 2

Step 1: Given a parameter $0 < \alpha < \frac{1}{T}$, one can obtain the solution of ν_i, υ_i, ϑ_i and ρ_i by solving optimization Problem 2.

Step 2: Compute the observer gain matrices K_i by $\rho_i = K_i^T \nu_i$.

Step 3: Check the condition (65). If it holds, the desire observer gain matrices K_i are obtained. Otherwise, return to Step 1.

Step 4: Compute the values of μ and τ_a^* by (18) and (20).

4. NUMERICAL EXAMPLE

In this section, two examples are presented to check the validity of the proposed results.

Example 1: Consider system (1) with $w(k) = 0$. The parameters of the system are as follows:

Subsystem 1:

$$A_1 = \begin{bmatrix} -3.8 & 1.3 \\ 1.5 & -3.1 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0.0 \\ 0.1 & 0.0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1.16 \\ 1.20 \end{bmatrix},$$
$$C_1 = [0.12 \quad 0.13], \quad H_1 = 0.13,$$

Subsystem 2:

$$A_2 = \begin{bmatrix} -3.8 & 1.5 \\ 1.3 & -3.2 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.1 & 0.0 \\ 0.1 & 0.0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1.25 \\ 1.30 \end{bmatrix},$$
$$C_2 = [0.14 \quad 0.15], \quad H_2 = 0.12,$$

and $\bar{d} = 10$, $\underline{d} = 0$, $\alpha = 1.5$, $T = 0.2$. Then, by solving optimization Problem 1, we can obtain the following solutions:

$$\nu_1 = \begin{bmatrix} 0.3567 \\ 0.2958 \end{bmatrix}, \quad \nu_2 = \begin{bmatrix} 0.1782 \\ 0.1742 \end{bmatrix}, \quad \upsilon_1 = \begin{bmatrix} 0.3278 \\ 0.1834 \end{bmatrix},$$
$$\upsilon_2 = \begin{bmatrix} 0.1868 \\ 0.1204 \end{bmatrix}, \quad \vartheta_1 = \begin{bmatrix} 0.2226 \\ 0.2160 \end{bmatrix}, \quad \vartheta_2 = \begin{bmatrix} 0.1424 \\ 0.1521 \end{bmatrix},$$
$$\rho_1 = 5.3619, \quad \rho_2 = 2.9715, \quad \beta^* = 0.0132.$$

By $\rho_i = K_i^T \nu_i$, the observer gain matrices K_i can be obtained as follows:

$$K_1 = \begin{bmatrix} 8.9080 \\ 7.3857 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 8.5257 \\ 8.3359 \end{bmatrix}.$$

Obviously, the condition (61) is satisfied. Moreover, by (18) and (20), we get $\mu = 2.0460$ and $\tau_a^* = 2.0071$. Then by Theorem 4, we can conclude that the considered system has a positive l_- fault detection observer.

In this example, the initial states are as follows:

$$x(0) = [0.3 \quad 0.5]^T, \; x(k) = [0 \quad 0]^T, \; k = -10, -9, \ldots, 0.$$

The fault signal $f(k)$ is set up as

$$f(k) = \begin{cases} 0.1k, & 3 \le k \le 8 \\ 0, & \text{others}. \end{cases}$$

The switching signal with average dwell time $\tau_a = 3$ is shown in Fig. 1. The generated residual $r(k)$ is shown in Fig. 2. Fig. 3 shows the evolution of residual evaluation function $J_r(k)$, where the solid line is fault-free case, the dashed line is the case with the fault $f(k)$. The threshold can be determined as $J_{th} = 0.027$ for $k = 15$. The simulation results shows that $J_r(k) = 0.031 > 0.027$ when $k = 6$, which means that the fault $f(k)$ can be detected 3 times after its occurrence.

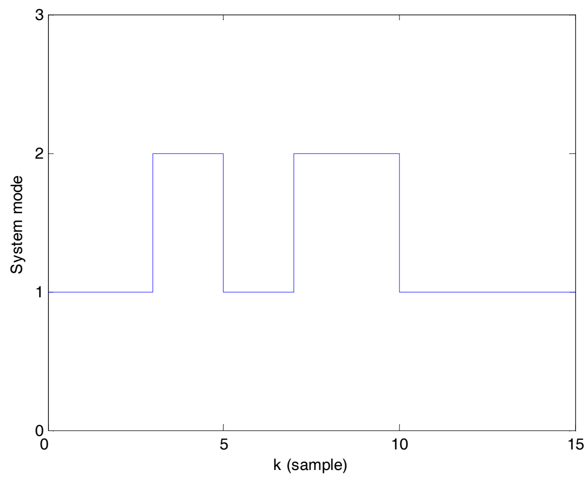

Fig. 1. Switching signal.

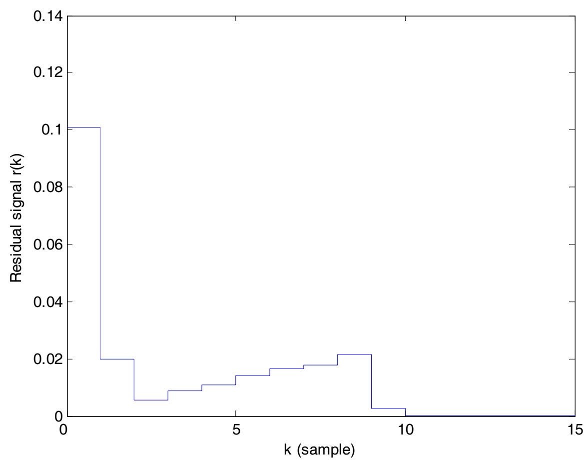

Fig. 2. Residual signal $r(k)$.

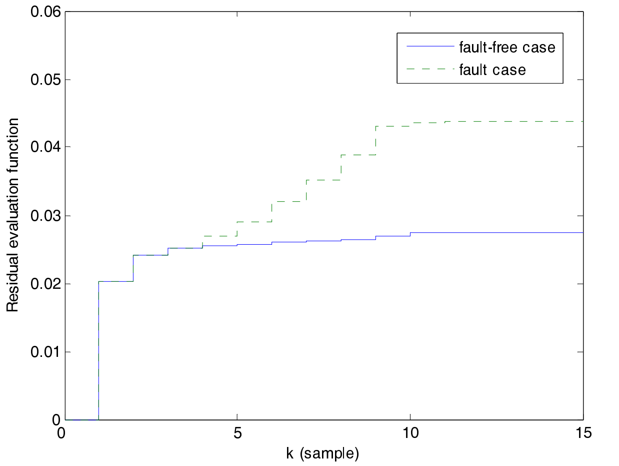

Fig. 3. Evolution of the residual evaluation function $J_r(k)$.

Example 2: Consider system (1) with parameters as follows:

Subsystem 1:

$$A_1 = \begin{bmatrix} -3.5 & 1.4 \\ 1.5 & -3.1 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0.0 \\ 0.1 & 0.0 \end{bmatrix},$$

$$C_1 = [0.12 \quad 0.13], \quad D_1 = 0.1,$$

$$E_1 = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1.5 \\ 1.6 \end{bmatrix}, \quad H_1 = 0.12;$$

Subsystem 2:

$$A_2 = \begin{bmatrix} -3.9 & 1.5 \\ 1.3 & -3.2 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.1 & 0.0 \\ 0.1 & 0.0 \end{bmatrix},$$

$$C_2 = [0.12 \quad 0.15], \quad D_2 = 0.1,$$

$$E_2 = \begin{bmatrix} 1.6 \\ 1.4 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1.4 \\ 1.5 \end{bmatrix}, \quad H_2 = 0.12,$$

and $\bar{d} = 10$, $\underline{d} = 0$, $\alpha = 1.5$, $T = 0.2$. Then, by solving optimization Problem 2, we can obtain the following solution:

$$\nu_1 = \begin{bmatrix} 0.5283 \\ 0.4963 \end{bmatrix}, \quad \nu_2 = \begin{bmatrix} 0.1356 \\ 0.1280 \end{bmatrix}, \quad \upsilon_1 = \begin{bmatrix} 0.4840 \\ 0.4206 \end{bmatrix},$$

$$\upsilon_2 = \begin{bmatrix} 0.1120 \\ 0.0573 \end{bmatrix}, \quad \vartheta_1 = \begin{bmatrix} 0.3720 \\ 0.5404 \end{bmatrix}, \quad \vartheta_2 = \begin{bmatrix} 0.0601 \\ 0.0839 \end{bmatrix},$$

$$\rho_1 = 12.4301, \ \rho_2 = 2.3487, \ \beta^* = 0.0095, \ \gamma^* = 2.1516.$$

By $\rho_i = K_i^T \nu_i$, the observer gain matrices K_i can be obtained as follows:

$$K_1 = \begin{bmatrix} 11.4987 \\ 11.7406 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 9.1578 \\ 8.6711 \end{bmatrix}.$$

Obviously, the condition (65) is satisfied. Moreover, by (18) and (20), we get $\mu = 7.7202$ and $\tau_a^* = 5.7303$. Thus by Theorem 5, we can conclude that the considered system has a mixed l_-/l_1 fault detection observer.

In this example, the external disturbance and the initial state are as follows:

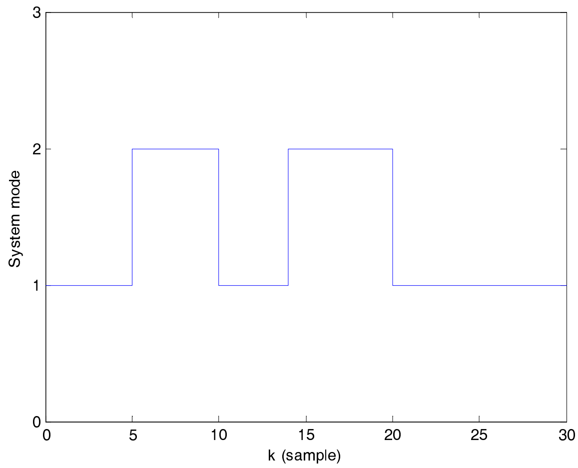

Fig. 4. Switching signal.

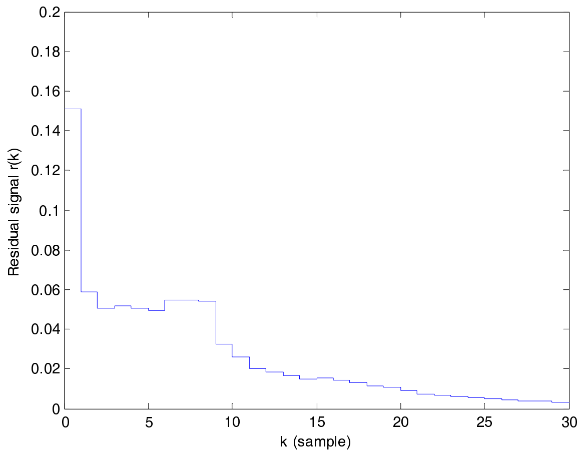

Fig. 5. Residual signal $r(k)$.

$$w(k) = 0.5e^{-0.5k}, \quad x(0) = [0.3 \quad 0.5]^T,$$

$$x(k) = [0 \quad 0]^T, \quad k = -10, -9, \ldots, 0.$$

The fault signal $f(k)$ is set up as

$$f(k) = \begin{cases} 0.1k, & 3 \le k \le 8 \\ 0, & \text{others.} \end{cases}$$

The switching signal with average dwell time $\tau_a = 6$ is shown in Fig. 4. The generated residual $r(k)$ is shown in Fig. 5. Fig. 6 shows the evolution of residual evaluation function $J_r(k)$, where the solid line is fault-free case, the dashed line is the case with the fault $f(k)$. The threshold can be determined as $J_{th} = 0.149$ for $k = 30$. The simulation results shows that $J_r(k) = 0.152 > 0.149$ when $k = 19$, which means that the fault $f(k)$ can be detected 16 times after its occurrence.

5. CONCLUSIONS

In this paper, we have presented a solution to the problem of fault detection observer for positive switched systems with time-varying delay via delta operator. The l_- index, as a new fault sensitivity measure, is proposed. By using the average dwell time approach and the

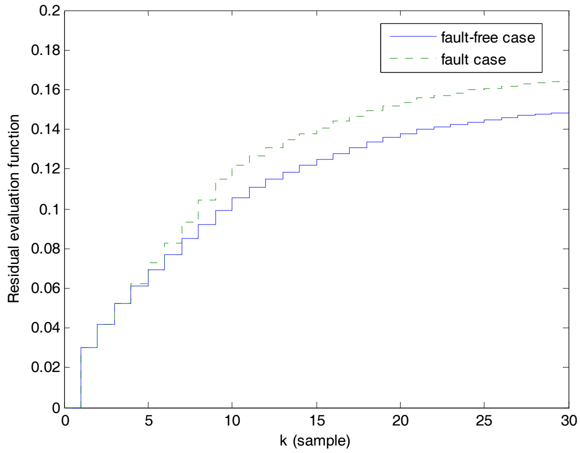

Fig. 6. Evolution of the residual evaluation function $J_r(k)$.

piecewise co-positive type Lyapunov-Krasovskii functional method, sufficient conditions for the existence of a mixed L_-/l_1 fault detection observer are given. Finally, two examples are provided to show the effectiveness and applicability of the proposed method. Our future work will focus on the design of robust L_-/l_1 fault detection filter for positive switched systems with parameter uncertainties via the delta operator approach.

REFERENCES

[1] R. Shorten, F. Wirth, and D. Leith, "A positive systems model of TCP-like congestion control: Asymptotic results," *IEEE/ACM Trans. on Networking*, vol. 14, no. 3, pp. 616-629, June 2006.

[2] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. on Automatic Control*, vol. 48, no. 6, pp. 988-1001, June 2003.

[3] X. Liu, "Constrained control of positive systems with delays," *IEEE Trans. on Automatic Control*, vol. 54, no. 7, pp. 1596-1600, July 2009.

[4] L. Benvenuti, A. Santis, and L. Farina, *Positive Systems, Lecture Notes in Control and Information Sciences*, Springer-Verlag, Berlin, 2003.

[5] M. Rami, F. Tadeo, and A. Benzaouia, "Control of constrained positive discrete systems," *Proc. of American Control Conference*, pp. 5851-5856, 2007.

[6] M. Rami and F. Tadeo, "Positive observation problem for linear discrete positive systems," *Proc. of the 45th IEEE Conf. on Decision and Control*, pp. 4729-4733, 2006.

[7] E. Fornasini and M. E. Valcher, "Linear copositive Lyapunov functions for continuous-time positive switched systems," *IEEE Trans. on Automatic Control*, vol. 55, no. 8, pp. 1933-1937, August 2010.

[8] X. Liu, "Stability analysis of switched positive systems: A switched linear co-positive Lyapunov function method," *IEEE Trans. on Circuits and Systems II: Express Briefs*, vol. 56, no. 5, pp. 414-418, May 2009.

[9] F. Knorn, O. Mason, and R. Shorten, "On linear co-positive Lyapunov functions for sets of linear positive systems," *Automatica*, vol. 45, no. 8, pp. 1943-1947, August 2009.

[10] Y. Ebihara, D. Peaucelle, and D. Arzelier, "L_1 gain analysis of linear positive systems and its application," *Proc. of the 50th IEEE Conf. on Decision and Control and European Control Conference*, pp. 4029-4034, 2011.

[11] X. Chen, J. Lam, P. Li, and Z. Shu, "L_1-induced norm and controller synthesis of positive systems," *Automatica*, vol. 49, no. 5, pp. 1377-1385, May 2013.

[12] M. Xiang and Z. Xiang, "Stability, L_1-gain and control synthesis for positive switched systems with time-varying delay," *Nonlinear Analysis: Hybrid Systems*, vol. 9, no. 1, pp. 9-17, January 2013.

[13] M. Xiang and Z. Xiang, "Reliable L_1 control of positive switched systems with time-varying delays," *Advances in Difference Equations*, vol. 2013, no. 25, pp. 1-15, January 2013.

[14] J. H. Park, O. M. Kwon, and S. M. Lee, "State estimation for neural networks of neutral-type with interval time-varying delays," *Applied Mathematics and Computation*, vol. 203, no. 1, pp. 217-223, September 2001.

[15] H. R. Karimi and H. Gao, "New delay-dependent exponential H_∞ synchronization for uncertain neural networks with mixed time delays," *IEEE Trans. on Systems, Man and Cybernetics Part B: Cybernetics*, vol. 40, no. 1, pp. 173-185, February 2010.

[16] X. Liu and C. Dang, "Stability analysis of positive switched linear systems with delays," *IEEE Trans. on Automatic Control*, vol. 56, no. 7, pp. 1684-1690, July 2011.

[17] X. Zhao, L. Zhang, and P. Shi, "Stability and a class of switched positive linear time-delay systems," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 5, pp. 578-589, March 2013.

[18] M. Xiang and Z. Xiang, "Observer design of switched positive systems with time-varying delays," *Circuits, Systems, and Signal Processing*, vol. 32, no. 5, pp. 2171-2184, October 2013.

[19] P. M. Frank and X. Ding, "Survey of robust residual generation and evaluation methods in observer-based fault detection systems," *Journal of Process Control*, vol. 7, no. 6, pp. 403-424, December 1997.

[20] F. Rambeaux, F. Hamelin, and D. Sauter, "Optimal thresholding for robust fault detection of uncertain Systems," *International Journal of Robust and Nonlinear Control*, vol. 10, no. 14, pp. 1155-1173, December 2000.

[21] S. K. Nguang, P. Shi, and S. Ding, "Fault detection for uncertain fuzzy systems: an LMI approach," *IEEE Trans. on Fuzzy Systems*, vol. 15, no. 6, pp. 1251-1262, December 2007.

[22] D. Wang, P. Shi, and W. Wang, "Robust fault detection for continuous-time switched delay systems: a linear matrix inequality approach," *IET Control Theory & Applications*, vol. 4, no. 1, pp. 100-108, January 2010.

[23] B. Jiang, D. Du, and V. Cocquempot, "Fault detection for discrete-time switched systems with interval time-varying delays," *International Journal of Control, Automation and Systems*, vol. 9, no. 2, pp. 396-401, April 2011.

[24] A. Abdo, W. Damlakhi, J. Saijai, and S. X. Ding, "Design of robust fault detection filter for hybrid switched systems," *Proc. of Conference on Control and Fault-Tolerant Systems*, pp. 161-166, 2010.

[25] M. L. Rank and H. Niemann, "Norm based design of fault detectors," *International Journal of Control*, vol. 72, no. 9, pp. 773-783, November 1999.

[26] S. X. Ding, T. Jeinsh, P. M. Frank, and E. L. Ding, "A unified approach to the optimization of fault detection systems," *International Journal of Adaptive Control and Signal Processing*, vol. 14, no. 7, pp. 725-745, November 2000.

[27] M. Hou and R. J. Patton, "An LMI approach to H_-/H_∞ fault detection observers," *Proc. of UKACC International Conference on Control*, pp. 305-310, 1996.

[28] J. Liu, J. L. Wang, and G. H. Yang, "An LMI approach to minimum sensitivity analysis with application to fault detection," *Automatica*, vol. 41, no. 11, pp. 1995-2004, November 2005.

[29] X. J. Li and G. H. Yang, "Fault detection observer design in low frequency domain for linear time-delay systems," *Acta Automatica Sinica*, vol. 35, no. 11, pp. 1465-1470, November 2009.

[30] J. L. Wang, G. H. Yang, and J. Liu, "An LMI approach to H-index and mixed H_-/H_∞ fault detection observer design," *Automatica*, vol. 43, no. 9, pp. 1656-1665, September 2007.

[31] Z. Zhang and I. M. Jaimoukha, "An optimal solution to an H_-/H_∞ fault detection problem," *Proc. of the 50th IEEE Conf. on Decision and Control and European Control Conference*, pp. 903-908, 2011.

[32] G. C. Goodwin, R. Lozano Leal, D. Q. Mayne, and R. H. Middleton, "Rapprochement between continuous and discrete model reference adaptive control," *Automatica*, vol. 22, no. 2, pp. 199-207, March 1986.

[33] S. Chen, R. H. Istepanian, J. Wu, and J. Chu, "Comparative study on optimizing closed-loop stability bounds of finite-precision controller structures with shift and delta operators," *Systems & Control Letters*, vol. 40, no. 3, pp. 153-163, July 2000.

[34] K. Premaratne, R. Salvi, N. R. Habib, and J. P. LeGall, "Delta operator formulated discrete-time approximations of continuous-time systems," *IEEE Trans. on Automatic Control*, vol. 39, no. 3, pp. 581-585, March 1994.

[35] R. H. Middleton and G. C. Goodwin, "Improved finite word length characteristics in digital control using delta operators," *IEEE Trans. on Automatic Control*, vol. 31, no. 11, pp. 1015-1021, November 1986.

[36] C. P. Neuman, "Properties of the delta operator model of dynamic physical systems," *IEEE Trans. on Systems, Man and Cybernetics*, vol. 23, no. 1, pp. 296-301, February 1993.

[37] J. Xing, R. Wang, P. Wang, and Q. Yang, "Robust control for a class of uncertain switched time delay systems using delta operator," *Proc. of 12th International Conference on Control Automation Robotics & Vision (ICARCV)*, pp. 518-523, 2012.

[38] R. Wang, J. Xing, P. Wang, and Q. Yang, "Non-fragile observer design for nonlinear time delay systems using delta operator," *UKACC International Conference on Control*, pp. 387-393, 2012.

[39] J. Zhang, Z. Han, H. Wu, and J. Huang, "Robust stabilization of discrete-time positive switched systems with uncertainties and average dwell time switching," *Circuits, Systems, and Signal Processing*, vol. 33, no. 1, pp. 71-95, January 2014.

[40] Z. Shu, J. Lam, H. Gao, B. Du, and L. Wu, "Positive observers and dynamic output-feedback controllers for interval positive linear systems," *IEEE Trans. on Circuits and Systems I: Regular Papers*, vol. 55, no. 10, pp. 3209-3222, April 2008.

[41] X. M. Sun, W. Wang, G. P. Liu, and J. Zhao, "Stability analysis for linear switched systems with time-varying delay," *IEEE Trans. on Systems, Man and Cybernetics Part B: Cybernetics*, vol. 38, no. 2, pp. 528-533, April 2008.

[42] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," *Proc. of the 38th IEEE Conf. on Decision and Control*, pp. 2655-2660, 1999.

[43] J. Liu, J. L. Wang, and G. H. Yang, "An LMI approach to worst case analysis for fault detection observers," *Proc. of the American Control Conference*, pp. 2985-2990, 2003.