

# Random Fuzzy Extension of the Universal Generating Function Approach for the Availability/Reliability Assessment of Multi-State Systems under Aleatory and Epistemic Uncertainties

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## Abstracts

Many engineering systems can perform their intended tasks with various levels of performance, which are modeled as multi-state systems (MSS) for system availability/reliability assessment problems. Uncertainty is an unavoidable factor in MSS modeling and it must be effectively handled. In this work, we extend the traditional universal generating function (UGF) approach for multi-state system (MSS) availability/reliability assessment to account for both aleatory and epistemic uncertainties. First, a theoretical extension, named hybrid UGF (HUGF), is made to introduce the use of random fuzzy variables (RFVs) in the approach; second, the composition operator of HUGF is defined by considering simultaneously the probabilistic convolution and the fuzzy extension principle; finally, an efficient algorithm is designed to extract probability boxes ( $p$ -boxes) from the system HUGF, which allow quantifying different levels of imprecision in system availability/reliability estimation. The HUGF approach is demonstrated on a numerical example and applied to study a distributed generation system, with a comparison to the widely used Monte Carlo simulation method.

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## Acronyms

BSS	binary-state system
CDF	cumulative distribution function
FV	fuzzy variable
HUGF	hybrid universal generating function
MCS	Monte Carlo simulation
MSS	multi-state system
PDF	probability density function
PMF	probability mass function
RV	RV
RFV	random fuzzy variable
UGF	universal generating function

## Notations

$M_i$	the highest state of component $i$
$G_i$	the performance variable of component $i$
$g_{i,j}$	the performance level of component $i$ at its state $j$
$n$	number of components of MSS
$G_S$	the power output of a solar generator
$G_{SYS}$	the performance variable of MSS
$\varphi(\cdot)$	the system structure function of MSS
$w$	the demand presented to MSS
$D$	the system adequacy variable defined as $G_{SYS} - w$
$d_j$	the system adequacy level at state $j$
$A(w)$	the availability function of MSS given $w$
$[\underline{A}_\alpha, \overline{A}_\alpha]$	$p$ -box of system availability of level $\alpha$
$\Omega$	probability sample space
$\Theta$	possibility sample space
$X$	a random variable
$\tilde{X}$	a fuzzy variable
$\ddot{X}$	a random fuzzy variable
$u_X(z)$	the u-function of a variable $X$

## 1. Introduction

Multi-state system (MSS) modeling has been widely applied to resolve system availability/reliability assessment problems [1, 2]. Under this framework, the performance of each component is discretized into more than two exclusive states from perfect functioning to complete failure, and each state is characterized by a probability of occurrence. In general, the intermediate state can be decided by component degradation situation and/or system function requirements, because many components are subject to natural deteriorations which can render them being partially functioning, and the system function requirement might force the component to reduce its performance level even if it bears no degradation. Compared to binary-state system (BSS) models, the MSS models offer higher flexibility in the description of the system state distribution and evolution, for more precise approximations of real-world systems. MSS is a modeling framework capable of handling both availability and reliability assessments. In this paper, we focus on availability assessment assuming that the system is repairable.

In general, the target of the MSS availability assessment is to derive the system availability as the probability that the system performance is no less than the demand  $w$ ,  
( ). is determined by the MSS system structure, which is a function of the  $n$  component performance variables, , where is the  $i$ -th component performance variable that takes values from the finite set where is the performance level of component  $i$  at its state  $j = 0, \dots, M_i$  and is the highest possible state of component  $i$ . Typically,  $g_{i,0}$  and represent the performance levels at complete failure and perfect functioning conditions, respectively. **In this study, we assume that the state values in the set  $\{g_{i,0}, \dots, g_{i,j}, \dots, g_{i,M_i}\}$  are ascendingly ordered.** For MSS availability assessment, a number of methods have been proposed: minimal cuts/paths [3], universal generating function (UGF) [4], multi-valued decision diagram [5], Monte Carlo simulation (MCS) [6], etc. Among them, UGF has been shown to be a flexible tool capable to represent the component performance probability distribution and derive the system performance probability distribution algebraically [7].

Uncertainty is an unavoidable factor in MSS availability assessment [2]. Conventionally, the uncertain behavior of  $G_i$  is described by its discrete probability distribution  $P(G_i = g_{i,j})$ , such

that  $\sum_{j=0}^{M_i} P(G_i = g_{i,j}) = 1$ . The probability distribution is sufficient to describe the state randomness, i.e. uncertainty of objective and aleatory type [8] due to the natural variability or stochasticity of the component behavior [9]. Another type of uncertainty to account for is that due to the incomplete or imprecise knowledge about the component performance [10-15]. This type of uncertainty is often referred to as subjective and epistemic [8, 16].

Recently, epistemic uncertainty in MSS model has been treated by a fuzzy UGF approach [17-19] which assumes that the state probabilities and the state performances of components to be FVs, respectively. This approach has been further extended to the time domain for dynamic fuzzy MSS by assuming the state transition rates and the state performances to be FVs [20, 21]. Later, interval values have been used in [22] to represent the imprecision at both state probability and performance. It can be observed that in most existing fuzzy UGF studies the imprecision of the state probability (or state transition rate in case of dynamic fuzzy UGF) and the state performance are treated separately, and represented as different fuzzy variables. Indeed it is a generalized approach of hybrid uncertainty representation.

On the other hand, the theoretical and practical developments in the area of reliability and risk assessment [23-26] reveal that a single entity, namely random fuzzy variable (RFV) [25] or hybrid number [26], is sufficient to represent and propagate both types of uncertainties in the system. RFV is a random distribution of fuzzy numbers [25]. One simple example of RFV is the perceived cost of automobile repair: suppose the actual cost of repair is a RV defined on positive real numbers, given little information about its exact sample values one can only perceive it through a set of ‘windows’ such as ‘cheap’, ‘moderate’, or ‘expensive’ [27]. By definition, the sum of the probability masses attached to all fuzzy numbers in the sample space of a RFV must be equal to 1. This property has not been considered in the original works of fuzzy UGF [17-19]. In dynamic fuzzy UGF papers [20, 21] it has been imposed as one constraint of the non-linear programming formulation for solving the system availability metrics. Differently, RFV possesses this property in nature. Due to the discussions above, we propose the UGF representation of RFV, namely hybrid UGF (HUGF). It is also noted that in a very recent work [28], UGF has been extended to represent an interval-valued random variable.

The uncertainty propagation process [23], which is analogous to the process of MSS availability assessment, propagates the uncertainties associated to the elementary variables onto the system-

level function with the least possible loss of information. It is typically realized by the MCS method [10, 23, 24], which however can be quite time-consuming [29] and can have difficulties in obtaining stable results [23]. Based upon the HUGF, the analytical results of uncertainty propagation can be achieved by combining the RFVs with a modified UGF composition operator. The efficiency of uncertainty propagation can be thus improved and the results stabilized.

The contributions of this work are summarized as follows: 1) RFV **is** introduced to represent both randomness and fuzziness in the MSS; 2) HUGF **is** defined to represent the RFV whose random dimension is discrete for the multi-state case; 3) composition operators of HUGF **is** defined for joint uncertainty propagation; 4) to extract useful information from the propagation result, an algorithm **is** designed to obtain the probability boxes ( $p$ -boxes) of system availability from the HUGF of system adequacy, defined as  $G_{SYS} - w$ .

The rest of this paper is organized as follows. Section 2 illustrates, through a multi-state model of solar generation, the co-existence of aleatory and epistemic uncertainty in MSS and presents the assumptions made for MSS modeling. In Section 3, the concept of RFV is recalled and HUGF is proposed as theoretical extension of UGF for RFV representation. In Section 4, the MCS algorithm of joint uncertainty propagation in MSS is presented and the algebraic operators of HUGF are defined. In Section 5, the algorithm extracting the probability boxes ( $p$ -boxes) of MSS availability is proposed. Section 6 presents two case studies with the comparisons to MCS method. Section 7 concludes this work and points out some possible future research directions.

## 2. MSS with Aleatory and Epistemic Uncertainties

As mentioned in Section 1, the multi-state model of a component might contain both types of uncertainties, aleatory and epistemic. We take the solar generator model from [30] as an illustrative example. This model consists of two RVs (RVs), solar irradiation and mechanical condition, a set of generation parameters and an energy conversion function (which transfers the irradiation to power output). In practice, there is usually sufficient historical data to capture the variability in the solar irradiation and mechanical condition. In multi-state setting, solar irradiation  $r_s$  is discretized into several exclusive states ranging from zero irradiation to maximum irradiation; the mechanical condition  $m_s$  is a binary RV taking values from the set  $\{0,$

1}, where ‘0’ means complete failure and ‘1’ means perfect functioning. The power output of one solar generator is given by the following functions [31]:

$$G_S = g_S(r_S, \boldsymbol{\theta}_S, m_S) = N_{sc} \cdot I_y \cdot V_y \cdot FF \cdot m_S \quad (1.a)$$

$$I_y = r_S \cdot [I_{sc} + k_c(T_c - 25)] \quad (1.b)$$

$$V_y = V_{oc} - k_v \cdot T_c \quad (1.c)$$

$$T_c = T_a + r_S \cdot \frac{N_{ot} - 20}{0.8} \quad (1.d)$$

$$FF = \frac{V_{MPP} \cdot I_{MPP}}{V_{oc} \cdot I_{sc}} \quad (1.e)$$

where  $G_S$  is the power output,  $g_S(\cdot)$  is the solar energy conversion function,  $\boldsymbol{\theta}_S = (I_{sc}, k_c, V_{oc}, k_v, T_a, N_{ot}, V_{MPP}, I_{MPP})$  is the vector of operation parameters,  $N_{sc}$  is the total number of solar cells consisting the solar generator,  $I_{sc}$  is the short circuit current in A,  $k_c$  is the current temperature coefficient  $A/^\circ C$ ,  $V_{oc}$  is the open-circuit voltage in V,  $k_v$  is the voltage temperature coefficient  $V/^\circ C$ ,  $T_c$  is the cell temperature in  $^\circ C$ ,  $T_a$  is the ambient temperature in  $^\circ C$ ,  $N_{ot}$  is the nominal operating temperature in  $^\circ C$ ,  $V_{MPP}$  is the voltage at maximum power point in V, and  $I_{MPP}$  is the current at maximum power point in A.

In literature, the operation parameters are typically treated as constants. In practice, they often change during the generator operation due to the degradation of materials, changes in the operating environments, etc [32]. However, there are seldom sufficient information to model them as RVs, due to the unwillingness of the manufacturers to disclose the commercially sensitive data [10]. In this situation, the fuzzy variables (FVs) are one promising alternative. It can be seen from eq. (1) that each realization of  $G_S$  is a fuzzy number. Essentially,  $G_S$  is a RFV which we will show in Sections 3 and 4. It should be noted that  $G_S$  can also be referred to as a fuzzy random variable [27]. These two concepts are interchangeable since they lead to equivalent representations, and complementary interpretations and calculation strategies [25].

Based on the example above, the following assumptions are made for our MSS modeling:

1. For any component  $i$ , it has  $M_i + 1$  different states  $\{0, 1, \dots, M_i\}$  where state  $M_i$  and 0 are the perfect functioning and the complete failure states, respectively. The generic

intermediate state  $j$  ( $0 < j < M_i$ ) is a degradation state where the component is partially functioning. The state index  $j$  is a crisp value.

2. In the model of a component  $i$ , the FVs are used to represent the model parameters if they are tainted with imprecision.
3. Following assumption 2, the performance of a component  $i$  is a discrete RV  $G_i$  if there is sufficient data to eliminate all the imprecision in its parameters; otherwise it will be a RFV  $\tilde{G}_i$  (or a pure FV  $\tilde{G}_i$  if only FVs are involved in the component model).
4. The state of the system is completely determined by the state of its components.
5. All components are repairable.

### 3. HUGF for Hybrid Uncertainty Representation in MSS

In this Section, the definition of RFV is first recalled. Then the UGF representation of RFV, named HUGF, is formally defined and the theoretical connection is drawn by proving that the first derivative of HUGF at  $z = 1$  equals the expectation of its corresponding RFV.

#### 3.1 RFV

RFV was first introduced by Kaufmann and Gupta [26] as a tool to express jointly the epistemic and aleatory uncertainties. Later on, RFV were extended by Cooper et al. [33] and Baudrit et al. [23] for hybrid uncertainty propagation in the area of risk analysis. Given the monotonicity of the cumulative distribution functions (CDFs) of the RVs and the nestedness of the possibility distribution functions of the FVs, the formal definition of RFV proposed by Ferson and Ginzburg [25] is presented as follows.

**Definition 1** (Ferson and Ginzburg [25]) Let  $\mathbf{F}$  denote the set of all CDFs defined on the real number set  $\mathfrak{R}$  and each element  $F \in \mathbf{F}$  is an onto function  $F: \mathfrak{R} \rightarrow [0, 1]$  such that  $F(x_1) \geq F(x_2)$  whenever  $x_1 > x_2$ . A RFV is a set of closed intervals, each characterized by a pair of functions from  $\mathbf{F}$ :

$$H: [0, 1] \rightarrow \mathbf{F} \times \mathbf{F}: \alpha \mapsto [F_\alpha, \bar{F}_\alpha]$$

such as for  $\alpha_1, \alpha_2 \in [0, 1]$ ,  $\bar{F}_{\alpha_1}(x) \geq \bar{F}_{\alpha_2}(x) \geq \underline{F}_{\alpha_2}(x) \geq \underline{F}_{\alpha_1}(x)$  whenever  $\alpha_1 < \alpha_2$ , where  $\alpha_1$  and  $\alpha_2$  represent fuzzy membership values of  $x$ .

**Example:** Figure 1(a) depicts the three-dimension representation of a RFV. The  $x$ -axis is the real number line,  $F$ -axis has the cumulative probability values, and  $\pi$ -axis contains the possibility values. The shaded area at  $\alpha \in (0,1)$  level includes all the closed probability intervals characterized by  $\underline{F}_\alpha$  as the lower bound and  $\bar{F}_\alpha$  as the upper bound. Figure 1(c) shows the two-dimension representation of this RFV and its  $\alpha$  level probability intervals. Figure 1(b) depicts the intersection of the RFV with the plane  $F(x) = p$ , which is essentially a fuzzy number. Similarly, Figure 1(d) depicts this intersection in the two-dimension representation.

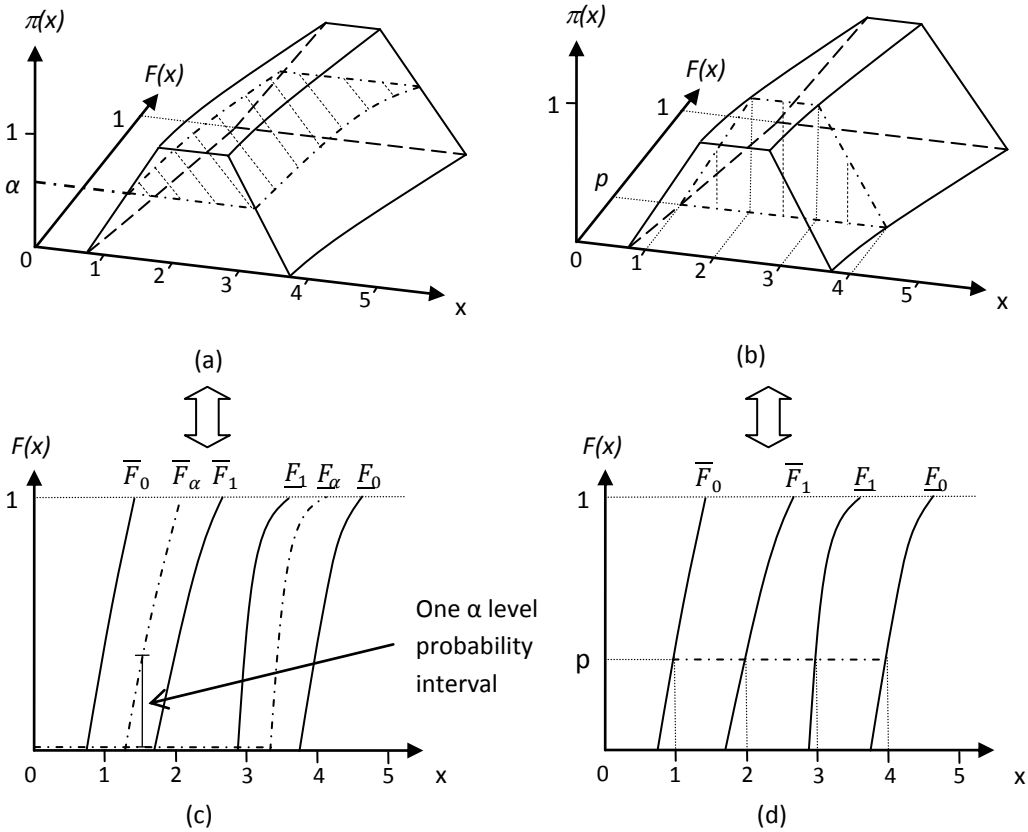


Figure 1. Three-dimension and two-dimension representations of an example RFV

### 3.2 HUGF representation of RFV



The UGF for a discrete RV  $X$  [34] is defined as:

$$u_X(z) = \sum_{j=0}^J p_j z^{x_j} \quad (2)$$

where  $z$  is the base of the  $z$ -transform,  $J + 1$  is the sample space size of  $X$ ,  $x_j$  is the  $j$ -th sample of  $X$ , and  $p_j$  is the probability mass attached to  $x_j$  satisfying  $\sum_{j=0}^J p_j = 1$ . The u-function is useful in representing the PMF of discrete RV because it preserves some basic properties of the moment-generating function, which uniquely determines its PMF. The readers could refer to [34], where the details about UGF are presented.

Beside *Definition 1*, RFV can also be regarded as a random distribution of fuzzy numbers [33]. In the context of MSS, the random distribution is defined on a finite set of elements, e.g. crisp numbers or fuzzy numbers. Figure 2 shows such a RFV. It is seen that the quantity  $\Delta P_j = F(\tilde{X}_j) - F(\tilde{X}_{j-1})$  for  $j \geq 1$  or  $\Delta P_0 = F(\tilde{X}_0)$  for  $j = 0$ , is the probability of occurrence of the fuzzy number  $\tilde{X}_j$ .

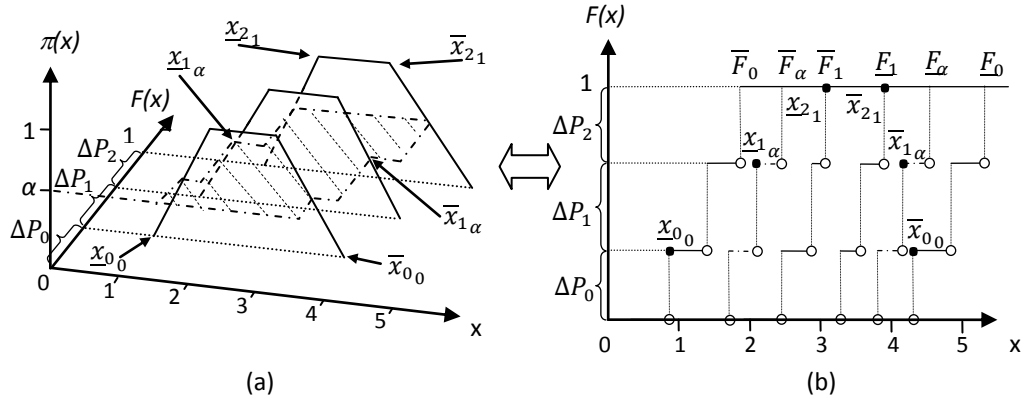


Figure 2. An example RFV defined on finite fuzzy numbers

**Definition 2.** For a RFV  $\tilde{X}$  defined on a finite set of fuzzy numbers  $\boldsymbol{\pi}$ ,  $|\boldsymbol{\pi}| = J + 1$ , its u-function (i.e. HUGF), denoted by  $u_{\tilde{X}}(z)$ , is written as follows:

$$u_{\tilde{X}}(z) = \sum_{j=0}^J p_j z^{\tilde{X}_j} = \sum_{j=0}^J p_j z^{[x_{j\alpha}, \bar{x}_{j\alpha}]} \quad (3)$$

It is noted that this definition satisfies the basic property of UGF: the coefficient and exponent are not necessarily scalar variables but can be other mathematical objects (i.e. FV) [2]. It is seen that (2) is the special case of (3): if all the exponents of  $z$  in (3) are crisp values (i.e. sufficient information is collected to eliminate the imprecision in state values), then (3) will reduce to (2). On the other hand, if there is only one term of  $z$ , with its coefficient equal to 1, then (4) will reduce to the following expression,

$$u_{\tilde{X}}(z) = z^{\tilde{X}} = z^{[\underline{x}_\alpha, \bar{x}_\alpha]} \quad (4)$$

which is the u-function of a pure FV. Recall that  $\pi_{\tilde{X}}(x)$  can be uniquely determined by its  $\alpha$ -cut  $[\underline{x}_\alpha, \bar{x}_\alpha]$  set, thus (4) defines a one-to-one correspondence to  $\tilde{X}$ .

To confirm that HUGF possesses the basic property of UGF, the two propositions presented in Appendix proof that the expectation of a RFV equal to the first derivative of HUGF (at  $z = 1$ ), which represents the PMF of this variable [34].

#### 4. Joint Uncertainty Propagation in MSS

This Section first presents the conventional simulation procedures for joint uncertainty propagation. The HUGF composition operators are then defined to combine different types of uncertain variables. Based on the HUGF composition operators, the method for joint uncertainty propagation in MSS availability assessment is proposed.

##### 4.1 Simulation approach for joint uncertainty propagation

Considering the case in eq. (1), the performance level of a solar generator model  $G_S$  is a function of  $r_S, m_S$  as RVs and  $I_{MPP}, I_{SC}, k_c, k_v, N_{ot}, T_a, V_{MPP}, V_{oc}$  as FVs. A general model for the MSS generation  $G_{SYS}$  versus the demand  $w$  can be written as  $Y = g(X_1, \dots, X_k, \tilde{X}_{k+1}, \dots, \tilde{X}_N)$ , function of  $N$  uncertain variables  $X_i, i = 1, \dots, N$  (possibly including  $w$ ), ordered in such a way that the first  $k$  RVs are described by PMFs  $(p_{X_1}(x), \dots, p_{X_k}(x))$ , whereas the last  $N-k$  ones are FVs represented by possibility distributions  $(\pi_{\tilde{X}_{k+1}}(x), \dots, \pi_{\tilde{X}_N}(x))$ . The MCS method proposed in

[25, 33, 35] propagates both types of uncertainties into a RFV according to their respective calculus: convolution principle for RV and extension principle for FV [36]. The detailed procedures are summarized as follows [35]:

For  $h = 1, 2, \dots, m$  (the outer loop processing aleatory uncertainty), do:

- Sample the  $h$ -th realization  $(x_1^h, x_2^h, \dots, x_k^h)$  of the RV vector  $(X_1, X_2, \dots, X_k)$  using sampling techniques such as Monte Carlo, Latin Hyper Cube, etc.
- For  $\alpha = 0, \Delta\alpha, 2 \cdot \Delta\alpha, \dots, 1$  (the inner loop processing epistemic uncertainty;  $\Delta\alpha$  is the step size, e.g.  $\Delta\alpha=0.05$ ), do:
  - Calculate the corresponding  $\alpha$ -cuts of possibility distributions  $(\pi_{\tilde{X}_{k+1}}, \dots, \pi_{\tilde{X}_N})$  as the intervals of the FVs  $(\tilde{X}_{k+1}, \dots, \tilde{X}_N)$ .
  - Compute the minimal and maximal values of the outputs of the model  $g(X_1, \dots, X_k, \tilde{X}_{k+1}, \dots, \tilde{X}_N)$ , denoted by  $\underline{g}_\alpha^h$  and  $\overline{g}_\alpha^h$ , respectively. In this computation, the RVs are fixed at the sampled values  $(x_1^h, x_2^h, \dots, x_k^h)$  whereas the FVs take all values within the ranges of the  $\alpha$ -cuts of their possibility distributions  $(\pi_{\tilde{X}_{k+1}}, \dots, \pi_{\tilde{X}_N})$ .
  - Record the extreme values  $\underline{g}_\alpha^h$  and  $\overline{g}_\alpha^h$  as the lower and upper limits of the  $\alpha$ -cuts of  $g(x_1^h, x_2^h, \dots, x_k^h, \tilde{X}_{k+1}, \dots, \tilde{X}_N)$ .

End

- Cumulate all the lower and upper limits of different  $\alpha$ -cuts of  $g(x_1^h, x_2^h, \dots, x_k^h, \tilde{X}_{k+1}, \dots, \tilde{X}_N)$  to establish an approximated possibility distribution (denoted by  $\hat{\pi}_i^f$ ) of the model output. Assign a probability mass  $1/m$  to each obtained distribution  $\hat{\pi}_i^f$ .

End

The resulting  $m$  possibility distributions are in fact the realizations of the RFV. It is noted that this procedure requires to store  $m \times \frac{1}{\Delta\alpha}$  intervals (with  $\Delta\alpha$  typically taken equal to 0.05 in our applications). The time complexity of this algorithm is  $O\left(m \times \frac{1}{\Delta\alpha} \times n_o\right)$ , where  $n_o$  is the number of operations needed to obtain the minimal and maximal values of the output of  $g(\cdot)$ .

## 4.2 HUGF composition operator for joint uncertainty propagation

Because RFV treats the two types of uncertainties separately, the composition operator of HUGF has to equip the properties of both probabilistic UGF composition operator [4] and fuzzy extension principle [36]. In Appendix, we show the definitions of HUGF composition operator in three basic cases: composition of two FVs, composition of one FV and one RV, and composition of two RFVs, respectively.

In general, the HUGF composition operator of  $N$  u-functions, i.e. uncertain variables, is defined as follows

$$\otimes_f \left( u_{\tilde{X}_1}(z), u_{\tilde{X}_2}(z), \dots, u_{\tilde{X}_N}(z) \right) = \sum_{j_1=0}^{J_1} \dots \sum_{j_N=0}^{J_N} \prod_{k=1}^N p_{kj_k} z^{f(\tilde{X}_{1j_1}, \tilde{X}_{2j_2}, \dots, \tilde{X}_{Nj_N})} \quad (5)$$

It is noted that for the case of two arguments, the following two interchangeable notations can be used:

$$\otimes_f \left( u_{\tilde{X}_1}(z), u_{\tilde{X}_2}(z) \right) = u_{\tilde{X}_1}(z) \otimes_f u_{\tilde{X}_2}(z) \quad (6)$$

Two basic properties of  $\otimes_f$ , namely the associative and communicative properties, are recalled for the reduction of composition computation time. If the function  $f(\cdot)$  possesses the associative property for any of its variables, then  $\otimes_f$  also possesses this property

$$\begin{aligned} \otimes_f \left( u_{\tilde{X}_1}(z), \dots, u_{\tilde{X}_k}(z), u_{\tilde{X}_{k+1}}(z), \dots, u_{\tilde{X}_N}(z) \right) = \\ \otimes_f \left( \otimes_f \left( u_{\tilde{X}_1}(z), \dots, u_{\tilde{X}_k}(z) \right), \otimes_f \left( u_{\tilde{X}_{k+1}}(z), \dots, u_{\tilde{X}_N}(z) \right) \right) \end{aligned} \quad (7)$$

If the function  $f(\cdot)$  possesses the communicative property for any of its variables, then  $\otimes_f$  also possesses this property

$$\begin{aligned} \otimes_f \left( u_{\tilde{X}_1}(z), \dots, u_{\tilde{X}_k}(z), u_{\tilde{X}_{k+1}}(z), \dots, u_{\tilde{X}_N}(z) \right) = \\ \otimes_f \left( u_{\tilde{X}_1}(z), \dots, u_{\tilde{X}_{k+1}}(z), u_{\tilde{X}_k}(z), \dots, u_{\tilde{X}_N}(z) \right) \end{aligned} \quad (8)$$

By applying these two properties, the elementary RVs and FVs might be separated:

$$\begin{aligned} \otimes_f \left( u_{X_1}(z), \dots, u_{\tilde{X}_N}(z) \right) = \\ \otimes_f \left( \otimes_f \left( u_{X_1}(z), \dots, u_{X_k}(z) \right), \otimes_f \left( u_{\tilde{X}_{k+1}}(z), \dots, u_{\tilde{X}_N}(z) \right) \right) \end{aligned} \quad (9)$$

In this way, the u-functions of FVs can be processed prior to the combination with the u-function of RVs which involves multiplication to the polynomials. Using the combination rules presented above, we can obtain the HUGF of (1) through the following bottom-up way:

$$\begin{aligned}
u_{i_y}(z) &= u_{r_s}(z) \otimes_{\times} \left[ u_{i_{sc}}(z) \otimes_{+} \left( u_{\tilde{k}_c}(z) \otimes_{\times} u_{\tilde{t}_a-25}(z) \right) \otimes_{+} \left( u_{\tilde{k}_c}(z) \otimes_{\times} u_{\frac{\tilde{N}_{ot}-20}{0.8}}(z) \otimes_{\times} u_{r_s}(z) \right) \right] \\
u_{\tilde{v}_y}(z) &= \left[ u_{\tilde{v}_{oc}}(z) \otimes_{-} \left( u_{\tilde{k}_v}(z) \otimes_{\times} u_{\tilde{t}_a}(z) \right) \right] \otimes_{-} \left[ \left( u_{\tilde{k}_v}(z) \otimes_{\times} u_{\frac{\tilde{N}_{ot}-20}{0.8}}(z) \right) \otimes_{\times} u_{r_s}(z) \right] \\
u_{\tilde{F}\tilde{F}}(z) &= \left( u_{\tilde{v}_{MPP}}(z) \otimes_{\times} u_{i_{MPP}}(z) \right) \otimes_{\div} \left( u_{\tilde{v}_{oc}}(z) \otimes_{\times} u_{i_{sc}}(z) \right) \\
u_{\tilde{G}_S}(z) &= \left( u_N(z) \otimes_{\times} u_{m_s}(z) \right) \otimes_{\times} u_{\tilde{F}\tilde{F}}(z) \otimes_{\times} \left( u_{\tilde{v}_y}(z) \otimes_{\times} u_{i_y}(z) \right)
\end{aligned} \tag{10}$$

Based on the example above, the procedures of computing the MSS adequacy index  $\tilde{D} = \tilde{G}_S - w$  given arbitrary demand  $w$  are presented as follows:

- (1) Build the u-function for each component. For component  $i$  affected by both types of uncertainties, obtain  $u_{\tilde{G}_i}(z)$  by combining the elementary FVs or RVs using  $\otimes_f$  with the consideration of the communicative and associative rules;
- (2) Obtain the system performance HUGF  $u_{\tilde{G}_{SYS}}(z)$  using  $\otimes_f$  to combine the component u-functions according to the system structure function  $\tilde{G}_{SYS} = \varphi(\tilde{G}_1, \dots, \tilde{G}_n)$ , where the communicative and associative rules also apply;
- (3) Compute the HUGF of MSS adequacy  $\tilde{D}$ ,  $u_{\tilde{D}}(z) = u_{\tilde{G}_{SYS}}(z) \otimes_{-} u_w(z)$ .

This method involves both the fuzzy arithmetic and probabilistic convolution operations, either of which could lead to high computational cost. To reduce the computational complexity of this method, approximation techniques have to be applied especially when the MSS contains a large number of uncertain variables. In the next Section the computational issues are addressed in further details.

### 4.3 Computation issues

As shown in eq. (10), the non-linear fuzzy arithmetic operators (e.g. multiplication) could produce complex polynomials that are difficult to evaluate and computationally expensive. In the

literature, the efficient standard approximation proposed by Dubois and Prade [37] has been widely used to reduce the computation time of fuzzy arithmetic operations. Take the fuzzy multiplication as an example: let  $\tilde{X}_1 = [(b_1 - a_1)\alpha + a_1, -(c_1 - d_1)\alpha + c_1]$  and  $\tilde{X}_2 = [(b_2 - a_2)\alpha + a_2, -(c_2 - d_2)\alpha + c_2]$ , then their actual product is  $[(b_1 - a_1)(b_2 - a_2)\alpha^2 + (b_1 - a_1)a_2\alpha + (b_2 - a_2)a_1\alpha + a_1a_2, (c_1 - d_1)(c_2 - d_2)\alpha^2 - (c_1 - d_1)c_2\alpha - (c_2 - d_2)c_1\alpha + c_1c_2]$  and the standard approximation of this product is  $[(b_1b_2 - a_1a_2)\alpha + a_1a_2, -(c_1c_2 - d_1d_2)\alpha + c_1c_2]$ . Figure 3 shows the actual and approximated products of the FV obtained in eq. (B.2). It should be noted that the standard approximation also has some limits, for instances it is adequate only when the spread of the FV is small and the membership value near to 1, so that too frequent use of it may lead to wrong results [37]. To tackle these problems, more advanced techniques have been proposed; interested readers can refer to [38-40] for detailed information.

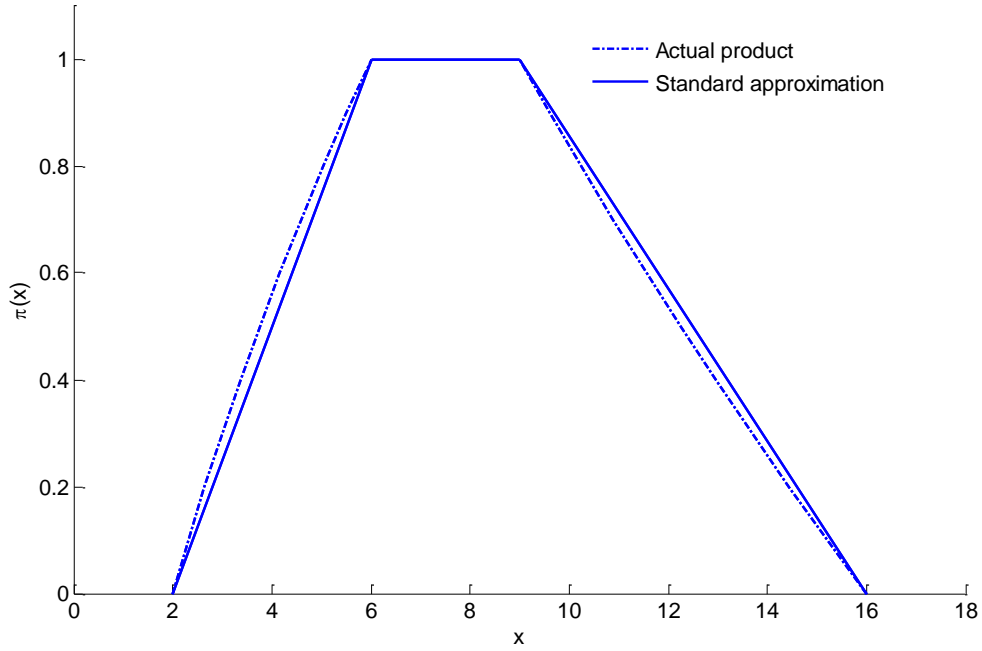


Figure 3. Actual and approximated products of the FV obtained in eq. (6)

Given the standard approximation method, the computation complexity of the proposed HUGF approach is presented as follows. In conventional MSS, the UGF approach has  $O((M_{max} + 1)^n)$  time complexity in the worst case, where  $M_{max}$  is the maximum highest state across all components and  $n$  is the number of components. In our MSS formulation, the component model

might contain more than one constituent RV so that the worst case time complexity is mainly dependent on the number of RVs,  $k$  and the maximum sample size of the RVs,  $s_{max}$ :  $O((s_{max})^k)$ . When  $k$  or  $s_{max}$  are large, the clustering technique introduced in [18] can be applied to control the number of resulting states of each composition operation between two RVs or two RFVs. The time complexity (in worst case) of each clustering operation is  $O(l \cdot s_{max} \cdot s_c(s_c + 1))$  [41], where  $l$  is the number of required iterations in the clustering algorithm, and  $s_c$  is the number of clusters. Thus, the time complexity (in worst case) of the whole UGF approach is  $O(k \cdot l \cdot s_{max} \cdot s_c(s_c + 1))$ . Recall the time complexity of the MCS method  $O\left(m \times \frac{1}{\Delta\alpha} \times n_o\right)$ , its parameters  $m$  and  $\Delta\alpha$  have to be chosen by the users and  $n_o$  is relevant to the total number of uncertain variables  $N$ . It is seen that when  $k$  and  $s_{max}$  are relatively small, the HUGF approach without clustering is preferable as it can produce the exact results of uncertainty propagation with the computation time comparable to that of the MCS method. When  $k$  or  $s_{max}$  is large, the clustering technique can be applied in the HUGF approach.

## 5. Extracting Information from System Adequacy HUGF

As shown in Section 4, the MSS adequacy index  $\ddot{D}$  is a RFV. Thus the MSS availability  $A(w) = \Pr(\ddot{D} \geq 0) = 1 - \Pr(\ddot{D} < 0)$  is no longer a precise value but a set of probability intervals, one for each  $\alpha$  level. They are often too complex to be utilized by the decision maker. In order to extract useful information from these probability intervals, the post-treatment methods are proposed. In this Section, we present two widely used post-treatment methods,  $p$ -boxes [42] and homogenous post-processing [23], and propose one efficient algorithm to produce them from the system adequacy HUGF.

### 5.1 $p$ -boxes

The concept of  $p$ -box is similar to that of RFV. Ferson and Ginzburg [42] proposed to fix the  $\alpha$  level and then to build the lower and upper probability bounds  $[E_\alpha(B), \bar{F}_\alpha(B)]$  of an event  $B$ , i.e.  $\ddot{D} < 0$ . Two representative cases of the  $p$ -boxes are  $\alpha = 0$  and  $\alpha = 1$ . The  $p$ -box  $[E_0(B), \bar{F}_0(B)]$

corresponds to a pessimistic condition where the imprecision is maximized while the  $p$ -box  $[E_1(B), \bar{F}_1(B)]$  corresponds to an optimistic situation where the imprecision is minimized. It is noted that even in the optimistic case, there still can be imprecision if the  $\alpha = 1$  level of each FV is not a single number.

## 5.2 Homogenous post-processing

Baudrit *et al.* [23] proposed this method to extract only one lower and one upper probability bounds, which takes the fuzzy mean [43] over all  $p$ -boxes:

$$E_{av}(B) = \int_0^1 \underline{E}_\alpha(B) d\alpha \text{ and } \bar{F}_{av}(B) = \int_0^1 \bar{F}_\alpha(B) d\alpha \quad (11)$$

It is shown that  $\forall B \subseteq \mathfrak{R}, \underline{E}_0(B) \leq \underline{E}_{av}(B) \leq \underline{E}_1(B)$  and  $\bar{F}_1(B) \leq \bar{F}_{av}(B) \leq \bar{F}_0(B)$ . Note that Baudrit *et al.* [23] has established the link between the average  $p$ -box  $[E_{av}(B), \bar{F}_{av}(B)]$  and the belief functions in evidence theory, under the condition that there are finite elements in the probability sample and possibility sample spaces, which is not true in our case. Figure 4 depicts the CDF curves of the  $p$ -boxes at the  $\alpha$  levels equal to 0 and 1, and the average  $p$ -boxes.

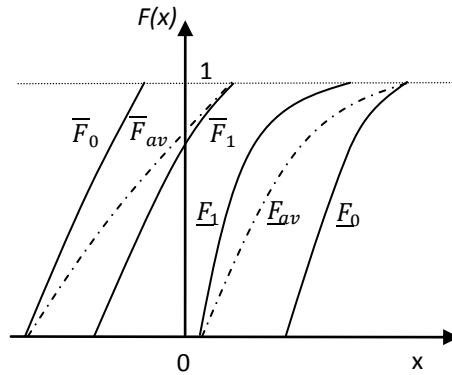


Figure 4. CDF curves of  $[E_0(x), \bar{F}_0(x)]$ ,  $[E_{av}(x), \bar{F}_{av}(x)]$ , and  $[E_1(x), \bar{F}_1(x)]$

## 5.3 Algorithm for the system availability $p$ -boxes extraction



Let  $B$  denote the event  $\ddot{D} < 0$ ; we have the system availability  $p$ -box:  $[\underline{A}_\alpha, \overline{A}_\alpha]$  where  $\underline{A}_\alpha = 1 - \overline{F}_\alpha(B)$  and  $\overline{A}_\alpha = 1 - \underline{F}_\alpha(B)$ . To show the extraction of  $[\underline{A}_\alpha, \overline{A}_\alpha]$  (at a fixed  $\alpha$  level), we take  $\underline{A}_\alpha$  as an example. By definition, we have  $\underline{A}_\alpha = \sum_{\underline{d}_{j_\alpha} \geq 0} p(\underline{d}_{j_\alpha})$ , where  $j = 0, \dots, J_{\ddot{D}}$  and  $J_{\ddot{D}}$  is the highest state of  $\ddot{D}$ . Its computation is straightforward and  $\overline{A}_\alpha$  can be calculated similarly. To show the extraction of the average availability  $p$ -box  $[\underline{A}_{av}, \overline{A}_{av}]$ , we take  $\underline{A}_{av}$  as an example. By definition we have  $\underline{A}_{av} = \int_0^1 \sum_{\underline{d}_{j_\alpha} \geq 0} p(\underline{d}_{j_\alpha}) d\alpha$ . For its computation, at a particular state  $j$  the following mutually exclusive conditions are identified: 1)  $\underline{d}_{j_\alpha} \geq 0$  for any  $\alpha \in [0,1]$ , then we have  $\int_0^1 p(\underline{d}_{j_\alpha}) d\alpha = p(\underline{d}_{j_\alpha})$  because  $p(\underline{d}_{j_\alpha})$  is a constant for any  $\alpha$ ; 2)  $\underline{d}_{j_\alpha} < 0$  for any  $\alpha \in [0,1]$ , then we have  $\int_0^1 p(\underline{d}_{j_\alpha}) d\alpha = 0$ ; 3)  $\underline{d}_{j_{\alpha_1}} < 0$  and  $\underline{d}_{j_{\alpha_2}} \geq 0$  for certain  $\alpha_1, \alpha_2 \in [0,1]$  and  $\alpha_1 < \alpha_2$ , then we have  $\int_0^1 p(\underline{d}_{j_\alpha}) d\alpha = (1 - \alpha_L^*) \times p(\underline{d}_{j_\alpha})$  where  $\underline{d}_{j_{\alpha_L^*}} = 0$  (See Fig. 5).  $\overline{A}_{av}$  can be obtained similarly.

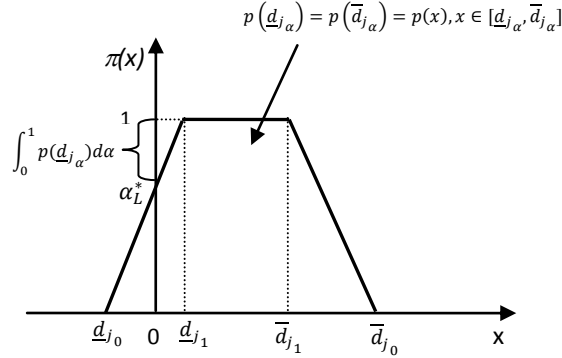


Figure 5. The computation of  $\int_0^1 p(\underline{d}_{j_\alpha}) d\alpha$  when  $\underline{d}_{j_{\alpha_1}} < 0$  and  $\underline{d}_{j_{\alpha_2}} \geq 0$  for certain  $\alpha_1, \alpha_2 \in [0,1]$  and  $\alpha_1 < \alpha_2$ , for a particular state  $j$

Based upon the discussions above, the following algorithm is proposed for the  $p$ -boxes extraction:

*Initialize:* set  $\underline{A}_\alpha = \overline{A}_\alpha = \underline{A}_{av} = \overline{A}_{av} = 0$

**For**  $j = 0$  to  $J_{\ddot{D}}$  **do**

    Obtain  $\underline{x}_{j_\alpha}$  and  $\overline{x}_{j_\alpha}$  by substituting the given  $\alpha$  value into the fuzzy number expression.

**If**  $\underline{d}_{j_\alpha} \geq 0$ , **then**  $\underline{A}_\alpha \leftarrow \underline{A}_\alpha + p_j$ .

**If  $\bar{d}_{j_\alpha} \geq 0$ , then  $\bar{A}_\alpha \leftarrow \bar{A}_\alpha + p_j$ .**  
**If  $\underline{d}_{j_0} \geq 0$ , then  $\underline{A}_{av} \leftarrow \underline{A}_{av} + p_j$ ;**  
**Else-if  $\underline{d}_{j_0} < 0$  and  $\underline{d}_{j_1} \geq 0$ , then calculate  $\alpha_L^*$  and  $\underline{A}_{av} \leftarrow \underline{A}_{av} + p_j \times (1 - \alpha_L^*)$ .**  
**If  $\bar{d}_{j_1} \geq 0$ , then  $\bar{A}_{av} \leftarrow \bar{A}_{av} + p_j$ ;**  
**Else-if  $\bar{d}_{j_1} < 0$  and  $\bar{d}_{j_0} \geq 0$ , then calculate  $\alpha_U^*$  and  $\bar{A}_{av} \leftarrow \bar{A}_{av} + p_j \times \alpha_U^*$  (where  $\bar{d}_{j_{\alpha_U^*}} = 0$ , similar to the definition of  $\alpha_L^*$ ).**  
**End**

## 6. Case Studies

This Section presents two application examples. The first example is relatively small in size. It intends to clearly show the steps of the proposed methods for joint uncertainty propagation and  $p$ -boxes extraction. The second example is more practical in terms of size and complexity. The HUGF approach is compared with the MCS method. All experiments in this example are performed in MATLAB 7.11 on a PC with the Intel CPU of 2.67GH and the memory of 4.00 GB.

### 6.1 Flow transmission system

In this Section, we demonstrate the proposed HUGF method on the three-element flow transmission system, whose block diagram is shown in Fig. 6.

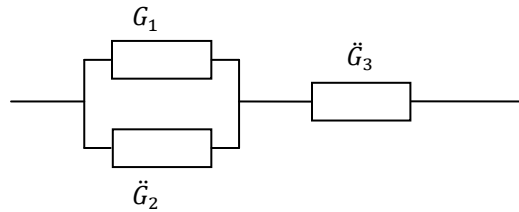


Figure 6. A three component flow transmission system

The u-function of each component performance variable is presented as follows,

$$u_{G_1}(z) = 0.1z^1 + 0.4z^3 + 0.3z^4 + 0.2z^5$$

$$u_{\ddot{G}_2}(z) = 0.1z^{[\alpha,3-\alpha]} + 0.5z^{[2+\alpha,4-\alpha]} + 0.4z^{[3+\alpha,5-\alpha]}$$

$$u_{\ddot{G}_3}(z) = 0.3z^{[6+\alpha,8-\alpha]} + 0.3z^{[7+\alpha,9-\alpha]} + 0.4z^{[8+\alpha,10-\alpha]}$$

Then, HUGF of the system can be written as:

$$\begin{aligned} u_{S\ddot{Y}S}(z) &= (u_{G_1}(z) \otimes_+ u_{\ddot{G}_2}(z)) \otimes_{\min} u_{\ddot{G}_3}(z) \\ &= (0.01z^{[1+\alpha,4-\alpha]} + 0.05z^{[3+\alpha,5-\alpha]} + 0.04z^{[4+\alpha,6-\alpha]} + 0.04z^{[4+\alpha,6-\alpha]} + 0.03z^{[4+\alpha,7-\alpha]} \\ &\quad + 0.2z^{[5+\alpha,7-\alpha]} + 0.02z^{[5+\alpha,8-\alpha]} + 0.31z^{[6+\alpha,8-\alpha]} + 0.22z^{[7+\alpha,9-\alpha]} \\ &\quad + 0.08z^{[8+\alpha,10-\alpha]}) \otimes_{\min} (0.3z^{[6+\alpha,8-\alpha]} + 0.3z^{[7+\alpha,9-\alpha]} + 0.4z^{[8+\alpha,10-\alpha]}) \\ &= 0.01z^{[1+\alpha,4-\alpha]} + 0.05z^{[3+\alpha,5-\alpha]} + 0.04z^{[3+\alpha,6-\alpha]} + 0.04z^{[4+\alpha,6-\alpha]} + 0.03z^{[4+\alpha,7-\alpha]} + 0.2z^{[5+\alpha,7-\alpha]} \\ &\quad + 0.02z^{[5+\alpha,8-\alpha]} + 0.4z^{[6+\alpha,8-\alpha]} + 0.178z^{[7+\alpha,9-\alpha]} + 0.032z^{[8+\alpha,10-\alpha]} \end{aligned}$$

Suppose that the load demand is a constant value 4.25, then the HUGF of system adequacy is:

$$\begin{aligned} u_{\ddot{D}}(z) &= 0.01z^{[-3.25+\alpha, -0.25-\alpha]} + 0.05z^{[-1.25+\alpha, 0.75-\alpha]} + 0.04z^{[-1.25+\alpha, 1.75-\alpha]} + 0.04z^{[-0.25+\alpha, 1.75-\alpha]} \\ &\quad + 0.03z^{[-0.25+\alpha, 2.75-\alpha]} + 0.2z^{[0.75+\alpha, 2.75-\alpha]} + 0.02z^{[0.75+\alpha, 3.75-\alpha]} + 0.4z^{[1.75+\alpha, 3.75-\alpha]} \\ &\quad + 0.178z^{[2.75+\alpha, 4.75-\alpha]} + 0.032z^{[3.75+\alpha, 5.75-\alpha]} \end{aligned}$$

Based on this u-function, the useful quantities for  $p$ -boxes constructions are presented in Table 1.

Table 1. Quantities for constructing  $p$ -boxes

Term	1	2	3	4	5	6	7	8	9	10
$\underline{d}_{j_0}$	-3.25	-1.25	-1.25	-0.25	-0.25	0.75	0.75	1.75	2.75	3.75
$\overline{d}_{j_0}$	-0.25	0.75	1.75	1.75	2.75	2.75	3.75	3.75	4.75	5.75
$\underline{d}_{j_1}$	-2.25	-0.25	-0.25	0.75	0.75	1.75	1.75	2.75	3.75	4.75
$\overline{d}_{j_1}$	-1.25	-0.25	0.75	0.75	1.75	1.75	2.75	2.75	3.75	4.75
$\alpha_L^*$				0.25	0.25					
$\alpha_U^*$		0.75								
Probability	0.01	0.05	0.04	0.04	0.03	0.2	0.02	0.4	0.178	0.032

According to our algorithm, the upper and lower bounds of system availability  $p$ -boxes (including the average  $p$ -box of as the results of homogeneous post-processing) are computed as follows:

$$\underline{A}_0 = 0.2 + 0.02 + 0.4 + 0.178 + 0.032 = 0.83$$

$$\bar{A}_0 = 0.05 + 0.04 + 0.04 + 0.03 + 0.2 + 0.02 + 0.4 + 0.178 + 0.032 = 0.99$$

$$\underline{A}_1 = 0.04 + 0.03 + 0.2 + 0.02 + 0.4 + 0.178 + 0.032 = 0.9$$

$$\bar{A}_1 = 0.04 + 0.04 + 0.03 + 0.2 + 0.02 + 0.4 + 0.178 + 0.032 = 0.94$$

$$\underline{A}_{av} = 0.75 \times 0.04 + 0.75 \times 0.03 + 0.2 + 0.02 + 0.4 + 0.178 + 0.032 = 0.8825$$

$$\bar{A}_{av} = 0.75 \times 0.05 + 0.04 + 0.04 + 0.03 + 0.2 + 0.02 + 0.4 + 0.178 + 0.032 = 0.9775$$

Therefore,  $[\underline{A}_0, \bar{A}_0] = [0.83, 0.99]$ ,  $[\underline{A}_{av}, \bar{A}_{av}] = [0.8825, 0.9775]$ , and  $[\underline{A}_1, \bar{A}_1] = [0.9, 0.94]$ .

## 6.2 Multi-state distributed generation system availability assessment

This Section presents a relative larger scale case study concerning a distributed generation (DG) system of literature [30], with a comparison to the MCS method. The system considered is modified from the IEEE 34 node distribution test feeder [44], and is a radial distribution network downscaled to 4.16 kV via the in-line transformer. The rated power of the transformer is 5000 kW. A number of renewable generators are added onto the network. The ratio of renewable energy to conventional energy is 25%. Within the renewable energy, wind, solar, and electric vehicle (EV) occupy a share of 60%, 30% and 10%, respectively. The DG system infrastructure consists of 5 identical wind turbines with rated power of 150 kW, 5 solar generators/arrays (each one containing 1000 solar cells), and 25 identical EVs with rated power 5 kW. It is noted that the EVs are treated as a single aggregation due to their similar daily charging and discharging patterns [30]. Figure 7 shows the reliability block diagram of this system.

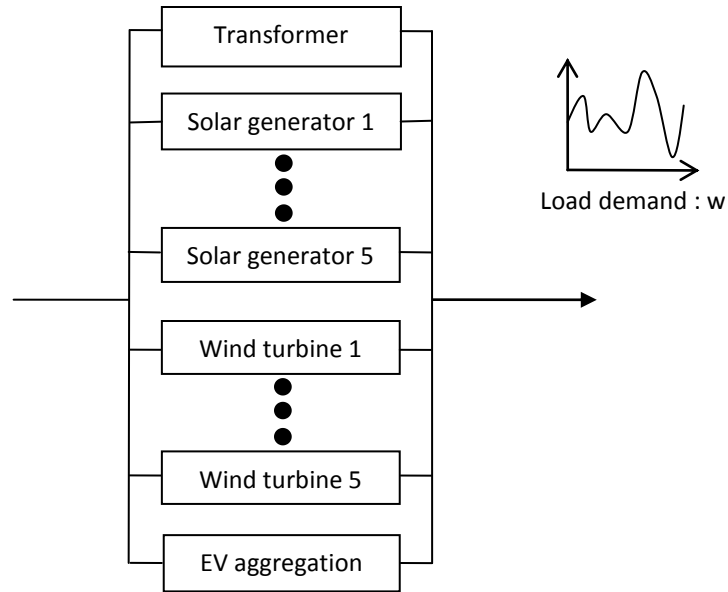


Figure 7. Reliability block diagram of the distributed generation system [30]

Table 2 summarizes the classifications of the uncertainties in all components. More details regarding these classifications can be found in [10].

Table 2. Uncertainties in the DG system model

<i>Component</i>	<i>Parameter</i>	<i>Source of uncertainty</i>	<i>Type of Information available</i>	<i>Uncertainty representation</i>
Solar generator	Solar irradiation $r_S$	Irradiation variability	Historical data	Probabilistic
	Operation parameters $\theta_S$	Incomplete knowledge	Experts' judgments, users' experiences	Possibilistic
	Mechanical state $m_S$	Mechanical failure	Historical data	Probabilistic
Wind turbine	Wind speed $v_W$	Speed variability	Historical data	Probabilistic
	Operation parameters $\theta_W$	Incomplete knowledge	Experts' judgments, users' experiences	Possibilistic
	Mechanical state $m_W$	Mechanical failure	Historical data	Probabilistic
EV aggregation	Power output $G_{EV}$	Incomplete knowledge, subjective decisions	Experts' judgments, users' experiences	Possibilistic
Transformer	Grid power $G_T$	Power fluctuations	Historical data	Probabilistic
	Mechanical state $m_T$	Mechanical failure date	Historical data	Probabilistic

Loads	Load value $W$	Consumption variability	Historical data	Probabilistic
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The single solar generator model is presented in eq. (1). The single wind generator model is presented as follows

$$G_W = g_W(v_W, \theta_W, m_W) = \begin{cases} 0 & v_W < v_{ci} \\ P_r \cdot \frac{(v_W - v_{ci})}{(v_r - v_{ci})} \cdot m_W & v_{ci} \leq v_W < v_r \\ P_r \cdot m_W & v_r \leq v_W < v_{co} \\ 0 & v_{co} \leq v_W \end{cases} \quad (12)$$

The transformer power output is  $G_T = P_T \cdot m_T$ . The HUGF of the system adequacy can be expressed as follows:

$$u_{\bar{D}}(z) = u_{\bar{G}_S}(z) \otimes_+ u_{\bar{G}_W}(z) \otimes_+ u_{\bar{G}_{EV}}(z) \otimes_+ u_{G_T}(z) \otimes_- u_w(z)$$

where

$$u_{\bar{G}_S}(z) = u_{\bar{G}_{S_1}}(z) \otimes_+ u_{\bar{G}_{S_2}}(z) \otimes_+ u_{\bar{G}_{S_3}}(z) \otimes_+ u_{\bar{G}_{S_4}}(z) \otimes_+ u_{\bar{G}_{S_5}}(z)$$

and

$$u_{\bar{G}_W}(z) = u_{\bar{G}_{W_1}}(z) \otimes_+ u_{\bar{G}_{W_2}}(z) \otimes_+ u_{\bar{G}_{W_3}}(z) \otimes_+ u_{\bar{G}_{W_4}}(z) \otimes_+ u_{\bar{G}_{W_5}}(z)$$

where  $u_{\bar{G}_{S_i}}(z)$  is the u-function of the  $i$ -th solar generator and  $u_{\bar{G}_{W_i}}(z)$  is the u-function of the  $i$ -th wind turbine. It is noted that because the DG system is located in a relatively small region the renewable resource variables  $r_S$  and  $v_W$  are identical in each of the solar and wind generators. The possibility and probability distributions of all the parameters in the DG system availability assessment are presented in Table 3.

Table 3. Possibility and probability distributions of the parameters in the DG system

Components	FVs	Core	Support
Solar generator	$\tilde{I}_{MPP}$ (A)	[4.56, 4.86]	[4.36, 5.06]
	$\tilde{V}_{MPP}$ (V)	[16.32, 18.02]	[15.32, 18.32]
	$\tilde{V}_{oc}$ (V)	[20.98, 21.98]	[19.98, 22.98]
	$\tilde{I}_{SC}$ (A)	[5.12, 5.42]	[4.82, 5.62]
	$\tilde{T}_a$ (°C)	[29, 30.5]	[27, 32]
	$\tilde{N}_{ot}$ (°C)	[41, 44]	[39, 46]
	$\tilde{k}_c$ (A/°C)	[0.00112, 0.00132]	[0.00102, 0.00152]
	$\tilde{k}_v$ (V/°C)	[0.0134, 0.0144]	[0.0124, 0.0164]

Wind turbine	$\tilde{v}_{ci}$ (m/s)	[3.2, 3.4]	[3.0, 3.5]
	$\tilde{v}_{co}$ (m/s)	[48, 51]	[45, 54]
	$\tilde{v}_r$ (m/s)	[11, 11.5]	[10, 12]
	$\tilde{P}_r$ (kW)	[145, 155]	[140, 160]
EV	$\tilde{G}_{EV}$ (kW)	[-75, 75]	[-125, 125]
	<b><math>RVs</math></b>	<b>State performance value</b>	<b>State probability</b>
Solar generator	$r_S$ (kW/m <sup>2</sup> )	0.05	5.36E-01
		0.15	8.90E-02
		0.25	6.11E-02
		0.35	4.91E-02
		0.45	4.26E-02
		0.55	3.91E-02
		0.65	3.74E-02
		0.75	3.77E-02
		0.85	4.12E-02
	0.95	6.64E-02	
$m_S$	0	4.00E-02	
	1	9.60E-01	
Wind turbine	$v_W$ (m/s)	0.75	4.36E-02
		2.25	1.54E-01
		3.75	2.30E-01
		5.25	2.33E-01
		6.75	1.75E-01
		8.25	1.00E-01
		9.75	4.42E-02
		11.25	1.50E-02
		12.75	3.94E-03
	14.25	7.93E-04	
$m_W$	0	4.00E-02	
	1	9.60E-01	
Transformer	$P_T$ (kW)	4050	1.00E-01
		4150	1.00E-01
		4250	1.00E-01
		4350	1.00E-01
		4450	1.00E-01
		4550	1.00E-01
		4650	1.00E-01
		4750	1.00E-01
		4850	1.00E-01
	4950	1.00E-01	
$m_T$	0	3.00E-02	
	1	9.70E-01	
Load demand	$w$ (KW)	1673	4.41E-02
		1971	1.37E-01
		2268	1.74E-01
		2566	1.31E-01
		2864	1.61E-01
		3161	1.24E-01
		3459	1.10E-01
		3756	8.78E-02
		4054	2.88E-02
		4351	4.00E-03

The results from the HUGF approach are compared to those obtained by the MCS method (with  $\Delta\alpha = 0.02$ ). To investigate the convergence property of MCS, different number of simulation runs: 10000, 100000, and 1000000, have been performed and all realizations are subdivided into 10 subsamples of equal size. The sample mean and standard deviations of the estimated  $p$ -boxes are presented in Table 4. The comparisons are made on the absolute errors between the upper and lower bounds of the  $p$ -boxes obtained by HUGF and the mean upper and lower bounds of the  $p$ -boxes (i.e. the belief functions) obtained by the MCS method with different numbers of runs. It is clearly seen that the MCS  $p$ -boxes are getting closer to the HUGF  $p$ -boxes when the number of simulation runs increases. In addition, the HUGF approach is in general much more efficient than the MCS method. It should be noted that the standard approximation method has been applied due to the large number of FVs in this case study.

Table 4 The system availability  $p$ -boxes and the comparisons

<i>Methods</i>		<i>HUGF</i>	<i>MCS</i>		
			1000 runs	10000 runs	100000 runs
$[\underline{A}_{av}, \overline{A}_{av}]$	<i>Mean</i>	<b>[0.9680, 0.9697]</b>	<b>[0.9689, 0.9711]</b>	<b>[0.9684, 0.9701]</b>	<b>[0.9682, 0.9699]</b>
	<i>Std*</i>	N.A.	0.0043, 0.0055	0.0020, 0.0019	0.0005, 0.0005
	<i>AE*</i>	N.A.	0.0009, 0.0009	0.0004, 0.0004	0.0002, 0.0002
$[\underline{A}_0, \overline{A}_0]$	<i>Mean</i>	<b>[0.9674, 0.9697]</b>	<b>[0.9682, 0.9712]</b>	<b>[0.9679, 0.9701]</b>	<b>[0.9676, 0.9699]</b>
	<i>Std</i>	N.A.	0.0042, 0.0055	0.0021, 0.0019	0.0005, 0.0005
	<i>AE</i>	N.A.	0.0009, 0.0009	0.0005, 0.0004	0.0002, 0.0002
$[\underline{A}_1, \overline{A}_1]$	<i>Mean</i>	<b>[0.9684, 0.9696]</b>	<b>[0.9696, 0.9711]</b>	<b>[0.9688, 0.9700]</b>	<b>[0.9686, 0.9698]</b>
	<i>Std</i>	N.A.	0.0044, 0.0054	0.0020, 0.0020	0.0005, 0.0005
	<i>AE</i>	N.A.	0.0009, 0.0008	0.0004, 0.0004	0.0002, 0.0002
<i>Computation time (Sec)</i>		0.8403	4.1194	41.1859	413.4156

Std\*: standard deviation

AE\*: absolute error

The MATLAB source code of this case study is available upon request to the first author.

## 7. Conclusions

Aleatory and epistemic uncertainties always co-exist in the models of the assessment of industrial systems. How to properly handle them poses challenges to the reliability engineers. In this work, we have proposed an efficient approach based on UGF for joint uncertainty representation, propagation and exploitation in availability assessments of MSS. Drawing from the well-established RFV theory, HUGF has shown to be adequate for the representation of RFVs defined on a finite set of FVs. Based upon this foundation, the composition operator of HUGF



has been defined by combining probabilistic convolution with the fuzzy extension principle. The computation complexity of the propagation procedure has been evaluated and reduction methods are presented. Finally, an efficient algorithm has been designed to extract availability  $p$ -boxes from the system adequacy HUGF. The case studies show the effectiveness of the HUGF approach in comparison to the widely used MCS method. However, the computational efficiency and accuracy of the HUGF can be still improved by, for example, using advanced approximation techniques for FV arithmetic operations and more efficient clustering algorithms for fuzzy state reduction.

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## Appendix

**Proposition 1.** For a RFV  $\check{X}$  defined on a finite set of fuzzy numbers  $\boldsymbol{\pi}$ ,  $|\boldsymbol{\pi}| = J + 1$ , its statistical expectation  $E(\check{X})$  is a nested FV expressed as  $\left[ \sum_{j=0}^J \underline{x}_{j\alpha} \Delta P_j, \sum_{j=0}^J \bar{x}_{j\alpha} \Delta P_j \right]$ .

**Proof:** Let  $[\underline{x}_{j\alpha}, \bar{x}_{j\alpha}]$  denote the  $j$ -th fuzzy number in the finite set  $\boldsymbol{\pi}$  such that at any  $\alpha$ -cut level  $\underline{x}_{j+1\alpha} \geq \underline{x}_{j\alpha} \geq \underline{x}_{j-1\alpha}$  and  $\bar{x}_{j-1\alpha} \leq \bar{x}_{j\alpha} \leq \bar{x}_{j+1\alpha}$  for any  $1 \leq j \leq J - 1, j \in \mathbb{N}$ .

According to *Definition 1*,  $E(\check{X}) = \left[ \int x d\underline{F}_\alpha(x), \int x d\bar{F}_\alpha(x) \right]$ . Because  $\boldsymbol{\pi}$  is finite, at any  $\alpha$  cut level the PMFs of the two boundary values  $\underline{x}_{j\alpha}$  and  $\bar{x}_{j\alpha}$  can be described by the 2-tuples  $[\underline{x}_{j\alpha}, p(\underline{x}_{j\alpha})]$  and  $[\bar{x}_{j\alpha}, p(\bar{x}_{j\alpha})]$ , respectively. Recall that the CDF of a discrete RV  $X$  can be written as  $F(x) = \sum_{x_j \leq x} p(x_j)$ . Then we have  $\underline{F}_\alpha(x) = \sum_{\bar{x}_{j\alpha} \leq x} p(\bar{x}_{j\alpha})$  and  $\bar{F}_\alpha(x) = \sum_{\underline{x}_{j\alpha} \leq x} p(\underline{x}_{j\alpha})$ . For the  $j$ -th fuzzy number, we have  $p(\underline{x}_{j\alpha}) = p(\bar{x}_{j\alpha}) = p(x)$ , where  $x \in [\underline{x}_{j\alpha}, \bar{x}_{j\alpha}]$ . Let  $\Delta P_j = p(\bar{x}_{j\alpha}) = p(\underline{x}_{j\alpha})$ , then  $\int x d\underline{F}_\alpha(x) = \sum_{j=0}^J \bar{x}_{j\alpha} \Delta P_j$  and  $\int x d\bar{F}_\alpha(x) = \sum_{j=0}^J \underline{x}_{j\alpha} \Delta P_j$ .

For any fuzzy membership value  $\beta \in [0,1]$  and  $\alpha < \beta$ , due to the nestedness of the possibility distribution we have  $\underline{x}_{j\alpha} \leq \underline{x}_{j\beta}$  and  $\bar{x}_{j\alpha} \geq \bar{x}_{j\beta}$ . Then,  $\sum_{j=0}^J \underline{x}_{j\alpha} \Delta P_j \leq \sum_{j=0}^J \underline{x}_{j\beta} \Delta P_j$  and  $\sum_{j=0}^J \bar{x}_{j\alpha} \Delta P_j \geq \sum_{j=0}^J \bar{x}_{j\beta} \Delta P_j$ . Therefore, the  $E(\check{X})$  is a nested FV. ■

**Proposition 2.** For a RFV  $\check{X}$  defined on a finite set of fuzzy numbers  $\boldsymbol{\pi}$ ,  $|\boldsymbol{\pi}| = J + 1$ , the first derivative of  $u_{\check{X}}(z)$  at  $z = 1$  equals to  $E(\check{X})$ .

**Proof:** The first derivative of  $u_{\check{X}}(z)$  is  $\frac{du_{\check{X}}(z)}{dz} = \frac{d}{dz} \left( \sum_{j=0}^J p_j z^{\check{X}_j} \right) = \sum_{j=0}^J p_j \check{X}_j z^{\check{X}_j - 1}$ , hence  $\frac{du_{\check{X}}(1)}{dz} = \sum_{j=0}^J p_j \check{X}_j = \left[ \sum_{j=0}^J p_j \underline{x}_{j\alpha}, \sum_{j=0}^J p_j \bar{x}_{j\alpha} \right]$ . Let  $p_j = \Delta P_j$ ; we, then, obtain  $E(\check{X}) = \left[ \sum_{j=0}^J p_j \underline{x}_{j\alpha}, \sum_{j=0}^J p_j \bar{x}_{j\alpha} \right]$ . ■

**Case 1:**  $\otimes_f$  between the  $u$ -functions of two FVs  $\check{X}_1$  and  $\check{X}_2$ ,

$$u_{\tilde{X}_1}(z) \otimes_f u_{\tilde{X}_2}(z) = z^{f(\tilde{X}_1, \tilde{X}_2)} \quad (\text{B.1})$$

The extension principle [36] reads that  $\pi_{\tilde{Y}}(y) = \sup_{y=f(x_1, x_2)} \min(\pi_{\tilde{X}_1}(x_1), \pi_{\tilde{X}_2}(x_2))$ . For example, in the denominator of eq. (1.e) if we have  $\tilde{I}_{sc} = [1 + \alpha, 4 - \alpha]$  and  $\tilde{V}_{oc} = [2 + \alpha, 4 - \alpha]$  then u-function of the denominator can be written as

$$u_{\tilde{I}_{sc}}(z) \otimes_{\times} u_{\tilde{V}_{oc}}(z) = z^{\tilde{I}_{sc} \times \tilde{V}_{oc}} = z^{[(1+\alpha) \times (2+\alpha), (4-\alpha) \times (4-\alpha)]} \quad (\text{B.2})$$

It is noted that the fuzzy arithmetic assumes the total dependence between the  $\alpha$ -cuts [23].

**Case 2:**  $\otimes_f$  between one RV  $X_1$  and one FV  $\tilde{X}_2$ ,

$$u_{X_1}(z) \otimes_f u_{\tilde{X}_2}(z) = \sum_{j_1=0}^{J_1} p_{1j_1} z^{f(x_{1j_1}, \tilde{X}_2)} \quad (\text{B.3})$$

For example, on the right hand side of eq. (1.b) the first term is  $r_S \cdot \tilde{I}_{sc}$ . Suppose that  $r_S$  has three state levels (0, 0.2, 0.8) with the probability vector (0.4, 0.4, 0.2), then the u-function of this term can be written as

$$u_{r_S}(z) \otimes_{\times} u_{\tilde{I}_{sc}}(z) = 0.4z^{[0,0]} + 0.4z^{[0.2(1+\alpha), 0.2(4-\alpha)]} + 0.2z^{[0.8(1+\alpha), 0.8(4-\alpha)]} \quad (\text{B.4})$$

**Case 3:**  $\otimes_f$  between two RFVs  $\tilde{X}_1$  and  $\tilde{X}_2$ ,

$$u_{\tilde{X}_1}(z) \otimes_f u_{\tilde{X}_2}(z) = \sum_{j_1=0}^{J_1} \sum_{j_2=0}^{J_2} p_{1j_1} p_{2j_2} z^{f(\tilde{X}_{1j_1}, \tilde{X}_{2j_2})} \quad (\text{B.5})$$

For example, by substituting eq. (1.d) into eq. (1.b) we have the first and second terms to be  $r_S \cdot \tilde{I}_{sc}$  and  $r_S \cdot \tilde{k}_c \cdot \tilde{T}_a$ . Let  $\tilde{k}_c = [1 + \alpha, 4 - \alpha]$  and  $\tilde{T}_a = [2 + \alpha, 5 - \alpha]$ ; then, we have the following u-function for the addition of these two terms

$$\begin{aligned} u_{r_S \cdot \tilde{I}_{sc}}(z) \otimes_{+} u_{r_S \cdot \tilde{k}_c \cdot \tilde{T}_a}(z) &= (0.4z^{[0,0]} + 0.4z^{[0.2(1+\alpha), 0.2(4-\alpha)]} + 0.2z^{[0.8(1+\alpha), 0.8(4-\alpha)]}) \otimes_{+} \\ & (0.4z^{[0,0]} + 0.4z^{[0.2(1+\alpha)(2+\alpha), 0.2(4-\alpha)(5-\alpha)]} + 0.2z^{[0.8(1+\alpha)(2+\alpha), 0.8(4-\alpha)(5-\alpha)]}) \\ &= 0.16z^{[0,0]} + 0.16z^{[0.2(1+\alpha)(2+\alpha), 0.2(4-\alpha)(5-\alpha)]} + 0.08z^{[0.8(1+\alpha)(2+\alpha), 0.8(4-\alpha)(5-\alpha)]} + \\ & 0.16z^{[0.2(1+\alpha), 0.2(4-\alpha)]} + 0.16z^{[0.2(1+\alpha)(3+\alpha), 0.2(4-\alpha)(6-\alpha)]} + \\ & 0.08z^{[0.2(1+\alpha)(9+4\alpha), 0.2(4-\alpha)(21-4\alpha)]} + 0.08z^{[0.8(1+\alpha), 0.8(4-\alpha)]} + \\ & 0.08z^{[0.2(1+\alpha)(6+\alpha), 0.2(4-\alpha)(9-\alpha)]} + 0.04z^{[0.8(1+\alpha)(3+\alpha), 0.8(4-\alpha)(6-\alpha)]} \end{aligned}$$