Monotonicity and 1-dimensional symmetry for solutions of an elliptic system arising in Bose-Einstein condensation

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Abstract

We study monotonicity and 1-dimensional symmetry for positive solutions with algebraic growth of the following elliptic system:

$$\begin{cases} -\Delta u = -uv^2 & \text{in } \mathbb{R}^N \\ -\Delta v = -u^2 v & \text{in } \mathbb{R}^N, \end{cases}$$

for every dimension $N \geq 2$. In particular, we prove a Gibbons-type conjecture proposed by H. Berestycki, T. C. Lin, J. Wei and C. Zhao.

Keywords: elliptic system; phase-separation; 1-dimensional symmetry; blow-down sequence; moving planes method.

1 Introduction

This paper concerns monotonicity and 1-dimensional symmetry for entire solutions with algebraic growth of the following semilinear elliptic system:

$$\begin{cases} -\Delta u = -uv^2 & \text{in } \mathbb{R}^N \\ -\Delta v = -u^2 v & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \geq 2$. System (1) has been intensively studied during the last years, starting from the seminal papers [2] and [10]. Therein, (1) appears in the analysis of phase-separation phenomena for Bose-Einstein condensates with multiple states (we refer to [2, 3] and to the references therein for more details concerning the physical motivations). In particular, in [2] is emphasized the relationship between system (1) and the celebrated Allen-Cahn equation. This relationship induced the authors to formulate a De Giorgi's-type and a Gibbons'-type conjecture for the solutions of (1) (we refer to [8] for a review on the De Giorgi's conjecture and some related problems). In this paper we address precisely the following Gibbons'-type conjecture:

Conjecture (section 7 of [2]). *Let $N \geq 2$, let (u,v) be a solution of (1) satisfying*

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \textit{and} \quad \lim_{x_N \to +\infty} u(x', x_N) = +\infty$$
$$\lim_{x_N \to -\infty} v(x', x_N) = +\infty \quad \textit{and} \quad \lim_{x_N \to +\infty} v(x', x_N) = 0,$$

the limits being uniform in $x' \in \mathbb{R}^{N-1}$. Then (u,v) is 1-dimensional.

Clearly, with respect to the original counterparts, major difficulties arise from the fact that in the present case we have to deal with a system of equations instead of with a single equation, and with unbounded solutions.
In what follows, we review the main achievements concerning the existence and the 1-dimensional symmetry of entire solutions to (1). In [10], it is showed that there is not a positive solution which is globally α-Hölder continuous for some $\alpha \in (0,1)$. On the other hand, in [2] the authors proved the existence of a non-constant solution for (1) when $N = 1$ (in this case we have a system of ODEs). This solution has linear growth: there exists $C > 0$ such that

$$u(t) + v(t) \leq C(1 + |t|) \qquad \forall t \in \mathbb{R};$$

moreover, it is reflectionally symmetric with respect to a certain $t_0 \in \mathbb{R}$, in the sense that

$$u(t_0 + t) = v(t_0 - t) \qquad \forall t \in \mathbb{R}.$$

In [3] it is proved that this is the unique positive entire solution (up to translations and scalings) in case $N = 1$. On the other hand, always in [3], the authors constructed for every $N \geq 2$ entire solutions with arbitrary integer algebraic growth; here and in the rest of the paper we say that (u,v) has algebraic growth if there exist $p \geq 1$ and $C > 0$ such that

$$u(x) + v(x) \leq C(1 + |x|^p) \qquad \forall x \in \mathbb{R}^N. \tag{h1}$$

These solutions, which depend on more then one variable, are constructed exploiting the deep relationship between entire solutions of (1) and entire harmonic functions. This relationship has been established in [5, 10, 12]. Recently, a similar argument has been exploited in [11] to prove the existence of solutions to (1) having exponential growth in one direction.
Concerning symmetry results, we say that (u,v) is 1-dimensional if there exists $\nu \in \mathbb{R}^N$ such that

$$u(x) = \bar{u}(\langle \nu, x \rangle) \quad \text{and} \quad v(x) = \bar{v}(\langle \nu, x \rangle),$$

for some $\bar{u}, \bar{v} : \mathbb{R} \to \mathbb{R}$. In [2] the authors proved that if $N = 2$, (u,v) has linear growth and is monotone in the e_N direction, in the sense that

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{and} \quad \frac{\partial v}{\partial x_N} < 0 \quad \text{in } \mathbb{R}^N,$$

then (u,v) is 1-dimensional. An improvement of this result has been recently obtained by the first author in [7]: he replaced the linear growth condition with an arbitrary algebraic growth condition (i.e. (h1)), and weakened the monotonicity assumption requiring that only one component between u and v is monotone in x_N. Always in case $N = 2$, in [3] it is showed that if (u,v) has linear growth and is stable then (u,v) is 1-dimensional. As far as the case $N \geq 2$ is concerned, we refer to the recent contribution [13]: the author proved that for any $N \geq 2$, if (u,v) has linear growth and is a local minimizer for the energy functional, then (u,v) is 1-dimensional.
Our main result is the following:

Theorem 1.1. *Let $N \geq 2$, let (u,v) be a solution of system (1) having algebraic growth (i.e. satisfying (h1)) and such that*

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \textit{and} \quad \lim_{x_N \to +\infty} u(x', x_N) = +\infty$$
$$\lim_{x_N \to -\infty} v(x', x_N) = +\infty \quad \textit{and} \quad \lim_{x_N \to +\infty} v(x', x_N) = 0, \tag{h2}$$

the limit being uniform in $x' \in \mathbb{R}^{N-1}$. *Then* (u,v) *depends only on the* x_N *variable, and*

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{and} \quad \frac{\partial v}{\partial x_N} < 0 \quad \textit{in } \mathbb{R}^N.$$

Some remarks are in order: the conjecture proposed by H. Berestycki, T. C. Lin, J. Wei and C. Zhao in [2] was formulated without assumption (h1). Nevertheless, at this stage it seems really hard to deal without an algebraic growth condition, because most of the results which are present in the literature rest strongly on it (concerning symmetry results, except the work [7] all the quoted achievements are obtained under the linear growth assumption). As far as we know, the unique contribution going beyond the algebraic growth is given in [11], where the authors proved the existence of solutions to (1) with exponential growth. Therein, it is often remarked the striking difference between solutions having algebraic growth and solutions having exponential growth, which reflects the difference between harmonic polynomial and harmonic function with exponential growth. For us, the main problem to deal with solutions not satisfying the algebraic growth condition would be the lack of the blow-down technology, see Theorem 1.4 of [3].
On the other hand, in light of the strongly coupled nature of system (1), we can weaken assumption (h2) obtaining again monotonicity and 1-dimensional symmetry.

Corollary 1.2. *Let* $N \geq 2$, *and let* (u,v) *be a solution of system* (1) *having algebraic growth (i.e. satisfying* (h1)*), and such that*

$$\lim_{x_N \to \pm\infty} (u(x', x_N) - v(x', x_N)) = \pm\infty, \tag{h3}$$

the limits being uniform in $x' \in \mathbb{R}^{N-1}$. *Then* (u,v) *depends only on the* x_N *variable, and*

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{and} \quad \frac{\partial v}{\partial x_N} < 0 \quad \textit{in } \mathbb{R}^N.$$

Notations. Let (u,v) be a solution of (1). We recall some notation that are by now standard. Given $x \in \mathbb{R}^N$ and $r > 0$, we set

$$\begin{aligned} H(x,r) &:= \frac{1}{r^{N-1}} \int_{\partial B_r(x)} u^2 + v^2, \\ E(x,r) &:= \frac{1}{r^{N-2}} \int_{B_r(x)} |\nabla u|^2 + |\nabla v^2| + u^2 v^2, \end{aligned} \tag{2}$$

and $N(x,r) := \dfrac{E(x,r)}{H(x,r)}$. The function N is called *Almgren frequency function*, or *Almgren quotient.*
For every $x_0 \in \mathbb{R}^N$ and $R > 0$, we introduce

$$(u_{x_0,R}(x), v_{x_0,R}(x)) := \left(\frac{1}{\sqrt{H(x_0,R)}} u(x_0 + Rx), \frac{1}{\sqrt{H(x_0,R)}} v(x_0 + Rx) \right). \tag{3}$$

The family $\{(u_{x_0,R}, v_{x_0,R}) : R > 0\}$ is called *the blow-down family of* (u,v) *centered in* x_0.
Finally, we will consider the function

$$J(x,r) := \frac{1}{r^4} \int_{B_r(x)} \frac{|\nabla u(y)|^2 + u^2(y) v^2(y)}{|x-y|^{N-2}}\, dy \int_{B_r(x)} \frac{|\nabla v(y)|^2 + u^2(y) v^2(y)}{|x-y|^{N-2}}\, dy.$$

For some properties related to the Almgren quotient, the blow-down family and the function J, we refer to the appendix and to the references therein.

We will use the notation $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ for a point of $\mathbb{R}^N$.
The directional derivative with respect to $\mu \in \mathbb{S}^{N-1}$ will be denoted by $\frac{\partial}{\partial \mu}$ or by ∂_μ. When we integrate

by parts, we denote by ∂_ν the normal derivative. The i-th coordinate direction will be denoted by e_i. We will use the notation $\langle \cdot, \cdot \rangle$ or $|\cdot|$ for the usual scalar product or the usual euclidean norm in any euclidean space.
Throughout the paper $C, C_1, C_2, \ldots$ will denote positive constants which may refer to different quantities from line to line. On the other hand, we will fix the value of some constants. In these cases we will use the over-lined notation $\bar{C}_1, \bar{C}_2, \ldots$.

Plan of the paper We wish to prove the 1-dimensional symmetry of the solution (u, v) by means of the moving planes method. First of all, in section 2, we will provide some estimates which will be useful in the rest of the paper.
In section 3 we will make rigorous the intuitive fact that, under assumption (h2), x_N is the privileged variable of the solution (u, v): to be precise, by means of the blow-down technology, we will show that independently on the base point $x_0 \in \mathbb{R}^N$ the entire blow-down family converges to the same function $(\gamma x_N^+, \gamma x_N^-)$, with $\gamma > 0$.
In section 4 we will show that, under our assumptions, $\partial_N u(x) > 0$ in $\{x_N \gg 1\}$ and $\partial_N v(x) < 0$ in $\{x_N \ll 1\}$. This does not follow directly from the results of section 3, because the quantitative information given by the convergence of the blow-down family get worse as $R \to +\infty$ (we refer to section 4 for more details).
In section 5 we will use the moving planes method to deduce that $\partial_N u > 0$ and $\partial_N v < 0$ in $\mathbb{R}^N$; firstly, by the fact that $\partial_N u > 0$ for $x_N \gg 1$ we will deduce that in the same region $\partial_N v < 0$; this can be done thanks to a version of the maximum principle in unbounded domains, and allow us to start the moving planes method. We point out that it is not possible to proceed separately on u and on v (that is, it is not possible to show that $\partial_N u > 0$ and, in a second time, that $\partial_N v < 0$ in $\mathbb{R}^N$); this reflects the strongly coupled nature of system (1), and introduce a lot of complications with respect to the case of a single equation.
In section 6, we will complete the proof of Theorem 1.1, passing from the monotonocity in the e_N direction to the monotonocity in all the directions of the upper hemisphere $\mathbb{S}_+^{N-1} := \{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu \rangle > 0\}$; we will follow the line of reasoning introduced by the first author in [6], with the obvious complications which come from the fact that we are working with a system and not on a single equation, and that we are dealing with unbounded solutions.
Finally, in section 7 we will give the proof of Corollary 1.2; to be precise, we will show that under (h1) and (h3), the assumption (h2) is satisfied, so that Corollary 1.2 follows from our main theorem.
We reported some known results in the appendix at the end of the paper; this appendix can be considered as an easy-to-read introduction to the study of system (1).

2 Preliminary results

In [13], the author introduced an Alt-Caffarelli-Friedman monotonicity formula for solutions of (1) (we reported it in the appendix). This formula gives a lower bound for some integral quantities related to solutions having linear growth (cf. the results of section 4 of [13]). In this section we prove some new results and we refine some estimates of the quoted paper, in order to use them in the next sections.

In Corollary 4.5 of [13], the author used the linear growth of the solution (u, v) to obtain a lower bound for the growth of the function

$$r \mapsto \int_{\partial B_r(0)} u^2 + v^2.$$

We think that it is interesting to note that an equivalent estimate holds true assuming only that (u, v) has algebraic growth. Clearly, this requires some extra-work.

Corollary 2.1. *Let (u, v) be a solution of (1) satisfying (h1). There exists $C > 0$ such that*

$$\int_{B_r(0)} u^2 + v^2 \geq C r^{N+2} \qquad \forall r \geq 1.$$

Proof. Assume by contradiction that the statement is not true: there exists $\varepsilon_n \to 0$ and $(r_n) \subset [1, +\infty)$ such that

$$\int_{B_{r_n}(0)} u^2 + v^2 \le \varepsilon_n r_n^{N+2}. \tag{4}$$

Step 1) $\liminf_{n\to\infty} r_n = +\infty$.
If not, up to a subsequence $r_n \to \bar{r} \ge 1$. By the dominated convergence theorem, we deduce

$$\int_{B_{\bar{r}}(0)} u^2 + v^2 = 0 \quad \Rightarrow \quad (u, v) \equiv (0, 0),$$

a contradiction.

Step 2) *Conclusion of the proof.*
To simplify the notation, we denote by $j_u(r)$ the quantity

$$\frac{1}{r^2} \int_{B_r(0)} \frac{|\nabla u(y)|^2 + u^2(y) v^2(y)}{|y|^{N-2}} \, dy,$$

and by $j_v(r)$ the same quantity for the component v. Now, by Theorem A.15 there exists $C > 0$ such that $J(0, r) \ge C$ for every $r \ge 1$, that is, $j_u(r) j_v(r) \ge C$ for every $r \ge 1$. In particular, this holds true for every r_n. Up to a subsequence, we can assume $j_u(r_n) \ge C$ for every n. By means of (49) (we remark that the constant appearing is independent on r) plus our absurd assumption (4), we obtain

$$0 < C \le j_u(r_n) \le \frac{C}{r_n^{N+2}} \int_{B_{2r_n}(0)} u^2 \le C\varepsilon_n \to 0$$

as $n \to \infty$, a contradiction. □

Under the linear growth assumption of (u, v), that is, there exists $C > 0$ such that

$$u(x) + v(x) \le C(1 + |x|) \qquad \forall x \in \mathbb{R}^N, \tag{5}$$

we obtain a uniform (in both $x \in \mathbb{R}^N$ and $r \ge 1$) lower bound for the values $\{H(x, r)\}$.

Lemma 2.2. *Let (u, v) be a solution of (1) with linear growth. There exists $\bar{C}_1 > 0$ such that*

$$H(x, r) \ge \bar{C}_1$$

for every $x \in \mathbb{R}^N$ and $r \ge 1$.

Proof. By the monotonicity of $H(x, \cdot)$, it is sufficient to show that $H(x, 1) \ge C$ with C independent on $x \in \mathbb{R}^N$. By contradiction, assume that there exists $(x_i) \subset \mathbb{R}^N$ such that

$$\lim_{i\to\infty} \int_{\partial B_1(x_i)} u^2 + v^2 = 0. \tag{6}$$

By Corollary 2.1, we know that there exists $C > 0$ such that

$$\int_{B_r(0)} u^2 + v^2 \ge C r^{N+2} \qquad \forall r \ge 1,$$

Let $r \ge 1$; for every i we have

$$\int_{B_{r+|x_i|}(x_i)} u^2 + v^2 \ge \int_{B_r(0)} u^2 + v^2 \ge C r^{N+2}. \tag{7}$$

Note that
$$\int_{B_{r+|x_i|}(x_i)} u^2+v^2 = \int_{B_{r+|x_i|}(x_i)\setminus B_1(x_i)} u^2+v^2 + \int_{B_1(x_i)} u^2+v^2;$$
thanks to Lemma A.9 we know that $N(x_i,r) \le 1$ for every $r \ge 1$, for every i. Hence, by means of Corollary A.7, we deduce
$$\begin{aligned}\int_{B_{r+|x_i|}(x_i)\setminus B_1(x_i)} u^2+v^2 &= \int_1^{r+|x_i|}\left(\int_{\partial B_s(x_i)} u^2+v^2\right) ds \le e\left(\int_{\partial B_1(x_i)} u^2+v^2\right)\int_1^{r+|x_i|} s^{N+1}\,ds\\ &\le e\left(\int_{\partial B_1(x_i)} u^2+v^2\right)(r+|x_i|)^{N+2}.\end{aligned}$$
Therefore
$$\int_{B_{r+|x_i|}(x_i)} u^2+v^2 \le e\left(\int_{\partial B_1(x_i)} u^2+v^2\right)(r+|x_i|)^{N+2} + \int_{B_1(x_i)} u^2+v^2. \tag{8}$$
We observe that from the linear growth of (u,v) it follows also
$$\int_{B_1(x_i)} u^2+v^2 \le C(1+|x_i|)^2,$$
where C does not depend on i. Plugging into the (8) and choosing $r = r_i \ge |x_i|$, $r_i \to +\infty$ as $i \to \infty$ (here i is fixed, so this choice is possible), we deduce
$$\begin{aligned}\int_{B_{r_i+|x_i|}(x_i)} u^2+v^2 &\le e\left(\int_{\partial B_1(x_i)} u^2+v^2\right)(r_i+|x_i|)^{N+2} + C\left(1+|x_i|^2\right)\\ &\le C\left(\int_{\partial B_1(x_i)} u^2+v^2\right) r_i^{N+2} + C(1+r_i^2).\end{aligned}$$
A comparison with (7) yields
$$Cr_i^{N+2} \le C\left(\int_{\partial B_1(x_i)} u^2+v^2\right) r_i^{N+2} + Cr_i^2.$$
Dividing for r_i^{N+2} and passing to the limit as $i \to \infty$, we finally obtain a contradiction:
$$0 < C \le C \lim_{i\to\infty}\int_{\partial B_1(x_i)} u^2+v^2 = 0,$$
where we used our absurd assumption, equation (6). □

Where $|u-v|$ is not too large, it is natural to expect that this provides a lower bound on the integrals of both u^2 and v^2. To be precise:

Lemma 2.3. *Let (u,v) be a solution of (1) having linear growth. For every $C_1 < \sqrt{\bar{C}_1|\mathbb{S}^{N-1}|}$ (where $\bar{C}_1$ has been defined in Lemma 2.2) there exists $\bar{C}_2 > 0$ such that*
$$\int_{\partial B_1(x_0)} u^2 \ge \bar{C}_2 \quad \text{and} \quad \int_{\partial B_1(x_0)} v^2 \ge \bar{C}_2$$
for every $x_0 \in \{|u-v| < C_1\}$.

Proof. Without loss of generality, we can assume by contradiction that, for a sequence $(x_i) \subset \{|u-v| < C_1\}$, we have

$$\lim_{i\to\infty} \int_{\partial B_1(x_i)} u^2 = 0.$$

We claim that under this assumption

$$\lim_{i\to\infty} \int_{\partial B_1(x_i)} v^2 = 0.$$

If not, up to a subsequence there exists $\delta > 0$ such that $\lim_i \int_{\partial B_1(x_i)} v^2 \geq \delta^2$. We introduce the sequence

$$(u_i(x), v_i(x)) = \left(\frac{1}{\sqrt{H(x_i,1)}} u(x_i + x), \frac{1}{\sqrt{H(x_i,1)}} v(x_i + x) \right).$$

Note that $\int_{\partial B_1(0)} u_i^2 + v_i^2 = 1$ for every i. Each (u_i, v_i) solves

$$\begin{cases} -\Delta u_i = H(x_i,1) u_i v_i^2 & \text{in } \mathbb{R}^N \\ -\Delta v_i = H(x_i,1) u_i^2 v_i & \text{in } \mathbb{R}^N; \end{cases}$$

By Corollary A.7 (which we can apply, see Remark A.11), we deduce that

$$\int_{\partial B_r(0)} u_i^2 + v_i^2 \leq e r^{N+1} \qquad \forall r, \ \forall i. \tag{9}$$

As u_i and v_i are subharmonic, the (9) gives a uniform bound on the $L^\infty(B_{r/2}(0))$ norm of the family $\{(u_i, v_i)\}$, for every $r \geq 1$. Now, we have to distinguish between

(i) the sequence $\{H(x_i, 1)\}$ is bounded.

(ii) the sequence $\{H(x_i, 1)\}$ is unbounded.

In case (i), up to a subsequence $H(x_i, 1) \to H_\infty$. Also, $\{u_i\}, \{v_i\}, \{\Delta u_i\}, \{\Delta v_i\}$ are uniformly bounded in every compact subset K of $\mathbb{R}^N$. By standard gradient estimates for elliptic equations (see [9]) we deduce that $\{\nabla u_i\}, \{\nabla v_i\}$ are uniformly locally bounded in $\mathbb{R}^N$, so that up to a subsequence $(u_i, v_i) \to (u_\infty, v_\infty)$ in $\mathcal{C}^2_{loc}(\mathbb{R}^N)$ (to pass from the uniform convergence to the $\mathcal{C}^2$ convergence, we refer to the regularity theory for elliptic equations, e.g. [9]). From the absurd assumption and our normalization it follows

$$\int_{\partial B_1(0)} u_\infty^2 = 0 \quad \text{and} \quad \int_{\partial B_1(0)} v_\infty^2 = 1. \tag{10}$$

Moreover, u_∞ and v_∞ are subharmonic and nonnegative. This implies $u_\infty \equiv 0$ in $B_1(0)$, which in turns yields (apply the strong maximum principle) $u_\infty \equiv 0$ in $\mathbb{R}^N$. Hence, v_∞ is harmonic and nonnegative in $\mathbb{R}^N$ (this follows by the $\mathcal{C}^2$ convergence): by the Liouville theorem for harmonic functions, $v_\infty \equiv$ *const.* Now, since $x_i \in \{|u-v| < C_1\}$ with $C_1 < \sqrt{\bar{C}_1 |\mathbb{S}^{N-1}|}$, and in light of Lemma 2.2, we deduce

$$v_\infty(0) = \lim_{i\to\infty} \left(\frac{1}{\sqrt{H(x_i,1)}} |v(x_i) - u(x_i)| + u_i(0) \right) \leq \lim_{i\to\infty} \left(\frac{1}{\sqrt{\bar{C}_1}} C_1 + u_i(0) \right) < \sqrt{|\mathbb{S}^{N-1}|}.$$

But since v_∞ is constant and (10) holds true, necessarily $v_\infty(0)^2 |\mathbb{S}^{N-1}| = 1$, a contradiction.

In case (ii), up to a subsequence $H(x_i, 1) \to +\infty$ as $i \to \infty$. Due to the fact the $\{(u_i, v_i)\}$ is uniformly bounded in every compact subset of $\mathbb{R}^N$, we are in position to apply Theorems A.2 and A.3: for every $K \subset\subset \mathbb{R}^N$, the sequence $\{(u_i, v_i)\}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(K)$ for every $\alpha \in (0,1)$, and, up to a subsequence, $(u_i, v_i) \to (u_\infty, v_\infty)$ in $\mathcal{C}^0(K) \cap H^1(K)$, where $u_\infty - v_\infty$ is harmonic, u_∞ and v_∞ are subharmonic and (10) holds true. As in the previous case, by subharmonicity, nonnegativity, and the fact that $\int_{\partial B_1(0)} u_\infty^2 = 0$ we deduce $u_\infty \equiv 0$ in $B_1(0)$. So, v_∞ is nonnegative and harmonic in $B_1(0)$; moreover,

$$v_\infty(0) = \lim_{i\to\infty} \left(\frac{1}{\sqrt{H(x_i,1)}} |v(x_i) - u(x_i)| + u_i(0) \right) \le \lim_{i\to\infty} \left(\frac{1}{\sqrt{H(x_i,1)}} C_1 + u_i(0) \right) = 0;$$

this implies $v_\infty \equiv 0$ in $B_1(0)$, and gives a contradiction with (10).

We proved that if $\int_{\partial B_1(x_i)} u^2 \to 0$, then $H(x_i,1) \to 0$ as $i \to \infty$. But this is contradiction with Lemma 2.2. □

Remark 2.4. From now on we will denote as $\bar{C}_3$ a fixed positive constant strictly smaller then $\sqrt{C_1|\mathbb{S}^{N-1}|}$.

Let's come back to the Alt-Caffarelli-Friedman monotonicity formula, see Theorem A.15. In some cases it is possible to get rid of the dependence of the constant $C(x_0)$ on x_0. This is the purpose of the following general result, which holds true for solutions with arbitrary algebraic growth and allows x_0 to vary in a set of full measure.

Proposition 2.5. *Let (u,v) be a solution of (1) satisfying (h1). Assume that*

$$\int_{\partial B_1(x_0)} u^2 \ge C_1 \quad \textit{and} \quad \int_{\partial B_1(x_0)} v^2 \ge C_1 \quad \forall x_0 \in \{|u-v| < \delta\}, \tag{11}$$

where $C_1, \delta > 0$. Then there exists $C_2 > 0$ such that

$$r \mapsto e^{-C_2 r^{-1/2}} J(x_0, r) \quad \textit{is nondecreasing in } r$$

for every $r \ge 1$, for every $x_0 \in \{|u-v| < \delta\}$.

Proof (cf. proof of Theorem 4.3 and the observation before Corollary 4.8 in [13]). For any $x_0 \in \{|u-v| < \delta\}$ and $r \ge 1$, we denote

$$(\bar{u}_{x_0,r}(x), \bar{v}_{x_0,r}(x)) = (u(x_0 + rx), v(x_0 + rx)) \qquad \text{with } x \in \partial B_1(0).$$

As in the proof of Lemma 2.5 of [10], it results

$$\frac{d}{dr} \log J(x_0, r) \ge -\frac{4}{r} + \frac{2}{r} \left[\Gamma\left(\Lambda_1(x_0,r)\right) + \Gamma\left(\Lambda_2(x_0,r)\right) \right], \tag{12}$$

where $\Gamma(t) = \sqrt{\left(\frac{N-2}{2}\right)^2 + t} - \left(\frac{N-2}{2}\right)$,

$$\Lambda_1(x_0,r) = \frac{r^2 \int_{\partial B_r(x_0)} |\nabla_\theta u|^2 + u^2 v^2}{\int_{\partial B_r(x_0)} u^2} = \frac{\int_{\partial B_1(0)} |\nabla_\theta \bar{u}_{x_0,r}|^2 + r^2 \bar{u}_{x_0,r}^2 \bar{v}_{x_0,r}^2}{\int_{\partial B_1(0)} \bar{u}_{x_0,r}^2}$$

$$\Lambda_2(x,r) = \frac{r^2 \int_{\partial B_r(x_0)} |\nabla_\theta v|^2 + u^2 v^2}{\int_{\partial B_r(x_0)} v^2} = \frac{\int_{\partial B_1(0)} |\nabla_\theta \bar{v}_{x_0,r}|^2 + r^2 \bar{u}_{x_0,r}^2 \bar{v}_{x_0,r}^2}{\int_{\partial B_1(0)} \bar{v}_{x_0,r}^2},$$

and $|\nabla_\theta u|^2 = |\nabla u|^2 - (\partial_\nu u)^2$.

Step 1) *There exist* $\tilde{C}_1, \tilde{C}_2 > 0$ *such that*

$$\tilde{C}_1 \leq \frac{\int_{\partial B_1(0)} \bar{u}^2_{x_0,r}}{\int_{\partial B_1(0)} \bar{v}^2_{x_0,r}} \leq \tilde{C}_2$$

for every $x_0 \in \{|u-v| < \delta\}$ *and* $r \geq 1$.
By contradiction, there are sequences $(x_i) \subset \{|u-v| < \delta\}$ and $(r_i) \subset [1, +\infty)$ such that

$$\lim_{i\to\infty} \frac{\int_{\partial B_1(0)} \bar{u}^2_{x_i,r_i}}{\int_{\partial B_1(0)} \bar{v}^2_{x_i,r_i}} = +\infty$$

(if the limit were 0 we can argue in a similar way). By assumption (11), we have

$$\frac{\int_{\partial B_1(0)} \bar{u}^2_{x_i,r_i}}{\int_{\partial B_1(0)} \bar{v}^2_{x_i,r_i}} \leq \frac{\int_{\partial B_1(0)} \bar{u}^2_{x_i,r_i}}{C_1}.$$

Consequently, $\int_{\partial B_1(0)} \bar{u}^2_{x_i,r_i} \to +\infty$ as $i \to \infty$, which in turns implies $H(x_i, r_i) \to +\infty$ as $i \to \infty$. Note that

$$\frac{\int_{\partial B_1(0)} \bar{u}^2_{x_i,r_i}}{\int_{\partial B_1(0)} \bar{v}^2_{x_i,r_i}} = \frac{\int_{\partial B_1(0)} u^2_{x_i,r_i}}{\int_{\partial B_1(0)} v^2_{x_i,r_i}} \quad \Rightarrow \quad \lim_{i\to\infty} \frac{\int_{\partial B_1(0)} u^2_{x_i,r_i}}{\int_{\partial B_1(0)} v^2_{x_i,r_i}} = +\infty \tag{13}$$

where we recall that the notation $(u_{x,r}, v_{x,r})$ has been introduced in (3). We set $(u_i, v_i) := (u_{x_i,r_i}, v_{x_i,r_i})$. By definition

$$\begin{cases} -\Delta u_i = -H(x_i, r_i) r_i^2 u_i v_i^2 & \text{in } \mathbb{R}^N \\ -\Delta v_i = -H(x_i, r_i) r_i^2 u_i^2 v_i & \text{in } \mathbb{R}^N. \end{cases}$$

and

$$\int_{\partial B_1(0)} u_i^2 + v_i^2 = 1, \tag{14}$$

which, by means of Corollary A.7, provides a uniform-in-i bound on $\int_{\partial B_r(0)} u_i^2 + v_i^2$ for every $r \geq 1$. In light of the subharmonicity of (u_i, v_i) this yields a uniform-in-i bound on the L^∞ norm of $\{(u_i, v_i)\}$ in every compact set of $\mathbb{R}^N$. As the competition parameter tends to $+infty$, we are in position to apply the local segregation Theorem A.3, deducing that up to a subsequence $(u_i, v_i) \to (u_\infty, v_\infty)$ in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$, where $u_\infty - v_\infty$ is harmonic and both u_∞ and v_∞ are subharmonic. By (13)

$$\int_{\partial B_1(0)} v_\infty^2 = \lim_{i\to+\infty} \int_{\partial B_1(0)} v_i^2 = \lim_{i\to\infty} \frac{\int_{\partial B_1(0)} v_i^2}{\int_{\partial B_1(0)} u_i^2 + v_i^2} = 0.$$

As v_∞ is subharmonic and nonnegative, $v_\infty \equiv 0$. This implies that u_∞ is harmonic and nonnegative in $B_1(0)$. Also, from (14) it follows $\int_{\partial B_1(0)} u_\infty^2 = 1$. On the other hand, since $x_i \in \{|u-v| < \delta\}$ and $H(x_i, r_i) \to +\infty$ it results

$$u_\infty(0) = \lim_{i\to\infty} \left(\frac{1}{\sqrt{H(x_i, r_i)}} |u(x_i) - v(x_i)| + v_i(0) \right) = 0$$

and by the strong maximum principle we obtain $u_\infty \equiv 0$, a contradiction.

Step 2) *Conclusion of the proof.*
For $x_0 \in \{|u-v| < \bar{\delta}\}$ and $r \geq 1$, we consider the functions

$$\tilde{u}_{x_0,r}(y) := \frac{\bar{u}_{x_0,r}(y)}{\left(\int_{\partial B_1(0)} \bar{u}^2_{x_0,r}\right)^{\frac{1}{2}}} \quad \text{and} \quad \tilde{v}_{x_0,r}(y) := \frac{\bar{v}_{x_0,r}(y)}{\left(\int_{\partial B_1(0)} \bar{u}^2_{x_0,r}\right)^{\frac{1}{2}}},$$

which are obtained by $\bar{u}_{x_0,r}$ and $\bar{v}_{x_0,r}$ after a normalization with respect to the L^2 norm of $\bar{u}_{x_0,r}$ on $\partial B_1(0)$. In light of assumption (11)

$$\Lambda_1(x_0,r) = \int_{\partial B_1(0)} |\nabla_\theta \tilde{u}_{x_0,r}|^2 + r^2 \left(\int_{\partial B_1(0)} \bar{u}^2_{x_0,r} \right) \tilde{u}^2_{x_0,r}\tilde{v}^2_{x_0,r} \geq \int_{\partial B_1(0)} |\nabla_\theta \tilde{u}_{x_0,r}|^2 + C_1 r^2 \tilde{u}^2_{x_0,r}\tilde{v}^2_{x_0,r}$$
$$\Lambda_1(x_0,r) = \int_{\partial B_1(0)} |\nabla_\theta \tilde{v}_{x_0,r}|^2 + r^2 \left(\int_{\partial B_1(0)} \bar{u}^2_{x_0,r} \right) \tilde{u}^2_{x_0,r}\tilde{v}^2_{x_0,r} \geq \int_{\partial B_1(0)} |\nabla_\theta \tilde{v}_{x_0,r}|^2 + C_1 r^2 \tilde{u}^2_{x_0,r}\tilde{v}^2_{x_0,r}.$$

As Γ is monotone nondecreasing, we deduce

$$\Gamma\left(\Lambda_1(x_0,r)\right) + \Gamma\left(\Lambda_2(x_0,r)\right) \\ \geq \Gamma\left(\int_{\partial B_1(0)} |\nabla_\theta \tilde{u}_{x_0,r}|^2 + C_1 r^2 \tilde{u}^2_{x_0,r}\tilde{v}^2_{x_0,r} \right) + \Gamma\left(\int_{\partial B_1(0)} |\nabla_\theta \tilde{v}_{x_0,r}|^2 + C_1 r^2 \tilde{u}^2_{x_0,r}\tilde{v}^2_{x_0,r} \right).$$

Thanks to the first step, we are in position to apply Lemma 4.2 in [13] in order to obtain

$$\Gamma\left(\Lambda_1(x_0,r)\right) + \Gamma\left(\Lambda_2(x_0,r)\right) \geq 2 - \frac{C}{r^{\frac{1}{2}}},$$

where C is a positive constant independent on $x_0 \in \{|u-v| < \delta\}$ and $r \geq 1$. Coming back to (12), we deduce that there exists $C > 0$ such that

$$\frac{d}{dr} \log J(x_0,r) \geq -C r^{-\frac{3}{2}}$$

for every $x_0 \in \{|u-v| < \delta\}$, for every $r \geq 1$. An integration gives the desired result. □

In light of Lemma 2.3, if (u,v) is a solution of (1) having linear growth then Proposition 2.5 holds true. By means of this uniform monotonicity formula, we deduce the following statement.

Corollary 2.6. *Let (u,v) be a solution of (1) having linear growth. Then there exists $\bar{C}_4 > 0$ such that*

$$\frac{1}{\bar{C}_4} \leq J(x_0,r) \leq \bar{C}_4, \qquad \int_{\partial B_1(x_0)} u^2 + v^2 \leq \bar{C}_4, \tag{15}$$

and

$$\sup_{x \in B_R(x_0)} u(x) + v(x) \leq \bar{C}_4(1+R)$$

for every $x_0 \in \{|u-v| < \bar{C}_3\}$ and $r \geq 1$ (where $\bar{C}_3$ has been defined Remark 2.4).

Proof. In light of Proposition 2.5, it is possible ti adapt the proof of Corollary 4.9 in [13] (see also the discussion at the end of the proof) replacing L_i with $\int_{\partial B_1(x_i)} u^2 + v^2$, where for us $(x_i) \subset \{|u-v| < \bar{C}_3\}$. In the quoted statement it is used the fact that $u(x_i) = v(x_i)$. As in this case $u(x_i) \neq v(x_i)$ in general, we obtain a contradiction with the same argument already used in the proof of Lemma 2.3. This permits to deduce the existence of $\bar{C}_4 > 0$ such that (15) holds. Now, Corollary A.7 and the subharmonicity of u and v permits to obtain also the pointwise estimate of the thesis. □

3 Uniqueness of the asymptotic profile

In this section we show that, under assumptions (h1) and (h2) (in fact it is sufficient to assume much less), any solution to (1) having algebraic growth is a solution with linear growth. Moreover, we will show that for every $x_0 \in \mathbb{R}^N$, the *entire* blow-down family $\{(u_{x_0,R}, v_{x_0,R}) : R > 0\}$ converges, as $R \to +\infty$, to the same harmonic function.

Proposition 3.1. *Let (u, v) be a solution of (1) satisfying assumptions (h1) and such that*

$$\lim_{x_N \to +\infty} v(x', x_N) = 0 \quad \text{uniformly in } x' \in \mathbb{R}^{N-1}. \tag{16}$$

Then $N(x_0, r) \leq 1$ for every $r > 0$, and consequently (u, v) has linear growth. Furthermore, there exists a constant $\gamma > 0$ such that, for every $x_0 \in \mathbb{R}^N$, the blow-down family $\{(u_{x_0,R}, v_{x_0,R}) : R > 0\}$ converges to the pair $(\gamma x_N^+, \gamma x_N^-)$ as $R \to +\infty$, in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ and in $H^1_{loc}(\mathbb{R}^N)$.

Remark 3.2. It is possible to replace assumption (16) with

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \text{uniformly in } x' \in \mathbb{R}^{N-1}.$$

Proof. As (u, v) has algebraic growth, thanks to Lemma A.9 Theorem A.13 applies: for every $x_0 \in \mathbb{R}^N$ there exists

$$\lim_{r \to +\infty} N(x_0, r) = d_{x_0} \in \mathbb{N} \setminus \{0\},$$

and there exists a subsequence $(u_{x_0,R_n}, v_{x_0,R_n})$ of the blow-down family which is convergent (in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ and in $H^1_{loc}(\mathbb{R}^N)$) to $(\Psi^+_{x_0}, \Psi^-_{x_0})$, where Ψ_{x_0} is a homogeneous harmonic polynomial of degree $d_{x_0} \geq 1$. As showed in Corollary A.14, this implies that $\lim_{r \to \infty} H(x_0, r) = +\infty$.
Now, let $K \subset\subset \mathbb{R}^N_+$. Since

$$\inf\{x_N : x \in K\} > 0,$$

in light of assumption (16) there holds

$$\lim_{R \to +\infty} v_R(x) = 0 \quad \text{uniformly in } K.$$

As K has been arbitrarily chosen, it follows that $v_{x_0,R_n}(x) \to 0$ pointwise in $\mathbb{R}^N_+$. By the uniqueness of the limit, we deduce $\Psi^-_{x_0} = 0$ in $\mathbb{R}^N_+$. Thus, Ψ_{x_0} is an homogeneous harmonic polynomial (hence $\Psi_{x_0}(0) = 0$) which is nonnegative in $\mathbb{R}^N_+$ and is not identically 0 (this follows simply from the fact that $d_{x_0} \geq 1$):

$$\begin{cases} -\Delta \Psi_{x_0} = 0 & \text{in } \mathbb{R}^N_+ \\ \Psi_{x_0} \geq 0, \ \Psi_{x_0} \not\equiv 0 & \text{in } \mathbb{R}^N_+ \\ \Psi_{x_0}(0) = 0. \end{cases}$$

By the strong maximum principle, we deduce that $\Psi_{x_0} > 0$ in $\mathbb{R}^N_+$; hence, the Hopf' Lemma guarantees that $\nabla \Psi_{x_0}(0) \neq 0$. The unique (up to a constant factor) homogeneous harmonic polynomial satisfying these properties is the linear one: $\Psi_{x_0}(x) = C_{x_0} x_N$; but $C_{x_0} > 0$ is uniquely determined (independently on x_0) by the condition

$$\int_{\partial B_1(0)} C^2_{x_0} x_N^2 = \lim_{n \to \infty} \int_{\partial B_1(0)} u^2_{x_0,R_n} + v^2_{x_0,R_n} = 1.$$

Hence, for every x_0 the blow-down family converges (up to a subsequence) to the same pair $(\gamma x_N^+, \gamma x_N^-)$, for a constant $\gamma > 0$. By Theorem A.13, the fact that the degree of the limiting profile is 1 means that $d_{x_0} = 1$ for every $x_0 \in \mathbb{R}^N$, and this gives the linear growth of (u, v), see Corollary A.8.
It remains to show that, for every $x_0 \in \mathbb{R}^N$, the entire blow-down family converges to γx_N. Assume by contradiction that this is not true: there exist a compact $K \subset \mathbb{R}^N$, a $\bar{\varepsilon} > 0$ and a subsequence $\{(u_{x_0,R_m}, v_{x_0,R_m})\}$ with $R_m \to +\infty$ as $m \to \infty$, such that

$$\|u_{x_0,R_m} - \gamma x_N^+\|_{\mathcal{C}^0(K)} + \|u_{x_0,R_m} - \gamma x_N^+\|_{H^1(K)} \\ + \|v_{x_0,R_m} - \gamma x_N^-\|_{\mathcal{C}^0(K)} + \|v_{x_0,R_m} - \gamma x_N^-\|_{H^1(K)} \geq \bar{\varepsilon} \tag{17}$$

for every m. But now it is possible to repeat step by step the proof of Theorem A.13 obtaining that, up to a subsequence, $\{(u_{x_0,R_m}, v_{x_0,R_m})\}$ converges, as $m \to +\infty$ to a homogeneous harmonic polynomial of degree $d_{x_0} \geq 1$. Following the above line of reasoning, we find that the limit is nothing but the function $(\gamma x_N^+, \gamma x_N^-)$, in contradiction with (17). □

4 Monotonicity at infinity

We aim at proving the following statement.

Proposition 4.1. *Let (u,v) be a solution of (1) satisfying (h1) and (h2). For every*

$$\nu \in \{\nu \in \mathbb{S}^{N-1} : \langle \nu, e_N\rangle > 0\},$$

there exists $M_\nu > 0$ such that

$$x \in \{x_N > M_\nu\} \Rightarrow \partial_\nu u(x) > 0 \quad \textit{and} \quad x \in \{x_N < -M_\nu\} \Rightarrow \partial_\nu v(x) < 0.$$

The achievement of section 3 says that (u,v) behaves at infinity as $(\gamma x_N^+, \gamma x_N^-)$; thus, the idea is that u has to be increasing in the e_N direction for $x_N \gg 1$ and v has to be decreasing in the e_N direction for $x_N \ll -1$. In order to prove this conjecture, we wish to apply the standard gradient estimate for the Poisson equation (see e.g. [9]) on u minus "a suitable linear function": this idea is corroborated by the fact that Δu can be uniformly bounded by an exponentially decaying function for x_N sufficiently large. An analogous bound holds for Δv when x_N is sufficiently large and negative.

Lemma 4.2. *Let (u,v) be a solution of (1) satisfying (h2). For every $p,q \geq 1$ there exist $M_1(p,q) > 0$ and a positive constant $C = C(p,q) > 0$ such that*

$$u^p(x)v^q(x) \leq Ce^{-C|x_N|} \qquad \forall x \in \{|x_N| > M_1(p,q)\}.$$

Proof. We consider the bound on u^pv^q in $x_N \gg 1$, the same argument applies for $x_N \ll -1$. Given $K > 0$ and $\delta > 0$, by (h2) there exists $M > 0$ such that

$$u(x) > K \quad \text{and} \quad v(x) < \delta \quad \text{if } x \in \{x_N > M/2\}.$$

For every $x \in \{x_N > M\}$ the ball $B_x := B_{x_N/4}(x)$ is contained in $\{x_N > M/2\}$. Consequently,

$$\begin{cases} u(y) \geq K_x := \inf_{B_x} u \geq K \\ v(y) \leq \delta \end{cases} \qquad \forall y \in B_x,\ \forall x \in \{x_N > M\},$$

so that

$$\begin{cases} -\Delta v \leq -K_x^2 v & \text{in } B_x \\ v \geq 0 & \text{in } B_x \\ v \leq \delta & \text{in } B_x. \end{cases}$$

We are in position to apply Lemma A.1:

$$\sup_{B_x'} v \leq C\delta e^{-CK_x x_N}, \tag{18}$$

where B_x' denotes the ball $B_{x_N/8}(x)$. On the other hand, it is possible to apply the Harnack inequality (Theorem 8.20 in [9], see also the subsequent observation concerning the estimate on the constant) on u in B_x, with potential v^2:

$$\sup_{B_x} u \leq Ce^{C\delta x_N} K_x. \tag{19}$$

The inequalities (18) and (19) yields

$$u^p(x)v^q(x) \leq CK_x^p\delta^q e^{-C_1 qK_x x_N + C_2 p\delta x_N} \qquad \forall x \in \{x_N > M\}.$$

A suitable choice of $K \leq K_x$ and δ permits to obtain the desired result. □

Remark 4.3. From now on we will denote as $M_1 := \max\{M_1(1,2), M_1(2,1)\}$, where $M_1(1,2)$ and $M_1(2,1)$ have been defined in Lemma 4.2.

If we could show that the function u can be approximated in $\{x_N > M_1\}$ by a linear function with positive slope in the e_N direction, the gradient estimates for the Poisson equation would give the desired monotonicity for u. So far we showed that for given $x_0 \in \mathbb{R}^N$ and $\varepsilon > 0$ there exists $R_{x_0,\varepsilon} > 0$ such that

$$\sup_{x \in B_1(0)} |u_{x_0,R}(x) - \gamma x_N^+| + |v_{x_0,R}(x) - \gamma x_N^-| < \varepsilon \tag{20}$$

for every $R > R_{x_0,\varepsilon}$. This means that

$$\sup_{x \in B_R(x_0)} \left| u(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^+ \right| + \left| v(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^- \right| < \sqrt{H(x_0,R)}\varepsilon$$

whenever $R > R_{x_0,\varepsilon}$. This reveals that we have to face two problems: the first one is the fact that we have not a unique candidate to approximate u for $x_N \gg 1$ and v for $x_N \ll -1$, the second one is that this approximation, which holds for R sufficiently large, get worse as R increases (recall that the function $H(x_0, \cdot)$ is nondecreasing and tends to $+\infty$ as $R \to +\infty$, see Corollary A.14). In order to overcome the second problem, we wish to find a uniform estimate (in both x_0 and R) on the ratio $\frac{\sqrt{H(x_0,R)}}{R}$; in the forthcoming Lemma 4.6, we show that this is possible if $x_0 \in \{|u - v| < \bar{C}_3\}$, where $\bar{C}_3$ has been defined in Remark 2.4. Before, we deduce some useful information about this special set.

Lemma 4.4. *Under the assumption (h2), the set $\{|u - v| < \bar{C}_3\}$ is bounded in the e_N direction and unbounded in all the other directions $\{e_1, \dots, e_{N-1}\}$. In particular, for every $x' \in \mathbb{R}^{N-1}$ there exists $\tilde{x} \in \{|u - v| < \bar{C}_3\}$ such that $\tilde{x}' = x'$.*

Proof. The properties follow easily by our main assumption (h2). Indeed, by considering the function $u - v$ one sees that

$$\lim_{x_N \to \pm\infty} (u(x', x_N) - v(x', x_N)) = \pm\infty,$$

uniformly in $x' \in \mathbb{R}^{N-1}$. This immediately implies that the level set $\{|u - v| \le M\}$ is bounded in the e_N direction for every $M > 0$ (in particular, this holds for $\bar{C}_3$). On the other hand, for a given $x' \in \mathbb{R}^{N-1}$ we can consider the map $s \in \mathbb{R} \mapsto u(x', s) - v(x', s)$. This is a continuous function which tends to $\pm\infty$ as $s \to \pm\infty$, thus there exist $\tilde{s} \in \mathbb{R}$ such that $|u(x', \tilde{s}) - v(x', \tilde{s})| < \bar{C}_3$. □

Remark 4.5. From now on, we denote $\zeta := \sup\{|x_{0,N}| : x_0 \in \{|u - v| < \bar{C}_3\}\} < +\infty$.

In the next Lemma we give uniform upper and lower bounds on the ratio $\frac{\sqrt{H(x_0,R)}}{R}$ for $x_0 \in \{|u-v| < \bar{C}_3\}$ and $R \ge 1$.

Lemma 4.6. *Let (u, v) be a solution of (1) satisfying (h1) and (h2). There exists $\bar{C}_5, \bar{C}_6 > 0$ such that*

$$\bar{C}_5 \le \frac{\sqrt{H(x_0,R)}}{R} \le \bar{C}_6$$

for every $x_0 \in \{|u - v| < \bar{C}_3\}$ and $R \ge 1$.

Proof. By Proposition 3.1, we know that under (h1) and (h2) the solution (u, v) has linear growth. Hence, we can invoke Corollary 2.6; combining this result with Corollary A.7 we deduce

$$\frac{H(x_0,R)}{R^2} \le eH(x_0,1) \le e\bar{C}_4 \qquad \forall x_0 \in \{|u - v| < \bar{C}_3\},\ R \ge 1.$$

For the lower bound, we show that the quantity

$$J_{x_0,R}(0,1) := \int_{B_1(0)} \frac{|\nabla u_{x_0,R}(y)|^2 + H(x_0,R)R^2 u_{x_0,R}^2(y) v_{x_0,R}^2(y)}{|y|^{N-2}}\, dy \cdot \int_{B_1(0)} \frac{|\nabla v_{x_0,R}(y)|^2 + H(x_0,R)R^2 u_{x_0,R}^2(y) v_{x_0,R}^2(y)}{|y|^{N-2}}\, dy$$

is bounded above by a positive constant C independent on $x_0 \in \mathbb{R}^N$ and $R \geq 1$. We use the (49): there exists $C > 0$ independent on $x_0 \in \mathbb{R}^N$ and on $R \geq 1$ such that

$$\int_{B_1(0)} \frac{|\nabla u_{x_0,R}(y)|^2 + H(x_0,R)R^2 u_{x_0,R}^2(y) v_{x_0,R}^2(y)}{|y|^{N-2}}\, dy = \frac{1}{H(x_0,R)} \int_{B_R(x_0)} \frac{|\nabla u(y)|^2 + u^2(y)v^2(y)}{|y-x_0|^{N-2}}\, dy$$
$$\leq \frac{C}{H(x_0,R)R^N} \int_{B_{2R}(x_0)} u^2 = C \int_{B_2(0)} u_{x_0,R}^2. \quad (21)$$

We point out that, as $N(x_0, r) \leq 1$ for every $x_0 \in \mathbb{R}^N$ and $r \geq 1$, the same estimate holds true for the Almgren quotient associated to $(u_{x_0,R}, v_{x_0,R})$, for every $x_0 \in \mathbb{R}^N$ and $R \geq 1$ (see Remark A.12). As a consequence, the normalization $\int_{\partial B_1(0)} u_{x_0,R}^2 + v_{x_0,R}^2 = 1$ gives, by Corollary A.7, a uniform (in both x_0 and R) upper bound for $\int_{\partial B_3(0)} u_{x_0,R}^2 + v_{x_0,R}^2$. Due to the subharmonicity of $(u_{x_0,R}, v_{x_0,R})$, we obtain a uniform bound for $\{(u_{x_0,R}, v_{x_0,R})\}$ in $L^\infty(B_2(0))$, so that we can estimate the right hand side of (21) obtaining

$$\int_{B_1(0)} \frac{|\nabla u_{x_0,R}(y)|^2 + H(x_0,R)R^2 u_{x_0,R}^2(y) v_{x_0,R}^2(y)}{|y|^{N-2}}\, dy \leq C$$

for every $x_0 \in \mathbb{R}^N$ and $R \geq 1$. Arguing in the same way on the second factor of $J_{x_0,R}(0,1)$ we obtain the desired upper bound: there exists $C > 0$ such that

$$J_{x_0,R}(0,1) \leq C \qquad \forall x_0 \in \mathbb{R}^N, \ \forall R \geq 1.$$

A simple change of variable shows that $J_{x_0,R}(0,1) = \frac{R^4}{H^2(x_0,R)} J(x_0,R)$, so that

$$J(x_0,R) \leq C \frac{H^2(x_0,R)}{R^4} \qquad \forall x_0 \in \mathbb{R}^N, \ \forall R \geq 1. \quad (22)$$

A comparison between (22) and the uniform lower estimate of Corollary 2.6 provides the desired result:

$$\frac{H^2(x_0,R)}{R^4} \geq \frac{C}{\bar{C}_4} \qquad \forall x_0 \in \{|u-v| < \bar{C}_3\}, \ \forall R \geq 1. \qquad \square$$

We are ready to improve the estimate given by (20). Firstly, we get rid of the dependence of $R_{x_0,\varepsilon}$ on x_0 for $x_0 \in \{|u-v| < \bar{C}_3\}$.

Lemma 4.7. *Let (u,v) be a solution of (1) satisfying (h1) and (h2). For every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that*

$$\sup_{x \in B_1(0)} |u_{x_0,R}(x) - \gamma x_N^+| + |v_{x_0,R}(x) - \gamma x_N^-| < \varepsilon$$

for every $R > R_\varepsilon$ and $x_0 \in \{|u-v| < \bar{C}_3\}$, where γ and $\bar{C}_3$ have been defined in Proposition 3.1 and Remark 2.4 respectively.

Proof. Assume by contradiction that there exist $\bar{\varepsilon} > 0$ and a sequence (x_j, R_j) with $x_j \in \{|u-v| < \bar{C}_3\}$ for every j, $R_j \to +\infty$, and

$$\sup_{x \in B_1(0)} |u_{x_j,R_j}(x) - \gamma x_N^+| + |v_{x_j,R_j}(x) - \gamma x_N^-| \geq \bar{\varepsilon} \quad (23)$$

for every j. Let us denote $(u_j, v_j) = (u_{x_j,R_j}, v_{x_j,R_j})$. We know that (u_j, v_j) solves

$$\begin{cases} -\Delta u_j = -H(x_j,R_j) R_j^2 u_j v_j^2 & \text{in } \mathbb{R}^N \\ -\Delta v_j = -H(x_j,R_j) R_j^2 u_j^2 v_j & \text{in } \mathbb{R}^N \end{cases} \qquad \forall j.$$

In light of Lemma 4.6, we know that

$$\lim_{j\to+\infty} H(x_j,R_j) \geq \lim_{j\to+\infty} \bar{C}_5 R_j^2 = +\infty; \tag{24}$$

a fortiori the competition parameter $H(x_j,R_j)R_j^2$ tends to $+\infty$ as $j\to+\infty$. Note that the normalization $\int_{\partial B_1(0)} u_j^2+v_j^2 = 1$ implies, by means of Corollary A.7 (which we can apply on (u_j,v_j), see Remark A.12), that

$$\int_{\partial B_r(0)} u_j^2+v_j^2 \leq e r^{N+1} \qquad \forall r>1,\ \forall j.$$

By subharmonicity, the sequence $\{(u_j,v_j)\}$ is uniformly bounded in every compact set K of $\mathbb{R}^N$, and in light of Theorem A.2 it is also uniformly bounded in $\mathcal{C}^{0,\alpha}(K)$, for every $\alpha\in(0,1)$. The local segregation Theorem A.3 implies that, up to a subsequence, $(u_j,v_j)\to(u_\infty,v_\infty)$ in $\mathcal{C}^0_{loc}(\mathbb{R}^N)\cap H^1_{loc}(\mathbb{R}^N)$, and

- (i) $u_\infty v_\infty \equiv 0$ in $\mathbb{R}^N$,
- (ii) $H(x_j,R_j)R_j^2u_j^2v_j^2 \to 0$ as $j\to\infty$ in $L^1_{loc}(\mathbb{R}^N)$,
- (iii) $u_\infty - v_\infty$ is harmonic in $\mathbb{R}^N$,
- (iv) by (24) and the fact that $x_j\in\{|u-v|<\bar{C}_3\}$

$$|u_\infty(0)-v_\infty(0)| = \lim_{j\to+\infty}\frac{1}{\sqrt{H(x_j,R_j)}}|u(x_j)-v(x_j)| = 0$$

- (v) by uniform convergence the normalization on $\partial B_1(0)$ pass to the limit:

$$\int_{\partial B_1(0)} u_\infty^2+v_\infty^2 = 1, \tag{25}$$

- (vi) by H^1 and uniform convergence and the point (ii)

$$\frac{r\int_{B_r(0)}|\nabla u_\infty|^2+|\nabla v_\infty|^2}{\int_{\partial B_r(0)}u_\infty^2+v_\infty^2} = \lim_{j\to+\infty}\frac{r\int_{B_r(0)}|\nabla u_j|^2+|\nabla v_j|^2+H(x_j,R_j)R_j^2u_j^2v_j^2}{\int_{\partial B_r(0)}u_j^2+v_j^2}$$
$$= \lim_{j\to+\infty} N(x_j,R_jr)\leq 1 \qquad \forall r\in(0,1), \tag{26}$$

where the upper bound on N follows from the fact that, under assumptions (h1) and (h2), Proposition 3.1 applies and guarantees that (u,v) has linear growth.

Note that

$$N_\infty(0,r) := \frac{r\int_{B_r(0)}|\nabla u_\infty|^2+|\nabla v_\infty|^2}{\int_{\partial B_r(0)}u_\infty^2+v_\infty^2}$$

is the Almgren quotient of the harmonic function $u_\infty - v_\infty$, and it is nondecreasing. As $u_\infty(0)-v_\infty(0)=0$, it results

$$N_\infty(0,r) \geq \lim_{s\to 0^+} N_\infty(0,s) = \deg(u_\infty-v_\infty,0)\geq 1 \tag{27}$$

for every $r>0$. Here, $\deg(u_\infty-v_\infty,0)$ denotes the degree of vanishing of the harmonic function $u_\infty-v_\infty$ in 0, and is greater then 1 because it has to be a positive integer (this result is by now well known). By monotonicity, a comparison between (26) and (27) yields $N_\infty(0,r)=1$ for every $r\in(0,1)$, which implies (see Proposition 3.9 in [10], which we can apply, as explained in Remark A.4) that $u_\infty - v_\infty$ is a linear function, that is, $(u_\infty(x),v_\infty(x)) = (\langle e,x\rangle^+,\langle e,x\rangle^-)$ for some $e\in\mathbb{R}^N$. We claim that

$$e = \gamma e_N, \tag{28}$$

which gives a contradiction with (23) and completes the proof of the statement. To prove the claim, we note that under our assumptions we have

$$v_j(x) = \frac{1}{\sqrt{H(x_j, R_j)}} v(x'_j + R_j x', x_{j,N} + R_j x_N) \to 0$$

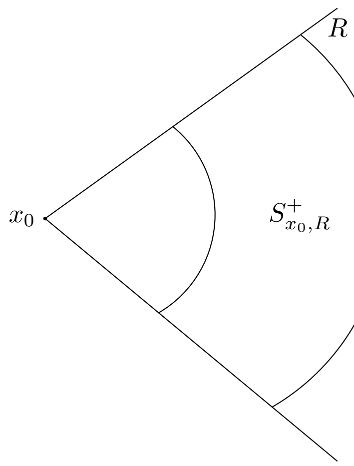
as $j \to +\infty$, uniformly in every compact subset of $B_1(0) \cap \mathbb{R}^N_+$; to pass to the limit, we used the fact that $H(x_j, R_j) \geq \bar{C}_1$ (see Lemma 2.2) and the boundedness of the set $\{|u - v| < \bar{C}_3\}$ in the e_N direction (see Lemma 4.4), which guarantees that $x_{j,N} + R_j x_N \to +\infty$ as $j \to +\infty$. By the uniqueness of the limit, we deduce $e = Ce_N$ for some $C > 0$. The normalization (25) yields $C = \gamma$, which concludes the proof of the claim (28). □

Definition 4.1. Let us fix $\tau > 0$ not too small (to be determined in the following Lemma). For a given $x_0 \in \mathbb{R}^N$ and $R > 0$ we introduce the conical sectors

$$S^+_{x_0,R} := \left\{ x = (x', x_N) \in \mathbb{R}^N : \frac{R}{2} < |x - x_0| < R, |x' - x'_0| < \tau(x_N - x_{0,N}) \right\}$$
$$S^-_{x_0,R} := \left\{ x = (x', x_N) \in \mathbb{R}^N : \frac{R}{2} < |x - x_0| < R, |x' - x'_0| < \tau(x_{0,N} - x_N) \right\},$$

and their union $S_{x_0,R}$.

The following picture represents the set $S^+_{x_0,R}$ for a given $x_0 \in \mathbb{R}^N$.



The geometry of the set $\{|u - v| < \bar{C}_3\}$ allows to show that the union of $S_{x_0,R}$ with R sufficiently large and $x_0 \in \{|u - v| < \bar{C}_3\}$ contains, and it is contained in, the union of two half-spaces.

Lemma 4.8. *Let (u, v) be a solution of (1) satisfying (h1) and (h2). There exists $\bar{R} > 0$ such that, for every $\widehat{R} \geq \bar{R}$ there exists $M_2 = M_2(\widehat{R}) > \zeta$ such that*

$$\{|x_N| > M_2\} \subset \bigcup_{\substack{x_0 \in \{|u-v|<\bar{C}_3\} \\ R > \widehat{R}}} S_{x_0,R} \subset \{|x_N| > \zeta\},$$

where ζ has been defined in Remark 4.5. Furthermore, for every $N \geq 2$ we can choose $\tau > 0$ such that, if $x \in \{|x_N| > M_2\}$, there exist $\tilde{x} \in \{|u - v| < \bar{C}_3\}$ and $\tilde{R} > \widehat{R}$ such that

$$\overline{Q}_x \subset S_{\tilde{x},\tilde{R}},$$

where Q_x denotes the open cube centered in x with side $\dfrac{x_N}{100}$.

Proof. Thanks to Lemma 4.4, it is not difficult to see that, provided $\bar{R}$ is sufficiently large and $\widehat{R} > \bar{R}$, it results

$$\bigcup_{\substack{x_0\in\{|u-v|<\bar{C}_3\}\\ R>\widehat{R}}} S_{x_0,R} \subset \bigcup_{\substack{x_0\in\{|u-v|<\bar{C}_3\}\\ R>\bar{R}}} S_{x_0,R} \subset \{|x_N| > \zeta\}.$$

Now we argue in $\mathbb{R}^N_+$ showing that there exists $M_2 = M_2(\widehat{R}) > \zeta$ such that

$$\{x_N > M_2\} \subset \bigcup_{\substack{x_0\in\{|u-v|<\bar{C}_3\}\\ R>\widehat{R}}} S^+_{x_0,R},$$

and that for every $x \in \{x_N > M_2\}$ there exist the desired $\tilde{x}$ and $\tilde{R}$. For $x \gg 1$, let $\tilde{x}$ the point of $\{|u-v| < \bar{C}_3\}$ such that $\tilde{x}' = x'$ ($\tilde{x}$ exists, see Lemma 4.4). Provided τ is not too small, the cube centered in x with side $\frac{x_N}{100}$ is contained in the conical sector $S^+_{\tilde{x},\tilde{R}}$ for $\tilde{R} := \frac{3}{2}(x_N - \tilde{x}_N)$. Note that,

$$\frac{3}{2}(x_N - \tilde{x}_N) \geq \frac{3}{2}(x_N - \zeta) \geq \frac{5}{4}x_N > \widehat{R}.$$

whenever $x_N > M_2 := \max\left\{6\zeta, \frac{4}{5}\widehat{R}\right\}$. The same argument works in the half-space $\mathbb{R}^N_-$. □

Remark 4.9. From the previous proof we see that, fixed $\widehat{R} > \bar{R}$, it is possible to associate to every $x \in \{|x_N| > M_2\}$ the conical sector $S_{\tilde{x},\tilde{R}}$ which contains the cube Q_x; that is, $\tilde{x}$ is a point of $\{|u-v| < \bar{C}_3\}$ such that $\tilde{x}' = x'$ and

$$\tilde{R} = \begin{cases} \frac{3}{2}(x_N - \tilde{x}_N) & \text{if } x_N > M_2 \\ \frac{3}{2}(\tilde{x}_N - x_N) & \text{if } x_N < -M_2. \end{cases}$$

In each $S_{x_0,R}$ we can obtain a further improvement, by means of Lemma 4.6, of the estimates of Lemma 4.7.

Lemma 4.10. *Let (u,v) be a solution of (1) satisfying (h1) and (h2). For every $\varepsilon > 0$, if $R > R_\varepsilon$ and $x_0 \in \{|u-v| < \bar{C}_3\}$ then*

$$\sup_{x\in S_{x_0,R}} \left| \frac{u(x) - \gamma\frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^+}{|x-x_0|} \right| + \left| \frac{v(x) - \gamma\frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^-}{|x-x_0|} \right| < \varepsilon,$$

with $\bar{C}_5 \leq \frac{\sqrt{H(x_0,R)}}{R} \leq \bar{C}_6$. We recall that $\bar{C}_3, \bar{C}_5, \bar{C}_6$ and R_ε have been defined in Remark 2.4, Lemma 4.6 and Lemma 4.7 respectively.

Proof. Lemma 4.7 ensures that for every $R > R_\varepsilon$, for every $x_0 \in \{|u-v| < \bar{C}_3\}$

$$\sup_{x\in S_{0,1}} \left| \frac{u(x_0+Rx)}{\sqrt{H(x_0,R)}} - \gamma x_N^+ \right| + \left| \frac{v(x_0+Rx)}{\sqrt{H(x_0,R)}} - \gamma x_N^- \right| < \varepsilon,$$

that is,

$$\left| u(x_0+Rx) - \gamma\sqrt{H(x_0,R)}x_N^+ \right| + \left| v(x_0+Rx) - \gamma\sqrt{H(x_0,R)}x_N^- \right| < \sqrt{H(x_0,R)}\varepsilon$$

for every $x \in S_{0,1}$. Consequently, dividing both the sides for R we obtain

$$|x|\left(\left| \frac{u(x_0+Rx)}{|Rx|} - \gamma\frac{\sqrt{H(x_0,R)}}{R}\frac{Rx_N^+}{|Rx|} \right| + \left| \frac{v(x_0+Rx)}{|Rx|} - \gamma\frac{\sqrt{H(x_0,R)}}{R}\frac{Rx_N^-}{|Rx|} \right| \right) < \frac{\sqrt{H(x_0,R)}}{R}\varepsilon$$

for every $x \in S_{0,1}$, provided $R > R_\varepsilon$ and $x_0 \in \{|u-v| < \bar{C}_3\}$. In turns, this gives

$$\sup_{x \in S_{x_0,R}} \left| \frac{u(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^+}{|x - x_0|} \right| + \left| \frac{v(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^-}{|x - x_0|} \right| < 2\frac{\sqrt{H(x_0,R)}}{R}\varepsilon$$

for every $R > R_\varepsilon$ and $x_0 \in \{|u-v| < \bar{C}_3\}$. Finally, we can use the upper bound on $\frac{\sqrt{H(x_0,R)}}{R}$, see Lemma 4.6. $\square$

We are ready to apply the gradient estimates for the Poisson equation in a half-space $x_N \gg 1$; we will show that if $x_N > 0$ is sufficiently large then there exists a linear functions φ_x (depending on x) which approximate u in a $\mathcal{C}^1$-sense in x. In light of the uniform control given in Lemma 4.6, the slope of φ_x will turn to be uniformly bounded from below in an entire half-space (the same holds for v in $x_N \ll -1$), allowing to conclude the proof of Proposition 4.1. It is essential to work in conical sectors, because in this way we can control the quantity $|x - x_0|$ with the privileged component $|x_N - x_{0,N}|$.

Lemma 4.11. *Let (u,v) be a solution of (1) satisfying (h1) and (h2). For every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that*

$$\left| \nabla u(x) - \gamma \frac{\sqrt{H(\tilde{x},\tilde{R})}}{\tilde{R}} e_N \right| < \varepsilon \qquad \forall x \in \{x_N > M_\varepsilon\},$$

where $\tilde{x}$ and $\tilde{R}$ have been defined in Remark 4.9. Analogously,

$$\left| \nabla v(x) - \gamma \frac{\sqrt{H(\tilde{x},\tilde{R})}}{\tilde{R}} e_N \right| < \varepsilon \qquad \forall x \in \{x_N < -M_\varepsilon\}.$$

Proof. For every $\varepsilon > 0$, let R_ε be defined in Lemma 4.7. Let $M_{2,\varepsilon} := M_2(\max\{\bar{R}, R_\varepsilon\})$, where M_2 has been defined in Lemma 4.8. Let $M_\varepsilon := \max\{M_1, M_{2,\varepsilon}\}$, where M_1 has been defined in Remark 4.3. For $x \in \{x_N > M_\varepsilon\}$, there are $\tilde{R} > R_\varepsilon$ and $\tilde{x} \in \{|u-v| < \bar{C}_3\}$ such that $\overline{Q}_x \subset S^+_{\tilde{x},\tilde{R}}$, see Lemma 4.8 and Remark 4.9. By the gradient estimates for the Poisson equation (see [9], section 3.4) plus Lemmas 4.2 and 4.10, we deduce that

$$\begin{aligned} \left| \nabla u(x) - \gamma \frac{\sqrt{H(\tilde{x},\tilde{R})}}{\tilde{R}} e_N \right| &\le \frac{C}{x_N} \sup_{y \in \overline{Q}_x} \left| u(y) - \gamma \frac{\sqrt{H(\tilde{x},\tilde{R})}}{\tilde{R}}(y_N - \tilde{x}_N) \right| + \frac{x_N}{2} \sup_{y \in \overline{Q}_x} v^2(y) u(y) \\ &\le \frac{C}{x_N} \sup_{y \in \overline{Q}_x} \varepsilon |y - \tilde{x}| + C x_N e^{-C x_N}. \end{aligned} \tag{29}$$

As $\overline{Q}_x \subset S^+_{\tilde{x},\tilde{R}}$, for every $y \in \overline{Q}_x$ it results

$$\begin{aligned} |y - \tilde{x}| &< (\tau+1)(y_N - \tilde{x}_N) \le (\tau+1)(y_N - x_N) + (\tau+1)(x_N - \tilde{x}_N) \\ &\le C x_N + (\tau+1)(x_N + \zeta) \le C x_N, \end{aligned}$$

where we recall that $\zeta = \sup\{x_{0,N} : x_0 \in \{u = v\}\} < M_\varepsilon < x_N$. Plugging this estimate into the (29), we obtain

$$\left| \nabla u(x) - \gamma \frac{\sqrt{H(\tilde{x},\tilde{R})}}{\tilde{R}} e_N \right| \le C\varepsilon + C x_N e^{-C x_N}$$

whenever $x_N > M_\varepsilon$; if necessary, we can replace M_ε with a larger quantity, obtaining the thesis for u. A similar argument can be carried on for v. $\square$

Conclusion of the proof of Proposition 4.1. Given $\nu \in \{\nu \in \mathbb{S}^{N-1} : \langle \nu, e_N \rangle > 0\}$, we choose

$$0 < \varepsilon(\nu) \leq \frac{\gamma \bar{C}_5}{2} \langle e_N, \nu \rangle.$$

where $\bar{C}_5$ has been defined in Lemma 4.6. It results

$$\begin{aligned} \partial_\nu u(x) &= \left\langle \nabla u(x) - \gamma \frac{\sqrt{H(\tilde{x}, \tilde{R})}}{\tilde{R}} e_N, \nu \right\rangle + \gamma \frac{\sqrt{H(\tilde{x}, \tilde{R})}}{\tilde{R}} \langle e_N, \nu \rangle \\ &\geq -\varepsilon(\nu) + \gamma \bar{C}_5 \langle e_N, \nu \rangle > 0 \end{aligned}$$

for every $x \in \{x_N > M_\nu\}$, where $M_\nu := M_{\varepsilon(\nu)}$ has been defined in Lemma 4.11. The same argument gives the monotonicity of v for $x_N \ll 1$. □

With a slightly modification of the conclusion of the proof, we obtain also the

Corollary 4.12. *If we consider $\Theta := \{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu \rangle \geq \hat{C}\}$ with $\hat{C} \in (0, 1]$, then there exists $M_\Theta > 0$ such that*

$$\begin{aligned} &x \in \{x_N > M_\Theta\} \Rightarrow \partial_\nu u(x) > 0 \qquad \forall \nu \in \Theta \\ &x \in \{x_N < -M_\Theta\} \Rightarrow \partial_\nu v(x) < 0 \qquad \forall \nu \in \Theta. \end{aligned}$$

5 Monotonicity in the e_N direction

We are going to apply the moving planes method in order to show that u and v are monotone in the e_N direction in the whole $\mathbb{R}^N$. To be precise:

Proposition 5.1. *Let (u, v) be a solution of (1) satisfying (h1) and (h2). Then*

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{and} \quad \frac{\partial v}{\partial x_N} < 0 \quad \text{in } \mathbb{R}^N.$$

In what follows we will use many times the following version of the maximum principle in unbounded domains, Lemma 2.1 in [1].

Lemma 5.2. *Let D be an open connected subset of $\mathbb{R}^N$, possibly unbounded. Assume that $\overline{D}$ is disjoint from the closure of an infinite open connected cone. Suppose that, for a function $c \in L^\infty_{loc}(D)$, $c \leq 0$ a.e. in D, we have*

$$\begin{cases} \Delta v + c(x) v \geq 0 & \text{in } D \\ v \leq 0 & \text{on } \partial D, \end{cases}$$

where $v \in \mathcal{C}^0(\overline{D}) \cap W^{2,N}_{loc}(D)$ and $v^+ \in L^\infty(D)$, that is, v is bounded above. Then $v \leq 0$ in D.

We postpone the proof of Proposition 5.1 after the following Lemma, which is a consequence of the uniform estimate given in Corollary 2.6.

Lemma 5.3. *Let (u, v) be a solution of (1) satisfying (h1) and (h2). Then for every $M > 0$ there exists $\bar{C}_M > 0$ such that*

$$\begin{aligned} &u(x) + |\nabla u(x)| \leq \bar{C}_M \quad \forall x \in \mathbb{R}^{N-1} \times (-\infty, M], \\ &v(x) + |\nabla v(x)| \leq \bar{C}_M \quad \forall x \in \mathbb{R}^{N-1} \times [-M, +\infty). \end{aligned}$$

Proof. We prove only the first inequality. Under our assumptions, we know that (u, v) has linear growth (see Proposition 3.1). For any $x \in \mathbb{R}^N$, let $\tilde{x} \in \{|u - v| < \bar{C}_3\}$ such that $\tilde{x}' = x'$ and let $\tilde{R} = \frac{3}{2}|x_N - \tilde{x}_N|$, so that $x \in B_{\tilde{R}}(\tilde{x})$ ($\tilde{x}$ exists, see Lemma 4.4). By means of Corollary 2.6 we deduce that

$$u(x) + v(x) \leq \sup_{y \in B_{\tilde{R}}(\tilde{x})} \bar{C}_4 \left(1 + \frac{3}{2}|x_N - \tilde{x}_N|\right) \leq \frac{3}{2}\bar{C}_4 \left(\frac{2}{3} + \zeta + |x_N|\right) \qquad \forall x \in \mathbb{R}^N, \tag{30}$$

where ζ has been defined in Remark 4.5. Now, let M_1 be defined in Remark 4.3, so that $uv^2 \leq Ce^{-C|x_N|}$ in $\{x_N < -M_1\}$. Moreover, by (h2) there exist $M_3 > 0$ such that $u \leq 1$ in $\mathbb{R}^{N-1} \times (-\infty, -M_3 + \frac{1}{2}]$. we set $M_4 := \max\{M_1, M_3\}$ and we take any $M > M_4$.

By (30), it results

$$u(x', x_N) \leq \begin{cases} \frac{3}{2}\bar{C}_4 \left(\frac{2}{3} + \zeta + M\right) & \text{if } x \in \{|x_N| \leq M\} \\ 1 & \text{if } x \in \{x_N \leq -M\} \end{cases} \leq 1 + \frac{3}{2}\bar{C}_4 \left(\frac{2}{3} + \zeta + |x_N|\right) =: C_{1,M}$$

whenever $(x', x_N) \in \mathbb{R}^{N-1} \times (-\infty, M]$. Clearly, if $M \leq M_4$ the same bound holds.

Let's pass to the estimate on the gradient. In $\mathbb{R}^{N-1} \times \left[-M - \frac{1}{2}, M + \frac{1}{2}\right]$ both u and uv^2 are uniformly bounded thanks to (30). Also, by definition of M_1 and M_3 both u and uv^2 are uniformly bounded in $\mathbb{R}^{N-1} \times (-\infty, -M]$. Altogether, this means that u and uv^2 are uniformly bounded in $\mathbb{R}^{N-1} \times \left(-\infty, M + \frac{1}{2}\right]$, so that we can apply the standard gradient estimates for the Poisson equation (see [9], section 3.4) in cubes of side 1, obtaining the existence of $C_{2,M} > 0$ such that $|\nabla u(x)| \leq C_{2,M}$ for every $x \in \mathbb{R}^{N-1} \times (-\infty, M]$.

The thesis is then satisfied with $\bar{C}_M := \max\{C_{1,M}, C_{2,M}\}$. $\square$

Proof of Proposition 5.1. We introduce the classical notation for the moving planes method: for $\lambda \in \mathbb{R}$, we set

$$u_\lambda(x', x_N) := u(x', 2\lambda - x_N) \quad \text{and} \quad T_\lambda := \{x_N > \lambda\}.$$

We aim at proving that

$$u_\lambda(x) \leq u(x) \quad \text{and} \quad v_\lambda(x) \geq v(x) \quad \forall x \in T_\lambda,\ \forall \lambda \in \mathbb{R}, \tag{31}$$

This and the strong maximum principle give the desired monotonicity.
To prove that (31) is satisfied, we show that

$$\Sigma := \{\lambda \in \mathbb{R} : u_\theta \leq u \text{ and } v_\theta \geq v \text{ in } T_\theta \text{ for every } \theta \geq \lambda\} = \mathbb{R}. \tag{32}$$

Step 1) *There exists $\bar{M} > 0$ such that if $\lambda > \bar{M}$ then $u_\lambda \leq u$ and $v_\lambda \geq v$ in T_λ.*
Let $M_N := M_{e_N}$, where M_{e_N} has been defined in Proposition 4.1. Let $K := \sup\{u : x_N < M_N\} < +\infty$. By assumption (h2), for every $\delta > 0$ there exists $\bar{M} > 0$ such that

$$u(x) > K \quad \text{and} \quad v(x) < \delta \quad \text{in } \{x_N > 2\bar{M} - M_N\}. \tag{33}$$

Let $\lambda > \bar{M}$. If $x \in \{x_N \geq 2\lambda - M_N\}$ then $x_N \geq 2\bar{M} - M_N$ and $2\lambda - x_N \leq M_N$, so that by definition

$$u_\lambda(x) = u(x', 2\lambda - x_N) \leq K \leq u(x).$$

To prove that $u_\lambda \leq u$ in T_λ for every $\lambda > \bar{M}$, it remains to show that if $\lambda > \bar{M}$ then $u_\lambda \leq u$ in $\{\lambda < x_N < 2\lambda - M_N\}$. If $x \in \{\lambda < x_N < 2\lambda - M_N\}$, then $x_N > 2\lambda - x_N > M_N$, so that the fact that $u_\lambda(x) \leq u(x)$ follows directly from the monotonicity of u in the e_N direction for $\{x_N > M_N\}$.
Now, let us show that if $\lambda > \bar{M}$ then $v_\lambda \geq v$ in T_λ. Since $u_\lambda \leq u$ in T_λ, we have

$$\begin{cases} \Delta(v - v_\lambda) - u_\lambda^2(v - v_\lambda) \geq 0 & \text{in } T_\lambda \\ v - v_\lambda = 0 & \text{on } \partial T_\lambda, \end{cases}$$

and $(v - v_\lambda)^+ \leq v \leq \delta$ in T_λ (see equation (33)). Consequently, we are in position to apply Lemma 5.2, obtaining $v - v_\lambda \leq 0$ in T_λ.

Step 2) $\Sigma = \mathbb{R}$.
In the first step we showed that $\Sigma \neq \emptyset$. Note that Σ is a closed interval and contains the unbounded interval $(\bar{M}, +\infty)$. Assume by contradiction that $\Sigma \neq \mathbb{R}$, that is, $\Lambda := \inf \Sigma > -\infty$. Then there exist sequences $(\lambda_i) \subset \mathbb{R}$ and $(x^i) \subset T_{\lambda_i}$ such that $\lambda_i < \Lambda$ and $\lambda_i \to \Lambda$ as $i \to \infty$, and at least one between

$$u_{\lambda_i}(x^i) > u(x^i) \quad \forall i \tag{34a}$$

$$v_{\lambda_i}(x^i) < v(x^i) \quad \forall i, \tag{34b}$$

holds true.

Assume that (34a) holds true. We claim that the sequence $(x_N^i) \subset \mathbb{R}$ is bounded. If not, as $x_N^i > \lambda_i$ and λ_i is bounded, up to a subsequence $x_N^i \to +\infty$ as $i \to \infty$. It follows that $2\lambda_i - x_N^i \to -\infty$, and in light of assumption (h2) we obtain

$$\lim_{i\to\infty} u_{\lambda_i}(x^i) = \lim_{i\to\infty} u((x^i)', 2\lambda_i - x_N^i) = 0 \quad \text{and} \quad \lim_{i\to\infty} u(x^i) = +\infty,$$

in contradiction with (34a) for i sufficiently large. Hence the claim is proved and, up to a subsequence, $x_N^i \to x_N^\infty$ as $i \to \infty$.
Let us set

$$u^i(x) := u((x^i)' + x', x_N) \quad \text{and} \quad v^i(x) := v((x^i)' + x', x_N).$$

From Lemma 5.3 it follows that $\{(u^i, v^i)\}$ is uniformly bounded and equi-Lipschitz-continuous in any compact subset of $\mathbb{R}^N$, so that the standard regularity theory for elliptic equations (see again [9]) implies that up to a subsequence (u^i, v^i) converges in $\mathcal{C}^2_{loc}(\mathbb{R}^N)$ to a pair (u^∞, v^∞), still solution of (1) in $\mathbb{R}^N$.

We wish to show that $x_N^\infty = \Lambda$. From the absurd assumption, equation (34a), we get

$$\begin{aligned} u_\Lambda^\infty(0', x_N^\infty) &= u^\infty(0', 2\Lambda - x_N^\infty) = \lim_{i\to\infty} u((x^i)', 2\lambda_i - x_N^i) \\ &= \lim_{i\to\infty} u_{\lambda_i}(x^i) \geq \lim_{i\to\infty} u(x^i) = u^\infty(0', x_N^\infty). \end{aligned} \tag{35}$$

Let us observe that $((x^i)' + x', x_N) \in T_\Lambda$ whenever $(x', x_N) \in T_\Lambda$. By definition of Λ, $u_\Lambda \leq u$ in T_Λ. Consequently, by the convergence of u^i to u^∞ we deduce

$$\begin{aligned} u_\Lambda^\infty(x', x_N) &= \lim_{i\to\infty} u^i(x', 2\Lambda - x_N) = \lim_{i\to\infty} u((x^i)' + x', 2\Lambda - x_N) \\ &\leq \lim_{i\to\infty} u((x^i)' + x', x_N) = \lim_{i\to\infty} u^i(x', x_N) = u^\infty(x', x_N) \end{aligned}$$

for every $(x', x_N) \in T_\Lambda$. Analogously, as $v_\Lambda \geq v$ in T_Λ, we have $v_\Lambda^\infty \geq v^\infty$ in T_Λ.
Now,

$$\begin{cases} -\Delta(u^\infty - u_\Lambda^\infty) + (v^\infty)^2(u^\infty - u_\Lambda^\infty) = ((v_\Lambda^\infty)^2 - (v^\infty)^2)u_\Lambda^\infty \geq 0 & \text{in } T_\Lambda \\ u^\infty - u_\Lambda^\infty \geq 0 & \text{in } T_\Lambda \\ u^\infty - u_\Lambda^\infty = 0 & \text{on } \partial T_\lambda. \end{cases} \tag{36}$$

Furthermore, $u^\infty - u_\Lambda^\infty$ is not identically 0: indeed by assumption (h2)

$$\lim_{x_N\to+\infty} u^\infty(x', x_N) - u_\Lambda^\infty(x', x_N) = +\infty.$$

Hence, the strong maximum principle implies that necessarily $u^\infty - u_\Lambda^\infty > 0$ in T_Λ. A comparison with (35) reveals that

$$x_N^\infty = \Lambda. \tag{37}$$

Now, by the absurd assumption (34a) we deduce

$$0 < u_{\lambda_i}(x^i) - u(x^i) = u^i(x', 2\lambda_i - x_N^i) - u^i(x', x_N) = 2\partial_N u^i(x', \xi^i)(\lambda_i - x_N^i) \qquad \forall i;$$

As $\lambda_i < x_N^i$ for every i this implies $\partial_N u^i(x', \xi_N^i) < 0$ for every i. As $\lambda_i \to \Lambda$ and $x_N^i \to \Lambda$ as $i \to \infty$, passing to the limit as $i \to \infty$ we deduce

$$\partial_N u^\infty(0', \Lambda) \leq 0. \tag{38}$$

On the other hand, thanks to the (36) and the fact that $u^\infty - u_\Lambda^\infty > 0$ in T_Λ, we are in position to apply the Hopf' Lemma:

$$\partial_\nu (u^\infty(0', \Lambda) - u_\Lambda^\infty(0', \Lambda)) < 0,$$

which means

$$2\partial_N u^\infty(0', \Lambda) > 0,$$

in contradiction with (38).

The above argument says that (34a) cannot occur. With minor changes, we can show that also (34b) is not verified, so that $\Sigma = \mathbb{R}$, which completes the proof. □

6 1-dimensional symmetry

In this section we complete the proof of our main result, Theorem 1.1. We will follow the technique introduced by the first author in [6]: we will show that, starting from Proposition 5.1, it is possible to prove that $\partial_\nu u > 0$ and $\partial_\nu v < 0$ for every $\nu \in \mathbb{S}_+^{N-1} = \{\nu \in \mathbb{S}^{N-1} : \nu_N > 0\}$. The conclusion will follow easily.

Proposition 6.1. *Let (u, v) be a solution of (1) satisfying (h1) and (h2). Then (u, v) depends only on x_N.*

Proof. We divide the proof in several steps.

Step 1) *For every $\sigma > 0$ there exists $\varepsilon = \varepsilon(\sigma) > 0$ such that*

$$\partial_N u(x) \geq \varepsilon \quad \textit{and} \quad \partial_N v(x) \leq -\varepsilon \quad \forall x \in \overline{S}_\sigma,$$

where $S_\sigma := \mathbb{R}^{N-1} \times (-\sigma, \sigma)$.
By contradiction, fixed $\sigma > 0$, assume that there exists $(x^i) \subset S_\sigma$ such that at least one between

$$\lim_{i \to +\infty} \frac{\partial u}{\partial x_N}(x^i) = 0 \tag{39a}$$

$$\lim_{i \to +\infty} \frac{\partial u}{\partial x_N}(x^i) = 0 \tag{39b}$$

holds true. Only to fix our minds, assume that (39a) holds. We define

$$u^i(x) := u(x + x^i) \quad \text{and} \quad v^i(x) := v(x + x^i).$$

Note that $|x_N^i| \leq \sigma$ for every i, so that for any compact set $K \subset \mathbb{R}^N$ there exists $M > 0$ such that $x + x^i \in S_M$ for every $x \in K$. Lemma 5.3 and standard elliptic estimates say that, up to a subsequence, $(u^i, v^i) \to (u^\infty, v^\infty)$ in $\mathcal{C}^2_{loc}(\mathbb{R}^N)$, where (u^∞, v^∞) is still a solution to (1). By the convergence, we have

$$\frac{\partial u^\infty}{\partial x_N} \geq 0 \quad \text{and} \quad \frac{\partial v^\infty}{\partial x_N} \leq 0 \quad \text{in } \mathbb{R}^N,$$

and $\partial_N u^\infty(0) = 0$. Furthermore,

$$-\Delta (\partial_N u^\infty) + (v^\infty)^2 (\partial_N u^\infty) = -2u^\infty v^\infty (\partial_N v^\infty) \geq 0 \quad \text{in } \mathbb{R}^N.$$

The strong maximum principle implies that either $\partial_N u^\infty > 0$ or $\partial_N u^\infty \equiv 0$. The former one is in contradiction with the fact that $\partial_N u^\infty(0) = 0$, the latter one is in contradiction with assumption (h2), which is also satisfied by the limiting profile (u^∞, v^∞). Thus, (39a) cannot occur. A similar argument shows that also (39b) does not hold.

Step 2) *For every $\sigma > 0$, the map $\nu \mapsto (\partial_\nu u, \partial_\nu v)$ is in $\mathcal{C}^{0,1}\left(\mathbb{S}^{N-1}, \left(\mathcal{C}^0(\overline{S}_\sigma)\right)^2\right)$.*
By Lemma 5.3, we know that $|\nabla u| + |\nabla v| \le \bar{C}_\sigma$ in $\overline{S}_\sigma$. Hence

$$\left|\frac{\partial u}{\partial \nu_1}(x) - \frac{\partial u}{\partial \nu_2}(x)\right| + \left|\frac{\partial v}{\partial \nu_1}(x) - \frac{\partial v}{\partial \nu_2}(x)\right| \le 2\bar{C}_\sigma |\nu_1 - \nu_2|$$

for every $x \in \overline{S}_\sigma$.

Step 3) *u is strictly increasing and v is strictly decreasing with respect to all the unit vectors of an open neighborhood of e_N in $\mathbb{S}^{N-1}$.*
Let $\Theta := \left\{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu\rangle \ge \frac{1}{2}\right\}$. By Corollary 4.12, we know that there exists M_Θ such that

$$\frac{\partial u}{\partial \nu} > 0 \quad \text{in } \{x_N > M_\Theta\} \quad \text{and} \quad \frac{\partial v}{\partial \nu} < 0 \quad \text{in } \{x_N < -M_\Theta\},$$

for every $\nu \in \Theta$. Let $\sigma > M_\Theta$. Using steps 1) and 2), we deduce that there exists an open neighborhood $\mathcal{O}_{e_N}$ of e_N in $\mathbb{S}^{N-1}$ such that

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}(x) < 0 \quad \forall x \in S_\sigma,\ \forall \nu \in \mathcal{O}_{e_N}. \tag{40}$$

We can assume that $\mathcal{O}_{e_N} \subset \Theta$ (if not, we replace $\mathcal{O}_{e_N}$ with a smaller neighborhood). This means that, for every $\nu \in \mathcal{O}_{e_N}$, it results

$$\frac{\partial u}{\partial \nu} > 0 \quad \text{in } \{x_N > -\sigma\} \quad \text{and} \quad \frac{\partial v}{\partial \nu} < 0 \quad \text{in } \{x_N < \sigma\},$$

Furthermore, for every $\nu \in \mathcal{O}_{e_N}$

$$\begin{cases} \Delta(-\partial_\nu u) - v^2(-\partial_\nu u) = -2uv\partial_\nu v \ge 0 & \text{in } \mathbb{R}^{N-1} \times (-\infty, -\sigma) \\ -u_\nu \le 0 & \text{on } \partial\left(\mathbb{R}^{N-1} \times (-\infty, -\sigma)\right) \\ -\partial_\nu u \in L^\infty\left(\mathbb{R}^{N-1} \times (-\infty, -\sigma)\right), \end{cases}$$

where the last one follows from Lemma 5.3. We are then in position to apply Lemma 5.2, obtaining $\partial_\nu u \ge 0$ in $\mathbb{R}^{N-1} \times (-\infty, -\sigma)$. Together with (40), this gives $\partial_\nu u \ge 0$ in $\mathbb{R}^N$ for every $\nu \in \mathcal{O}_{e_N}$. Similarly, from

$$\begin{cases} \Delta(\partial_\nu v) - u^2(\partial_\nu v) = 2uv\partial_\nu u \ge 0 & \text{in } \mathbb{R}^{N-1} \times (\sigma, +\infty) \\ v_\nu \le 0 & \text{on } \partial\left(\mathbb{R}^{N-1} \times (\sigma, +\infty)\right) \\ \partial_\nu v \in L^\infty\left(\mathbb{R}^{N-1} \times (\sigma, +\infty)\right), \end{cases}$$

we deduce $\partial_\nu v \le 0$ in $\mathbb{R}^N$ for every $\nu \in \mathcal{O}_{e_N}$. Finally, the strong maximum principle provides $\partial_\nu u > 0$ and $\partial_\nu v < 0$ in $\mathbb{R}^N$, for every $\nu \in \mathcal{O}_{e_N}$.

Step 4) *u is strictly increasing and v is strictly decreasing with respect to all the directions of the upper hemisphere $\mathbb{S}^{N-1}_+ = \{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu\rangle > 0\}$.*
Let Ω be the set of $\nu \in \mathbb{S}^{N-1}_+$ for which there exists an open neighborhood $\mathcal{O}_\nu \subset \mathbb{S}^{N-1}$ of ν such that

$$\frac{\partial u}{\partial \mu} > 0 \quad \text{and} \quad \frac{\partial v}{\partial \mu} < 0 \quad \text{in } \mathbb{R}^N,\ \forall \mu \in \mathcal{O}_\nu.$$

The set Ω is open by definition, and contains e_N for the previous step. If we show that it is closed with respect to the topology of $\mathbb{S}^{N-1}_+$, then $\Omega = \mathbb{S}^{N-1}_+$ and the claim is proved. Let $\bar{\nu}$ be a cluster point of Ω (note that $\langle e_N, \bar{\nu}\rangle > 0$), that is, there exists $(\nu_n) \subset \Omega$ such that $\nu_n \to \bar{\nu}$. As

$$\frac{\partial u}{\partial \nu_n} > 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu_n} < 0 \quad \text{in } \mathbb{R}^N,\ \forall n,$$

by continuity

$$\frac{\partial u}{\partial \bar\nu} \geq 0 \quad \text{and} \quad \frac{\partial v}{\partial \bar\nu} \leq 0 \quad \text{in } \mathbb{R}^N.$$

The strong maximum principle implies that or $\partial_{\bar\nu} u \equiv 0$ or $\partial_{\bar\nu} u > 0$ in $\mathbb{R}^N$; analogously, $\partial_{\bar\nu} v \equiv 0$ or $\partial_{\bar\nu} v < 0$ in $\mathbb{R}^N$. As $\bar\nu$ is not orthogonal to e_N, assumption (h2) says that neither $\partial_{\bar\nu} u \equiv 0$ nor $\partial_{\bar\nu} v \equiv 0$ can be satisfied, thus $\partial_{\bar\nu} u > 0$ and $\partial_{\bar\nu} v < 0$ in $\mathbb{R}^N$. It remains to show that there exists an open neighborhood $\mathcal{O}_{\bar\nu}$ of $\bar\nu$ in $\mathcal{S}_+^{N-1}$ such that for every $\mu \in \mathcal{O}_{\bar\nu}$

$$\frac{\partial u}{\partial \mu} > 0 \quad \text{and} \quad \frac{\partial v}{\partial \mu} < 0 \quad \text{in } \mathbb{R}^N.$$

It is possible to adapt the same proof of steps 1) to 3) with minor changes, in order to deduce the existence of $\mathcal{O}_{\bar\nu}$ (in the third step we replace Θ with $\{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu\rangle \geq \frac{1}{2}\langle e_N, \bar\nu > 0\rangle\}$). Consequently, $\bar\nu \in \Omega$ and Ω is closed with respect to the topology of $\mathbb{S}_+^{N-1}$.

Step 5) *Conclusion of the proof.*
Since $\Omega = \mathbb{S}_+^{N-1}$, by continuity we have

$$\frac{\partial u}{\partial \nu} \geq 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu} \leq 0 \quad \text{in } \mathbb{R}^N$$

for every ν which is orthogonal to e_N. But also $-\nu$ is orthogonal to e_N, so that

$$\frac{\partial u}{\partial \nu} \equiv 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu} \equiv 0 \quad \text{in } \mathbb{R}^N$$

for every ν orthogonal to e_N. In particular

$$\frac{\partial u}{\partial x_i} \equiv 0 \quad \text{and} \quad \frac{\partial v}{\partial x_i} \equiv 0 \quad \text{in } \mathbb{R}^N, \text{for } i = 1, \dots, N-1. \qquad \square$$

7 Proof of Corollary 1.2

We will show that if (u, v) is a solution of (1) with algebraic growth and (h3) holds true, then (h2) is satisfied.

Proof of Corollary 1.2. Firstly, let us observe that, since $u, v > 0$, (h3) implies

$$\lim_{x_N \to +\infty} u(x', x_N) = +\infty \quad \text{and} \quad \lim_{x_N \to -\infty} v(x', x_N) = +\infty \tag{41}$$

uniformly in $x' \in \mathbb{R}^{N-1}$. Thus, in order to obtain the thesis it remains to show that under (h1) and (h3) we have

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \text{and} \quad \lim_{x_N \to +\infty} v(x', x_N) = 0 \tag{42}$$

We prove only the second one in (42), for the first one it is possible to argue in the same way.

Step 1) *under (h1) and (h3), (u, v) has linear growth.*
Given $K > 0$, by (h3) there exists $M > 0$ such that $u(x) > K$ if $x \in \{x_N > M/2\}$. For an arbitrary $\theta > 1$, if $x \in \{x_N > M, \ |x'| < \theta x_N\}$ the ball $B_x := B_{x_N/100}(x)$ is contained in $\{x_N > M/2, \ |x'| < 2\theta x_N\}$. Consequently, if $x \in \{x_N > M, \ |x'| < \theta x_N\}$ we have

$$u(y) \geq K_x := \inf_{z \in B_x} u(z) \geq K \qquad \forall y \in B_x,$$

and

$$v(y) \leq C(1 + |y|^p) \leq C\left(1 + (2\theta + 1)^p y_N^p)\right) \leq C(1 + x_N^p) \qquad \forall y \in B_x.$$

The latter one gives $\delta_x := \sup_{y \in B_x} v(y) \le C(1 + x_N^p)$. Now,

$$\begin{cases} -\Delta v \le -K^2 v & \text{in } B_x \\ v \ge 0 & \text{in } B_x \\ v \le \delta_x & \text{in } B_x, \end{cases}$$

and we are in position to apply Lemma A.1: it follows

$$v(x) \le C\delta_x e^{-CKx_N} \le C(1 + x_N^p) e^{-CKx_N} \qquad \forall x \in \{x_N > M,\ |x'| < \theta x_N\}\,. \tag{43}$$

Let us consider the blow-down family $(u_{0,R}, v_{0,R}) =: (u_R, v_R)$. In light of the algebraic growth of (u, v), Theorem A.13 applies: there exists a homogeneous harmonic polynomial Ψ of degree $d \in \mathbb{N} \setminus \{0\}$ such that, up to a subsequence, (u_R, v_R) converges to (Ψ^+, Ψ^-) in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ as $R \to +\infty$. On the other hand, let $x \in \{|x'| < \theta x_N\}$; there exists $R_x > 0$ such that $Rx \in \{x_N > M,\ |x'| < \theta \pi x_N\}$ for every $R > R_x$. By means of (43), we deduce that

$$\lim_{R \to +\infty} v_R(x) = \lim_{R \to +\infty} \frac{1}{\sqrt{H(0,R)}} v(Rx) = 0 \qquad \forall x \in \{|x'| < \theta x_N\}\,,$$

where we used also Corollary A.14 to ensure that $H(0, R)$ does not tend to 0. As θ has been arbitrarily chosen, we deduce that $v_R \to 0$ pointwise in $\mathbb{R}^N_+$. By the uniqueness of the limit, Ψ has to be a homogeneous harmonic polynomial which vanishes in the entire half-space $\mathbb{R}^N_+$: as showed in the proof of Proposition 3.1, necessarily Ψ is a linear function and $d = 1$. By means of Corollary A.8, we deduce that (u, v) has linear growth.

Step 2) *Conclusion of the proof.*
As (u, v) has linear growth, we can choose $\bar{C}_3$ as in Remark 2.4. Assumption (h3) it is sufficient to ensure that the geometry of the set $\{|u - v| < \bar{C}_3\}$ is described by Lemma 4.4: $\{|u - v| < \bar{C}_3\}$ is bounded in the e_N direction and unbounded in all the other directions. Consequently, also Lemma 4.8 applies: for $\hat{R} \ge \bar{R}$ we can find M_2 as in the quoted statement.
Given $K > 0$, by (41) there exists $M > 0$ such that if $x \in \{x_N > \frac{M}{2}\}$ then $u(x) \ge K$. Let $M_5 := \max\{M, M_2\}$, so that

$$\{x_N > M_5\} \subset \bigcup_{\substack{x_0 \in \{|u-v| < \bar{C}_3\} \\ R > \hat{R}}} S^+_{x_0,R}.$$

If $x \in \{x_N > M_5\}$ then the ball $B_x := B_{x_N/100}(x)$ is contained in $\{x_N > \frac{M}{2}\}$, so that

$$\begin{cases} -\Delta v \le -K^2 v & \text{in } B_x \\ v \ge 0 & \text{in } B_x \\ v \le \delta_x & \text{in } B_x, \end{cases}$$

where $\delta_x := \sup_{B_x} v < +\infty$, because $v \in L^\infty_{loc}(\mathbb{R}^N)$. From Lemma A.1 we obtain

$$v(x) \le C \left(\sup_{y \in B_x} v(y) \right) e^{-CKx_N}. \tag{44}$$

To control $\sup_{B_x} v$, we consider $\tilde{x}$ and $\tilde{R}$ defined in Lemma 4.8 and Remark 4.9. As $B_x \subset Q_x$, a fortiori $B_x \subset S^+_{\tilde{x},\tilde{R}} \subset B_{\tilde{R}}(\tilde{x})$. We are then in position to apply Corollary 2.6:

$$\begin{aligned} \sup_{y \in B_x} v(y) &\le \bar{C}_4(1 + \tilde{R}) = \bar{C}_4 \left(1 + \frac{3}{2}(x_N - \tilde{x}_N) \right) \\ &\le \bar{C}_4 \left(1 + \frac{3}{2}\zeta + \frac{3}{2} x_N \right) \le C x_N \end{aligned}$$

provided x_N is sufficiently large (recall the definition of ζ, Remark 4.5). Plugging into (44), we see that for every x such that $x_N \gg 1$ is sufficiently large it results

$$v(x) \leq C x_N e^{-CKx_N},$$

which gives the second limit in (42). □

A Appendix

For the reader's convenience, we report some known and few new results which we used many times in our work. We prefer to write down explicitly the statements below, because in the literature they do not appear always in this form, and because sometimes the proofs are missing. In such a case, we will write them for the sake of completeness.

The exponential decay

It is by now well known that, if (u, v) solves (1) and u is very large in a ball $B_{2r}(x_0)$, then v has to be exponentially small with respect to u in a smaller ball.

Lemma A.1 (Lemma 4.4 in [4]). *Let $x_0 \in \mathbb{R}^N$ and $r > 0$. Let $u \in H^1(B_{2r}(x_0))$ be such that*

$$\begin{cases} -\Delta v \leq -Kv & \text{in } B_{2r}(x_0) \\ v \geq 0 & \text{in } B_{2r}(x_0) \\ v \leq A & \text{on } \partial B_{2r}(x_0), \end{cases}$$

where K and A are two positive constants. Then for every $\alpha \in (0,1)$ there exists $C_\alpha > 0$, not depending on A, K, R and x_0, such that

$$\sup_{x \in B_r(x_0)} v(x) \leq \alpha A e^{-C_\alpha K^{1/2} r}.$$

We will always apply this result with $\alpha = 1/2$ to simplify the notation.

The segregation theorem

Let us consider the problem

$$\begin{cases} -\Delta u_\beta = -\beta u_\beta v_\beta^2 \\ -\Delta v_\beta = -\beta u_\beta^2 v_\beta \\ u_\beta > 0, v_\beta > 0, \end{cases} \tag{45}$$

where β is a positive parameter tending to $+\infty$. The following is the local version of the uniform Hölder estimates obtained in [10], which has been proved in [13].

Theorem A.2. *Let $\{(u_\beta, v_\beta)\}$ be a family of solutions to (45) in a ball $B_{2r}(x_0) \subset \mathbb{R}^N$ (where $x_0 \in \mathbb{R}^N$ and $r > 0$). Assume that, as $\beta \to +\infty$, $\{(u_\beta, v_\beta)\}$ is uniformly bounded in $L^\infty(B_{2r}(x_0))$. Then $\{(u_\beta, v_\beta)\}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(B_r(x_0))$, for every $\alpha \in (0,1)$.*

As a consequence, one can easily adapt the proof of Theorem 1.2 of [10] and obtain a local segregation theorem, see also [5, 12].

Theorem A.3. *Let $\{(u_\beta, v_\beta)\}$ be a family of solutions to (45) in a ball $B_{2r}(x_0) \subset \mathbb{R}^N$ (where $x_0 \in \mathbb{R}^N$ and $r > 0$). Assume that, as $\beta \to +\infty$, $\{(u_\beta, v_\beta)\}$ is uniformly bounded in $L^\infty(B_{2r}(x_0))$. Then there exists a pair (u_∞, v_∞) such that, up to a subsequence, there holds*

(i) $u_\beta \to u_\infty$ and $v_\beta \to v_\infty$ in $\mathcal{C}^0(B_r(x_0)) \cap H^1(B_r(x_0))$,

(ii) $u_\infty v_\infty \equiv 0$ *in* $B_r(x_0)$ *and*

$$\lim_{\beta\to+\infty} \int_{B_r(x_0)} \beta u_\beta^2 v_\beta^2 = 0,$$

(iii) the limiting profile satisfies

$$\begin{cases} -\Delta u_\infty = 0 & \text{in } \{u_\infty > 0\} \cap B_r(x_0) \\ -\Delta v_\infty = 0 & \text{in } \{v_\infty > 0\} \cap B_r(x_0), \end{cases}$$

(iv) $u_\infty - v_\infty$ *is harmonic and both* u_∞ *and* v_∞ *are subharmonic in* $B_r(x_0)$*.*

Remark A.4. In [10] it is considered a different system with some additional terms. In particular, the term u^3 appear in the equation for u, and v^3 in the equation for v. Since it is required that these powers are subcritical for the Sobolev embedding, this imposes a restriction on the dimension N. However, as explained in the introduction of the quoted paper, all the results are valid in any dimension provided u^3 and v^3 are replaced by subcritical terms; this is clearly the case of system (45).

The Almgren monotonicity formula

We recall some properties of the functions H and N, defined in (2). Firstly

Remark A.5. A direct computation shows that

$$\frac{\partial}{\partial r} H(x_0, r) = 2r^{1-N} \int_{B_r(x_0)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2 \geq 0 :$$

for every $x_0 \in \mathbb{R}^N$ and $r > 0$ the function $H(x_0, r)$ is nondecreasing in r.

Proposition 5.2 of [3] says that also the Almgren quotient is nondecreasing as function of r.

Proposition A.6 (Almgren monotonicity formula)**.** *Let* (u, v) *be a solution of* (1)*, let* $x_0 \in \mathbb{R}^N$*. The Almgren frequency function* $N(x_0, r)$ *is well defined for* $r \in (0, +\infty)$*, nonnegative and nondecreasing in* r*.*

A control on the Almgren frequency function gives useful information about the growth of the function H with respect to the radial variable. The proof of the following result is a straightforward modification of the proof of Proposition 5.3 in [3]

Corollary A.7. *Let* (u, v) *be a solution of* (1)*, let* $x_0 \in \mathbb{R}^N$*, and assume that* $d_1 \leq N(x_0, r) \leq d_2$ *for* $0 < R_1 < r < R_2$*. Then*

$$\frac{r_2^{2d_1}}{r_1^{2d_1}} \leq \frac{H(x_0, r_2)}{H(x_0, r_1)} \leq e^{d_2} \frac{r_2^{2d_2}}{r_1^{2d_2}}$$

for every $R_1 < r_1 < r_2 < R_2$*.*

In light of the subharmonicity of (u, v), it is not difficult to deduce a pointwise estimate on the growth of the solution (u, v).

Corollary A.8. *Let* (u, v) *be a solution of* (1)*, let* $x_0 \in \mathbb{R}^N$ *and* $p \geq 1$*, and assume that* $N(x_0, r) \leq p$ *for every* $r > 0$*. Then there exists* $C > 0$ *such that*

$$u(x) + v(x) \leq C(1 + |x|^p) \qquad \forall x \in \mathbb{R}^N.$$

Proof. The thesis follows if we show that there exists $C > 0$ such that

$$u(x) + v(x) \le C(1 + |x - x_0|^p) \qquad \forall x \in \mathbb{R}^N.$$

Suppose by contradiction that our claim is not true. Then there exists $r_n \to +\infty$ such that

$$\lim_{n\to+\infty} \frac{u(x_0 + r_n x)}{r_n^p} = +\infty \tag{46}$$

for some $x \in \mathbb{S}^{N-1}$ and $r_n \to +\infty$. In light of Corollary A.7, we have

$$\frac{H(x_0, 2r_n)}{(2r_n)^{2p}} \le e^p H(x_0, 1) \quad \Rightarrow \quad \int_{\partial B_{2r}(x_0)} u^2 + v^2 \le C r_n^{2p+N-1}. \tag{47}$$

As u is subharmonic, $u \le \varphi_n$ in $B_{2r_n}(x_0)$, where φ_n is the solution of

$$\begin{cases} -\Delta \varphi_n = 0 & \text{in } B_{2r_n}(x_0) \\ \varphi_n = u & \text{on } \partial B_{2r_n}(x_0). \end{cases}$$

By the representation formula for harmonic functions we know that for every $x \in \overline{B}_{r_n}(x_0)$

$$\begin{aligned} \varphi_n(x) &= \frac{4r_n^2 - |x - x_0|^2}{2N|\mathbb{S}^{N-1}|r_n} \int_{\partial B_{2r_n}(x_0)} \frac{u(x)}{|x-y|^N}\, d\sigma_y \\ &\le C r_n \left(\int_{\partial B_{2r_n}(x_0)} \frac{d\sigma_y}{r_n^{2N}} \right)^{\frac12} \left(\int_{\partial B_{2r_n}(x_0)} u^2 \right)^{\frac12} \le C r_n^{-\frac{N-1}{2} + p + \frac{N-1}{2}} = C r_n^p, \end{aligned}$$

where C depends only on the dimension N, and for the last inequality we used the (47). Thus, for every $x \in \mathbb{S}^{N-1}$ we obtain

$$u(x_0 + r_n x) \le \varphi_n(x) \le C r_n^p \qquad \forall n,$$

in contradiction with equation (46). □

As proved in [7], the converse holds true.

Lemma A.9 (Lemma 2.1 in [7])**.** *Let (u, v) be a solution of (1), let $x_0 \in \mathbb{R}^N$, and assume that there exist $p \ge 1$ and $C > 0$ such that*

$$u(x) + v(x) \le C(1 + |x|^p) \qquad \forall x \in \mathbb{R}^N.$$

Then $N(x_0, r) \le p$ for every $x_0 \in \mathbb{R}^N$ and for every $r > 0$.

Remark A.10. Combining Corollary A.8 and Lemma A.9, we deduce that if for a single $x_0 \in \mathbb{R}^N$ we know that $N(x_0, r) \le p$ for every $r > 0$, then

$$u(x) + v(x) \le C(1 + |x|^p) \qquad \forall x \in \mathbb{R}^N,$$

so that $N(x, r) \le p$ for every $x \in \mathbb{R}^N$. That is, a bound of the Almgren quotient centered in a point $x_0 \in \mathbb{R}^N$ provides the same bound for the quotients $N(x, \cdot)$ for every $x \in \mathbb{R}^N$.

Remark A.11. We point out that all these results hold true for a solution (u_β, v_β) of (45), with $E(x_0, r)$ replaced by the corresponding energy function, that is,

$$\frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla u_\beta|^2 + |\nabla v_\beta|^2 + \beta u_\beta^2 v_\beta^2.$$

The blow-down family

By means of the previous monotonicity formulae, in [3] it is proved that the asymptotic information about $\{(u_\beta, v_\beta)\}$ can be improved for particular sequences. Let (u,v) be a solution of (1). For every $x_0 \in \mathbb{R}^N$ and $R > 0$, recall that we introduced the blow-down family

$$(u_{x_0,R}(x), v_{x_0,R}(x)) := \left(\frac{1}{\sqrt{H(x_0,R)}} u(x_0 + Rx), \frac{1}{\sqrt{H(x_0,R)}} v(x_0 + Rx) \right).$$

By definition, $\int_{\partial B_1(0)} u_{x_0,R}^2 + v_{x_0,R}^2 = 1$ for every $x_0 \in \mathbb{R}^N$ and $R > 0$. Also, $(u_{x_0,R}, v_{x_0,R})$ solves

$$\begin{cases} -\Delta u_{x_0,R} = -H(x_0,R)R^2\, u_{x_0,R}\, v_{x_0,R}^2 & \text{in } \mathbb{R}^N \\ -\Delta v_{x_0,R} = -H(x_0,R)R^2\, u_{x_0,R}^2\, v_{x_0,R} & \text{in } \mathbb{R}^N \\ u_{x_0,R}, v_{x_0,R} > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{48}$$

Remark A.12. A direct computation shows that if $N(x_0, r) \le p$ for every $r \ge 1$, the same estimate holds true for the Almgren quotient associated to the function $(u_{x_0,R}, v_{x_0,R})$ (for every $x_0 \in \mathbb{R}^N$ and $R > 0$):

$$\frac{\frac{1}{r^{N-2}} \int_{B_r(0)} |\nabla u_{x_0,R}|^2 + |\nabla v_{x_0,R}|^2 + H(x_0,R)R^2\, u_{x_0,R}^2\, v_{x_0,R}^2}{\frac{1}{r^{N-1}} \int_{\partial B_r(0)} u_{x_0,R}^2 + v_{x_0,R}^2} = N(x_0, Rr) \le p \qquad \forall r \ge 1.$$

As a consequence, if we can bound $N(x_0, \cdot)$, we can apply Corollary A.7 on $(u_{x_0,R}, v_{x_0,R})$.

Theorem 1.4 in [3] says, roughly speaking, that if the Almgren frequency function is bounded, then the limit of $N(x_0, r)$ as $r \to +\infty$ (which exists by monotonicity) is a positive integer and the limiting profile is a homogeneous harmonic polynomial. It is straightforward to check that, although therein it is considered the case $x_0 = 0$, the result holds true for any $x_0 \in \mathbb{R}^N$.

Theorem A.13. *Let (u,v) be a solution of (1), let $x_0 \in \mathbb{R}^N$, and assume that*

$$\lim_{r \to +\infty} N(x_0, r) =: d_{x_0} < +\infty.$$

Then d_{x_0} is a positive integer. There exist a subsequence of the blow down family $\{(u_{x_0,R}, v_{x_0,R}) : R > 0\}$, denoted $\{(u_{x_0,R_n}, v_{x_0,R_n})\}$, and a homogeneous harmonic polynomial of degree d_{x_0}, denoted by Ψ_{x_0}, such that $(u_{x_0,R_n}, v_{x_0,R_n}) \to (\Psi_{x_0}^+, \Psi_{x_0}^-)$ as $R \to +\infty$ in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ and in $H^1_{loc}(\mathbb{R}^N)$. Moreover,

$$H(x_0,R)R^2\, u_{x_0,R_n}^2\, v_{x_0,R_n}^2 \to 0 \qquad \text{in } L^1_{loc}(\mathbb{R}^N).$$

This achievement permits to say something more on the asymptotic of $H(x_0, \cdot)$ in case (u,v) has algebraic growth.

Corollary A.14. *Let (u,v) be a solution of (1) with algebraic growth. For $x_0 \in \mathbb{R}^N$, let $d_{x_0} = \lim_{r\to+\infty} N(x_0, r)$, which is a positive integer by the previous statement. For every $\varepsilon > 0$ it results*

$$\lim_{r \to +\infty} \frac{H(x_0, r)}{r^{2d_{x_0}(1-\varepsilon)}} = +\infty.$$

Proof. As $d_{x_0} \ge 1$, using the Almgren monotonicity formula (Theorem A.6) we deduce that for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that if $r > r_\varepsilon$ then

$$N(x_0, r) \ge d_{x_0} \left(1 - \frac{\varepsilon}{2}\right).$$

Hence, we can use Corollary A.7 to obtain

$$H(x_0, r) \ge C r^{2d_{x_0}\left(1-\frac{\varepsilon}{2}\right)} \qquad \forall r > r_\varepsilon,$$

with $C > 0$. Therefore

$$\lim_{r \to +\infty} \frac{H(x_0, r)}{r^{2d_{x_0}(1-\varepsilon)}} \ge \lim_{r \to +\infty} C \frac{r^{2d_{x_0}\left(1-\frac{\varepsilon}{2}\right)}}{r^{2d_{x_0}(1-\varepsilon)}} = +\infty.$$

□

An Alt-Caffarelli-Friedman monotonicity formula

For a solution (u,v) to (1), recall the definition

$$J(x_0,r)=\frac{1}{r^4}\int_{B_r(x_0)}\frac{|\nabla u(y)|^2+u^2(y)v^2(y)}{|y-x_0|^{N-2}}\,dy\int_{B_r(x_0)}\frac{|\nabla v(y)|^2+u^2(y)v^2(y)}{|y-x_0|^{N-2}}\,dy.$$

First of all, we report the useful formula (4.11) in [13]: there exists $C>0$ independent on $x_0\in\mathbb{R}^N$ and on $r\geq 1$ such that

$$\frac{1}{r^2}\int_{B_r(x_0)}\frac{|\nabla u(y)|^2+u^2(y)v^2(y)}{|y-x_0|^{N-2}}\,dy\leq\frac{C}{r^{N+2}}\int_{B_{2r}(x_0)}u^2. \tag{49}$$

Recently, K. Wang proved an Alt-Caffarelli-Friedman monotonicity formula which enhances a previous similar result in [10].

Theorem A.15 (Theorem 4.3 in [13])**.** *Let (u,v) be a solution of (1) satisfying (h1), let $x_0\in\mathbb{R}^N$. There exists $C(x_0)>0$ such that*

$$r\mapsto e^{-C(x_0)r^{-1/2}}J(x_0,r)\quad\textit{is nondecreasing in } r$$

for every $r\geq 1$.

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