

Stabilization of Positive Switched Systems with Time-varying Delays under Asynchronous Switching

Mei Xiang, Zhengrong Xiang*, and Hamid Reza Karimi

Abstract: This paper investigates the state feedback stabilization problem for a class of positive switched systems with time-varying delays under asynchronous switching in the frameworks of continuous-time and discrete-time dynamics. The so-called asynchronous switching means that the switches between the candidate controllers and system modes are asynchronous. By constructing an appropriate co-positive type Lyapunov-Krasovskii functional and further allowing the functional to increase during the running time of active subsystems, sufficient conditions are provided to guarantee the exponential stability of the resulting closed-loop systems, and the corresponding controller gain matrices and admissible switching signals are presented. Finally, two illustrative examples are given to show the effectiveness of the proposed methods.

Keywords: Asynchronous switching, average dwell time, positive systems, switched systems, time-varying delays.

1. INTRODUCTION

Positive systems, whose states and outputs are positive (at least nonnegative) whenever the initial conditions and inputs are nonnegative, spread almost all over the areas. For instance, biomedicine [1], ecology [2], industrial engineering [3], TCP-like Internet congestion control [4], and so on. Very recently, positive switched systems, which contain a family of positive subsystems and a switching signal governing the switching among them, have been highlighted by many researchers due to their broad applications in communication systems [5], formation flying [6], and systems theory [7]. It should be pointed out that studying positive switched systems is more challenging because, in order to obtain some results, we have to combine the features of positive systems [8,9] and switched systems [10-12].

In practice, time-delay phenomenon exists widely in engineering and social systems, for example, long-distance transportation systems, hydraulic pressure systems, networked control systems and so on. Time-delay often causes undesirable performance, even makes systems unstable. Therefore, many results have been

reported for time-delay systems [13-15].

On the other hand, stability and stabilization problems of positive switched systems have been extensively studied in the past years, see for instance [16-25] and the references therein. It is worth pointing out that the aforementioned results are based on an assumption that the switching instants of the controllers coincide with those of system modes. However, as pointed out in [26,27], there inevitably exists asynchronous switching in actual operation, i.e., the switching instants of the controllers exceed or lag behind those of system modes. Thus, it is necessary to consider asynchronous switching for efficient control design. Some results on non-positive switched systems under asynchronous switching have been proposed in [26-33]. For positive switched systems, the asynchronous stabilization is more challenging because the designer concerns not only the stability of the system, but also the positive property of the system. However, to the best of the authors' knowledge, the asynchronous control problem for positive switched systems has not been fully investigated. Moreover, the asynchronous switching controller design method developed in the existing literature cannot be directly applied to positive switched systems, and this constitutes the main motivation of the present study.

In this paper, we are interested in studying the asynchronous stabilization problem for positive switched systems with time-varying delays. The main contribution of this paper is that: 1) by constructing a novel co-positive type Lyapunov-Krasovskii functional which is allowed to increase during the running time of active subsystems with the mismatched controller, sufficient conditions for the existence of the stabilizing controllers for the underlying systems in both continuous-time and discrete-time contexts are developed; 2) The inequalities derived in this paper are decoupled and can be easily solved.

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Mei Xiang and Zhengrong Xiang are with the School of Automation, Nanjing University of Science and Technology, Nanjing, 210094, P. R. China (e-mails: xiangmei1009@163.com, xiangzr@mail.njust.edu.cn).

Hamid Reza Karimi is with the Department of Engineering, Faculty of Engineering and Science, University of Agder, N-4898 Grimstad, Norway (e-mails: hamid.r.karimi@uia.no).

* Corresponding author.

The remainder of this paper is organized as follows. In Section 2, problem statements and necessary lemmas are given. In Section 3, based on the average dwell time approach, the stability and stabilization problems for both continuous-time and discrete-time positive switched systems with time-varying delays are developed. Two numerical examples are provided to show the effectiveness of the proposed approaches in Section 4. Concluding remarks are given in Section 5.

Notations: In this paper, $A \succ \succ 0$ ($A \succ \succ 0$) means that all elements of the matrix A are positive (nonnegative). $A \succ \succ B$ ($A \succ \succ B$) means that $A - B \succ \succ 0$ ($A - B \succ \succ 0$). The notation $X \geq Y$ ($X > Y$) means that $X - Y$ is positive semi-definite (positive definite). R^n is an n -dimensional real vector space and R_+^n is a set of vectors whose elements are all positive; $R^{n \times s}$ is the set of all real matrices of $(n \times s)$ -dimension. A^T denotes the transpose of the matrix A . $\|x\| = \sum_{s=1}^n |x_s|$, where x_s is the s -th element of $x \in R^n$. The set of all nonnegative integers is represented by Z^+ .

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following continuous-time switched linear system with time-varying delay:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d(t)) + B_{\sigma(t)}u(t), \\ x(t_0 + \theta) = \varphi(\theta), \theta \in [-\tau, 0], \end{cases} \quad (1)$$

where $x(t) \in R^n$ and $u(t) \in R^m$ denote the system state and the control input, respectively; $\sigma(t): [0, \infty) \rightarrow \underline{N} = \{1, 2, \dots, N\}$ is a piecewise constant function of time, called a switching signal; N is the number of subsystems; Corresponding to the switching signal, we have the switching sequence $\Sigma = \{(t_0, \sigma(t_0)), (t_1, \sigma(t_1)), \dots, (t_q, \sigma(t_q)), \dots\}$, where t_0 is the initial instant and t_q denotes the q -th switching instant; A_i, A_{di} and $B_i, i \in \underline{N}$, are known constant matrices with appropriate dimensions; $d(t)$ denotes the time-varying delay satisfying $0 \leq d(t) \leq \tau$ and $\dot{d}(t) \leq d < 1$ for known constants τ and d ; $\varphi(\theta)$ is a continuous vector-valued initial function defined on the interval $[-\tau, 0]$.

Definition 1: System (1) is said to be a positive switched system if for any switching signals $\sigma(t)$ and any initial conditions $\varphi(\theta) \succeq 0, \theta \in [-\tau, 0], x(t) \succeq 0$ for all $t \geq t_0$.

Definition 2 [34]: A is called a Metzler matrix, if the off-diagonal entries of the matrix A are non-negative.

Definition 3 [35]: System (1) is said to be exponentially stable under switching signal $\sigma(t)$, if for initial conditions $x(t_0 + \theta) = \varphi(\theta), \theta \in [-\tau, 0]$, there exist constants $\zeta > 0$ and $\rho > 0$ such that the solution of the system satisfies

$$\|x(t)\| \leq \zeta \|x(t_0)\|_C e^{-\rho(t-t_0)}, \quad \forall t \geq t_0,$$

$$\text{where } \|x(t_0)\|_C = \sup_{-\tau \leq \theta \leq 0} \|x(\theta + t_0)\|.$$

Definition 4 [36]: For a switching signal $\sigma(t)$ and any $T_2 > T_1 \geq t_0$, let $N_\sigma(T_1, T_2)$ be the switching number of $\sigma(t)$ over the interval $[T_1, T_2)$. If $N_\sigma(T_1, T_2) \leq N_0 + (T_2 - T_1)/T_a$ holds for $T_a > 0$ and $N_0 \geq 0$, then T_a is called the average dwell time and N_0 is called the chattering bound.

Remark 1: As stated in [36], $N_0 = 0$ implies that $\sigma(t)$ cannot switch at all on any interval of length smaller than T_a . In general, if we discard the first N_0 switches (more precisely, the smallest integer greater than N_0), the average time between consecutive switches is at least T_a . Without loss of generality, we choose $N_0 = 0$ in this paper.

A state feedback controller $u(t) = K_{\sigma(t)}x(t)$ for system (1) is considered, where $K_i (i \in \underline{N})$ are the controller gain matrices to be determined. However, due to the existence of asynchronous switching, the switching instants of the controller do not coincide exactly with those of the system mode. Without loss of generality, the asynchronous switching considered here means that the switching instants of controller lag behind those of the system mode. Then, the real control input will become

$$u(t) = K_{\sigma(t-\Delta_q)}x(t), \quad \forall t \in [t_q, t_{q+1}), \quad (2)$$

where $\Delta_0 = 0$, and $0 \leq \Delta_q < t_{q+1} - t_q$ ($q = 1, 2, \dots$) represents the switching delay.

Remark 2: The delay time $0 \leq \Delta_q < t_{q+1} - t_q$ implies that there exists a period that the system mode and the controller operate synchronously during the interval $[t_q, t_{q+1})$.

Denote $\sigma'(t)$ as the real switching signal of the controller (4), that is, $\sigma'(t) = \sigma(t - \Delta_q)$. Let the i -th subsystem be activated at the switching instant t_q and the j -th subsystem be activated at the switching instant t_{q+1} , then the corresponding switching controllers are activated at the switching instants $t_q + \Delta_q$ and $t_{q+1} + \Delta_{q+1}$, respectively. Applying the controller (2) to system (1), the resulting closed-loop system is given by:

$$\begin{cases} \dot{x}(t) = \bar{A}_i x(t) + A_{di} x(t-d(t)), t \in [t_q + \Delta_q, t_{q+1}) \\ \dot{x}(t) = \bar{A}_{i,j} x(t) + A_{dj} x(t-d(t)), t \in [t_{q+1}, t_{q+1} + \Delta_{q+1}), \end{cases} \quad (3)$$

where $\bar{A}_i = A_i + B_i K_i$ and $\bar{A}_{i,j} = A_j + B_j K_i$.

Lemma 1: System (3) is positive if and only if \bar{A}_i and $\bar{A}_{i,j}$ are Metzler matrices, and $A_{di} \succeq 0, \forall i, j \in \underline{N}, i \neq j$.

Proof: This Lemma can be directly obtained from Proposition 1 in [16].

Furthermore, we consider the following discrete-time switched system with time-varying delay:

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k-d(k)) + B_{\sigma(k)}u(k), \\ x(k+k_0) = \varphi(k), k = -\bar{d}, -\bar{d}+1, \dots, -1, 0, \end{cases} \quad (4)$$

where $x(k) \in R^n$ and $u(k) \in R^m$ denote the system state and the control input, respectively; $\sigma(k): Z^+ \rightarrow \underline{N} = \{1, 2, \dots, N\}$ is the switching signal; $\sigma(k) = i \in \underline{N}$ means that the i -th subsystem is active; N is the number of subsystems; Corresponding to the switching signal, we have the switching sequence $\bar{\Sigma} = \{(k_0, \sigma(k_0)), (k_1, \sigma(k_1)), \dots, (k_l, \sigma(k_l)), \dots\}$, where $k_0 = 0$ is the initial instant and k_l denotes the l -th switching instant; A_i, A_{di} and $B_i, i \in \underline{N}$, are constant matrices with appropriate dimensions; $d(k)$ denotes the time-varying delay satisfying $0 < \underline{d} \leq d(k) \leq \bar{d}$ for known constants \underline{d} and \bar{d} ; $\varphi(k)$ is a discrete vector-valued initial function.

Definition 5: System (4) is said to be positive if, for any initial conditions $\varphi(k) \succeq 0, k = -\bar{d}, -\bar{d} + 1, \dots, -1, 0$, and any switching signals $\sigma(k), x(k) \succeq 0$ holds for all $k \geq k_0$.

Definition 6 [36]: System (4) is said to be exponentially stable under switching signal $\sigma(k)$ if for initial conditions $x(k + k_0) = \varphi(k), k = -\bar{d}, -\bar{d} + 1, \dots, -1, 0$, there exist constants $\varepsilon > 0$ and $\xi > 0$ such that $\|x(k)\| \leq \varepsilon e^{-\xi(k-k_0)} \|x(k_0)\|_C, \forall k \geq k_0$, where $\|x(k_0)\|_C = \sup_{-\bar{d} \leq \theta \leq 0} \|x(k_0 + \theta)\|$.

Definition 7 [37]: For any $k_2 > k_1 \geq k_0$, let $N_\sigma(k_1, k_2)$ be the switching number of $\sigma(k)$ over the interval $[k_1, k_2]$. If $N_\sigma(k_1, k_2) \leq N_0 + (k_2 - k_1) / \tau_a$ holds for $\tau_a > 0$ and $N_0 \geq 0$, then τ_a is called the average dwell time and N_0 is called the chattering bound.

The asynchronous state feedback controller for system (4) can be written as

$$u(k) = K_{\sigma(k-\Delta_l)} x(k), \quad \forall k \in [k_l, k_{l+1}), \quad (5)$$

where $\Delta_0 = 0$ and $0 \leq \Delta_l < k_{l+1} - k_l (l = 1, 2, \dots)$ represents the switching delay.

Denote $\sigma'(k)$ as the real switching signal of the controller (5), then $\sigma'(k) = \sigma(k - \Delta_l)$. Letting $\sigma(k_l) = i$ and $\sigma(k_{l+1}) = j$, we have $\sigma(k_l + \Delta_l) = i, \sigma(k_{l+1} + \Delta_{l+1}) = j$, and the resulting closed-loop system is given by

$$\begin{cases} x(k+1) = \bar{A}_i x(k) + A_{di} x(k-d(k)), k \in [k_l + \Delta_l, k_{l+1}) \\ x(k+1) = \bar{A}_{i,j} x(k) + A_{dj} x(k-d(k)), k \in [k_{l+1}, k_{l+1} + \Delta_{l+1}), \end{cases} \quad (6)$$

where $\bar{A}_i = A_i + B_i K_i$ and $\bar{A}_{i,j} = A_j + B_j K_i$.

The following lemma is an extension of Lemma 3 in [38].

Lemma 2: System (6) is positive if and only if $\bar{A}_i \succeq 0, \bar{A}_{i,j} \succeq 0$ and $A_{di} \succeq 0, \forall i, j \in \underline{N}, i \neq j$.

3. MAIN RESULTS

3.1. Continuous-time case

Before giving the main results, we first present the following lemmas which will be essential for our later development.

Lemma 3: Consider the following positive system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d(t)), \\ x(t_0 + \theta) = \varphi(\theta), \theta \in [-\tau, 0], \end{cases} \quad (7)$$

where A is a Metzler constant matrix and $A_d \succeq 0$ is a constant matrix; $d(t)$ denotes the time-varying delay satisfying $0 \leq d(t) \leq \tau$ and $\dot{d}(t) \leq d < 1$; $\varphi(\theta) \succeq 0, \theta \in [-\tau, 0]$. For a given positive scalar α , if there exist vectors $v, v \in R_+^n$, such that

$$A^T v + \alpha v + v \preceq 0, \quad (8)$$

$$A_d^T v - (1-d)e^{-\alpha\tau} v \preceq 0, \quad (9)$$

then along the trajectory of system (7), we have

$$V(t) \leq e^{-\alpha(t-t_0)} V(t_0). \quad (10)$$

Proof: Applying Lemma 2 in [39,40], and letting $\zeta = 0$ and $\vartheta = 0$, the lemma can be directly obtained.

Lemma 4: Consider system (7), for a given positive scalar β , if there exist vectors $v, v \in R_+^n$, such that

$$A^T v - \beta v + v \preceq 0, \quad (11)$$

$$A_d^T v - (1-d)v \preceq 0, \quad (12)$$

then along the trajectory of system (7), we have

$$V(t) \leq e^{\beta(t-t_0)} V(t_0). \quad (13)$$

Proof: Applying Lemma 3 in [39], and letting $\zeta = 0$ and $\vartheta = 0$, this lemma can be directly obtained.

Remark 3: Lemmas 3 and 4 provide the decay estimation and growth estimation of the co-positive type Lyapunov-Krasovskii functional, respectively. They will be used for the asynchronous controller design of positive switched system (1).

The following theorem presents a sufficient condition for the existence of an asynchronous stabilizing controller for positive switched system (1).

Theorem 1: Consider positive switched system (1). For given positive scalars $\alpha, \beta, \lambda_{i,j}$, and vectors $\tilde{v}_i \in R_+^m$, if there exist vectors $v_i, v_i, v_{i,j}, v_{i,j} \in R_+^n$, and $h_i \in R^n$, such that, $\forall (i, j) \in \underline{N} \times \underline{N}, i \neq j$,

$$A_i^T v_i + \alpha v_i + h_i + v_i \preceq 0, \quad (14)$$

$$A_{di}^T v_i - (1-d)e^{-\alpha\tau} v_i \preceq 0, \quad (15)$$

$$\lambda_{i,j} \tilde{v}_i^T B_i^T v_i \leq \tilde{v}_i^T B_j^T v_{i,j}, \quad h_i \preceq 0, \quad (16)$$

$$\tilde{v}_i^T B_i^T v_i > 0, \quad (17)$$

$$A_j^T v_{i,j} - \beta v_{i,j} + \lambda_{i,j} h_i + v_{i,j} \preceq 0, \quad (18)$$

$$A_{dj}^T v_{i,j} - (1-d)v_{i,j} \preceq 0, \quad (19)$$

$$\mathbf{I}_g^T (\tilde{v}_i^T B_i^T v_i A_i^T + h_i \tilde{v}_i^T B_i^T) \mathbf{I}_l \geq 0, \quad g \neq l, \quad (20)$$

$$\mathbf{I}_g^T (\tilde{v}_i^T B_i^T v_i A_j^T + h_i \tilde{v}_i^T B_j^T) \mathbf{I}_l \geq 0, \quad g \neq l, \quad (21)$$

where $\mathbf{I}_g = [\overbrace{0 \dots 0}^{g-1} \ 1 \ \overbrace{0 \dots 0}^{n-g}]$ and $\mathbf{I}_l = [\overbrace{0 \dots 0}^{l-1} \ 1 \ \overbrace{0 \dots 0}^{n-l}]$, then

under the controller (2) with

$$K_i = \frac{1}{\tilde{v}_i^T B_i^T v_i} \tilde{v}_i h_i^T, \quad (22)$$

the closed-loop system (3) is exponentially stable for any switching signal $\sigma(t)$ with average dwell time satisfying

$$T_a > T_a^* = (\Delta(\alpha + \beta) + \ln(\mu_1 \mu_2 \mu)) / \alpha, \quad (23)$$

where $\Delta = \max\{\Delta_q\}$, $q = 1, 2, \dots$, $\mu = e^{\tau(\alpha + \beta)}$, μ_1 and μ_2 satisfy

$$\begin{aligned} v_j &\preceq \mu_1 v_{i,j}, \quad v_j \preceq \mu_1 v_{i,j}, \quad v_{i,j} \preceq \mu_2 \mu v_i, \\ v_{i,j} &\preceq \mu_2 v_i, \quad \forall (i, j) \in \underline{N} \times \underline{N}, \quad i \neq j. \end{aligned} \quad (24)$$

Proof: When $t \in [t_q + \Delta_q, t_{q+1})$, the closed-loop system (3) can be written as

$$\dot{x}(t) = \bar{A}_i x(t) + A_{di} x(t-d(t)). \quad (25)$$

Equation (20) implies that $\tilde{v}_i^T B_i^T v_i A_i^T + h_i \tilde{v}_i^T B_i^T$ is a Metzler matrix. Then it follows from (17) that

$$A_i^T + \frac{1}{\tilde{v}_i^T B_i^T v_i} h_i \tilde{v}_i^T B_i^T$$

is a Metzler matrix. From (22), one has $\bar{A}_i = A_i + B_i K_i$ is a Metzler matrix.

Consider the following co-positive type Lyapunov-Krasovskii functional for system (25)

$$V_i(t) = x^T(t) v_i + \int_{t-d(t)}^t e^{\alpha(-t+s)} x^T(s) v_i ds. \quad (26)$$

Using (22), we have $h_i = K_i^T B_i^T v_i$, then according to Lemma 3, it can be obtained from (14)-(15) that, for $t \in [t_q + \Delta_q, t_{q+1})$,

$$V_i(t) \leq e^{-\alpha(t-t_q-\Delta_q)} V_i(t_q + \Delta_q). \quad (27)$$

When $t \in [t_{q+1}, t_{q+1} + \Delta_{q+1})$, the closed-loop system can be written as

$$\dot{x}(t) = \bar{A}_{i,j} x(t) + A_{dj} x(t-d(t)). \quad (28)$$

Similarly, one can obtain from (16), (17), and (21) that $\bar{A}_{i,j} = A_j + B_j K_i$ is a Metzler matrix.

Choose a co-positive type Lyapunov-Krasovskii functional for system (28) as follows:

$$V_{i,j}(x(t)) = x^T(t) v_{i,j} + \int_{t-d(t)}^t e^{\beta(t-s)} x^T(s) v_{i,j} ds. \quad (29)$$

From (22), it is easy to get that

$$K_j^T B_j^T v_{i,j} = h_i \frac{\tilde{v}_i^T B_j^T v_{i,j}}{\tilde{v}_i^T B_j^T v_i}.$$

It follows from (16)-(18) that

$$A_j^T v_{i,j} - \beta v_{i,j} + K_j^T B_j^T v_{i,j} + v_{i,j} \preceq 0. \quad (30)$$

Using (19), (30), and applying Lemma 4, one obtains

$$V_{i,j}(t) \leq e^{\beta(t-t_{q+1})} V_{i,j}(t_{q+1}). \quad (31)$$

Denote t_1, \dots, t_f as the switching instants on the interval $[t_0, t)$, and consider the following Lyapunov-Krasovskii functional candidate for system (3): When $t \in [t_\kappa + \Delta_\kappa, t_{\kappa+1})$, $\kappa = 0, 1, 2, \dots, f$,

$$V_{\sigma(t)}(t) = x^T(t) v_{\sigma(t)} + \int_{t-d(t)}^t e^{\alpha(-t+s)} x^T(s) v_{\sigma(t)} ds.$$

When $t \in [t_\kappa, t_\kappa + \Delta_\kappa)$, $\kappa = 1, 2, \dots, f$,

$$\begin{aligned} V_{\sigma'(t), \sigma(t)}(t) &= x^T(t) v_{\sigma'(t), \sigma(t)} \\ &\quad + \int_{t-d(t)}^t e^{\beta(t-s)} x^T(s) v_{\sigma'(t), \sigma(t)} ds. \end{aligned}$$

From (24), (27) and (31), for $t \in [t_f + \Delta_f, t_{f+1})$, we obtain that

$$\begin{aligned} V_{\sigma(t)}(t) &\leq e^{-\alpha(t-t_f-\Delta_f)} V_{\sigma(t_f+\Delta_f)}(t_f + \Delta_f) \\ &\leq \mu_1 e^{-\alpha(t-t_f-\Delta_f)} e^{\beta \Delta_f} V_{\sigma'(t_f), \sigma(t_f)}(t_f) \\ &\leq \mu_1 \mu_2 \mu e^{-\alpha(t-t_f-\Delta_f)} e^{\beta \Delta_f} V_{\sigma(t_f^-)}(t_f^-) \\ &\leq \mu_1 \mu_2 \mu e^{-\alpha(t-t_f-\Delta)} e^{\beta \Delta} V_{\sigma(t_f^-)}(t_f^-) \\ &\leq \dots \\ &\leq (\mu_1 \mu_2 \mu)^{(t-t_0)/T_a} e^{-\alpha(t-t_0)} e^{\Delta(\alpha+\beta)(t-t_0)/T_a} \\ &\quad \times V_{\sigma(t_0)}(t_0). \end{aligned} \quad (32)$$

From (23) and (32), one has

$$V_{\sigma(t)}(t) \leq e^{-\rho(t-t_0)} V_{\sigma(t_0)}(t_0), \quad (33)$$

where $\rho = -(\Delta(\alpha + \beta) + \ln(\mu_1 \mu_2 \mu)) / T_a + \alpha > 0$.

Thus there exists a constant $\zeta > 0$ such that the system state satisfies

$$\|x(t)\| \leq \zeta e^{-\rho(t-t_0)} \|x(t_0)\|_C.$$

The proof is completed.

Remark 4: From Theorem 1, it can be seen that a smaller α and a larger β will be favorable to the feasibility of inequalities (14)-(21). In view of this, we put forward the following procedure for determining the controller gain matrices K_i and the desired T_a^* .

Algorithm 1

Step 1: Input the system matrices A_i , A_{di} and B_i , and given $\lambda_{i,j} > 0$ and vectors $\tilde{v}_i \in R_+^m$.

Step 2: Take α and β (For the first time, we can choose a larger $\alpha > 0$ and a smaller $\beta > 0$), and solve (14)-(21).

Step 3: If there is a feasible solution, go to Step 5.

Step 4: If there inequalities are infeasible, decrease α and increase β appropriately, and go to Step 2.

Step 5: Compute the controller gain matrices K_i by (22), and calculate μ_1 and μ_2 by the following optimization approach

$$\begin{aligned} &\text{minimize } \mu_1 + \mu_2 \\ &\text{s.t. (24).} \end{aligned}$$

Step 6: Calculate T_a^* by (23).

When $\Delta = 0$, i.e., the controllers are switched synchronously with the subsystems, we can get the following result.

Corollary 1: Consider positive switched system (1). For given positive scalars α, β , and vectors $\tilde{v}_i \in R_+^m$, if there exist vectors $v_i, \nu_i \in R_+^n$ and $h_i \in R^n$, such that (14), (15), (17), and (20) hold, then under the state feedback controller $u(t) = K_{\sigma(t)}x(t)$ with (22), the resulting closed-loop system (3) is exponentially stable for any switching signal $\sigma(t)$ with average dwell time satisfying $T_a > T_a^* = \ln \mu_1 / \alpha$, where $\mu_1 \geq 1$ satisfies $v_i \leq \mu_1 \nu_i, \nu_i \leq \mu_1 \nu_j, \forall (i, j) \in \underline{N} \times \underline{N}, i \neq j$.

In what follows, we will give the asynchronous switching controller design scheme for discrete-time positive switched system (4).

3.2. Discrete-time case

Before giving the result, we first present the following lemma.

Lemma 5: Consider the following positive system

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d(k)), \\ x(k+k_0) = \varphi(k), \quad k = -\bar{d}, -\bar{d}+1, \dots, -1, 0, \end{cases} \quad (34)$$

where $A \succeq 0$ and $A_d \succeq 0$ are constant matrices with appropriate dimensions; $d(k)$ denotes the time-varying delay satisfying $0 < \underline{d} \leq d(k) \leq \bar{d}$; $\varphi(k) \succeq 0, k = -\bar{d}, -\bar{d}+1, \dots, -1, 0$. For a given positive scalar η , if there exist vectors $v, \nu \in R_+^n$, such that

$$A^T v - e^{-\eta} v + (\bar{d} - \underline{d} + 1) \nu \leq 0, \quad (35)$$

$$A_d^T v - e^{-\eta \bar{d}} \nu \leq 0, \quad (36)$$

then along the trajectory of system (34), we have

$$V(k) \leq e^{-\eta(k-k_0)} V(k_0).$$

Proof: Consider the following co-positive type Lyapunov-Krasovskii functional:

$$\begin{aligned} V(k) = & x^T(k)v + \sum_{s=k-d(k)}^{k-1} e^{\eta(-k+s+1)} x^T(s)\nu \\ & + \sum_{\theta=-\bar{d}+1}^{-d} \sum_{s=k+\theta}^{k-1} e^{\eta(-k+s+1)} x^T(s)\nu. \end{aligned}$$

Along the trajectory of (34), we have

$$\begin{aligned} V(k+1) - e^{-\eta} V(k) \leq & x^T(k)(A^T v - e^{-\eta} v + (\bar{d} - \underline{d} + 1)\nu) \\ & + x^T(k-d(k))(A_d^T v - e^{-\eta \bar{d}} \nu). \end{aligned} \quad (37)$$

It follows from (35)-(36) that

$$V(k) \leq e^{-\eta(k-k_0)} V(k_0).$$

The proof is completed.

The following theorem presents a sufficient condition for the existence of an asynchronous stabilizing controller for system (4).

Theorem 2: Consider positive switched system (4).

For given positive scalars $\eta, \gamma, \lambda_{i,j}$, and vectors $\tilde{v}_i \in R_+^m$, if there exist vectors $v_i, \nu_i, \nu_{i,j}, \nu_{i,j} \in R_+^n$, and $h_i \in R^n$, such that, $\forall (i, j) \in \underline{N} \times \underline{N}, i \neq j$,

$$A_i^T v_i + h_i - e^{-\eta} v_i + (\bar{d} - \underline{d} + 1) \nu_i \leq 0, \quad (38)$$

$$A_{di}^T v_i - e^{-\eta \bar{d}} \nu_i \leq 0, \quad (39)$$

$$\lambda_{i,j} \tilde{v}_i^T B_i^T v_i \leq \tilde{v}_i^T B_j^T \nu_{i,j}, \quad h_i \leq 0, \quad (40)$$

$$\tilde{v}_i^T B_i^T v_i > 0, \quad (41)$$

$$A_j^T \nu_{i,j} - e^{-\gamma} \nu_{i,j} + \lambda_{i,j} h_i + (\bar{d} - \underline{d} + 1) \nu_{i,j} \leq 0, \quad (42)$$

$$A_{dj}^T \nu_{i,j} - e^{-\gamma \bar{d}} \nu_{i,j} \leq 0, \quad (43)$$

$$\tilde{v}_i^T B_i^T v_i A_i^T + h_i \tilde{v}_i^T B_i^T \geq 0, \quad (44)$$

$$\tilde{v}_i^T B_i^T v_i A_j^T + h_i \tilde{v}_i^T B_j^T \geq 0, \quad (45)$$

then under the controller (5) with

$$K_i = \frac{1}{\tilde{v}_i^T B_i^T v_i} \tilde{v}_i h_i^T, \quad (46)$$

the closed-loop system (6) is exponentially stable for any switching signal $\sigma(t)$ with average dwell time satisfying

$$\tau_a > \tau_a^* = (\Delta'(\eta + \gamma) + \ln(\mu_1 \mu_2 \mu)) / \eta, \quad (47)$$

where $\Delta' = \max\{\Delta_l\}, l = 1, 2, \dots, \mu_1, \mu_2$ and μ satisfy

$$\begin{aligned} \mu = & e^{\bar{d}(\eta + \gamma)}, \quad v_j \leq \mu_1 \nu_{i,j}, \quad \nu_j \leq \mu_1 \nu_{i,j}, \\ & \nu_{i,j} \leq \mu_2 \mu \nu_i, \quad \nu_{i,j} \leq \mu_2 \nu_i, \\ & \forall (i, j) \in \underline{N} \times \underline{N}, \quad i \neq j. \end{aligned} \quad (48)$$

Proof: Similarly, one can obtain from (41), (44)-(46) that $A_i + B_i K_i \succeq 0$ and $A_j + B_j K_j \succeq 0$.

Denote k_1, \dots, k_w the switching instants during the interval $[k_0, k)$, and consider the following Lyapunov-Krasovskii functional candidates for system (6): When $k \in [k_z + \Delta_z, k_{z+1}), z = 0, 1, 2, \dots, w$,

$$\begin{aligned} V_{\sigma(k)}(k) = & x^T(k)v_{\sigma(k)} + \sum_{s=k-d(k)}^{k-1} e^{\eta(-k+s+1)} x^T(s)\nu_{\sigma(k)} \\ & + \sum_{\theta=-\bar{d}+1}^{-d} \sum_{s=k+\theta}^{k-1} e^{\eta(-k+s+1)} x^T(s)\nu_{\sigma(k)}. \end{aligned}$$

When $k \in [k_z, k_z + \Delta_z), z = 1, 2, \dots, w$,

$$\begin{aligned} V_{\sigma'(k), \sigma(k)}(k) = & x^T(k)v_{\sigma'(k), \sigma(k)} \\ & + \sum_{s=k-d(k)}^{k-1} e^{\gamma(k-s-1)} x^T(s)\nu_{\sigma'(k), \sigma(k)} \\ & + \sum_{\theta=-\bar{d}+1}^{-d} \sum_{s=k+\theta}^{k-1} e^{\gamma(k-s-1)} x^T(s)\nu_{\sigma'(k), \sigma(k)}. \end{aligned}$$

Then similar to the proof of Theorem 1, we can obtain from (38)-(48) and Lemma 5 that

$$V_{\sigma(k)}(k) \leq e^{-\xi(k-k_0)} V_{\sigma(k_0)}(k_0), \quad (49)$$

where $\xi = \eta - (\ln(\mu_1 \mu_2 \mu) + \Delta'(\eta + \gamma)) / \tau_a > 0$.

It follows that there exists a constant $\varepsilon > 0$ such that the system state satisfies

$$\|x(k)\| \leq \varepsilon e^{-\xi(k-k_0)} \|x(k_0)\|_C.$$

The proof is completed.

When $\Delta = 0$, we can get the following result.

Corollary 2: Consider positive switched system (4).

For given positive scalars η, γ , and vectors $\tilde{v}_i \in R_+^m$, if there exist vectors $v_i, \nu_i \in R_+^n$ and $h_i \in R^n$, such that (38), (39), (41), and (44) hold, then under the state feedback controller $u(k) = K_{\sigma(k)} x(k)$ with (46), the closed-loop system (6) is exponentially stable for any switching signal $\sigma(k)$ with average dwell time satisfying $\tau_a > \tau_a^* = \ln \mu_1 / \eta$, where $\mu_1 \geq 1$ satisfies $v_i \leq \mu_1 \nu_j, \nu_i \leq \mu_1 \nu_j, \forall i, j \in \underline{N}, i \neq j$.

Based on Theorem 2, we give the following procedure to obtain the controller gain matrices K_i and the desired τ_a^* .

Algorithm 2

Step 1: Input the system matrices A_i, A_{di} and B_i , and given $\lambda_{i,j} > 0$ and vectors $\tilde{v}_i \in R_+^m$.

Step 2: Take η and γ (For the first time, we can choose a larger $\eta > 0$ and a smaller $\gamma > 0$), and solve (38)-(45).

Step 3: If there is a feasible solution, and go to Step 2; otherwise, go to Step 5.

Step 4: If there inequalities are infeasible, decrease η and increase γ appropriately, and go to Step 2.

Step 5: Compute the controller gain matrices K_i by (46), and calculate μ_1 and μ_2 by the following optimization approach

$$\begin{aligned} & \text{minimize} \quad \mu_1 + \mu_2 \\ & \text{s.t.} \quad (48). \end{aligned}$$

Step 6: Calculate τ_a^* by (47).

4. NUMERICAL EXAMPLES

In this section, two numerical examples will be presented to demonstrate the validity of our developed theoretical results.

Example 1: Consider system (1) with the following parameters

$$A_1 = \begin{bmatrix} -3.5 & 3 \\ 5.8 & -4.8 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.18 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -3 & 3.3 \\ 4.3 & -5 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix},$$

$$\tau = 0.1, \quad d = 0.1, \quad \Delta = 0.3.$$

Take $\alpha = 1.8, \beta = 1.5, \tilde{v}_1 = 0.4, \tilde{v}_2 = 0.6, \lambda_{1,2} = 0.4$ and $\lambda_{2,1} = 1$, then solving (14)-(21) in Theorem 1 gives rise to

$$v_1 = \begin{bmatrix} 18.5311 \\ 13.1778 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 10.1169 \\ 11.0389 \end{bmatrix}, \quad \nu_1 = \begin{bmatrix} 16.3290 \\ 13.7281 \end{bmatrix},$$

$$\nu_2 = \begin{bmatrix} 15.2088 \\ 14.5994 \end{bmatrix}, \quad \nu_{1,2} = \begin{bmatrix} 20.1997 \\ 15.6530 \end{bmatrix}, \quad \nu_{2,1} = \begin{bmatrix} 19.3433 \\ 16.2392 \end{bmatrix},$$

$$\nu_{2,1} = \begin{bmatrix} 28.6204 \\ 29.3711 \end{bmatrix}, \quad \nu_{1,2} = \begin{bmatrix} 27.0522 \\ 25.2969 \end{bmatrix},$$

$$h_1 = \begin{bmatrix} -71.6062 \\ -38.0025 \end{bmatrix}, \quad h_2 = \begin{bmatrix} -59.5055 \\ -21.6463 \end{bmatrix}.$$

By (22), the gain matrices $K_i, i=1,2$, can be obtained

$$K_1 = [-6.0167 \quad -3.1932], \quad K_2 = [-6.3426 \quad -2.3073].$$

Furthermore, according to (24), we have $\mu_1 = 0.9580, \mu_2 = 2.9698$ and $\mu = 1.3910$. From (23), it can be obtained that $T_a^* = 1.0979$. Choosing $T_a = 1.5$, the simulation results are shown in Figs. 1-2, where the initial conditions are $x(0) = [0.2 \quad 0.1]^T$ and $x(\theta) = [0 \quad 0]^T, \theta \in [-0.1, 0)$.

From Figs. 1-2, we can see that the resulting closed-loop system is positive and exponentially stable, which indicates that the proposed method is effective.

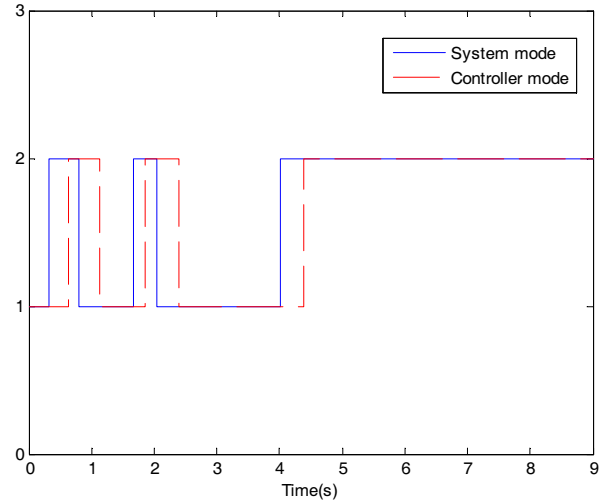


Fig. 1. Switching signal in Example 1.

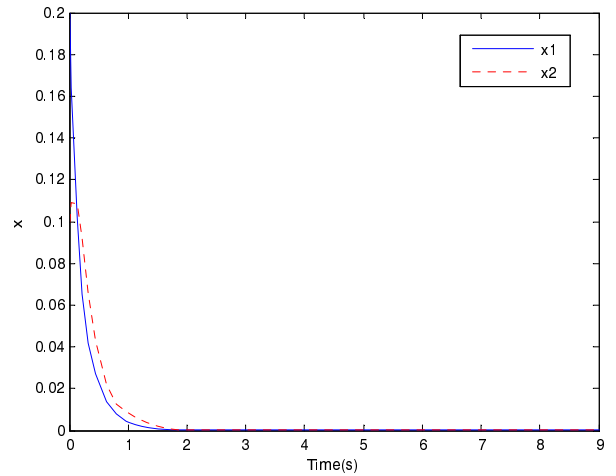


Fig. 2. State response of the closed-loop system in Example 1.

Example 2: Consider system (4) with the following parameters

$$A_1 = \begin{bmatrix} 0.6 & 0.48 \\ 0.6 & 0.5 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.5 & 0.45 \\ 0.62 & 0.54 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.25 \\ 0.32 \end{bmatrix},$$

$$\underline{d} = 1, \quad \bar{d} = 2, \quad \Delta' = 1.$$

Take $\eta = 0.32$, $\gamma = 0.02$, $\tilde{v}_1 = 1$, $\tilde{v}_2 = 1$, $\lambda_{1,2} = 1.2$ and $\lambda_{2,1} = 1.2$, then solving (38)-(45) in Theorem 2 gives rise to

$$v_1 = \begin{bmatrix} 14.5190 \\ 15.2754 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 15.0434 \\ 16.6506 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 2.9007 \\ 1.7853 \end{bmatrix},$$

$$v_{2,1} = \begin{bmatrix} 3.0949 \\ 1.8310 \end{bmatrix}, \quad v_{2,2} = \begin{bmatrix} 20.8460 \\ 20.6063 \end{bmatrix}, \quad v_{1,2} = \begin{bmatrix} 20.0610 \\ 22.5155 \end{bmatrix},$$

$$v_{2,1} = \begin{bmatrix} 4.8236 \\ 3.8481 \end{bmatrix}, \quad v_{1,2} = \begin{bmatrix} 4.8193 \\ 4.0905 \end{bmatrix},$$

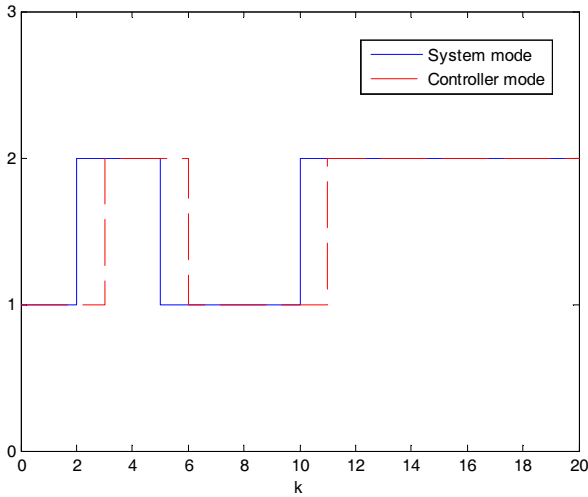


Fig. 3. Switching signal in Example 2.

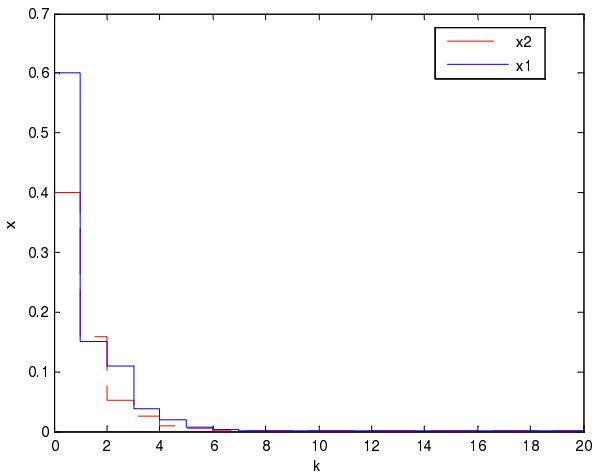


Fig. 4. State response of the closed-loop system in Example 2.

$$h_1 = \begin{bmatrix} -14.0503 \\ -8.7643 \end{bmatrix}, \quad h_2 = \begin{bmatrix} -14.0897 \\ -9.0480 \end{bmatrix}.$$

By (46), the gain matrices K_i , $i=1,2$, can be obtained

$$K_1 = [-1.5719 \quad -0.9805], \quad K_2 = [-1.5502 \quad -0.9955].$$

Furthermore, from (47)-(48), we have $\mu_1 = 0.7499$, $\mu_2 = 2.2913$, $\mu = 1.9739$ and $\tau_a^* = 4.8790$. Choosing $\tau_a = 5$, the simulation results are shown in Figs. 3-4, where the initial conditions are $x(0) = [0.6 \quad 0.4]^T$ and $x(k) = [0 \quad 0]^T$, $k = -2, -1$.

From Figs. 3-4, it is easy to see that the designed controller can guarantee that the resulting closed-loop system is positive and exponentially stable. This demonstrates the effectiveness of the proposed method.

5. CONCLUSIONS

The stabilization problems for both continuous-time and discrete-time positive switched systems with time-varying delays under asynchronous switching have been investigated in this paper. Sufficient conditions for the existence of the asynchronous controllers are established, and the desired controller gain matrices are obtained easily through the solutions of LMIs. Our future work will focus on studying the stability and stabilization problems for positive switched systems by using the mode-dependent average dwell time method proposed in [41].

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