Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities

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Abstract

We extend to a functional setting the concept of quermassintegrals, well-known within the Minkowski theory of convex bodies. We work in the class of quasi-concave functions defined on the Euclidean space, and with the hierarchy of their subclasses given by α-concave functions. In this setting, we investigate the most relevant features of functional quermassintegrals, and we show they inherit the basic properties of their classical geometric counterpart. As a first main result, we prove a Steiner-type formula which holds true by choosing a suitable functional equivalent of the unit ball. Then, we establish concavity inequalities for quermassintegrals and for other general hyperbolic functionals, which generalize the celebrated Prékopa-Leindler and Brascamp-Lieb inequalities. Further issues that we transpose to this functional setting are: integral-geometric formulae of Cauchy-Kubota type, valuation property and isoperimetric/Uryshon-like inequalities.

2010 MSC: 28B, 46G, 52A

Keywords: Quasi-concave functions, quermassintegrals, Prékopa-Leindler-type theorems

1 Introduction

For every K belonging to the class $\mathcal{K}^n$ of non-empty convex compact sets in $\mathbb{R}^n$, its quermassintegrals $W_i(K)$, for $i = 0, \ldots, n$, are defined as the coefficients in the polynomial expansion

$$\mathcal{H}^n(K + \rho B) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\, \rho^i, \tag{1.1}$$

where $\mathcal{H}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$ and $K + \rho B$ is the Minkowski sum of K plus ρ times the unit Euclidean ball B. As special cases, W_0 is the Lebesgue measure $\mathcal{H}^n$, nW_1 is the surface area, $2\kappa_n^{-1} W_{n-1}$ is the mean width, and $\kappa_n^{-1} W_n = 1$ is the Euler characteristic (being $\kappa_n = \mathcal{H}^n(B)$).

The aim of this paper is to develope the notion of quermassintegrals for *quasi-concave* functions, as well as to enlighten their basic properties. Quasi-concave functions f on $\mathbb{R}^n$ are defined by the inequality

$$f((1-\lambda)x_0 + \lambda x_1) \geq \min\{f(x_0), f(x_1)\}, \qquad \forall\, x_0, x_1 \in \mathbb{R}^n,\ \forall \lambda \in [0,1],$$

*Research is partially supported by NSF grant and Simons Fellowship

and may also be described via the property that their level sets $\{f \geq t\} = \{x \in \mathbb{R}^n : f(x) \geq t\}$ are convex. More precisely, we will work in the following class:

$$\mathcal{Q}^n = \Big\{ f : \mathbb{R}^n \to [0, +\infty] : \ f \not\equiv 0 \,, \ f \text{ is quasi-concave, upper semi-continuous, } \lim_{\|x\| \to +\infty} f(x) = 0 \Big\},$$

and also on the subclasses $\mathcal{Q}^n_\alpha$ of $\mathcal{Q}^n$ given by α-concave functions, for $\alpha \in [-\infty, +\infty]$ (see Section 2.4 for details). The class $\mathcal{Q}^n$ can be considered a natural functional counterpart of $\mathcal{K}^n$: in particular, for any $K \in \mathcal{K}^n$, its characteristic function χ_K lies in $\mathcal{Q}^n$.

When passing from sets to (integrable) functions, the role of the volume functional is played by the integral with respect to the Lebesgue measure:

$$I(f) = \int_{\mathbb{R}^n} f(x)\, dx. \tag{1.2}$$

This quite intuitive assertion, inspired by the equality $I(\chi_K) = \mathcal{H}^n(K)$, is commonly agreed and is also confirmed by several functional counterparts of geometric inequalities for convex bodies, in which the volume functional $\mathcal{H}^n(K)$ is replaced by the integral functional $I(f)$. As a significant example, one may indicate the celebrated Prékopa-Leindler inequality [12, 21, 26, 27, 28] (see also [4, 5, 8] for recent related papers), or the functional form of Blaschke-Santaló inequality [1, 2].

Less obvious is how to give a functional notion of the quermassintegrals W_i for $i > 0$. The goodness of such a notion should be evaluated through the possibility of exporting to the functional framework the more relevant properties enjoyed by the quermassintegrals on $\mathcal{K}^n$. The approach we propose goes exactly in this direction and relies on Cavalieri's principle: For every non-negative integrable function f on $\mathbb{R}^n$,

$$I(f) = \int_0^{+\infty} \mathcal{H}^n\big(\{f \geq t\}\big)\, dt.$$

With a full consistency with the abstract Measure Theory (including its part dealing with integration over non-additive set functions), we define analogously the functionals

$$W_i(f) = \int_0^{+\infty} W_i\big(\{f \geq t\}\big)\, dt, \qquad f \in \mathcal{Q}^n.$$

The above definition is well-posed, since the mappings $t \mapsto W_i\big(\{f \geq t\}\big)$ are monotone increasing, as a consequence of the monotonicity of the functionals $W_i(\cdot)$ with respect to set inclusion. Actually, one can adopt the same natural extension from sets to functions in more general situations: If Φ is any functional with values in $[0, +\infty)$, defined on $\mathcal{K}^n$ (or on the larger class of all Borel measurable subsets of $\mathbb{R}^n$), and if it is monotone increasing with respect to set inclusion, one can extend it to the class $\mathcal{Q}^n$ (respectively, to the class of all non-negative Borel measurable functions), by setting

$$\Phi(f) = \int_0^{+\infty} \Phi\big(\{f \geq t\}\big)\, dt. \tag{1.3}$$

Definition (1.3) may look somewhat naïve if compared with previous notions existing in the literature for special quermassintegrals, such as the perimeter or the mean width. These different definitions are rather based on the idea to mimic (1.1), by computing first order derivatives of the integral functional (1.2). More precisely, starting from the the equalities

$$\mathrm{Per}(K) = \lim_{\rho \to 0^+} \frac{\mathcal{H}^n(K + \rho B) - \mathcal{H}^n(K)}{\rho}\,, \qquad M(K) = \lim_{\rho \to 0^+} \frac{\mathcal{H}^n(B + \rho K) - \mathcal{H}^n(B)}{\rho}\,,$$

which are valid up to normalization constants for every $K \in \mathcal{K}^n$, the following definitions have been considered in the recent works [15, 20, 30, 31], dealing especially with log-concave functions:

$$\mathrm{Per}(f) = \lim_{\rho \to 0^+} \frac{I(f \oplus \rho \cdot \varphi_n) - I(f)}{\rho} \,, \qquad M(f) = \lim_{\rho \to 0^+} \frac{I(\varphi_n \oplus \rho \cdot f) - I(f)}{\rho},$$

where φ_n denotes the density of the standard Gaussian measure on $\mathbb{R}^n$. Some more comments are in order to correctly understand the meaning of the above equalities. Firstly, the symbols $\cdot$ and $\oplus$ denote respectively a suitable multiplication by a nonnegative scalar and a suitable addition of functions, which can be defined so as to provide a natural extension of the usual Minkowski algebraic structure on $\mathcal{K}^n$ to functions, see Section 2 for more details. Thus, the above definitions of perimeter and mean width, correspond to choose φ_n as the functional counterpart of the unit ball on $\mathbb{R}^n$. Now, this choice may be somehow disputable. To some extent, it is justified by the fact that the Gaussians turn out to be optimal in the functional version of meaningful geometric inequalities for which the Euclidean balls are optimal (see *e.g.* [1]).

Notwithstanding, the investigation of the functional quermassintegrals introduced in (1.3) carried on in this paper, suggests a different point of view. As a starting point of this investigation, we consider, for a given $f \in \mathcal{Q}^n$ and any $\rho > 0$, the functions

$$f_\rho(x) := \sup_{y \in B_\rho(x)} f(y) \,,$$

where $B_\rho(x)$ denotes the ball of radius r centered at x. In fact, this is equivalent to perturb f with the "unit ball" in the above mentioned algebraic structure, namely, if $f \in \mathcal{Q}^n_\alpha$, it holds

$$f_\rho = f \oplus \rho \cdot \Theta_\alpha(B) \,,$$

being $\Theta_\alpha(B)$ the image of the unit ball through a natural isomorphic embedding of $\mathcal{K}^n$ into $\mathcal{Q}^n_\alpha$. In particular, if $\alpha = -\infty$, meaning f is merely quasi-concave, $\Theta_\alpha(B)$ is simply the characteristic function χ_B. Therefore, in our perspective, χ_B is the most natural functional equivalent of the ball B in the class $\mathcal{Q}^n$. Actually, in Theorem 3.4, we prove that a Steiner-type formula holds true for the mapping

$$\rho \mapsto I(f_\rho). \tag{1.4}$$

More precisely, we prove that such mapping is polynomial in ρ, and its coefficients are precisely the quermassintegrals defined in (1.3), see Theorem 3.4. In particular, up to normalization constants, the notions of perimeter and mean width of f which are obtained from (1.3) with $i = 1$ and $i = n-1$, correspond respectively to the coefficients of ρ and of ρ^{n-1} in the polynomial $I(f_\rho)$:

$$I(f_\rho) = I(f) + \mathrm{Per}(f)\,\rho + \cdots + \frac{n\kappa_n}{2} M(f)\,\rho^{n-1} + \kappa_n (\max_{\mathbb{R}^n} f)\,\rho^n \,. \tag{1.5}$$

We then focus attention on the other main features of the quermassintegrals, dealing in particular with:

– concavity-like inequalities;

– integral-geometric formulae;

– valuation property;

– isoperimetric type inequalities.

It is well-known that each of the functionals W_i's satisfies on $\mathcal{K}^n$ the following Brunn-Minkowski type inequality:

$$W_i((1-\lambda)K_0+\lambda K_1) \geq \Big((1-\lambda)W_i(K_0)^{\frac{1}{n-i}}+\lambda W_i(K_1)^{\frac{1}{n-i}}\Big)^{\frac{1}{n-i}} \quad \forall\, K_0, K_1 \in \mathcal{K}^n\ ,\ \forall \lambda \in [0,1]\ . \quad (1.6)$$

For short, this may be expressed as the property that the functional $\Phi = W_i$ is α-concave on $\mathcal{K}^n$ with $\alpha = \frac{1}{n-i}$. For $i = 0$, namely for the Lebesgue measure, the functional counterpart of (1.7) is given by the dimension-free inequality due to Prékopa and Leindler and by its dimensional extension due to Brascamp and Lieb. We obtain a further generalization of these results (Theorems 4.2 and 4.7), which holds true for general monotone α-concave functionals Φ extended from $\mathcal{K}^n$ to $\mathcal{Q}^n$ according to the formula (1.4). As a special case, we thus obtain Prékopa-Leindler-type inequalities for the functional quermassintegrals introduced in (1.3). On the example of the surface area, *i.e.* for the functional $\Phi = W_1$, the possibility of such generalization was already demonstrated in [6]. As further examples of functionals satisfying a Brunn-Minkowski type inequality, let us mention the p-capacity of convex bodies in $\mathbb{R}^n$ for $1 \leq p < n$ (with $\alpha = \frac{1}{n-p}$, see [11, 16]), the first non-trivial eigenvalue of the Laplacian with the Dirichlet boundary condition (with $\alpha = -2$, see [12]) and other similar functionals (see for instance [14] and [32]). These results link the study of quasiconcave functions to the theory of elliptic PDE's; an example of the interaction between these subjects, particularly related to the matter treated here, can be found in [22].

Let us point out that our approach in order to prove Theorems 4.2 and 4.7 does not use induction on the dimension (nor mass transportation) as in the more typical proof of Prékopa-Leindler inequality, but is rather based on a new one-dimensional variant of it, inspired by a previous observation due to Ball [2]. It is also remarkable that, as we show by constructing suitable counterexamples, this kind of concavity property turns out to fail, if one defines the perimeter of a function along the different line sketched above, namely as the derivative of the volume functional under Gaussian-type perturbations.

For what concerns integral-geometric results, we show that the Cauchy-Kubota formula for the quermassintegrals on $\mathcal{K}^n$ can be suitably extended on $\mathcal{Q}^n$ (see Theorem 5.3). To that aim, we exploit as a crucial tool the concept of the functional projection introduced in [20]. By combining it with definition (1.3), the desired extension turns out to be quite straightforward. To the best of our knowledge, this is the first step moved in bringing integral-geometric properties of convex bodies into a functional framework.

One of the most important characterizations of quermassintegrals is given by the celebrated Hadwiger's Theorem, which asserts that they generate the space of rigid motion invariant valuations on $\mathcal{K}^n$ which are continuous with respect to the Hausdorff metric (see [33]). The valuation property can be transferred in a natural way from sets to functions (replacing union and intersection by max and min operations, respectively, see Section 5 for details). In Section 5 we check that the functionals defined in (1.3) are in fact valuations on $\mathcal{Q}^n$. Let us mention that recently some characterizations of valuations in various function spaces have been found, see for instance [23, 35].

Besides concavity inequalities, and partly as a consequence of them, quermassintergrals verify various inequalities of isoperimetric type; hence, having introduced a similar notion for functions, it is natural to ask for corresponding results in the functional setting. In Section 6 we derive two possible versions of the standard isoperimetric inequalities for quermassintegrals of quasi-concave and log-concave functions (see Theorems 6.1 and 6.2) along with a functional version of the Urysohn's inequality (Corollary 6.3).

The outline of the paper is as follows. After collecting some background material in Section 2, in Section 3 we set and discuss our notion of functional quermassintegrals, and prove the corresponding Steiner formula. In Section 4 we deal with generalized Prékopa-Leindler inequalities, while Section 5 is devoted to the integral-geometric formulae and the valuation property for functional quermassintergrals. Section 6 contains some concluding remarks on further properties related to isoperimetric and functional inequalities.

When this paper was in the final part of its preparation we learned by L. Rotem about the paper [25], where the authors present ideas and results, found independently, which partially overlap with those of the present paper.

Acknowledgment. We wish to thank Paolo Salani for several discussions on the theme of quasi-concave functions, which gave a strong impulse to some of the ideas contained in this paper.

2 Preliminaries

We work in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$, equipped with the usual Euclidean norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$. For $x \in \mathbb{R}^n$ and $r > 0$, we set $B_r(x) = B(x,r) = \{y \in \mathbb{R}^n \,:\, \|y-x\| \leq r\}$, and $B = B_1(0)$. We denote by $\mathrm{int}(E)$ and $\mathrm{cl}(E)$ the relative interior and the closure of a set $E \subset \mathbb{R}^n$ respectively.
The unit sphere in $\mathbb{R}^n$ will be denoted by $\mathbb{S}^{n-1}$. For $k = 0, 1, \dots, n$, $\mathcal{H}^k$ stands for the k-dimensional Hausdorff measure on $\mathbb{R}^n$. In particular, $\mathcal{H}^n$ denotes the usual Lebesgue measure on $\mathbb{R}^n$.

2.1 Convex bodies

We denote by $\mathcal{K}^n$ the class of all non-empty convex compact sets in $\mathbb{R}^n$ (called convex bodies). For the general theory of convex bodies, we refer the interested reader to the monograph [33].
For every $K \in \mathcal{K}^n$, we denote by χ_K and I_K respectively its characteristic and indicatrix functions, namely:

$$\chi_K(x) = \begin{cases} 1, & \text{if } x \in K, \\ \\ 0, & \text{if } x \notin K, \end{cases} \qquad I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ \\ +\infty, & \text{if } x \notin K. \end{cases}$$

Note that I_K is convex. We will also use the notion of support function h_K of a convex body K, defined by

$$h_K(x) = \sup_{y \in K} (x, y) \,.$$

The class $\mathcal{K}^n$ is endowed with the algebraic structure based on the Minkowski addition. For K and L in $\mathcal{K}^n$, we set

$$K + L = \{x + y \,|\, x \in K \,,\, y \in L\},$$

while for $\lambda \geq 0$ and $K \in \mathcal{K}^n$, we set

$$\lambda K = \{\lambda x \,|\, x \in K\} \,.$$

It is worth noticing the following property connecting the Minkowski addition and support functions: For every $K, L \in \mathcal{K}^n$, and for every $\alpha, \beta \geq 0$,

$$h_{\alpha K + \beta L} = \alpha h_K + \beta h_L \,.$$

$\mathcal{K}^n$ can be endowed with the Hausdorff metric. The Hausdorff distance between two convex bodies K and L can be simply defined as

$$\delta(K,L) = \|h_K - h_L\|_{L^\infty(\mathbb{S}^{n-1})}$$

(see [33, Sec. 1.8]).

2.2 Quermassintegrals of convex bodies

In this subsection we collect basic properties and relations satisfied by the quermassintegrals. Recall that, for every $K \in \mathcal{K}^n$, the quermassintegrals $W_i(K)$, $i = 0, \dots, n$, represent the corresponding coefficients in the polynomial expansion (1.1). In particular, $W_0(K) = \mathcal{H}^n(K)$ is the volume of K, $W_n(K) = \kappa_n := \mathcal{H}^n(B)$, $nW_1(K) = \mathcal{H}^{n-1}(\partial K)$ is the surface area of K, and $2\kappa_n^{-1}W_{n-1}(K)$ is the mean width, which is given by

$$\int_{\mathbb{S}^{n-1}} (h_K(u) + h_K(-u))\, d\mathcal{H}^{n-1}(u).$$

The quermassintegrals are invariant under rigid motions and continuous with respect to the Hausdorff distance. They also obey to the following remarkable properties (where K, K_0 and K_1 denote arbitrary convex bodies in $\mathcal{K}^n$).

(i) *Homogeneity.*

$$W_i(\lambda K) = \lambda^{n-i}\, W_i(K) \quad \forall\, \lambda \geq 0\,.$$

(ii) *Monotonicity.*

$$K_0 \subseteq K_1 \;\Rightarrow\; W_i(K_0) \leq W_i(K_1).$$

(iii) *Brunn-Minkowski-type inequality.* For every $\lambda \in [0,1]$,

$$W_i((1-\lambda)K_0 + \lambda K_1) \geq \Big((1-\lambda)W_i(K_0)^{1/(n-i)} + \lambda W_i(K_1)^{1/(n-i)}\Big)^{n-i}. \tag{2.1}$$

Equivalently, the map $\lambda \to W_i((1-\lambda)K_0 + \lambda K_1)^\alpha$ is concave on $[0,1]$, where $\alpha = \frac{1}{n-i}$. We will refer to this property as the α-concavity of W_i. Note that in each case, α represents the reciprocal of the homogeneity order of the relevant quermassintegral. The usual Brunn-Minkowski inequality corresponds to the case $i = 0$.

(iv) *Cauchy-Kubota integral formulae.* Given $k \in \{1, \dots, n-1\}$, let $\mathcal{L}^n_k$ be the set of all linear subspaces of $\mathbb{R}^n$ of dimension k, and let dL_k denote the integration with respect to the standard invariant probability measure on $\mathcal{L}^n_k$. Then, for every $i = 1, \dots, k$, we have

$$W_i(K) = c(i,k,n) \int_{\mathcal{L}^n_k} W_i(K|L_k)\, dL_k \tag{2.2}$$

with a suitable constant $c(i,k,n)$. Here $K|L_k$ denotes the orthogonal projection of K onto $L_k \in \mathcal{L}^n_k$. An exhaustive presentation of these formulas (along with an explicit expression of the constant $c(i,k,n)$) may be found for instance in [34]. In the particular case $i = k = 1$ we have the Cauchy integral formula for the perimeter:

$$W_1(K) = c \int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-1}(K|u^\perp)\, du\,,$$

where c is a constant depending on n and du indicates integration with respect to the invariant probability measure on the unit sphere.

(v) *Valuation property.* Every quermassintegral is a valuation on $\mathcal{K}^n$, *i.e.*, if K_0 and K_1 belong to $\mathcal{K}^n$ and are such that $K_0 \cup K_1 \in \mathcal{K}^n$, then

$$W_i(K_0) + W_i(K_1) = W_i(K_0 \cup K_1) + W_i(K_0 \cap K_1). \tag{2.3}$$

According to a celebrated theorem by Hadwiger, this additivity property together with rigid motion invariance and continuity with respect to the Hausdorff distance (or monotonicity), characterizes linear combinations of quermassintegrals; see, for instance, Theorems 4.2.6 and 4.2.7 in [33].

2.3 M-means and α-concave functions

In order to introduce the class of α-concave functions, we start with the definition of α-means. Given $\alpha \in (-\infty, +\infty)$ and $s, t > 0$, for every $u, v > 0$ we first define

$$M_\alpha^{(s,t)}(u,v) := \begin{cases} (su^\alpha + tv^\alpha)^{1/\alpha}, & \text{if } \alpha \neq 0, \\ u^s v^t, & \text{if } \alpha = 0. \end{cases} \tag{2.4}$$

For $\alpha \geq 0$, definition (2.4) extends to the case when at least one of u and v is zero. If $\alpha < 0$ and $uv = 0$ (with $u, v \geq 0$), we set $M_\alpha^{(s,t)}(u,v) = 0$. In the extreme cases $\alpha = \pm\infty$, we set

$$M_{-\infty}^{(s,t)}(u,v) := \min(u,v), \qquad M_{+\infty}^{(s,t)}(u,v) := \max(u,v).$$

The functions $u \to M_\alpha^{(s,t)}(u,v)$ and $v \to M_\alpha^{(s,t)}(u,v)$ are non-decreasing. If $u = +\infty$ or $v = +\infty$, the value $M_\alpha^{(s,t)}(u,v)$ is defined so that the monotonicity property is preserved. In particular, $M_\alpha^{(s,t)}(+\infty, v) = M_\alpha^{(s,t)}(u, +\infty) = +\infty$ for every v (including $v = +\infty$) in case $\alpha > 0$. We also put $M_\alpha^{(s,t)}(+\infty, 0) := M_\alpha^{(s,t)}(0, +\infty) = 0$ for $\alpha \leq 0$.

The α-mean of $u, v \geq 0$, with weight $\lambda \in (0,1)$ is defined as

$$M_\alpha^{(\lambda)}(u,v) = M_\alpha^{(1-\lambda,\lambda)}(u,v)\,.$$

The particular cases $\alpha = 1, 0, -1$ correspond to the arithmetic, geometric and harmonic mean, respectively. In general, the functions $\alpha \to M_\alpha^{(\lambda)}(u,v)$ are non-decreasing. Note, however, that this property fails for the functions $\alpha \to M_\alpha^{(s,t)}(u,v)$ with $s + t \neq 1$.

For $\alpha \in [-\infty, +\infty]$, we denote by $\mathcal{C}_\alpha$ the family of all functions $f : \mathbb{R}^n \to [0, +\infty]$ which are not identically zero and are α-concave, meaning that

$$f((1-\lambda)x + \lambda y) \geq M_\alpha^{(\lambda)}(f(x), f(y)), \qquad \forall\, x, y \text{ such that } f(x)f(y) > 0, \quad \forall\, \lambda \in (0,1).$$

The same definition may be given when f is defined on a convex subset of $\mathbb{R}^n$. Note that, as a straightforward consequence of the monotonicity property of the α-means with respect to α, we have $\mathcal{C}_\alpha \subseteq \mathcal{C}_{\alpha'}$ if $\alpha' \leq \alpha$.
The following particular cases of α describe canonical classes of α-concave functions:

$\mathcal{C}_{-\infty}$ is the largest class of quasi-concave functions;

$\mathcal{C}_0$ is the class of log-concave functions;

$\mathcal{C}_1$ is the class of concave functions on convex sets Ω (extended by zero outside Ω);

$\mathcal{C}_{+\infty}$ is the class of multiples of characteristic functions of convex sets $\Omega \subset \mathbb{R}^n$.

Any function $f \in \mathcal{C}_\alpha$ is supported on the (nonempty) convex set $K_f = \{f > 0\}$, and if $\alpha > -\infty$, it is continuous in the relative interior Ω_f of K_f. If α is finite and nonzero, it has the form $f = V^{1/\alpha}$, where V is concave on Ω_f in case $\alpha > 0$, and is convex in case $\alpha < 0$; for $\alpha = 0$, the general form is $f = e^{-V}$ for some convex function V on Ω_f.

2.4 Algebraic structure of the class of α-convex functions

For any $\alpha \in [-\infty, +\infty]$, we are going to introduce in $\mathcal{C}_\alpha$ an addition and a multiplication by positive reals, which extend the usual Minkowski algebraic structure on $\mathcal{K}^n$.
Let be given $f, g \in \mathcal{C}_\alpha$ and $s, t > 0$. If $\alpha \leq 0$, we put

$$(s \cdot f \oplus t \cdot g)(z) := \sup \left\{ M_\alpha^{(s,t)}(f(x), g(y)) \, : \, z = sx + ty \right\} ; \tag{2.5}$$

if $\alpha > 0$, we put

$$(s \cdot f \oplus t \cdot g)(z) := \begin{cases} \sup \left\{ M_\alpha^{(s,t)}(f(x), g(y)) \, : \, z = sx + ty, \; f(x)g(y) > 0 \right\} & \text{if} \quad z \in sK_f + tK_g, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Note that (2.6) is also applicable in case $\alpha \leq 0$, since $M_\alpha^{(s,t)}(u, v) = 0$, whenever $uv = 0$; in this sense (2.6) is more general than (2.5).
Clearly the operations $\oplus$ and $\cdot$ depend on α. However for simplicity we will not indicate this dependence explicitly, unless it is strictly needed. In particular, this abuse of notation is consistent with the following immediate relation: For all non-empty sets K and L in $\mathbb{R}^n$ and all $s, t > 0$,

$$s \cdot \chi_K \oplus t \cdot \chi_L = \chi_{sK+tL}$$

(in particular, in this case the left-hand side does not depend on α).
The operations $\oplus$ and $\cdot$ may also be used for arbitrary non-negative, not identically zero functions, without any convexity assumption. For any fixed $\alpha \in [-\infty, +\infty]$, they are easily checked to enjoy the following general properties:

(i) *Commutativity.* $s \cdot f \oplus t \cdot g = t \cdot g \oplus s \cdot f$.

(ii) *Associativity.* $(s \cdot f \oplus t \cdot g) \oplus u \cdot h = s \cdot f \oplus (t \cdot g \oplus u \cdot h)$.

(iii) *Homogeneity.* $s \cdot f \oplus t \cdot g = (s + t)^{1/\alpha} \left(\frac{s}{s+t} \cdot f \oplus \frac{t}{s+t} \cdot g \right) \quad (\alpha \neq 0)$.

(iv) *Measurability.* $s \cdot f \oplus t \cdot g$ is Lebesgue measurable as long as f and g are Borel measurable.

Next, we show that every class $\mathcal{C}_\alpha$ is closed under the introduced operations.

Proposition 2.1. *If $f, g \in \mathcal{C}_\alpha$ and $s, t > 0$, then $s \cdot f \oplus t \cdot g \in \mathcal{C}_\alpha$.*

Proof. First let α be non-zero. Using the homogeneity property (iii), it suffices to consider the case $s+t=1$. We set for brevity

$$u(x,y) = M_\alpha^{(s,t)}(f(x),g(y)), \qquad x \in K_f,\ y \in K_g,$$

and, for $z \in K = sK_f + tK_g$, let

$$h(z) = (s\cdot f \oplus t\cdot g)(z) = \sup\left\{u(x,y):\ z = sx+ty,\ x\in K_f, y\in K_g\right\},$$

putting $h=0$ outside K.
We claim that the function u is α-concave on the convex supporting set $K_f \times K_g$. Indeed, if additionally α is finite, taking $(x,y) = s'(x_1,y_1)+t'(x_2,y_2)$ with $s',t'>0$, $s'+t'=1$ and $x_1,x_2\in K_f$, $y_1,y_2\in K_g$, we have

$$\begin{aligned}
u(x,y) &= M_\alpha^{(s,t)}\left(f(s'x_1+t'x_2), g(s'y_1+t'y_2)\right)\\
&\geq M_\alpha^{(s,t)}\left(M_\alpha^{(s',t')}(f(x_1),f(x_2)), M_\alpha^{(s',t')}(g(y_1),g(y_2))\right)\\
&= \left(s\left(s'f(x_1)^\alpha + t'f(x_2)^\alpha\right) + t\left(s'g(y_1)^\alpha + t'g(y_2)^\alpha\right)\right)^{1/\alpha}\\
&= \left(s'\left(sf(x_1)^\alpha + tg(y_1)^\alpha\right) + t'\left(sf(x_2)^\alpha + tg(y_2)^\alpha\right)\right)^{1/\alpha}\\
&= M_\alpha^{(s',t')}\left(M_\alpha^{(s,t)}(f(x_1),g(y_1)), M_\alpha^{(s,t)}(f(x_2),g(y_2))\right)\\
&= M_\alpha^{(s',t')}(u(x_1,y_1),u(x_2,y_2)).
\end{aligned}$$

Thus,

$$u(s'(x_1,y_1)+t'(x_2,y_2)) \geq M_\alpha^{(s',t')}(u(x_1,y_1),u(x_2,y_2)),$$

which means α-concavity of u on $\mathbb{R}^{2n}$ (if we define it to be zero outside $K_f\times K_g$).
With corresponding modifications, or using continuity and monotonicity of the function M_α with respect to α, we have a similar property of the function u in the remaining cases.
Now, for $z\in K$, fix a decomposition $z = sz_1+tz_2$, $z_1,z_2\in K$. Using truncation, if necessary, we may assume that both f and g are bounded, so that h is bounded, as well. Then, given $\varepsilon>0$, choose $x_1,x_2\in K_f$, $y_1,y_2\in K_g$ such that $z_1 = sx_1+ty_1$, $z_2 = sx_2+ty_2$, and

$$h(z_1) \leq u(x_1,y_1)+\varepsilon, \qquad h(z_2)\leq u(x_2,y_2)+\varepsilon.$$

Since the function u is α-concave, setting $x = sx_1+tx_2$ and $y = sy_1+ty_2$, we get

$$u(x,y) \geq M_\alpha^{(s,t)}(u(x_1,y_1),u(x_2,y_2)) \geq M_\alpha^{(s,t)}\left((h(z_1)-\varepsilon)^+,(h(z_2)-\varepsilon)^+\right).$$

Letting $\varepsilon\to 0$, the latter yields

$$u(x,y)\geq M_\alpha^{(s,t)}(h(z_1),h(z_2)).$$

It remains to note that $sx+ty = sz_1+tz_2 = z$, which implies $u(x,y)\leq h(z)$.
Now, let $\alpha=0$, in which case we should work with

$$u(x,y) = f(x)^s g(y)^t, \qquad x,y\in\mathbb{R}^n,$$

and with a similarly defined function h. Again, for $(x,y) = s'(x_1,y_1)+t'(x_2,y_2)$, we have, using the log-concavity of f and g,

$$\begin{aligned} u(x,y) &= f(s'x_1+t'x_2)^s\, g(s'y_1+t'y_2)^t \\ &\geq f(x_1)^{ss'} f(x_2)^{st'}\, g(y_1)^{ts'} g(y_2)^{tt'} \;=\; M_0^{(s',t')}(u(x_1,y_1),u(x_2,y_2)). \end{aligned}$$

This means that u is log-concave on $\mathbb{R}^{2n}$. The rest of the proof is similar to the basic case. □

In the next remarks we collect further comments on the operations $\oplus$ and $\cdot$, more specifically on their relationship with the usual Minkowski structure in $\mathcal{K}^n$, and on their interpretation in the two special cases $\alpha = -\infty$ and $\alpha = 0$.

Remark 2.2. Equipped with quermassintegral in (2.5)-(2.6), and in view of Proposition 2.1, $\mathcal{C}_\alpha$ can be seen as an extension of $\mathcal{K}^n$ which preserves its algebraic structure. More precisely, the mappings $\Theta_\alpha : \mathcal{K}^n \to \mathcal{C}_\alpha$ defined by

$$\Theta_\alpha(K) := \begin{cases} e^{-I_K} & \text{if } \alpha = 0 \\ I_K^{-1} & \text{if } \alpha \neq 0 \end{cases}$$

are isomorphic embeddings of $\mathcal{K}^n$ (endowed with the Minkowski structure) into $\mathcal{C}_\alpha$ (endowed with the operations $\oplus$ and $\cdot$).

Remark 2.3. In $\mathcal{C}_{-\infty}$, quermassintegral in (2.5) can be characterized through the Minkowski addition of the level sets $K_f(r) = \{x \in \mathbb{R}^n : f(x) > r\}$. Namely, for $f, g \in \mathcal{C}_{-\infty}$ and $s, t > 0$, the functional equality

$$h(z) = (s \cdot f \oplus t \cdot g)(z) = \sup\{\min\{f(x), g(y)\} : sx + ty = z\}$$

is equivalent to the family of set equalities

$$K_h(r) = sK_f(r) + tK_g(r) \qquad \forall\, r > 0\,. \tag{2.7}$$

Note that for a general value of α, we only have the following set inclusion, valid if $s + t = 1$:

$$K_h(r) \supset sK_f(r) + tK_g(r) \qquad \forall\, r > 0\,. \tag{2.8}$$

Remark 2.4. In $\mathcal{C}_0$, the operation $\oplus$ (defined as in (2.5) with $t = s = 1$) is related to the operation introduced in 1991 by Maurey. More precisely, starting with $U, V : \mathbb{R}^n \to (-\infty, +\infty]$, we get

$$e^{-U} \oplus e^{-V} = e^{-W}, \tag{2.9}$$

where

$$W(z) = \inf_x\, [U(z-x) + V(x)]$$

represents the infimum-convolution of U and V. If these functions are convex, so is W (as we also know from Proposition 2.1). This fact is crucial in the study of the so-called "convex" concentration for product measures, *cf.* [24].

2.5 Prékopa-Leindler and Brascamp-Lieb Theorems

The following well-known result due to Prékopa and Leindler [21, 26, 27, 28] is a functional extension of the classical Brunn-Minkowski inequality.

Theorem 2.5. *Let $\lambda \in (0,1)$. Let f, g, h be non-negative measurable functions on $\mathbb{R}^n$. If*

$$h(M_1^{(\lambda)}(x,y)) \geq M_0^{(\lambda)}(f(x), g(y)) \qquad \forall x, y \in \mathbb{R}^n\,, \tag{2.10}$$

then

$$\int h \geq M_0^{(\lambda)}\Big(\int f, \int g\Big). \tag{2.11}$$

Given non-empty Borel sets $A, B \subset \mathbb{R}^n$, and $\lambda \in (0,1)$, by applying the above result with $f = \chi_A$, $g = \chi_B$, and $h = \chi_{(1-\lambda)A+\lambda B}$ (after noticing that h is Lebesgue measurable), one gets

$$\mathcal{H}^n((1-\lambda)A + \lambda B) \geq \mathcal{H}^n(A)^{1-\lambda}\mathcal{H}^n(B)^{\lambda}. \tag{2.12}$$

This is a multiplicative variant of the Brunn-Minkowski inequality

$$\mathcal{H}^n((1-\lambda)A + \lambda B) \geq \Big((1-\lambda)\mathcal{H}^n(A)^{1/n} + \lambda\mathcal{H}^n(B)^{1/n}\Big)^n \tag{2.13}$$

with convexity parameter $\alpha = 1/n$ (which is optimal). Though in principle (2.12) is weaker (2.13), using the homogeneity of the volume it is easy to derive (2.13) from (2.12). However, the difference between (2.13) and (2.12) suggests a different, dimension-dependent variant of Theorem 2.5, which would directly yield (2.13) when applied to characteristic functions. Such a variant is known and is recalled in Theorem 2.6 below. It was proposed by Brascamp and Lieb [12] and somewhat implicitly in Borell [9, 10]; *cf.* also [17] and [18].

Theorem 2.6. *Let $\lambda \in (0,1)$ and let $\alpha \in [-\frac{1}{n}, +\infty]$. Let f, g, h be non-negative measurable functions on $\mathbb{R}^n$. If*

$$h(M_1^{(\lambda)}(x,y)) \geq M_\alpha^{(\lambda)}(f(x), g(y)), \qquad \forall\, x, y \ \text{such that } f(x)g(y) > 0, \tag{2.14}$$

then

$$\int h \geq M_\beta^{(\lambda)}\Big(\int f, \int g\Big) \qquad \text{where} \ \ \beta := \frac{\alpha}{1+\alpha n}. \tag{2.15}$$

In the extreme cases $\alpha = -\frac{1}{n}$ and $\alpha = +\infty$, the definition of β in (2.15) is understood respectively as $\beta = -\infty$ and $\beta = \frac{1}{n}$.

Since $\beta = 0$ for $\alpha = 0$, Theorem 2.6 includes Theorem 2.5 as a particular case. Note also that, if A, B and λ are as above, by applying Theorem 2.6 with $\alpha = +\infty$, $f = \chi_A$, $g = \chi_B$ and $h = \chi_{(1-\lambda)A+\lambda B}$, one obtains directly the Brunn-Minkowski inequality in its dimension-dependent form (2.13).
We point out that, under additional assumptions on f and g, the value of β in (2.15) may be improved. For instance, in dimension $n = 1$, if $\operatorname{ess\,sup} f(x) = \operatorname{ess\,sup} g(x) = 1$, then one may take $\beta = 1$ regardless of α, see for instance [7]. Without additional constraints, the value of β in (2.15) is optimal. For instance, for $n = 1$ and $\alpha = 0$, take $f(x) = ae^{-x}\chi_{(0,+\infty)}(x)$ and $g(x) = be^{-x}\chi_{(0,+\infty)}(x)$, where a and b are positive parameters. In this case, the function $h(x) := M_0^{(\lambda)}(a,b)\, e^{-x}\chi_{(0,+\infty)}(x)$ satisfies (2.10), and (2.11) becomes equality.

As a further natural generalization of Theorem 2.6, one can consider the case when λ and $(1-\lambda)$ are replaced by arbitrary positive parameters s and t, not necessarily satisfying the condition $s+t = 1$.

Assume $\alpha \neq 0$, and $\alpha < +\infty$. If non-negative measurable functions f, g, h satisfy the inequality $h(M_1^{(s,t)}(x,y)) \geq M_\alpha^{(s,t)}(f(x), g(y))$ for all x, y such that $f(x)g(y) > 0$, then the function

$$\tilde{h}(z) := \frac{1}{(s+t)^{1/\alpha}}\, h((s+t)\, z)$$

is easily checked to satisfy the hypothesis (2.14) with $\lambda = \frac{t}{s+t}$. Hence, by applying Theorem 2.6, we arrive at the following statement (where also the case $\alpha = +\infty$ can be easily included as a limit):

Theorem 2.7. *Let $s, t > 0$ and let $\alpha \in [-\frac{1}{n}, +\infty]$, $\alpha \neq 0$. Let f, g, h be non-negative measurable functions on $\mathbb{R}^n$. If*

$$h(M_1^{(s,t)}(x,y)) \geq M_\alpha^{(s,t)}(f(x), g(y)), \qquad \forall\, x, y \;\; \text{such that } f(x)g(y) > 0,$$

then

$$\int h \geq M_\beta^{(s,t)}\Big(\int f, \int g\Big) \qquad \text{where} \;\; \beta := \frac{\alpha}{1+\alpha n}.$$

In the extreme cases $\alpha = -\frac{1}{n}$ and $\alpha = +\infty$, the value of β has to be understood as in Theorem 2.6.

We observe that, using the operations $\oplus$ and $\cdot$ introduced in the previous section, Theorem 2.7 (and similarly also Theorems 2.5 and 2.6) can be written in a more compact form as the inequality

$$\int \big(s \cdot f \oplus t \cdot g\big) \geq M_\beta^{(s,t)}\Big(\int f, \int g\Big), \qquad \text{where } \alpha \in [-\frac{1}{n}, +\infty]\,,\ \alpha \neq 0, \text{ and } \beta = \frac{\alpha}{1+\alpha n},$$

holding true for all non-negative Borel measurable functions f and g on $\mathbb{R}^n$, and for all $t, s > 0$ (the assumption $\alpha \neq 0$ may be removed when $t + s = 1$).
In particular, taking $s = t = 1$, and replacing first f, g respectively with $f^{1/\alpha}$, $g^{1/\alpha}$, and then α with $\frac{1}{\alpha}$, one gets the following inequality

$$\int \Big(\sup\{f(x) + g(y) \ :\ x + y = z\,, f(x)g(y) > 0\}\Big)^\alpha \geq \left[\Big(\int f^\alpha\Big)^{\frac{1}{\alpha+n}} + \Big(\int g^\alpha\Big)^{\frac{1}{\alpha+n}}\right]^{\alpha+n}, \quad (2.16)$$

where $\alpha \geq 0$ or $\alpha \leq -n$.
In dimension $n = 1$ and for the range $\alpha > 0$, this inequality was obtained in 1953 by Henstock and Macbeath as part of their proof of the Brunn-Minkowski inequality, *cf.* [19]. Indeed, stated in $\mathbb{R}^n$ for characteristic functions $f = \chi_A$, $g = \chi_B$, and with $\alpha = 0$, (2.16) gives back

$$\mathcal{H}^n(A+B) \geq \Big(\mathcal{H}^n(A)^{1/n} + \mathcal{H}^n(B)^{1/n}\Big)^n.$$

3 Functional notion of quermassintegrals and Steiner-type formula

Let us introduce the following class of admissible functions

$$\mathcal{Q}^n = \Big\{f : \mathbb{R}^n \to [0, +\infty] \ :\ f \not\equiv 0\,,\ f \text{ is quasi-concave, upper semicontinuous, } \lim_{\|x\|\to+\infty} f(x) = 0\Big\}.$$

We also consider the subclasses formed by the functions in $\mathcal{Q}^n$ which are α-concave:

$$\mathcal{Q}^n_\alpha = \mathcal{Q}^n \cap \mathcal{C}_\alpha\,, \qquad \alpha \in [-\infty, +\infty].$$

In particular, $\mathcal{Q}^n = \mathcal{Q}^n_{-\infty}$.
Note that, if f is quasi-concave, the property $\lim_{\|x\|\to+\infty} f(x) = 0$ is necessary to keep $I(f)$ finite (we recall that $I(f)$ is just the integral of f on $\mathbb{R}^n$). Indeed, the vanishing of f at infinity may be equivalently formulated as the boundedness of all the level sets $\{f \geq t\}$: if $I(f)$ is finite, then all such convex sets have finite Lebesgue measure and are therefore bounded.
We also observe that, if $f \in \mathcal{Q}^n$, the level sets $\{f \geq t\}$ are convex closed sets, because f is quasi-concave and upper semicontinuous; since f is vanishing at infinity, these sets are also compact. Hence, $\sup_x f(x)$ is attained at some point, and one may freely speak about the maximum value of f (which in general may be finite or not). In addition, all quermassintegrals of the sets $\{f \geq t\}$ are well-defined and finite, so that we are allowed to give the the following definition.

Definition 3.1. Let $f \in \mathcal{Q}^n$. For every $i = 0, \dots, n$, we define the *i-th quermassintegral of f* as

$$W_i(f) := \int_0^{+\infty} W_i\big(\{f \geq t\}\big)\, dt = \int_0^{+\infty} W_i\big(\operatorname{cl}\{f > t\}\big)\, dt. \tag{3.1}$$

In particular,

$$I(f) = W_0(f) = \int_0^{+\infty} \mathcal{H}^n\big(\{f \geq t\}\big)\, dt.$$

As further special cases, by analogy with convex bodies, we define the *perimeter*, the *mean width* and the *Euler characteristic* of $f \in \mathcal{Q}^n$ respectively as

$$\begin{aligned}
\operatorname{Per}(f) &= nW_1(f) = \int_0^{+\infty} \operatorname{Per}\big(\{f \geq t\}\big)\, dt,\\
M(f) &= 2\kappa_n^{-1}\, W_{n-1}(f) = \int_0^{+\infty} M\big(\{f \geq t\}\big)\, dt,\\
\chi(f) &= \kappa_n^{-1}\, W_n(f) = \max_{x\in\mathbb{R}^n} f(x).
\end{aligned}$$

Let us emphasize that the two integrals in (3.1) do coincide, so that we may use any of them at our convenience. To see this fact, one may use the inclusion $\operatorname{cl}\{f > t\} \subseteq \{f \geq t\}$, which ensures that the second integral in (3.1) is dominated by the first one (applying the monotonicity property of W_i). On the other hand, for any $\varepsilon > 0$, we have $\{f \geq t + \varepsilon\} \subseteq \{f > t\} \subseteq \operatorname{cl}\{f > t\}$, which yields

$$\int_\varepsilon^{+\infty} W_i\big(\{f \geq t\}\big)\, dt \leq \int_0^{+\infty} W_i\big(\operatorname{cl}\{f > t\}\big)\, dt.$$

Letting $\varepsilon \to 0$, we obtain that the first integral in (3.1) is dominated by the second one, as well.

3.1 Basic properties

Let us mention a few general properties of the functional quermassintegrals, which follow immediately from Definition 3.1.

(i) *Positivity.* $0 \leq W_i(f) \leq +\infty$.

(ii) *Homogeneity under dilations.* $W_i(f_\lambda) = \lambda^{n-i}\, W_i(f)$, where $f_\lambda(x) = f(x/\lambda)$, $\lambda > 0$.

(iii) *Monotonicity.* $W_i(f) \leq W_i(g)$, whenever $f \leq g$.

For what concerns the finiteness of the quermassintergals, the problem of characterizing those functions in $\mathcal{Q}^n$ whose all quermassintegrals are finite seems to be an interesting question. Let us examine what happens in this respect within the subfamily of radial functions.

Example 3.2. Let $f \in \mathcal{Q}^n$ be a spherically invariant function. Equivalently, it has the form

$$f(x) = F(|x|), \quad x \in \mathbb{R}^n,$$

where $F : [0, +\infty) \to [0, \Lambda]$ is a non-increasing upper semi-continuous function vanishing at infinity, with maximum $\Lambda = F(0)$, finite or not.
Incidentally, this example shows that quasi-concave functions do not need to be continuous on their domain, nor to be in $L^1(\mathbb{R}^n)$, so that it may be $I(f) = +\infty$.
Define the inverse function $F^{-1} : (0, \Lambda] \to [0, +\infty)$ canonically by

$$F^{-1}(t) = \min\{r > 0 : F(r) \geq t\}, \qquad 0 < t < \Lambda.$$

Since $\{f \geq t\} = F^{-1}(t)B$, we have $W_i(\{f \geq t\}) = \kappa_n \big(F^{-1}(t)\big)^{n-i}$. Integrating this equality over t, we arrive at the formula

$$W_i(f) = \kappa_n \int_0^{+\infty} r^{n-i}\, dF(r), \qquad i = 0, 1, \dots, n,$$

where F may be treated as an arbitrary positive measure on $(0, +\infty)$, finite on compact subsets of the positive half-axis. Hence, the quermassintegrals of the function f are described as the first n moments of F (up to the normalization constant κ_n).
In particular, we see that the finiteness of $W_n(f)$ is equivalent to the finiteness of the measure F (namely to the condition $\Lambda < +\infty$), whereas the finiteness of $W_0(f)$ is equivalent to $\int_0^{+\infty} r^n\, dF(r) < +\infty$. Thus we can conclude that the quermassintegrals $W_i(f)$ are finite for all $i = 0, \dots, n$, if and only if they are finite for $i = 0$ and $i = n$.

The above example suggests a simple way to find upper bounds on the quermassintegrals in the general case. Namely, the monotonicity property (iii) stated above readily yields:

Proposition 3.3. *Given a function $f \in \mathcal{Q}^n$, define $\mu_f(r) = \max_{\|x\| \geq r} f(x)$, $r > 0$. Then*

$$W_i(f) \leq \kappa_n \int_0^{+\infty} r^{n-i}\, d\mu_f(r), \qquad i = 0, 1, \dots, n.$$

In particular, all quermassintegrals of f are finite, provided f is bounded and $\int_0^{+\infty} r^n\, d\mu_f(r) < +\infty$.

3.2 Steiner formula

Let $f \in \mathcal{Q}^n$. For $\rho > 0$, consider the function

$$f_\rho(x) = \sup_{y \in B_\rho(x)} f(y)\,.$$

If $f \in \mathcal{Q}^n_\alpha$, using the operations $\oplus$ and $\cdot$ introduced in Section 2.4 on the class $\mathcal{C}_\alpha$, and the isomorphic embeddings Θ_α of Remark 2.2, the function f_ρ may also be rewritten as

$$f_\rho = f \oplus \rho \cdot \Theta_\alpha(B)$$

(recall that $B = B_1(0)$, $\Theta_0(B) = \chi_B$, and $\Theta_\alpha(B) = I_B^{-1}$ for $\alpha \neq 0$). Therefore, the function f_ρ can be seen as a perturbation of f through the unit ball. Actually, the next result provides a functional analogue of the Steiner formula, stating that the integral of f_ρ admits a polynomial expansion in ρ, with coefficients given precisely by the functional quermassintegrals $W_i(f)$'s.

Theorem 3.4. (Steiner-type formula) *Let $f \in \mathcal{Q}^n$. For every $\rho > 0$, there holds*

$$I(f_\rho) = \sum_{i=0}^{n} \binom{n}{i} W_i(f)\, \rho^i. \tag{3.2}$$

Before giving the proof of Theorem 3.4, let us point out that, as a consequence of (3.2), the following properties turn out to be equivalent to each other:

(i) $W_i(f) < +\infty \quad \forall i = 0, \dots, n$;

(ii) $I(f_\rho) < +\infty$ for some $\rho > 0$;

(iii) $I(f_\rho) < +\infty$ for all $\rho > 0$.

In particular, the condition $I(f) < +\infty$ is not sufficient to guarantee that $I(f_\rho) < +\infty$ (as the latter condition implies the boundedness of f). A simple sufficient condition is for instance that f is of class $C^1(\mathbb{R}^n)$, with $I(f) < +\infty$ and

$$\int_{\mathbb{R}^n} \max_{y \in B_\rho(x)} \|\nabla f(y)\|\, dx < +\infty\,;$$

indeed, by using the inequality $f_\rho(x) \leq f(x) + \max_{y \in B_\rho(x)} \|\nabla f(y)\|$, it follows that $I(f_\rho) < +\infty$. Whenever $I(f_\rho)$ is finite, as an immediate consequence of Theorem 3.4, the quermassintegrals $W_i(f)$ can be expressed through differential formulae involving $I(f_\rho)$. In particular, it holds

$$\mathrm{Per}(f) = \lim_{\rho \to 0^+} \frac{I(f_\rho) - I(f)}{\rho} \tag{3.3}$$

and

$$M(f) = \frac{2}{n\kappa_n} \lim_{\rho \to +\infty} \frac{I(f_\rho) - (\kappa_n \max_{\mathbb{R}^n} f)\, \rho^n}{\rho^{n-1}}. \tag{3.4}$$

Remark 3.5. Let $f \in \mathcal{Q}^n$. Denote by K_f the support set $\{f > 0\}$, by $|Df|(\mathbb{R}^n)$ the total variation of f as a BV function on $\mathbb{R}^n$, and by f_+ the interior trace of f on ∂K_f. Then

$$\mathrm{Per}(f) = \int_0^{+\infty} \mathrm{Per}(\{f \geq t\})\, dt = |Df|(\mathbb{R}^n) = \int_{K_f} |\nabla f|\, dx + \int_{\partial K_f} f_+\, d\mathcal{H}^{n-1}, \tag{3.5}$$

where we have used the definition of $\mathrm{Per}(f)$ and the coarea formula. This formula is simplified to

$$\mathrm{Per}(f) = \int_{\mathbb{R}^n} |\nabla f|\, dx,$$

if f is continuously differentiable on the whole $\mathbb{R}^n$ (which also follows from (3.3) in case $I(f_\rho) < +\infty$, for some $\rho > 0$). We point out that (3.5) may be seen as a variant of the integral representation formula given by Theorem 4.6 in [15]: in fact, (3.5) can be derived "formally" by applying Theorem 4.6 in [15] beyond its assumptions (more precisely, by taking therein $\psi(y) = |y|$).

Proof of Theorem 3.4. We start from the well-known elementary identity (which is often used in derivation of various Sobolev-type inequalities)

$$\{f_\rho > t\} = \{f > t\} + \rho B \qquad (\rho, t > 0). \tag{3.6}$$

Define the sets

$$\Omega^t = \{f > t\}, \qquad \Omega^t_\rho = \{f_\rho > t\}, \qquad K^t = \operatorname{cl}\Omega^t, \qquad K^t_\rho = \operatorname{cl}\Omega^t_\rho.$$

Since $f \in \mathcal{Q}^n$, the convex sets Ω^t are bounded, so are Ω^t_ρ, and one has $\mathcal{H}^n(\Omega^t_\rho) = \mathcal{H}^n(K^t_\rho)$. Then, by virtue of Cavalieri's principle, we can express $I(f_\rho)$ as

$$I(f_\rho) = \int_0^{+\infty} \mathcal{H}^n(\Omega^t_\rho)\, dt = \int_0^{+\infty} \mathcal{H}^n(K^t_\rho)\, dt.$$

By (3.6), we have

$$K^t_\rho = \operatorname{cl}(\Omega^t_\rho) = \operatorname{cl}(\Omega^t + \rho B) = \operatorname{cl}(\Omega^t) + \rho B = K^t + \rho B.$$

Hence,

$$I(f_\rho) = \int_0^{+\infty} \mathcal{H}^n(K^t + \rho B)\, dt.$$

Finally, using the Steiner formula for the convex bodies K^t, we obtain

$$I(f_\rho) = \int_0^{+\infty} \sum_{i=0}^{n} \rho^i \binom{n}{i} W_i(K^t)\, dt = \sum_{i=0}^{n} \rho^i \binom{n}{i} \int_0^{+\infty} W_i(K^t)\, dt,$$

which is (3.2). □

3.3 A dual expansion

One can observe that the functional notion of mean introduced in Definition 3.1 is not linear with respect to the sum in $\mathcal{Q}^n_\alpha$ (unless $\alpha = -\infty$), while this is always the case for the mean width of convex bodies. As the latter quantity can be also defined, up to a dimensional constant, as

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^n(B + \rho K) - \mathcal{H}^n(B)}{\rho} \qquad \forall\, K \in \mathcal{K}^n\,,$$

it is natural to ask what happens, if in place of considering the map $\rho \mapsto I\big(f \oplus \rho \cdot \Theta_\alpha(B)\big)$ as done in the previous section, one looks at its "dual" map $\rho \mapsto I\big(\Theta_\alpha(B) \oplus \rho \cdot f\big)$.
Here we focus attention on the case $\alpha = 0$, namely on the class $\mathcal{Q}^n_0$ of log-concave functions with the corresponding algebraic operation. As $\Theta_0(B) = \chi_B$, we set

$$\Psi(\rho) := I(\chi_B \oplus \rho \cdot f). \tag{3.7}$$

and

$$\widetilde{M}(f) := \lim_{\rho \to 0^+} \frac{\Psi(\rho) - \Psi(0)}{\rho} = \Psi'(0^+),$$

whenever the latter limit exists. The first derivative of the mapping $\rho \mapsto \Psi(\rho)$ is by construction linear in f (exactly as it occurs for the notion of the mean width introduced by Klartag and Milman in [20], mentioned in the Introduction). It turns out that $\widetilde{M}(f)$ is finite only when the support of f is compact: in this case it can be computed explicitly, and it is given precisely by the logarithm of the maximum of f plus the mean width of the support of f. More precisely we have the following result, which is somehow dual to Theorem 3.4. For this reason we call it "dual Steiner-type formula"; however we stress that using this expression is somehow an abuse, since in this case the function $\rho \mapsto \Psi(\rho)$ is *not* a polynomial in ρ.

Theorem 3.6. (Dual Steiner-type formula) *Let $f \in \mathcal{Q}_0^n$ and let Ψ be the mapping defined in (3.7). For every $\rho > 0$, there holds*

$$\Psi(\rho) = \sum_{j=0}^{n} \binom{n}{j} \rho^{j+1} \int_0^{+\infty} W_{n-j}(\mathrm{cl}\{f > t\})\, t^{\rho-1}\, dt. \tag{3.8}$$

In particular, setting $K_f := \{f > 0\}$, it holds

$$\widetilde{M}(f) = \begin{cases} \kappa_n \log(\max_{\mathbb{R}^n}(f)) + nW_{n-1}(K_f), & \text{if } K_f \in \mathcal{K}^n \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.9}$$

For the proof of Theorem 3.6 the following elementary Lemma is needed.

Lemma 3.7. *For every non-increasing function $g : (0, m] \to \mathbb{R}_+$,*

$$\lim_{\rho\to 0^+} \rho \int_0^m g(t) t^{\rho-1}\, dt = \lim_{t\to 0^+} g(t)\,.$$

Proof. Set $L := g(0+) = \lim_{t\to 0^+} g(t)$. With a change of variable, we have

$$\rho \int_0^m g(t) t^{\rho-1}\, dt = \int_0^{m^\rho} g\big(t^{1/\rho}\big)\, dt.$$

If $m \geq 1$, write

$$\int_0^{m^\rho} g\big(t^{1/\rho}\big)\, dt = \int_1^{m^\rho} g\big(t^{1/\rho}\big)\, dt + \int_0^1 g\big(t^{1/\rho}\big)\, dt. \tag{3.10}$$

We observe that the first integral in the r.h.s. of (3.10) is infinitesimal: since g is non-increasing, we have

$$\int_1^{m^\rho} g\big(t^{1/\rho}\big)\, dt \leq g(1)\big(m^\rho - 1\big).$$

Concerning the second integral in the r.h.s. of (3.10), we observe that, as $\rho \to 0^+$, the functions $g\big(t^{1/\rho}\big)$ do not decrease and converge pointwise to L on $(0,1)$. Hence, $\int_0^1 g\big(t^{1/\rho}\big)\, dt \to +\infty$, by the monotone convergence theorem. Thus, the statement is proved for $m \geq 1$.
If $0 < m < 1$, for any prescribed $\varepsilon > 0$, we have $m^\rho > 1 - \varepsilon$, for all ρ small enough. Then regardless of whether $L = +\infty$ or $L < +\infty$, we have

$$L \geq Lm^\rho \geq \int_0^{m^\rho} g\big(t^{1/\rho}\big)\, dt \geq \int_0^{1-\varepsilon} g\big(t^{1/\rho}\big)\, dt \to L(1-\varepsilon), \qquad \text{as } \rho \to 0^+,$$

where we used the monotone convergence theorem once more. The statement then follows by the arbitrariness of $\varepsilon > 0$. □

Proof of Theorem 3.6. Let us set for brevity $f^{(\rho)} := \chi_B \oplus \rho \cdot f$, which in explicit form reads

$$f^{(\rho)}(z) = \sup\big\{f(y)^\rho : x + \rho y = z,\ \|x\| \leq \rho\big\}, \qquad \forall\, z \in \mathbb{R}^n.$$

The above definition yields

$$\{f^{(\rho)} > t\} = \Big\{x : f^\rho\Big(\frac{x}{\rho}\Big) > t\Big\} + B = \rho\,\{f > t^{1/\rho}\} + B.$$

Therefore,

$$I\big(f^{(\rho)}\big) = \int_0^{m^\rho} \mathcal{H}^n\Big(\rho\,\{f > t^{1/\rho}\} + B\Big)\,dt = \rho \int_0^m \mathcal{H}^n\Big(\rho\,\{f > t\} + B\Big) t^{\rho-1}\,dt. \tag{3.11}$$

Letting $\Omega^t = \{f > t\}$ and $K^t = \mathrm{cl}(\Omega^t)$, we have

$$\mathcal{H}^n\Big(\rho\Omega^t + B\Big) = \mathcal{H}^n\Big(\rho K^t + B\Big) = \sum_{j=0}^n \rho^j \binom{n}{j} W_{n-j}(K^t). \tag{3.12}$$

Inserting (3.12) into (3.11), the equality (3.8) is proved.
Let us now prove (3.9). Set $m = \max_{\mathbb{R}^n} f$. We claim that all the terms corresponding to $j \geq 2$ on the right-hand side of (3.8) are $o(\rho)$, as $\rho \to 0$. To see this, recall that since the functions in $\mathcal{Q}_0^n$ are log-concave and are vanishing at infinity, they must decay exponentially fast (at least). Hence, there exist constants $\alpha > 0$, $\beta \in \mathbb{R}$, such that

$$f(x) \leq e^{-(\alpha|x|+\beta)} \qquad \forall x \in \mathbb{R}^n$$

(see Lemma 2.5 in [15]), which yields

$$K^t \subseteq \{x : e^{-(\alpha|x|+\beta)} \geq t\} = \Big\{x : |x| \leq -\frac{\beta + \log t}{\alpha}\Big\}.$$

Letting $R(t) = \max\{0, -\frac{\beta+\log t}{\alpha}\}$, we get

$$\rho^{j+1} \int_0^m W_{n-j}(\mathrm{cl}\{f > t\})\, t^{\rho-1}\,dt \leq \rho^{j+1} \int_0^m R(t)^j\, t^{\rho-1}\,dt = \rho^j \int_0^{m^\rho} R(t^{1/\rho})^j\ dt \leq C\rho^j,$$

and the claim is proved.
Next we observe that the terms corresponding to $j = 0$ and $j = 1$ in the sum of (3.8) are given respectively by

$$\rho \int_0^m W_n(K^t)\, t^{\rho-1}\,dt = \kappa_n\, \rho \int_0^m t^{\rho-1}\,dt = \kappa_n m^\rho = \kappa_n m^\rho = I(f_0)\, m^\rho$$

(where in the first equality we have exploited the identity $W_n(K^t) = \kappa_n$), and by

$$n\rho^2 \int_0^m W_{n-1}(K^t)\, t^{\rho-1}\,dt.$$

Summarizing, we have

$$I(f^{(\rho)}) = I(f_0)\, m^\rho + n\rho^2 \int_0^m W_{n-1}(K^t)\, t^{\rho-1}\,dt + o(\rho),$$

whence

$$\frac{I(f^{(\rho)}) - I(f_0)}{\rho} = \kappa_n \frac{m^\rho - 1}{\rho} + n\rho \int_0^m W_{n-1}(K^t)\, t^{\rho-1}\,dt + \frac{o(\rho)}{\rho}.$$

In the limit as $\rho \to 0^+$, the first addendum tends to $\kappa_n \log m$, whereas the second one tends to $nW_{n-1}(K_f)$ thanks to Lemma 3.7. ◻

4 Generalized Prékopa-Leindler inequalities

This section is entirely devoted to the study of generalized versions of the Prékopa-Leindler inequality. More precisely: in Section 4.1 we prove some variants of such inequality for functions of one variable; in Sections 4.2-4.3 we extend Prékopa-Leindler's Theorem from the usual case of the volume functional to the general case of arbitrary monotone concave functionals on $\mathcal{K}^n$ (including as special cases the functional quermassintegrals); in Section 4.4 we show that this generalized concavity fails to be true if one chooses to define the perimeter of quasi-concave functions in a different, though apparently natural, way.

4.1 Variant of Prékopa-Leindler inequality in dimension one

Let us return to Theorem 2.6, which we consider here in dimension one for non-negative functions defined on $(0,+\infty)$. In some situations it is desirable to replace the arithmetic mean $M_1^{(\lambda)}(x,y)$ on the left-hand side of (2.14) by more general means $M_\gamma^{(\lambda)}(x,y)$. In the (rather typical) case, when h is non-increasing (and if $\gamma<1$), this would give a strengthened one-dimensional variant of this theorem, since the hypothesis would be weaker (due to the inequality $M_\gamma^{(\lambda)}(x,y)\le M_1^{(\lambda)}(x,y)$). The case $\gamma=\alpha=0$ (and hence $\beta=0$) was considered by K. Ball [3], who showed that the hypothesis

$$h(M_0^{(\lambda)}(x,y)) \ge M_0^{(\lambda)}(f(x),g(y)), \qquad \forall\, x,y>0, \tag{4.1}$$

implies

$$\int_0^{+\infty} h \;\ge\; M_0^{(\lambda)}\left(\int_0^{+\infty} f, \int_0^{+\infty} g\right). \tag{4.2}$$

Actually, this assertion immediately follows from Prekopa-Leindler's Theorem 2.5, when it is applied in one dimension to the functions $f(e^{-x})e^{-x}$, $g(e^{-x})e^{-x}$ and $h(e^{-x})e^{-x}$.
Below we propose an extension of Ball's observation to general values $\gamma\le 1$.

Theorem 4.1. *Let $\lambda\in(0,1)$, $\gamma\in[-\infty,1]$ and $\alpha\in[-\gamma,+\infty]$. Let f,g,h be non-negative measurable functions on $(0,+\infty)$. If*

$$h(M_\gamma^{(\lambda)}(x,y)) \ge M_\alpha^{(\lambda)}(f(x),g(y)), \qquad \forall\, x,y>0 \text{ such that } f(x)g(y)>0\,, \tag{4.3}$$

then

$$\int_0^{+\infty} h \;\ge\; M_\beta^{(\lambda)}\left(\int_0^{+\infty} f, \int_0^{+\infty} g\right) \qquad \text{with } \beta=\frac{\alpha\gamma}{\alpha+\gamma}\,. \tag{4.4}$$

In the extreme cases $\alpha=-\gamma$ and $\alpha=+\infty$, the definition of β in (4.4) is understood respectively as $\beta=-\infty$ and $\beta=\gamma$. In addition, we put $\beta=-\infty$ in case $\gamma=-\infty$.

Before giving the proof of Theorem 4.1 let us recall that, as a consequence of the generalized Hölder inequality, we have the following elementary inequality: For all $u_1,u_2,v_1,v_2\ge 0$ and $\lambda\in(0,1)$, it holds

$$M_{\alpha_1}^{(\lambda)}(u_1,v_1)M_{\alpha_2}^{(\lambda)}(u_2,v_2) \;\ge\; M_{\alpha_0}^{(\lambda)}(u_1u_2,v_1v_2), \tag{4.5}$$

whenever

$$\alpha_1+\alpha_2>0, \qquad \frac{1}{\alpha_0}=\frac{1}{\alpha_1}+\frac{1}{\alpha_2}. \tag{4.6}$$

Inequality (4.5) also holds in the following cases:

- $\alpha_0 = \alpha_1 = 0, \ \ 0 \le \alpha_2 \le +\infty$;
- $\alpha_0 = \alpha_2 = 0, \ \ 0 \le \alpha_1 \le +\infty$;
- $\alpha_0 = -\infty, \ \ \alpha_1 + \alpha_2 \ge 0$.

The latter includes the cases $\alpha_1 = -\infty$, $\alpha_2 = +\infty$ and $\alpha_1 = +\infty$, $\alpha_2 = -\infty$. Clearly, $\alpha_0 > 0$ when $\alpha_1 > 0$ and $\alpha_2 > 0$; on the other hand if $\alpha_1 < 0 < \alpha_2$ or $\alpha_2 < 0 < \alpha_1$, then necessarily $\alpha_0 < 0$.

Proof of Theorem 4.1.
If $\gamma = 1$, we are reduced to Bracamp-Lieb's Theorem 2.6 in dimension one.
If $\gamma = 0$, then $\beta = 0$ regardless of $\alpha \ge 0$. But the hypothesis (4.3) is weaker for $\alpha = 0$, and this case corresponds to Ball's result (4.1) $\Rightarrow$ (4.2).
Hence, we may assume that $-\infty \le \gamma < 1$, $\gamma \neq 0$. Let $-\gamma \le \alpha \le +\infty$ with $\gamma > -\infty$. In terms of the functions

$$u(x) = f(x^{1/\gamma}), \quad v(x) = g(x^{1/\gamma}), \quad w(x) = h(x^{1/\gamma})$$

the hypothesis (4.3) may be rewritten as

$$w(z) \ge M_\alpha^{(\lambda)}(u(x), v(y)), \qquad z = (1-\lambda)x + \lambda y, \quad \forall\, x, y > 0 \ \text{ such that } \ u(x)v(y) > 0\ . \tag{4.7}$$

Here and below we omit for brevity the parameter λ and write just M_α instead of $M_\alpha^{(\lambda)}$.
We apply the inequality (4.5) with $\alpha_1 = \alpha$, $\alpha_2 = \gamma' = \frac{\gamma}{1-\gamma}$, in which case the condition (4.6) becomes $\alpha + \gamma' > 0$. Using (4.7), it gives

$$\begin{aligned} w(z) z^{1/\gamma'} &= w(z)\, M_{\gamma'}(x^{1/\gamma'}, y^{1/\gamma'}) \\ &\ge M_\alpha(u(x), v(y))\, M_{\gamma'}(x^{1/\gamma'}, y^{1/\gamma'}) \ \ge \ M_{\alpha_0}(u(x)x^{1/\gamma'}, v(y)y^{1/\gamma'}), \end{aligned}$$

where α_0 is defined by

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} + \frac{1}{\gamma'} = \frac{1}{\alpha} + \frac{1}{\gamma} - 1.$$

Here, in case $\alpha = +\infty$, we have $\alpha_0 = \gamma'$, and in case $\alpha = 0$, one should put $\alpha_0 = 0$ (with constraint $\gamma > 0$ in view of $\alpha + \gamma' > 0$).
Thus, the new three functions $u(x)x^{1/\gamma'}$, $v(x)x^{1/\gamma'}$ and $w(x)x^{1/\gamma'}$ satisfy the condition (2.14) in one-dimensional Brascamp-Lieb's Theorem with parameter α_0. Hence, if $\alpha_0 \ge -1$, we obtain the inequality (2.15) for these functions, that is,

$$\int_0^{+\infty} w(z) z^{1/\gamma'} dz \ \ge \ M_\beta^{(\lambda)}\left(\int_0^{+\infty} u(x)x^{1/\gamma'} dx, \int_0^{+\infty} v(y)y^{1/\gamma'} dy\right) \tag{4.8}$$

with $\beta = \frac{\alpha_0}{1+\alpha_0}$. But

$$\int_0^{+\infty} u(x)x^{1/\gamma'}\, dx = \int_0^{+\infty} f(x^{1/\gamma})\, x^{1/\gamma - 1}\, dx = |\gamma| \int_0^{+\infty} f(x)\, dx,$$

and similarly for the couples (v, g) and (w, h). In addition,

$$\beta = \frac{1}{\frac{1}{\alpha_0} + 1} = \frac{1}{\frac{1}{\alpha} + \frac{1}{\gamma}} = \frac{\alpha\gamma}{\alpha + \gamma}.$$

Here, $\beta = \gamma$ for $\alpha = +\infty$, and $\beta = 0$ for $\alpha = 0$ and $\gamma > 0$, and $\beta = -\infty$, for $\alpha = -\gamma$.

Thus, (4.8) yields the desired inequality (4.4) of Theorem 4.1, provided that:

a) $\alpha + \gamma' > 0$;
b) $\alpha_0 \geq -1$.

Case $0 < \gamma < 1$.
Then $\gamma' > 0$. If $\alpha > 0$, then $\alpha_0 > 0$, so both *a*) and *b*) are fulfilled. If $\alpha = 0$, then $\alpha_0 = 0$, so *a*) and *b*) are fulfilled, as well. If $\alpha < 0$, then necessarily $\alpha_0 < 0$ (as already noticed before). In this case,

$$\alpha + \gamma' > 0 \Leftrightarrow -\alpha < \gamma' \Leftrightarrow -\frac{1}{\alpha} > \frac{1}{\gamma'} \Leftrightarrow \frac{1}{\alpha} + \frac{1}{\gamma} < 1.$$

In addition, since *b*) may be rewritten as $-\frac{1}{\alpha_0} \geq 1$, this condition is equivalent to $-(\frac{1}{\alpha} + \frac{1}{\gamma'}) \geq 1 \Leftrightarrow \frac{1}{\alpha} + \frac{1}{\gamma} \leq 0 \Leftrightarrow \gamma \geq -\alpha$, which was assumed.

Case $-\infty < \gamma < 0$.
Then $\gamma' < 0$ and $\alpha > 0$ to meet *a*). Again $\alpha_0 < 0$, so *b*) may be written as $-\frac{1}{\alpha_0} \geq 1$. As before, we have

$$\alpha + \gamma' > 0 \Leftrightarrow \alpha > -\gamma' \Leftrightarrow \frac{1}{\alpha} > -\frac{1}{\gamma'} \Leftrightarrow \frac{1}{\alpha} + \frac{1}{\gamma} < 1.$$

In addition, *b*) is equivalent to $-(\frac{1}{\alpha} + \frac{1}{\gamma'}) \geq 1 \Leftrightarrow \frac{1}{\alpha} + \frac{1}{\gamma} \leq 0 \Leftrightarrow \gamma \geq -\alpha$.

Case $\gamma = -\infty$.
This case may be treated by a direct argument. Indeed, necessarily $\alpha = +\infty$, and the hypothesis (4.3) takes the form

$$h(\min(x,y)) \geq \max(f(x), g(y)) \qquad \forall\, x, y \text{ such that } f(x)g(y) > 0\ . \tag{4.9}$$

We may assume that both f and g are not identically zero. Put

$$a = \sup\{x > 0 : f(x) > 0\}, \qquad b = \sup\{y > 0 : g(y) > 0\},$$

and let for definiteness $a \leq b \leq +\infty$. If $0 < x < a$ and $f(x) > 0$, one may choose $y \geq x$ such that $g(y) > 0$, and then (4.9) gives $h(x) \geq f(x)$. Hence,

$$\begin{aligned} \int_0^{+\infty} h(x)\,dx \ &\geq\ \int_{\{0<x<a,\, f(x)>0\}} h(x)\,dx \\ &\geq\ \int_{\{0<x<a,\, f(x)>0\}} f(x)\,dx \ =\ \int_0^{+\infty} f(x)\,dx. \end{aligned}$$

As a result,

$$\int_0^{+\infty} h(x)\,dx \geq \min\Big\{\int_0^{+\infty} f(x)\,dx, \int_0^{+\infty} g(x)\,dx\Big\},$$

which is the desired inequality (4.4) with $\beta = -\infty$.
Theorem 4.1 is now proved. □

4.2 Prékopa-Leindler inequality for monotone γ-concave functionals.

We are now ready to extend Theorem 2.6 by Brascamp and Lieb to general monotone γ-concave set functionals Φ, mentioned in the Introduction. To be more precise, a functional Φ defined on the class of all Borel subsets of $\mathbb{R}^n$ with values in $[0, +\infty]$ will be said to be monotone, if

$$\Phi(K_0) \leq \Phi(K_1), \quad \text{whenever} \ \ K_0 \subseteq K_1,$$

and to be (γ, λ)-concave with parameters $\gamma \in [-\infty, +\infty]$ and $\lambda \in (0, 1)$, if

$$\Phi((1-\lambda)K_0 + \lambda K_1) \geq M_\gamma^{(\lambda)}\big(\Phi(K_0), \Phi(K_1)\big), \tag{4.10}$$

for all Borel sets K_0, K_1 such that $\Phi(K_0) > 0$ and $\Phi(K_1) > 0$. If (4.10) is fulfilled for an arbitrary $\lambda \in (0, 1)$, then we simply say that Φ is γ-concave.
We always assume that $\Phi(\emptyset) = 0$. In particular, the requirement $\Phi(K) > 0$ ensures that K is non-empty.
If Φ is monotone, we extend it canonically to the class of all Borel measurable non-negative functions on $\mathbb{R}^n$ by setting

$$\Phi(f) = \int_0^{+\infty} \Phi(\{f \geq r\})\, dr.$$

In case Φ is well-defined only on $\mathcal{K}^n$, the above definition remains well-posed in the class of all semi-continuous, quasi-concave non-negative functions on $\mathbb{R}^n$.

Theorem 4.2. *Let Φ be a monotone (γ, λ)-concave functional on Borel sets of $\mathbb{R}^n$ (respectively, on $\mathcal{K}^n$), with parameters $\gamma \in [-\infty, 1]$ and $\lambda \in (0, 1)$. Let $\alpha \in [-\gamma, +\infty]$, and let $f, g, h : \mathbb{R}^n \to [0, +\infty)$ be Borel measurable (respectively, semi-continuous quasi-concave) functions. If*

$$h((1-\lambda)x + \lambda y) \geq M_\alpha^{(\lambda)}(f(x), g(y)) \qquad \forall\, x, y \in \mathbb{R}^n \text{ such that } f(x)g(y) > 0, \tag{4.11}$$

then

$$\Phi(h) \geq M_\beta^{(\lambda)}(\Phi(f), \Phi(g)) \quad \text{where } \beta := \frac{\alpha\gamma}{\alpha + \gamma}\ . \tag{4.12}$$

Before giving the proof, several comments on the above statement are in order.

Remark 4.3. (i) Theorem 2.6 by Brascamp-Lieb can be recast as a special case from Theorem 4.2 by taking for the functional Φ the Lebesgue measure on $\mathbb{R}^n$, in which case $\gamma = \frac{1}{n}$.

(ii) In the extreme cases the interpretation of the parameter β in Theorem 4.2, as well as in the Corollaries hereafter, has to be the same as in Theorem 4.1.

(iii) In particular, $\beta = \gamma$ for $\alpha = +\infty$. Thus, if $f = \chi_{K_0}$, $g = \chi_{K_1}$, and $h = \chi_{(1-\lambda)K_0 + \lambda K_1}$, the inequality (4.11) is fulfilled, and (4.12) gives back the definition of γ-concavity of Φ. In other words, Theorem 4.2 does represent a functional form for the geometric inequality (4.10).

(iv) The proof of Theorem 4.2 given below is obtained without using an induction argument on the space dimension n, but just combining the γ-concavity inequality satisfied by assumption by Φ, with the one-dimensional functional result stated in Theorem 4.1.

(v) If a functional Φ is monotone and γ-concave on a given subclass of Borel sets (possibly different than $\mathcal{K}^n$), our proof of Theorem 4.2 shows that the implication (4.11) $\Rightarrow$ (4.12) holds true for all Borel measurable functions whose level sets belong to the class under consideration.

Proof of Theorem 4.2. Denote by $K_f(r)$ the level sets $\{f \geq r\}$, and similarly for g and h. By the hypothesis (4.11), we have the set inclusion

$$(1-\lambda)K_f(r) + \lambda K_g(s) \subseteq K_h\big(M_\alpha^{(\lambda)}(r, s)\big), \tag{4.13}$$

which makes sense and is valid for all $r, s > 0$ such that $\Phi(K_f(r)) > 0$ and $\Phi(K_g(s)) > 0$. Using (4.13), together with the monotonicity and (γ, λ)-concavity assumption on Φ, we see that the functions

$$u(r) := \Phi\big(\{f \geq r\}\big), \quad v(r) := \Phi\big(\{g \geq r\}\big), \quad w(r) := \Phi\big(\{h \geq r\}\big)$$

satisfy the relation

$$w\big(M_\gamma^{(\lambda)}(r,s)\big) \geq M_\alpha^{(\lambda)}(u(r), v(s)), \qquad \text{whenever} \;\; u(r)v(s) > 0\ .$$

Therefore, we are in position to apply Theorem 4.1 to the triple (u, v, w), which yields

$$\int_0^{+\infty} w(r)\, dr \;\geq\; M_\beta^{(\lambda)}\left(\int_0^{+\infty} u(r)\, dr, \int_0^{+\infty} v(r)\, dr\right)$$

with $\beta = \frac{\alpha\gamma}{\alpha+\gamma}$. This is exactly (4.12). □

4.3 Hyperbolic functionals.

Let us now specialize Theorem 4.2 to an important family of geometric functionals called hyperbolic or convex.

Definition 4.4. A monotone functional Φ defined on the class of all Borel subsets of $\mathbb{R}^n$ with values into $[0, +\infty]$ is said to be *hyperbolic*, if

$$\Phi((1-\lambda)K_0 + \lambda K_1) \geq \min\big\{\big(\Phi(K_0), \Phi(K_1)\big)\big\}, \tag{4.14}$$

for all $\lambda \in (0,1)$ and for all Borel sets K_0, K_1 in $\mathbb{R}^n$ such that $\Phi(K_0) > 0$ and $\Phi(K_1) > 0$.

We adopt a similar definition also if Φ is defined only on some sublass of Borel sets, such as $\mathcal{K}^n$. Thus, hyperbolic functionals are exactly $(-\infty)$-concave functionals, *i.e.*, they satisfy the inequality (4.10) with $\gamma = -\infty$.

Apparently, the application of Theorem 4.2 to hyperbolic functionals seems to be not so interesting. Indeed, when $\gamma = -\infty$, one has $\alpha = +\infty$, in which case the hypothesis (4.11) considerably restricts the range of applicability of the resulting inequality (4.12). Nevertheless, the situation is much more favorable if the hyperbolicity condition (4.14) is combined with some homogeneity property.

Definition 4.5. A functional Φ defined on the class of all Borel subsets of $\mathbb{R}^n$ (respectively on convex compact sets in $\mathbb{R}^n$) is said to be *homogeneous of order* ρ (with $\rho \in \mathbb{R} \setminus 0$), if

$$\Phi(\lambda K) = \lambda^\rho\, \Phi(K), \tag{4.15}$$

for all $\lambda > 0$ and for all Borel sets K in $\mathbb{R}^n$ (respectively, for all $K \in \mathcal{K}^n$).

Combining (4.14) and (4.15) yields the following observation, which is elementary and well-known, especially for the Lebesgue measure. However, because of its importance, we state it separately and in a general setting:

Proposition 4.6. *Any hyperbolic functional Φ, which is homogeneous of order ρ, is γ-concave for $\gamma = 1/\rho$.*

Proof. Let $\Phi(K_0) > 0$ and $\Phi(K_1) > 0$. We have to show that

$$\Phi(K_0 + K_1) \geq \big(\Phi(K_0)^\gamma + \Phi(K_1)^\gamma\big)^{1/\gamma}, \tag{4.16}$$

If $\Phi(K_0 + K_1) = +\infty$, then (4.16) is immediate. Otherwise, $0 < \Phi(K_0) < +\infty$ and $0 < \Phi(K_1) < +\infty$, by the monotonicity of Φ. In this case, set

$$K_0' := \frac{1}{\Phi(K_0)^\gamma}\, K_0 \qquad \text{and} \qquad K_1' := \frac{1}{\Phi(K_1)^\gamma}\, K_1,$$

so that, by the homogeneity property (4.15), $\Phi(K_0') = \Phi(K_1') = 1$. Next, applying the assumption (4.14) to K_0' and K_1', with

$$\lambda = \frac{\Phi(K_1)^\gamma}{\Phi(K_0)^\gamma + \Phi(K_1)^\gamma}\,,$$

and using once more (4.15), we arrive exactly at the desired inequality (4.16).
Finally, being applied to the sets $(1-\lambda)K_0$ and λK_1 with arbitrary $\lambda \in (0,1)$, (4.16) turns into (4.10), expressing the γ-concavity property of the functional Φ. □

As a consequence of Proposition 4.6, one may apply Theorem 4.2 to hyperbolic functionals Φ, which are homogeneous of order ρ, as long as $\gamma = \frac{1}{\rho} \leq 1$, that is, when $\rho < 0$ or $\rho \geq 1$. In that case, if $\lambda \in (0,1)$, $\alpha \in [-\gamma, +\infty]$, and if the functions $f, g, h \geq 0$ on $\mathbb{R}^n$ satisfy

$$h((1-\lambda)x + \lambda y) \geq M_\alpha^{(\lambda)}(f(x), g(y)) \qquad \forall\, x, y \in \mathbb{R}^n \text{ such that } f(x)g(y) > 0, \tag{4.17}$$

we obtain

$$\Phi(h) \,\geq\, M_\beta^{(\lambda)}(\Phi(f), \Phi(g)) \quad \text{ with } \beta = \frac{\alpha\gamma}{\alpha+\gamma} = \frac{\alpha}{1+\alpha\rho}. \tag{4.18}$$

Similarly as done for Theorem 2.7, one may develop a further generalization of this statement, involving the means $M_\alpha^{(s,t)}$ for arbitrary s and $t > 0$, not necessarily satisfying $s + t = 1$, and taking in (4.17) the "optimal" function

$$h = s \cdot f \oplus t \cdot g.$$

Here, the operations $\oplus$ and $\cdot$ are those in $\mathcal{C}_\alpha$ for a fixed value $\alpha \geq -\gamma$. Arguing as before, let for simplicity α be non-zero and finite. By its definition, for all $x, y \in \mathbb{R}^n$, the above function h satisfies

$$h(sx + ty) \geq M_\alpha^{(s,t)}(f(x), g(y)) = (s+t)^{1/\alpha}\Big(\frac{s}{s+t}\, f(x)^\alpha + \frac{t}{s+t}\, g(y)^\alpha\Big)^{1/\alpha},$$

which means that the triple $(f, g, \tilde{h})$, where

$$\tilde{h}(z) := (s+t)^{-1/\alpha}\, h((s+t)\, z),$$

satisfies the hypothesis (4.17) with $\lambda = \frac{t}{s+t}$. Hence, we obtain (4.18), *i.e.*,

$$\Phi\big(\tilde{h}\big) \,\geq\, \left[\frac{s}{s+t}\, \Phi(f)^\beta + \frac{t}{s+t}\, \Phi(g)^\beta\right]^{1/\beta}. \tag{4.19}$$

Changing the variable and using the homogeneity property (4.15), we find

$$\begin{aligned}
\Phi\big(\tilde{h}\big) &= \int_0^{+\infty} \Phi\big(\{z : h((s+t)\, z) \geq (s+t)^{1/\alpha} r\}\big)\, dr \\
&= (s+t)^{-1/\alpha} \int_0^{+\infty} \Phi\big(\{z : h((s+t)\, z) \geq r\}\big)\, dr \\
&= (s+t)^{-\rho-1/\alpha} \int_0^{+\infty} \Phi\big(\{z : h(z) \geq r\}\big)\, dr \\
&= (s+t)^{-\rho-1/\alpha}\; \Phi(h).
\end{aligned}$$

Taking into account that $\rho = \frac{1}{\alpha} + \frac{1}{\beta}$, the inequality (4.19) can be reformulated as in the following statement, where we include the limit case $\alpha = +\infty$ as well.

Theorem 4.7. *Let Φ be a hyperbolic functional defined on Borel sets of $\mathbb{R}^n$ (respectively, on $\mathcal{K}^n$), which is homogeneous of order ρ, with $\rho < 0$ or $\rho \geq 1$. Let $s, t > 0$, let $\alpha \in [-\frac{1}{\rho}, +\infty]$, and let $f, g : \mathbb{R}^n \to [0, +\infty]$ be measurable (respectively, semi-continuous quasi-concave) functions. Then*

$$\Phi(s \cdot f \oplus t \cdot g) \geq M_\beta^{(s,t)}\big(\Phi(f), \Phi(g)\big), \qquad \text{where } \beta := \frac{\alpha}{1 + \alpha\rho}. \tag{4.20}$$

In case $\alpha = 0$, the restriction $s + t = 1$ is necessary. In the extreme cases $\alpha = -\frac{1}{\rho}$ and $\alpha = +\infty$, the definition of β in (4.20) has to be understood respectively as $\beta = -\infty$ and $\beta = \frac{1}{\rho}$.

Note that the space dimension n is not involved in (4.20). In particular, when $t = s = 1$, such inequality becomes

$$\Phi(f \oplus g) \,\geq\, \Big[\Phi(f)^{\frac{\alpha}{1+\alpha\rho}} + \Phi(g)^{\frac{\alpha}{1+\alpha\rho}}\Big]^{\frac{1+\alpha\rho}{\alpha}}, \qquad \text{where } \alpha \neq 0, \ \alpha \geq -\frac{1}{\rho}.$$

In a similar way as already discussed in Section 2.5, this may be viewed as an extension to hyperbolic functionals in higher dimensions of the result of Henstock and Macbeath, who considered the case of the Lebesgue measure in dimension $n = 1$.

As a basic example illustrating Theorem 4.7, we apply it to the quermassintegrals $\Phi = W_i$, which are known to be hyperbolic and homogeneous of positive (integer) orders $\rho = n - i$.

Corollary 4.8. *Let $i = 0, 1, \ldots, n-1$. Let $s, t > 0$, let $\alpha \in [-\frac{1}{n-i}, +\infty]$, and let f, g belong to $\mathcal{Q}_\alpha^n$. Then*

$$W_i(s \cdot f \oplus t \cdot g) \geq M_\beta^{(s,t)}\big(W_i(f), W_i(g)\big), \qquad \beta = \frac{\alpha}{1 + \alpha(n-i)}. \tag{4.21}$$

In case $\alpha = 0$, the restriction $s + t = 1$ is necessary.

For $i = 1$, we recall that $nW_1(K)$ represents the perimeter of a set $K \in \mathcal{K}_n$, while according to the co-area formula (*cf.* Remark 3.5), the perimeter of any C^1-smooth function f, vanishing at infinity, can be expressed as the integral

$$\mathrm{Per}(f) = \int |\nabla f(x)|\, dx.$$

Hence, in this special case, and for $s + t = 1$, Corollary 4.8 can be rephrased as:

Corollary 4.9. *Let $\lambda \in (0, 1)$ and $\alpha \in [-\frac{1}{n-1}, +\infty]$, $n \geq 2$. Let $f, g, h : \mathbb{R}^n \to [0, +\infty)$ be C^1-smooth quasi-concave functions, such that $h(z) \to 0$ as $|z| \to \infty$. If*

$$h((1-\lambda)x + \lambda y) \geq M_\alpha^{(\lambda)}(f(x), g(y)) \qquad \forall\, x, y \in \mathbb{R}^n \text{ such that } f(x)g(y) > 0\ ,$$

then

$$\int |\nabla h(z)|\, dz \,\geq\, M_\beta^{(\lambda)}\Big(\int |\nabla f(x)|\, dx, \int |\nabla g(y)|\, dy\Big) \quad \text{with } \beta = \frac{\alpha}{1 + \alpha(n-1)}.$$

Here, the hypothesis that h vanishes at infinity guarantees that f and g vanish at infinity, as well. Moreover, the C^1-smoothness assumption may be relaxed to the property of being locally Lipschitz.

4.4 Counterexamples

Below we show that, choosing a different functional equivalent of the unit ball, may lead to a notion of perimeter which does not satisfy a concavity property like the one stated in Corollary 4.9. To be more precise, let us restrict ourselves to the case $\alpha = 0$, namely to the class $\mathcal{Q}_0^n$ of log-concave functions, endowed with its corresponding algebraic structure. Then, for a given function $f \in \mathcal{Q}_0^n$, the definition of the perimeter given in Section 2 amounts to

$$\mathrm{Per}(f) = \lim_{\rho \to 0^+} \frac{I(f \oplus \rho \cdot \chi_B) - I(f)}{\rho}\,.$$

In this definition, one might be willing to replace χ_B by another log-concave function acting as a unitary ball. A natural choice would be the Gaussian function

$$e^{-|x|^2/2}\,,$$

or, more generally,

$$g_q(x) = e^{-|x|^q/q}\,,$$

with $q \geq 1$. Note that this function tends to χ_B as $q \to +\infty$. In this case one could then define

$$\mathrm{Per}_q(f) := \lim_{\rho \to 0^+} \frac{I(f \oplus \rho \cdot g_q) - I(f)}{\rho},$$

whenever this limit exists. It was proved in [15] that, under suitable assumption of smoothness, decay at infinity and strict convexity of f (see Theorem 4.5 in [15] for the precise statement), the following representation formula holds:

$$\mathrm{Per}_q(f) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u(x)|^p\, f(x)\, dx\,,$$

where $p = \frac{q}{q-1}$ is the conjugate Hölder exponent of q, and $f = e^{-u}$. The aim of this section is to show that $\mathrm{Per}_q(f)$ does not have the same significant properties shown in the previous sections for $\mathrm{Per}(f)$, and in particular it does not verify a generalized Prékopa-Leindler inequality.
For simplicity, given $f \in \mathcal{Q}_0^n$ and $p \in (0, \infty)$ let

$$I_p(f) = \int_{\mathbb{R}^n} |\nabla u(x)|^p\, f(x)\, dx\,.$$

We want to show that, if $f_0, f_1 \in \mathcal{Q}_0^n$, $t \in [0,1]$, and $f_t := (1-t) \cdot f_0 \oplus t \cdot f_1$, the following inequality is in general false:

$$I_p(f_t) \geq (I_p(f_0))^{1-t} (I_p(f_1))^t\,. \tag{4.22}$$

We will consider log-concave functions of the form

$$f(x) = e^{-h_K(x)}, \tag{4.23}$$

where K is a convex body in $\mathbf{R}^n$ and h_K is the support function of K. We will always assume that K contains the origin as *interior point*. Since support functions are sub-linear and positively homogeneous of order one (see [33]), in particular they are convex. Thus a function of the form (4.23) is log-concave (and it is also real-valued and non-negative).
The following result is probably well-known; we include the proof for the sake of completeness.

Proposition 4.10. *Let K_0, K_1 be convex bodies in $\mathbf{R}^n$, let $f_0 = e^{-h_{K_0}}$, $f_1 = e^{-h_{K_1}}$, let $t \in [0,1]$, and let $f_t := (1-t) \cdot f_0 \oplus t \cdot f_1$. Then*

$$f_t = e^{-h_{K_0 \cap K_1}} .$$

Proof. Set $u_t := -\log(f_t)$; we want to prove that $u_t = h_{K_0 \cap K_1}$. For every $z \in \mathbf{R}^n$, setting for brevity $h_i = h_{K_i}$ for $i = 0, 1$, we have

$$\begin{aligned} u_t(z) &= \inf\{(1-t)h_0(x) + t h_1(y) \,:\, (1-t)x + ty = z\} \\ &= \inf\{h_0(x) + h_1(y) \,:\, x + y = z\} . \end{aligned}$$

This means that u_t is the *infimal convolution* of h_0 and h_1. By Theorem 16.4 in [29] we have

$$(u_t)^* = (h_0)^* + (h_1)^*$$

where u^* denotes the usual conjugate of convex functions:

$$u^*(z) = \sup_w [(z, w) - u(w)] .$$

On the other hand, it is easy to verify that, for every convex body K,

$$(h_K)^* = I_K \quad \text{and} \quad (I_K)^* = h_K$$

(recall that I_K denotes the indicatrix function of K). Hence we have

$$(u_t)^* = I_{K_0} + I_{K_1} = I_{K_0 \cap K_1} = (h_{K_0 \cap K_1})^* .$$

The proof is concluded taking the conjugates of the first and the last function of the above chain of equalities. ☐

For a convex body K with the origin in its interior (and for fixed $p > 1$), set

$$F_p(K) := I_p(e^{-h_K}) = \int_{\mathbf{R}^n} |\nabla h_K(x)|^p e^{-h_K(x)} \, dx .$$

By Proposition 4.10, inequality (4.22) restricted to functions of the form (4.23) becomes

$$F_p(K_0 \cap K_1) \geq F_p(K_0)^t \, F_p(K_1)^{1-t} , \quad \forall \, K_0, K_1 , \; \forall \, t \in [0,1] . \tag{4.24}$$

The above inequality is in turn equivalent to the the fact that the functional F_p is decreasing with respect to set inclusion, in the class of convex bodies having the origin as interior point:

$$F_p(K) \geq F_p(K') , \quad \forall K \subset K' . \tag{4.25}$$

Indeed, taking $K_0 = K$, $K_1 = K'$ and $t = 0$ in (4.24) we get (4.25). On the other hand, (4.25) implies that for every K_0 and K_1 and for every $t \in [0,1]$,

$$(F_p(K_0 \cap K_1))^t \geq F_p(K_0)^t , \quad (F_p(K_0 \cap K_1))^{1-t} \geq F_p(K_1)^{1-t} .$$

Multiplying these inequalities term by term we have (4.24). In Proposition 4.11 below, we construct examples of convex bodies K and K' for which (4.25) is false, under the assumptions $p > 1$. As an immediate consequence, we obtain that also inequality (4.22) fails to be true for $p > 1$.

Proposition 4.11. *For every $n \geq 2$ and every $p > 1$, there exist two convex bodies K and K' in $\mathcal{K}^n$ such that $0 \in \mathrm{int}(K \cap K')$ and $F_p(K) < F_p(K')$.*

Corollary 4.12. *For every $n \geq 2$ and every $p > 1$, there exist $f_0, f_1 \in \mathcal{Q}_0^n$ and $t \in [0,1]$ such that, if $f_t := (1-t) \cdot f_0 \oplus t \cdot f_1$, then*

$$\int |\nabla f_t(z)|^p \, dz < \left(\int |\nabla f_0(z)|^p \, dx \right)^{1-t} \left(\int |\nabla f_1(z)|^p \, dy \right)^{t}.$$

Proof of Proposition 4.11. We write an arbitrary point $x = (x_1, x_2, \dots, x_n)$ of $\mathbf{R}^n$ in polar coordinates $(r, \theta) = (r, \theta_1, \theta_2, \dots, \theta_n)$:

$$\begin{cases} x_1 = x_1(r,\theta) = x_1(r, \theta_1, \dots, \theta_{n-1}) = r \cos\theta_1 \\ x_2 = x_2(r,\theta) = x_2(r, \theta_1, \dots, \theta_{n-1}) = r \sin\theta_1 \cos\theta_2 \\ x_3 = x_3(r,\theta) = x_3(r, \theta_1, \dots, \theta_{n-1}) = r \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ \vdots \\ x_{n-1} = x_{n-1}(r,\theta) = x_{n-1}(r, \theta_1, \dots, \theta_{n-1}) = r \sin\theta_1 \dots \sin\theta_{n-2} \cos\theta_{n-1} \\ x_n = x_n(r,\theta) = x_n(r, \theta_1, \dots, \theta_{n-1}) = r \sin\theta_1 \dots \sin\theta_{n-2} \sin\theta_{n-1} \,. \end{cases}$$

Here $(r, \theta_1, \dots, \theta_{n-1}) \in [0, \infty) \times [0, \pi)^{n-2} \times [0, 2\pi)$. The Jacobian of the mapping $x = x(r, \theta)$ is $r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \dots \sin\theta_{n-2}$. For brevity we set $S = [0, \pi)^{n-2} \times [0, 2\pi)$. Let us also set

$$H_K(\theta) = h_K(x(1, \theta)) \,, \quad \theta \in S \,.$$

By the homogeneity of h_K we have

$$h_K(x(r,\theta)) = r H_K(\theta) \,, \quad \forall r \geq 0 \,,\ \theta \in S \,.$$

The gradient of h_K is positively homogeneous of order 0, so that $|\nabla h_K(x(r,\theta))|$ does not depend on r. Hence we put

$$N_K(\theta) = |\nabla h_K(x(r,\theta))| \,. \tag{4.26}$$

The functional $F_p(K)$ can now be written in the following form

$$F_p(K) = \int_S N_K(\theta)^p \left(\int_0^\infty r^{n-1} e^{-r H_K(\theta)} \, dr \right) \phi(\theta) \, d\theta \,,$$

where

$$\phi(\theta) = \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \dots \sin\theta_{n-2} \,.$$

After integration with respect to r, we get

$$F_p(K) = (n-1)! \int_S \frac{N_K(\theta)^p}{H_K(\theta)^n} \phi(\theta) \, d\theta \,. \tag{4.27}$$

Using the above formula, we can immediately deduce counterexamples to (4.25) for $p > n$. Indeed, from (4.27) we see that F_p is homogeneous of order $(p-n)$ with respect to homotheties. In particular, if $\alpha > 1$ and K is such that $F_p(K) > 0$ (for instance, if K is a ball centered at the origin), we have

$$F_p(\alpha K) = \alpha^{p-n} F_p(K) > F_p(K)\,,$$

and since $\alpha K \supset K$, this is in conflict with (4.25).
The construction of counterexamples for $p \leq n$ is still based on (4.27), but it is slightly more involved. We set

$$K_1 = B \qquad \text{and} \qquad K_2 = \text{conv}(B \cup l\, e_1)\,,$$

where conv denotes the convex hull, $l \geq 1$ and $e_1 = (1, 0, \ldots, 0)$. We will prove that, for every $p > 1$, there is a suitable choice of l such that $F_p(K_1) < F_p(K_2)$. Since clearly $K_1 \subset K_2$, and the origin is interior to both K_1 and K_2, this will provide a counterexample to (4.25). Note that the body K_2 is rotationally invariant with respect to the x_1-axis, so that the function H_{K_2} depends on θ_1 only. With abuse of notations we write

$$H_{K_2}(\theta_1, \theta_2, \ldots, \theta_{n-1}) = H_{K_2}(\theta_1)\,.$$

More precisely, an explicit expression for H_{K_2} can be written down. Let $\phi \in [0, \pi/2]$ be such that

$$l = \frac{1}{\cos\phi}\,.$$

Then

$$H_{K_2}(\theta_1) = \begin{cases} \dfrac{\cos\phi}{\cos\theta_1} & \text{if } \theta_1 \in [0, \phi]\,, \\ \\ 1 & \text{if } \theta_1 \in [\phi, \pi]\,. \end{cases}$$

Next we have to compute the function N_{K_2}. Due to the axial symmetry it is not hard to see that the following formula holds

$$N_{K_2}(\theta) = |\nabla h_{K_2}(x(r,\theta))| = \sqrt{H_{K_2}^2(\theta_1) + \left(\frac{dH_{K_2}}{d\theta_1}(\theta_1)\right)^2}\,.$$

Hence

$$N_{K_2}(\theta_1) = \begin{cases} \dfrac{1}{\cos\phi} & \text{if } \theta_1 \in [0, \phi]\,, \\ \\ 1 & \text{if } \theta_1 \in [\phi, \pi]\,. \end{cases}$$

Now we can compute $F_p(K_2)$. We have

$$\begin{aligned} F_p(K_2) &= (n-1)! \int_S \frac{N_{K_2}(\theta)^p}{H_{K_2}(\theta)^n} \phi(\theta)\, d\theta \\ &= 2\pi(n-1)! \int_{[0,\pi)^{n-2}} \frac{N_{k_2}(\theta_1)^p}{H_{K_2}(\theta_1)^n} \phi(\theta_1, \theta_2, \ldots \theta_{n-2}) d\theta_1 d\theta_2 \ldots d\theta_{n-2} \\ &= 2\pi(n-1)!\; C(n) \int_0^\pi \frac{N_{k_2}(\theta_1)^p}{H_{K_2}(\theta_1)^n} \sin^{n-2}(\theta_1)\, d\theta_1\,, \end{aligned}$$

where

$$C(n) = \prod_{i=1}^{n-3} \int_0^{\pi} \sin^i t \, dt \, .$$

Using the explicit expressions that we have found for H_{K_2} and N_{K_2} we obtain

$$F_p(K_2) = 2\pi(n-1)! \; C(n) \left[(\cos\phi)^{n-p} \int_0^{\phi} \frac{(\sin\theta_1)^{n-2}}{(\cos\theta_1)^n} \, d\theta_1 + \int_{\phi}^{\pi} \sin^{n-2}\theta_1 \, d\theta_1 \right] .$$

If $p > 1$ the following equality holds

$$\lim_{\phi \to \frac{\pi}{2}^-} (\cos\phi)^{n-p} \int_0^{\phi} \frac{(\sin\theta_1)^{n-2}}{(\cos\theta_1)^n} \, d\theta_1 = \infty$$

and consequently

$$\lim_{\phi \to \frac{\pi}{2}^-} F_p(K_2) = \infty \, .$$

Thus, $F_p(K_2)$ can be made arbitrarily large for a suitable choice of ϕ, and in particular, it can be made strictly bigger that $F_p(K_1)$ which is independent of ϕ. □

5 Integral geometric formulas and the valuation property

In this section we show that the quantities introduced in Definition 3.1 verify integral geometric formulas and a valuation type property, suitably reformulated in the functional case. In both cases, the proofs are straightforward consequences of the definition of the W_i's and the validity of the corresponding properties for convex bodies.

5.1 Integral geometric formulae

To begin with, we introduce a notion of *projection* for functions, which has already been considered in the literature, see for instance [20]. As in Section 2, for $1 \le k \le n$ we denote by $\mathcal{L}^n_k$ the set of linear subspaces of $\mathbb{R}^n$ of dimension k. Furthermore, for $L \in \mathcal{L}^n_k$, we denote by $L^\perp \in \mathcal{L}^n_{n-k}$ the orthogonal complement of L in $\mathbb{R}^n$.

Definition 5.1. Let $k \in \{1, \dots, n\}$, $L \in \mathcal{L}^n_k$ and $f \in \mathcal{Q}^n$. We define the *orthogonal projection of f onto L* as the function

$$f|L \, : \, E \mapsto [0, +\infty] \, , \quad f|L\,(x') = \sup\left\{ f(x' + y) \,|\, y \in L^\perp \right\} .$$

When f is the characteristic function of a convex body $K \in \mathcal{K}^n$, for any direction $L \in \mathcal{L}^n_k$, the projection $f|L$ agrees with the characteristic function of the projection of K onto H_ξ.
The following lemma, whose proof follows directly from Definition 5.1, shows that the projection of a quasi-concave function is quasi-concave, as well. We recall that for $A \subset \mathbb{R}^n$ and $L \in \mathcal{L}^n_k$, $A|L$ denotes the orthogonal projection of A onto L.

Lemma 5.2. *Let $f \in \mathcal{Q}^n$, $k \in \{1, \dots, n\}$ and $L \in \mathcal{L}^n_k$. For every $t \ge 0$,*

$$\left\{ x' \in L \, : \, f|L(x') > t \right\} = \left\{ x \in \mathbb{R}^n \, : \, f(x) > t \right\} |L \, .$$

As a consequence of the Cauchy-Kubota formulas for convex bodies, Definition 3.1, Lemma 5.2 and Fubini's Theorem, we have the following result.

Theorem 5.3. (Cauchy-Kubota integral formula for quasi-concave functions) *Given $f \in \mathcal{Q}^n$, for all integers $1 \le i \le k \le n$,*

$$W_i(f) = c(i,k,n) \int_{\mathcal{L}^n_k} W_i(f|L_k)\, dL_k \,,$$

where the constant $c(i,k,n)$ is the same as in in formula (2.2).

As a special case, we consider $i = k = 1$, which corresponds to the Cauchy formula.

Definition 5.4. For $\xi \in S^{n-1}$, let H_ξ denote the hyperplane through the origin orthogonal to ξ. For every $f \in \mathcal{A}$, we define the *projection of f in the direction ξ* as the function defined on H_ξ by

$$(f|\xi)(x') = \sup \big\{ f(x' + s\xi) : \; s \in \mathbb{R} \big\}, \qquad x' \in H_\xi \;.$$

Proposition 5.5. (Cauchy integral formula for quasi-concave functions) *For any $f \in \mathcal{Q}^n$,*

$$\mathrm{Per}(f) = c_n \int_{S^{n-1}} \Big\{ \int_{H_\xi} (f|\xi)(x')\, d\mathcal{H}^{n-1}(x') \Big\}\, d\mathcal{H}^{n-1}(\xi) \;. \tag{5.1}$$

5.2 Valuation property

The quermassintegrals of convex bodies are known to satisfy the following restricted additivity property: For every $i = 0, \dots, n$,

$$W_i(K) + W_i(L) = W_i(K \cup L) + W_i(K \cap L)\,, \tag{5.2}$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$. A real-valued functional defined on $\mathcal{K}$ for which (5.2) holds is called a valuation. The notion of valuation can be transposed into a functional setting, simply replacing union and intersection by maximum and minimum. At this regard, note that if f and g are quasi-concave function, then $f \vee g$ is quasi-concave, as well. Here we prove that all quermassintegrals of functions in $\mathcal{Q}^n$ are valuations in the above sense.

Proposition 5.6. (Valuation property) *Let $f, g \in \mathcal{Q}^n$ be such that $f \wedge g \in \mathcal{Q}^n$. Then for every $i = 0, 1, \dots, n-1$,*

$$W_i(f \wedge g) + W_i(f \vee g) = W_i(f) + W_i(g) \;.$$

Proof. We observe that, for every $t > 0$,

$$\big\{ f \wedge g \ge t \big\} = \big\{ f \ge t \big\} \cap \big\{ g \ge t \big\}$$

$$\big\{ f \vee g \ge t \big\} = \big\{ f \ge t \big\} \cup \big\{ g \ge t \big\} \;.$$

Since $f, g \in \mathcal{Q}^n$, one can easily check that also $f \vee g \in \mathcal{Q}^n$, whereas $f \wedge g \in \mathcal{Q}^n$, by the assumption. Therefore all the superlevels appearing in the above equalities belong to $\mathcal{K}^n$, and the valuation property (5.2) for the geometric quermassintegrals ensures that

$$W_i\big(\{f \wedge g \ge t\}\big) + W_i\big(\{f \vee g \ge t\}\big) = W_i\big(\{g \ge t\}\big) + W_i\big(\{g \ge t\}\big) \;.$$

Recalling Definition 3.1, the statement follows after integration over $(0, +\infty)$. □

6 Functional inequalities

As we have explicitly defined a notion of the perimeter for quasi-concave functions, it is natural to ask for related isoperimetric type inequalities. Below, we propose two different kind of inequalities in this direction.

Theorem 6.1. (Isoperimetric-type inequalities)

(i) *For every* $f \in \mathcal{Q}^n$,
$$\mathrm{Per}(f) \geq n\kappa_n^{1/n} \|f\|_{\frac{n}{n-1}} \,. \tag{6.1}$$

(ii) *For every* $f \in \mathcal{Q}_0^n$,
$$\mathrm{Per}(f) \geq nI(f) + \mathrm{Ent}(f)\,, \tag{6.2}$$

where
$$\mathrm{Ent}(f) = \int f(x) \log f(x)\, dx - I(f)\, \log I(f)\,.$$

Equality in (6.1) *and* (6.2) *is attained if and only if* f *is the characteristic function of an arbitrary ball.*

Inequality (6.1) is nothing but the Sobolev inequality in $\mathbb{R}^n$ for functions of bounded variation (for which the equality case is known to hold iff $f = \chi_B$ up to translations). Actually, it holds without the quasi-concavity assumption. Inequality (6.2), together with the corresponding equality case, can be obtained by applying Theorem 5.1 in [15] with $g = \chi_B$. The isoperimetric inequality (6.1) can naturally be extended to other functional quermassintegrals.

Theorem 6.2. *For every* $f \in \mathcal{Q}^n$, *and for all integers* $0 \leq i \leq k \leq n-1$,
$$W_k(f) \,\geq\, c\, W_i(f^p)^{1/p}, \quad \textit{where} \;\; p = \frac{n-i}{n-k},\; c = \kappa_n^{1-1/p}. \tag{6.3}$$

In particular,
$$W_k(f) \,\geq\, \kappa_n^{k/n} \|f\|_{\frac{n}{n-k}}. \tag{6.4}$$

Equality in (6.3) *and* (6.4) *is attained if and only if* f *is the characteristic function of an arbitrary ball.*

Note that inequality (6.4) corresponds to (6.3) in the particular case $i = 0$. Futhermore, taking $k = 1$ in (6.4), gives back the Sobolev inequality (6.1).

Proof. The following inequality holds for the quermassintegrals of convex bodies:
$$W_k(K) \geq c\, W_i(K)^{1/p},$$

with c and p as in (6.3), *cf.* [33]. Applying this bound to the level sets $K_f(t) = \{f \geq t\}$ and integrating over $t > 0$, we therefore obtain
$$W_k(f) \geq c \int_0^{+\infty} W_i\big(K_f(t)\big)^{1/p}\, dt. \tag{6.5}$$

To further bound from below the integral in (6.5), we use the following elementary inequality which is commonly applied in the derivation of the Sobolev inequality (6.1), see for instance [13]: If $u = u(t)$ is a non-negative, non-increasing function on $(0, +\infty)$, then for all $p \geq 1$,

$$\int_0^{+\infty} u(t)^{1/p} \, dt \geq \Big(\int_0^{+\infty} u(t) \, dt^p \Big)^{1/p}.$$

Choosing $u(t) = W_i(K_f(t))$, we see that the p-th power of the integral in (6.5) is greater than or equal to

$$\int_0^{+\infty} W_i\big(\{f \geq t\}\big) \, dt^p = \int_0^{+\infty} W_i\big(\{f^p \geq t^p\}\big) \, dt^p = \int_0^{+\infty} W_i\big(\{f \geq t\}\big) \, dt = W_i(f^p).$$

□

While we already noticed that the case $k = 1$ in (6.4) amounts to the isoperimetric inequality, the case $k = n - 1$ leads to the following functional version of Urysohn's inequality:

Corollary 6.3. *For every $f \in \mathcal{Q}^n$,*

$$M(f) \; \geq \; 2\kappa_n^{-1/n} \, \|f\|_n. \tag{6.6}$$

Equality in (6.6) *is attained if and only if f is the characteristic function of an arbitrary ball.*

For the characteristic functions of convex bodies, (6.6) reduces to the classical Urysohn's inequality. We point out that, for log-concave functions, a different functional version of the Urysohn inequality involving Gaussian densities, was earlier proposed by Klartag and Milman in [20]. In fact, (6.4) and its particular case (6.6) admits a further refinement in terms of radial functions. Below, for a given $K \in \mathcal{K}^n$, we denote by K^* the ball with the same mean width as K.

Theorem 6.4. *Given $f \in \mathcal{Q}^n$, denote by f^* the rearrangement of f obtained by replacing each of the level sets $\{f \geq t\}$ by $\{f \geq t\}^*$. Then, for every $k = 0, 1, \ldots, n-1$,*

$$W_k(f) \geq W_k(f^*) \; .$$

Proof. We have

$$\begin{aligned} W_k(f) \;\; &= \;\; \int_0^{+\infty} W_k\big(\{f \geq t\}\big) \, dt \\ &\geq \;\; \int_0^{+\infty} W_k\big(\{f \geq t\}\big)^* \, dt \;\; = \;\; \int_0^{+\infty} W_k\big(\{f^* \geq t\}\big) \, dt \;\; = \;\; W_k(f^*) \; . \end{aligned}$$

□

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