A VARIATIONAL ALGORITHM FOR THE DETECTION OF LINE SEGMENTS

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1. **Introduction.** Detection of edges, that is, points in a digital image at which the image intensity changes sharply is one of the most often performed steps in image processing. Ideally, the algorithm employed for solving this problem should provide a set of connected curves that indicate the edges of objects. In a recent paper [12], three of the authors have developed an iterative algorithm for edge detection using the concept of topological asymptotic analysis. The basic idea of this approach is to

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cover the expected edge set with balls of radius $\varepsilon > 0$ and use the number of balls, multiplied with 2ε, as an estimate for its length. It was shown that under certain conditions the proposed variational model approximates the Mumford–Shah functional [15] in the sense of Γ–limits, and, therefore, this algorithm may be considered as a computational method for the approximate minimization of the Mumford–Shah functional. A criterion for the optimal positioning of balls covering the edge set is provided by the leading term of a topological asymptotic expansion of the approximating functional. The (iterative) implementation of the algorithm selects edges successively according to certain rules. In a follow up paper [10], it was shown that this approach is useful for scale detection of edges.

In this paper, we consider again the problem of edge detection in the framework of topological asymptotic analysis. As opposed to the previous work, however, we consider now covering the edge set with *line segments* rather than with balls. There are several reasons: First, edges should rather be seen as a union of small line segments than as accumulations of points. Second, numerically, the resulting algorithm is expected to be faster, as in each iteration step a whole set of edge points (the segment) is detected and not a single point only. We admit here that there is still a conceptual misfit between the continuous formulation and the discrete setting. Theoretically, by our analysis, only edge segments can be detected that display a certain distance from the previously detected ones (this will be reflected in the constant δ_0 below). We believe that this technical problem can in fact be solved, but it seems that this requires a much more sophisticated analysis of the topological expansion. In fact, for practical realizations, it is not a severe restriction, since the distance can, theoretically, be chosen arbitrarily small, in particular below half of the pixel size, in which case the union of line segments appears closed. However, compared to [12] the effect is less pronounced, because the covering line segments are relatively larger than the balls.

The novelty of this paper is an algorithm for edge detection based on the asymptotic analysis for topological derivatives with respect to line segments. We note that the topological asymptotic expansion in [12] has been derived in the framework of potential theory [16]. However, in the present case, due to the more complex geometry of the inhomogeneities and the impossibility of introducing a uniform scaling, this approach fails. To avoid these difficulties, in this paper we build up on a geometry independent approach of Capdeboscq & Vogelius [6, 7]. To outline our method, we have to introduce some notation first.

Let Ω be an open and bounded subset of $\mathbb{R}^2$. We assume that a given image $f : \Omega \to \mathbb{R}$ is a bounded function that assigns to each point $x \in \Omega$ some gray value $f(x) \in \mathbb{R}$.

Definition 1.1. We denote by

$$\sigma_\varepsilon(y, \tau) := \{ x \in \mathbb{R}^2 \ : \ x = y + \rho\tau \, , \ -\varepsilon \leq \rho \leq \varepsilon \} \tag{1}$$

a line segment of length $2\varepsilon > 0$ centered at $y \in \mathbb{R}^2$ and with the unit tangent vector $\tau \in \mathbb{S}^1$. Moreover, we define a thin strip around $\sigma_\varepsilon(y, \tau)$ as

$$\omega_\varepsilon(y, \tau) := \{ x \in \mathbb{R}^2 \ : \ \mathrm{dist}(x, \sigma_\varepsilon(y, \tau)) \leq \varepsilon^2 \} \, . \tag{2}$$

If $K \subset \Omega$ is a closed subset and $0 < \kappa < 1$, we define the function $v_K : \Omega \to \mathbb{R}$ by

$$v_K(x) := \begin{cases} \kappa & \text{if } x \in K, \\ 1 & \text{else.} \end{cases} \tag{3}$$

In particular, we will apply this notion if K is the union of strips $\omega_\varepsilon(y, \tau)$. Finally, for every $v \in L^2(\Omega)$ we define

$$m_\varepsilon(v) := \inf\left\{|S| : S \subset \mathbb{R}^2 \times \mathbb{S}^1, \ v = v_K \text{ with } K = \bigcup_{(y,\tau) \in S} \omega_\varepsilon(y, \tau)\right\}. \qquad (4)$$

Here we set $m_\varepsilon(v) := +\infty$, if $v \neq v_K$ for every finite subset $S \subset \mathbb{R}^2 \times \mathbb{S}^1$ with $K = \bigcup_{(y,\tau)} \omega_\varepsilon(y, \tau)$.

With this notation at hand, we introduce the functional

$$\mathcal{J}_\varepsilon(u, v) := \frac{1}{2}\int_\Omega (u - f)^2\, dx + \frac{\alpha}{2}\int_\Omega v|\nabla u|^2\, dx + 2\beta\varepsilon\, m_\varepsilon(v), \qquad (5)$$

which is to be minimized over all functions $u \in H^1(\Omega)$ and $v \in L^\infty(\Omega)$. Here α and β are some positive parameters.

For the approximate numerical minimization of $\mathcal{J}_\varepsilon$ we use a topological asymptotic expansion. Defining

$$\mathcal{J}(u, v) = \frac{1}{2}\int_\Omega (u - f)^2\, dx + \frac{\alpha}{2}\int_\Omega v|\nabla u|^2\, dx,$$

we see that for general $\omega_\varepsilon(y, \tau) \cap K = \emptyset$ we have

$$\mathcal{J}_\varepsilon(u, v_{K \cup \omega_\varepsilon(y,\tau)}) - \mathcal{J}_\varepsilon(\hat{u}, v_K) = \mathcal{J}(u, v_{K \cup \omega_\varepsilon(y,\tau)}) - \mathcal{J}(\hat{u}, v_K) + 2\beta\varepsilon.$$

Thus the largest decrease of $\mathcal{J}_\varepsilon$ with respect to a strip $\omega_\varepsilon(y, \tau)$ can as well be found by optimizing $\mathcal{J}$ with respect to y and τ. Let now K be some subset of Ω; in particular, it can be the union of a finite number of thin strips. Now assume that we cut out a small strip $\omega_\varepsilon(y, \tau)$ of $\Omega \setminus K$ and denote by v_K and $v_{K \cup \omega_\varepsilon} := v_{K \cup \omega_\varepsilon(y,\tau)}$ the corresponding edge indicators. Denote moreover by u_K and $u_{K \cup \omega_\varepsilon}$ the minimizers of the functionals $\mathcal{J}(\cdot, v_K)$ and $\mathcal{J}(\cdot, v_{K \cup \omega_\varepsilon})$, respectively. Our main result in Section 2 is the derivation of an expansion of the form

$$\mathcal{J}(u_{K \cup \omega_\varepsilon}, v_{K \cup \omega_\varepsilon}) - \mathcal{J}(u_K, v_K) \approx -2\alpha(1 - \kappa)\varepsilon^3 \mathcal{M}\nabla u_K(y) \cdot \nabla u_K(y), \qquad (6)$$

where $\mathcal{M} = \frac{1}{\kappa}n \otimes n + \tau \otimes \tau$ and n, τ are the unit normal and tangent vectors to the segment σ_ε, and the intersection of K and $\omega_\varepsilon(y, \tau)$ is empty. The above difference (6) is asymptotically valid whenever a strip is removed from the potential edge set and can be used for finding the points of Ω where we can expect the largest decrease of $\mathcal{J}_\varepsilon$ by removing small strips.

2. **Asymptotic expansion.** We assume that $\Omega \subset \mathbb{R}^2$ is an open bounded smooth domain and $f : \Omega \to \mathbb{R}$ is a given function in $L^\infty(\Omega)$. We define the functional

$$\mathcal{J}(u, v) := \frac{1}{2}\int_\Omega (u - f)^2\, dx + \frac{\alpha}{2}\int_\Omega v|\nabla u|^2\, dx, \qquad (7)$$

for $u \in H^1(\Omega)$ and $v \in L^\infty(\Omega)$, and the parameter $\alpha > 0$.

Now assume that K is a fixed open subset of Ω and define the function $v\colon \Omega \to \mathbb{R}$ by

$$v(x) = \begin{cases} \kappa & x \in K, \\ 1 & x \in \Omega \setminus \overline{K}, \end{cases} \qquad (8)$$

with $0 < \kappa < 1$. Using standard results of calculus of variations, one can show that the unique minimizer $u \in H^1(\Omega)$ of $\mathcal{J}(\cdot, v)$ is the unique weak solution to the boundary value problem

$$
\begin{cases}
u - \alpha \operatorname{div}(v \nabla u) = f & \text{in } \Omega, \\
\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{9}
$$

In the remainder of this section, we will derive a variation of the functional $\mathcal{J}$ with respect to perturbation of the function v obtained by adding a small strip to the set K. More precisely, let us denote by L_0 a compact subset of $\Omega \setminus \overline{K}$ such that

$$
\operatorname{dist}(L_0, \partial \Omega \cup \overline{K}) \geq \delta_0 > 0.
$$

Let $y \in \operatorname{int}(L_0)$ and $\tau \in \mathbb{S}^1$. We choose $\varepsilon > 0$ small enough so that the thin strip $\omega_\varepsilon(y, \tau)$ defined as in (2) is contained in L_0. From now on, in order to simplify the notation, we set

$$
\omega_\varepsilon := \omega_\varepsilon(y, \tau) \text{ and } \sigma_\varepsilon := \sigma_\varepsilon(y, \tau).
$$

We define the function $v_\varepsilon \colon \Omega \to \mathbb{R}$ by

$$
v_\varepsilon(x) =
\begin{cases}
\kappa & x \in K \cup \omega_\varepsilon, \\
1 & x \in \Omega \setminus (\overline{K} \cup \overline{\omega_\varepsilon}).
\end{cases}
\tag{10}
$$

Similarly as above, we note that the unique minimizer $u_\varepsilon \in H^1(\Omega)$ of $\mathcal{J}(\cdot, v_\varepsilon)$ is the unique weak solution to the boundary value problem

$$
\begin{cases}
u_\varepsilon - \alpha \operatorname{div}(v_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\
\dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{11}
$$

Our goal is to establish an expansion for $\mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v)$ in powers of ε as $\varepsilon \to 0$. We will prove the following theorem:

Theorem 2.1. *We have*

$$
\lim_{\varepsilon \to 0} \max_{y \in L_0} \frac{1}{4\varepsilon^3} \left| \mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v) - \frac{\alpha(\kappa - 1)}{2} 4\varepsilon^3 \mathcal{M} \nabla u(y) \cdot \nabla u(y) \right| = 0,
$$

where $\mathcal{M} = \frac{1}{\kappa} n \otimes n + \tau \otimes \tau$ and n, τ are the unit normal and unit tangent vectors to the segment σ_ε.

In order to prove Theorem 2.1 we will follow the approach of Capdeboscq & Vogelius [6, 7]. We will need the set

$$
\hat{L}_0 := L_0 + \overline{B_{\delta_0/2}(0)},
$$

which is constructed in such a way that it satisfies

$$
L_0 \subset \hat{L}_0 \subset \Omega \setminus \overline{K} \text{ and } \operatorname{dist}(\hat{L}_0, \partial \Omega \cup \overline{K}) \geq \delta_0/2,
$$

and several auxiliary lemmas.

Lemma 2.2. *The following identity holds:*

$$
\mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v) = \frac{\alpha(\kappa - 1)}{2} \int_{\omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla u \, dx.
\tag{12}
$$

Proof of Lemma 2.2. Since u_ε and u are weak solutions of (9) and (11) they satisfy, respectively,

$$\int_\Omega (u_\varepsilon - f)\phi + \alpha v_\varepsilon \nabla u_\varepsilon \cdot \nabla \phi \, dx = 0 \qquad \text{for all } \phi \in H^1(\Omega), \qquad (13)$$

$$\int_\Omega (u - f)\phi + \alpha v \nabla u \cdot \nabla \phi \, dx = 0 \qquad \text{for all } \phi \in H^1(\Omega). \qquad (14)$$

Setting $\phi = u$ in (13) and $\phi = u_\varepsilon$ in (14) and subtracting (14) from (13) we get

$$\int_\Omega f(u_\varepsilon - u) \, dx = \alpha(1 - \kappa) \int_{\omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla u \, dx \qquad (15)$$

On the other hand, inserting $\phi = u_\varepsilon$ in (13) and $\phi = u$ in (14), we obtain, respectively,

$$\int_\Omega u_\varepsilon(u_\varepsilon - f) + \alpha v_\varepsilon |\nabla u_\varepsilon|^2 \, dx = 0 \,, \qquad (16)$$

$$\int_\Omega u(u - f) + \alpha v |\nabla u|^2 \, dx = 0 \,. \qquad (17)$$

Now

$$2(\mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v)) = \int_\Omega (u_\varepsilon - f)^2 + \alpha v_\varepsilon |\nabla u_\varepsilon|^2 \, dx - \int_\Omega (u - f)^2 - \alpha v |\nabla u|^2 \, dx$$

and, by (16) and (17), we have

$$2(\mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v)) = \int_\Omega (u_\varepsilon - f)^2 - (u_\varepsilon - f)u_\varepsilon - (u - f)^2 + (u - f)u \, dx$$

$$= - \int_\Omega f(u_\varepsilon - u) \, dx \,.$$

Finally, recalling (15), the claim follows.

Lemma 2.3. *The function u satisfies*

$$u \in C^{1,\lambda}(L_0) \qquad (18)$$

for every $0 < \lambda < 1$. Moreover there exists a constant $C = C(\delta_0, \Omega)$ such that

$$\|\nabla u\|_{L^\infty(L_0)} \le C(\|f\|_{H^{-1}(\Omega)} + \|f\|_{L^\infty(\Omega)}) \qquad (19)$$

Proof of Lemma 2.3. First we observe that, in $\Omega \setminus \overline{K}$, u solves

$$u - \alpha \Delta u = f.$$

Now, let $\tilde{x} \in L_0$ and let $\varphi \in C_0^\infty(\Omega)$ be a function with a compact support in L_0 such that $\varphi(x) \equiv 1$ in a neighborhood U of $\tilde{x}$. Since $u - f$ is bounded, we get that $w = (u - f)\varphi \in L^p(L_0)$ for all $p < +\infty$, and therefore, $\Delta^{-1}w \in W^{2,p}(L_0)$ for all $p < +\infty$. In particular we have that $u \in W^{2,p}(U)$ for all $p < +\infty$. From the Sobolev imbedding theorem and since $\tilde{x} \in L_0$ is arbitrary, we conclude that $u \in C^{1,\lambda}(L_0)$ with $\lambda \in (0, 1 - 2/p)$ for $2 < p < +\infty$. Moreover from [14] we have

$$\|u\|_{W^{2,p}(L_0)} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)})$$
$$\le C(\|f\|_{H^{-1}(\Omega)} + \|f\|_{L^\infty(\Omega)})$$

for any $p > 2$ and where C depends on δ_0, Ω. Finally the Sobolev imbedding theorem implies (19).

We now derive energy estimates for $u_\varepsilon - u$.

Lemma 2.4. *There exists a constant $C = C(\kappa, \delta_0)$ such that*

$$\|u_\varepsilon - u\|_{H^1(\Omega)} \le C(\|f\|_{H^{-1}(\Omega)} + \|f\|_{L^\infty(\Omega)})|\omega_\varepsilon|^{\frac{1}{2}} \qquad (20)$$

and

$$\|u_\varepsilon - u\|_{L^2(\Omega)} \le C(\|f\|_{H^{-1}(\Omega)} + \|f\|_{L^\infty(\Omega)})|\omega_\varepsilon|^{\frac{1}{2}+\eta} \qquad (21)$$

for some $\eta > 0$.

Proof of Lemma 2.4. Subtracting (14) from (13) we get

$$\int_\Omega (u_\varepsilon - u)\phi + \alpha v_\varepsilon \nabla(u_\varepsilon - u) \cdot \nabla\phi \, dx = \alpha(1-\kappa)\int_{\omega_\varepsilon} \nabla u \cdot \nabla\phi \, dx \quad \forall\phi \in H^1(\Omega).$$

Setting $\phi = u_\varepsilon - u$, inserting it in the last equality and applying Schwarz' inequality, we get

$$\alpha\kappa\|u_\varepsilon - u\|^2_{H^1(\Omega)} \le \alpha(1-\kappa)\|\nabla u\|_{L^2(\omega_\varepsilon)}\|u_\varepsilon - u\|_{H^1(\Omega)}.$$

Hence

$$\|u_\varepsilon - u\|_{H^1(\Omega)} \le \frac{1-\kappa}{\kappa}\|\nabla u\|_{L^2(\omega_\varepsilon)}$$

and by Schwarz' inequality and the regularity estimates proved in Lemma 2.3 for u we derive

$$\|u_\varepsilon - u\|_{H^1(\Omega)} \le C(\|f\|_{H^{-1}(\Omega)} + \|f\|_{L^\infty(\Omega)})|\omega_\varepsilon|^{\frac{1}{2}}.$$

To prove (21) we subtract (13) from (14) getting

$$\int_\Omega (u - u_\varepsilon)w + \alpha v\nabla(u - u_\varepsilon) \cdot \nabla w \, dx = \alpha(\kappa-1)\int_{\omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla w \, dx \quad \forall w \in H^1(\Omega). \quad (22)$$

Let $w \in H^1(\Omega)$ be the solution to

$$\begin{cases} w - \alpha\operatorname{div}(v\nabla w) = u - u_\varepsilon & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \qquad (23)$$

Since $w - \alpha\Delta w = u - u_\varepsilon$ in $\hat{L}_0$, by interior regularity results (cf. [11, Thm. 8.8]) we have

$$\|w\|_{H^2(\hat{L}_0)} \le C(\|u - u_\varepsilon\|_{L^2(\Omega)} + \|w\|_{H^1(\Omega)}).$$

Moreover since

$$\|w\|_{H^1(\Omega)} \le C\|u - u_\varepsilon\|_{L^2(\Omega)}$$

we have that

$$\|w\|_{H^2(\hat{L}_0)} \le C\|u - u_\varepsilon\|_{L^2(\Omega)}.$$

By the Sobolev imbedding theorem, the last inequality implies that $\nabla w \in L^p(\hat{L}_0)$ for any $p \in (1, +\infty)$ and

$$\|\nabla w\|_{L^p(\hat{L}_0)} \le C\|u - u_\varepsilon\|_{L^2(\Omega)}.$$

Let us now choose $q \in (1, 2)$ and p so that $\frac{1}{p} + \frac{1}{q} = 1$. Then, combining the variational formulation of the problem (23) with (22) and applying Hölder's inequality we get

$$\begin{aligned} \int_\Omega (u - u_\varepsilon)^2 \, dx &= \alpha(\kappa - 1)\int_{\omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla w \, dx \\ &\le C\|\nabla u_\varepsilon\|_{L^q(\omega_\varepsilon)}\|\nabla w\|_{L^p(\omega_\varepsilon)} \\ &\le C\|\nabla u_\varepsilon\|_{L^q(\omega_\varepsilon)}\|u - u_\varepsilon\|_{L^2(\Omega)} \end{aligned} \qquad (24)$$

and since $\frac{1}{q} > \frac{1}{2}$ the claim follows.

We recall here a general, geometry independent, result due to Capdeboscq & Vogelius (cf. [6, 7]). Let us indicate with $V^j := x_j - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} x_j \, d\sigma$, $j = 1, 2$, the so called corrector terms. Let

$$\gamma_\varepsilon(x) = \begin{cases} \kappa & x \in \omega_\varepsilon, \\ 1 & x \in \Omega \backslash \overline{\omega_\varepsilon}, \end{cases} \tag{25}$$

and let V_ε^j, $j = 1, 2$, be the solutions to

$$\begin{cases} \operatorname{div}(\gamma_\varepsilon \nabla V_\varepsilon^j) & = & 0 & \text{in } \Omega, \\ \dfrac{\partial V_\epsilon^j}{\partial \nu} & = & \nu_j & \text{on } \partial\Omega, \\ \int_{\partial\Omega} V_\epsilon^j \, d\sigma & = & 0. \end{cases} \tag{26}$$

Proceeding with similar arguments as in Lemma 2.4 one easily sees that there exists a constant $C = C(\kappa, \delta_0)$ such that

$$\|V_\varepsilon^j - V^j\|_{H^1(\Omega)} \leq C|\omega_\varepsilon|^{\frac{1}{2}} \tag{27}$$

and

$$\|V_\varepsilon^j - V^j\|_{L^2(\Omega)} \leq C|\omega_\varepsilon|^{\frac{1}{2}+\eta} \tag{28}$$

for $j = 1, 2$ and for some $\eta > 0$. Observe now that, as $\varepsilon \to 0$,

$$|\omega_\varepsilon|^{-1} 1_{\omega_\varepsilon}(\cdot) \text{ converges in the sense of measure to } \mu \tag{29}$$

and the Borel measure μ is concentrated on L_0. In fact, due to the form of the set ω_ε, it is immediate to see that $\mu = \delta_y$, where δ_y denotes the Dirac measure concentrated at y. Using (27) and from the analysis in [6] it follows that, possibly up to the extraction of a subsequence,

$$|\omega_\varepsilon|^{-1} 1_{\omega_\varepsilon} \frac{\partial V_\varepsilon^j}{\partial x_i}(\cdot) \text{ converges in the sense of measure to } \mathcal{M}_{ij} \text{ when } \varepsilon \to 0, \tag{30}$$

where $\mathcal{M}_{ij}$ is a Borel measure with support in L_0. Again, the fact that the set ω_ε shrinks to the point y implies that the measure $\mathcal{M}_{ij}$ is simply a multiple of δ_y. Hence, identifying $\mathcal{M}_{ij}$ with $\mathcal{M}_{ij}\delta_y$, we have

$$\mathcal{M}_{ij}\phi(y) = \lim_{\varepsilon\to 0} \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \frac{\partial V_\varepsilon^j}{\partial x_i}(x)\phi(x) \, dx \tag{31}$$

for every smooth function ϕ. Following [6] it is possible to show the following result:

Lemma 2.5. *Let $\varepsilon \to 0$ be such that (29) and (30) hold. Then*

$$\lim_{\varepsilon\to 0} |\omega_\varepsilon|^{-1} \int_\Omega 1_{\omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_j}\phi \, dx = \frac{\partial u}{\partial x_i}(y)\mathcal{M}_{ij}\phi(y) \quad \forall\phi \in C_0^1(\hat{L}_0) \tag{32}$$

for any $j = 1, 2$.

Proof of Lemma 2.5. From the energy estimates we have that, possibly extracting a subsequence that we do not relabel,

$$|\omega_\varepsilon|^{-1} 1_{\omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_j}(\cdot) \text{ converges in the sense of measure to } \bar{\nu}_j, \tag{33}$$

that is,

$$\lim_{\varepsilon\to 0} |\omega_\varepsilon|^{-1} \int_\Omega \frac{\partial u_\varepsilon}{\partial x_j}\phi \, dx = \int_\Omega \phi \, d\bar{\nu}_j$$

for all continuous function ϕ in Ω. In order to prove (32) we will prove the following relation for $\varepsilon \to 0$

$$|\omega_\varepsilon|^{-1} \int_{\omega_\varepsilon} \nabla u \cdot \nabla V_\varepsilon^j \phi \, dx = |\omega_\varepsilon|^{-1} \int_{\omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla V^j \phi \, dx + o(1), \quad \forall \phi \in C_0^1(\hat{L}_0). \quad (34)$$

Once (34) is proved, passing to the limit as $\varepsilon \to 0$, we get

$$\int_\Omega \phi \, d\bar{\nu}_j = \frac{\partial u}{\partial x_i}(y) \mathcal{M}_{ij} \phi(y) \quad \forall \phi \in C_0^1(\hat{L}_0),$$

from which (32) follows.

Hence let us prove (34). Let us notice that, if $\phi \in C_0^1(\hat{L}_0)$, then $\phi v_\varepsilon = \phi \gamma_\varepsilon$ and $\phi v = \phi \gamma_0 = \phi$, and we have

$$\int_\Omega (u_\varepsilon - f)\phi + \alpha \gamma_\varepsilon \nabla u_\varepsilon \cdot \nabla \phi \, dx = \int_\Omega (u - f)\phi + \alpha \nabla u \cdot \nabla \phi \, dx \quad (35)$$

and

$$\int_\Omega \gamma_\varepsilon \nabla V_\varepsilon^j \cdot \nabla \phi \, dx = \int_\Omega \nabla V^j \cdot \nabla \phi \, dx. \quad (36)$$

Using (35) and (36) and after some algebraic manipulations we get

$$\alpha \int_\Omega (1 - \gamma_\varepsilon) \nabla u \cdot \nabla V_\varepsilon^j \phi \, dx - \alpha \int_\Omega (1 - \gamma_\varepsilon) \nabla u_\varepsilon \cdot \nabla V^j \phi \, dx$$

$$= \alpha \int_\Omega \nabla u \cdot \nabla (V_\varepsilon^j \phi) - \gamma_\varepsilon \nabla V_\varepsilon^j \cdot \nabla (u\phi) \, dx - \alpha \int_\Omega (\nabla V^j \cdot \nabla(u_\varepsilon \phi) - \gamma_\varepsilon \nabla u_\varepsilon \cdot \nabla(V^j \phi)) \, dx$$

$$\quad - \alpha \int_\Omega (\nabla u \cdot V_\varepsilon^j \nabla \phi - \gamma_\varepsilon \nabla V_\varepsilon^j \cdot u \nabla \phi) \, dx + \alpha \int_\Omega (\nabla V^j \cdot u_\varepsilon \nabla \phi - \gamma_\varepsilon \nabla u_\varepsilon \cdot V^j \nabla \phi) \, dx$$

$$= \alpha \int_\Omega (\gamma_\varepsilon \nabla u_\varepsilon \cdot \nabla(V_\varepsilon^j \phi) - \nabla V^j \cdot \nabla(u\phi)) \, dx + \int_\Omega (u_\varepsilon - u) V_\varepsilon^j \phi \, dx$$

$$\quad - \alpha \int_\Omega (\gamma_\varepsilon \nabla V_\varepsilon^j \cdot \nabla(u_\varepsilon \phi) - \nabla u \cdot \nabla(V^j \phi)) \, dx - \int_\Omega (u_\varepsilon - u) V^j \phi \, dx$$

$$\quad - \alpha \int_\Omega (\nabla u \cdot V_\varepsilon^j \nabla \phi - \gamma_\varepsilon \nabla V_\varepsilon^j \cdot u \nabla \phi) \, dx + \alpha \int_\Omega (\nabla V^j \cdot u_\varepsilon \nabla \phi - \gamma_\varepsilon \nabla u_\varepsilon \cdot V^j \nabla \phi) \, dx$$

$$= \int_\Omega (u_\varepsilon - u)(V_\varepsilon^j - V^j)\phi \, dx + \alpha \int_\Omega (\gamma_\varepsilon \nabla u_\varepsilon \cdot V_\varepsilon^j \nabla \phi - \nabla V^j \cdot u \nabla \phi) \, dx$$

$$\quad - \alpha \int_\Omega (\gamma_\varepsilon \nabla V_\varepsilon^j \cdot u_\varepsilon \nabla \phi - \nabla u V^j \cdot \nabla \phi) \, dx - \alpha \int_\Omega (\nabla u \cdot V_\varepsilon^j \nabla \phi - \gamma_\varepsilon \nabla V_\varepsilon^j \cdot u \nabla \phi) \, dx$$

$$\quad + \alpha \int_\Omega (\nabla V^j \cdot u_\varepsilon \nabla \phi - \gamma_\varepsilon \nabla u_\varepsilon \cdot V^j \nabla \phi) \, dx$$

$$= \int_\Omega (u_\varepsilon - u)(V_\varepsilon^j - V^j)\phi \, dx + \alpha \int_\Omega \nabla V^j \cdot (u_\varepsilon - u)\nabla \phi \, dx - \alpha \int_\Omega \nabla u \cdot (V_\varepsilon^j - V^j)\nabla \phi \, dx$$

$$\quad - \alpha \int_\Omega \gamma_\varepsilon \nabla V_\varepsilon^j \cdot (u_\varepsilon - u)\nabla \phi \, dx + \alpha \int_\Omega \gamma_\varepsilon \nabla u_\varepsilon \cdot (V_\varepsilon^j - V^j)\nabla \phi \, dx$$

$$= \int_\Omega (u_\varepsilon - u)(V_\varepsilon^j - V^j)\phi \, dx + \alpha \int_\Omega \gamma_\varepsilon \nabla(u_\varepsilon - u) \cdot (V_\varepsilon^j - V^j)\nabla \phi \, dx$$

$$\quad - \alpha \int_\Omega \gamma_\varepsilon \nabla(V_\varepsilon^j - V^j) \cdot (u_\varepsilon - u)\nabla \phi \, dx + \alpha \int_{\omega_\varepsilon} (\kappa - 1)\nabla u \cdot (V_\varepsilon^j - V^j)\nabla \phi \, dx$$

$$\quad - \alpha \int_{\omega_\varepsilon} (\kappa - 1)\nabla V^j \cdot (u_\varepsilon - u)\nabla \phi \, dx.$$

Now, by Lemma 2.4, (27), (28), Schwarz inequality and using finally the regularity of u and of V^j we get (34) and the claim follows.

We now state several properties of the polarization tensor $\mathcal{M}$ established in [6, 7] that we will use in the sequel. From the definition of the tensor $\mathcal{M}$ given in [4], it is easy to see that it is symmetric and satisfies

$$|\xi|^2 \leq \mathcal{M}\xi \cdot \xi \leq \frac{1}{\kappa}|\xi|^2 \tag{37}$$

for any $\xi \in \mathbb{R}^2$. Moreover, from [7, Theorem 3] we have that

$$\operatorname{tr} \mathcal{M} \leq 1 + \frac{1}{\kappa}, \tag{38}$$

$$\operatorname{tr} \mathcal{M}^{-1} \leq 1 + \kappa. \tag{39}$$

Furthermore, in the case of constant coefficients by insertion of $\phi = \xi_i \xi_j$ in (31) we get

$$\mathcal{M}_{ij}\xi_i\xi_j = |\omega_\varepsilon|^{-1} \int_{\omega_\varepsilon} \nabla V_\varepsilon \cdot \xi \, dx + o(1) = |\omega_\varepsilon|^{-1} \int_{\omega_\varepsilon} \nabla V_\varepsilon \cdot \nabla V \, dx + o(1), \tag{40}$$

where $V_\varepsilon = V_\varepsilon^i \xi_i$ and $V = V^i \xi_i$. Hence, we can write

$$\mathcal{M}\xi \cdot \xi = |\xi|^2 + |\omega_\varepsilon|^{-1} \int_{\omega_\varepsilon} \nabla W_\varepsilon \cdot \xi \, dx + o(1), \tag{41}$$

where $W_\varepsilon = V_\varepsilon - V$ is the solution to

$$\begin{cases} \operatorname{div}(\gamma_\varepsilon \nabla W_\varepsilon) = \operatorname{div}((1 - \kappa)1_{\omega_\varepsilon}\xi) & \text{in } \Omega, \\ \gamma_\varepsilon \dfrac{\partial W_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{42}$$

We are now ready to prove the following result:

Proposition 1. *We have*

$$\mathcal{M}\tau \cdot \tau = 1, \qquad \mathcal{M}n \cdot n = \frac{1}{\kappa}. \tag{43}$$

Proof of Proposition 1. Without loss of generality we may assume that $\tau = e_1 = (1, 0)$ and $n = e_2 = (0, 1)$ are the standard basis vectors in $\mathbb{R}^2$.

Let us set $\xi = \tau = e_1$ and denote by W_ε^1 the corresponding solution of (42). We will first show that

$$|\omega_\varepsilon|^{-1} \int_{\omega_\varepsilon} \nabla W_\varepsilon^1 \cdot e_1 \, dx = o(1). \tag{44}$$

Let $\omega_\varepsilon' = \{x + \rho e_2 : x \in \sigma_\varepsilon, \, -\varepsilon^2 \leq \rho \leq \varepsilon^2\}$ and let us write

$$\int_{\omega_\varepsilon} \nabla W_\varepsilon^1 \cdot e_1 \, dx = \int_{\omega_\varepsilon'} \nabla W_\varepsilon^1 \cdot e_1 \, dx + \int_{\omega_\varepsilon \setminus \omega_\varepsilon'} \nabla W_\varepsilon^1 \cdot e_1 \, dx := I_1 + I_2.$$

Observe that

$$|I_2| \leq \|\nabla W_\varepsilon^1\|_{L^2(\Omega)} |\omega_\varepsilon \setminus \omega_\varepsilon'|^{1/2},$$

and by the energy estimates

$$|I_2| \leq |\omega_\varepsilon|^{1/2} |\omega_\varepsilon \setminus \omega_\varepsilon'|^{1/2} = o(|\omega_\varepsilon|).$$

Let us now estimate I_1

$$|I_1| = \left| \int_{-\varepsilon^2}^{\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} \frac{\partial W_\varepsilon^1}{\partial x_1} \, dx_1 dx_2 \right| = \left| \int_{-\varepsilon^2}^{\varepsilon^2} W_\varepsilon^1 |_{-\varepsilon}^{\varepsilon} \, dx_2 \right| .$$

Observe now that by standard regularity results

$$\|W_\varepsilon^1\|_{L^\infty(\omega_\varepsilon)} \leq C(\|W_\varepsilon^1\|_{H^1(\Omega)} + \|1_{\omega_\varepsilon} e_1\|_{L^q(\omega_\varepsilon)})$$

for $q > 2$. Hence

$$\|W_\varepsilon^1\|_{L^\infty(\omega_\varepsilon)} \leq C|\omega_\varepsilon|^{1/q}$$

and if $q \in (2,3)$ we get

$$|I_1| = o(|\omega_\varepsilon|).$$

Summarizing

$$\left| \int_{\omega_\varepsilon} \nabla W_\varepsilon^1 \cdot e_1 \, dx \right| \leq |I_1| + |I_2| = o(|\omega_\varepsilon|),$$

which proves (44). Finally, inserting (44) in (41) and letting $\varepsilon \to 0$,

$$\mathcal{M} e_1 \cdot e_1 = 1.$$

Recalling (38) and (39) we get that

$$\mathcal{M} e_2 \cdot e_2 = \frac{1}{\kappa}.$$

We are now ready to prove our main result:

Proof of Theorem 2.1. Let $\omega_\varepsilon{}'$ be defined as in the proof of Proposition 1. Then, by Lemma 2.2, we can write

$$\mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v) = \frac{\alpha(\kappa - 1)}{2} \int_{\omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla u \, dx$$

$$= \frac{\alpha(\kappa - 1)}{2} \int_{\omega_\varepsilon{}'} \nabla u_\varepsilon \cdot \nabla u \, dx + \frac{\alpha(\kappa - 1)}{2} \int_{\omega_\varepsilon \backslash \omega_\varepsilon{}'} \nabla u_\varepsilon \cdot \nabla u \, dx .$$

Observe now that

$$\int_{\omega_\varepsilon \backslash \omega_\varepsilon{}'} \nabla u_\varepsilon \cdot \nabla u \, dx = \int_{\omega_\varepsilon \backslash \omega_\varepsilon{}'} \nabla(u_\varepsilon - u) \cdot \nabla u \, dx + \int_{\omega_\varepsilon \backslash \omega_\varepsilon{}'} |\nabla u|^2 \, dx .$$

Using Schwarz inequality, the regularity estimates of Lemma 2.3 and Lemma 2.4 we get

$$\left| \int_{\omega_\varepsilon \backslash \omega_\varepsilon{}'} \nabla u_\varepsilon \cdot \nabla u \, dx \right| \leq C(\|u_\varepsilon - u\|_{H^1(\Omega)} |\omega_\varepsilon \backslash \omega_\varepsilon{}'|^{1/2} + |\omega_\varepsilon \backslash \omega_\varepsilon{}'|) \leq C|\omega_\varepsilon \backslash \omega_\varepsilon{}'| = o(|\omega_\varepsilon|) .$$

Hence

$$\mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v) = \frac{\alpha(\kappa - 1)}{2} \int_{\omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla u \, dx = \frac{\alpha(\kappa - 1)}{2} \int_{\omega_\varepsilon{}'} \nabla u_\varepsilon \cdot \nabla u \, dx + o(|\omega_\varepsilon|).$$

Let us choose some vector function $\Phi \in C_0^0(\Omega; \mathbb{R}^2)$ such that

$$\Phi(x) = \begin{cases} \nabla u & x \in L_0, \\ 0 & x \in \Omega \backslash \hat{L}_0. \end{cases} \tag{45}$$

Then, from Lemma 2.5 we get

$$|\omega_\varepsilon'|^{-1} \int_{\omega_\varepsilon'} \nabla u_\varepsilon \cdot \nabla u \, dx \to \mathcal{M} \nabla u(y) \cdot \nabla u(y)$$

as $\varepsilon \to 0$. Observing that $|\omega_\varepsilon'| = 4\varepsilon^3$ we get

$$\mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v) = \frac{\alpha(\kappa - 1)}{2} 4\varepsilon^3 \mathcal{M} \nabla u(y) \cdot \nabla u(y) + o(\varepsilon^3) \,.$$

Finally, observing that the remainder term is uniformly bounded with respect to $y \in L_0$, i.e., $|o(\varepsilon^3)| \le C\varepsilon^{3+\eta}$ for some $\eta > 0$ and C and η depend only on $\kappa, \delta_0, \|f\|_{H^{-1}(\Omega)}, \|f\|_{L^\infty(\Omega)}$ and ∇u is continuous on the compact set L_0 the claim follows.

Discussion. We note that information about position and orientation of edges in an image can also be obtained by considering an asymptotic expansion with respect to a small elliptic inhomogeneity instead of a thin stripe. More precisely, let $B_\varepsilon(y)$ be an ellipse centred in y with semi-axes of length εa and εb aligned with the coordinate system. Furthermore, assume that $a > b$. Proceeding in a similar way as in [12, Section 2], we obtain an asymptotic expansion of the form

$$\mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v) = \varepsilon^2 \frac{1}{2} \alpha(1 - \kappa) \mathcal{M}_B \nabla u(y) \cdot \nabla u(y) + O(\varepsilon^{5/2}) \,, \qquad (46)$$

where the matrix $\mathcal{M}_B$ is given by

$$\mathcal{M}_B := \left[I|B| + \left(\frac{1}{\kappa} - 1 \right) \int_{\partial B} \phi(x) n(x)^T ds \right] \,. \qquad (47)$$

In the above formula, ∂B denotes the boundary of the ellipse B centred in the origin with semi-axes equal to a and b, and the vector function ϕ is the unique solution of the problem

$$\begin{cases} \Delta \phi(x) = 0 & x \in \mathbb{R}^2 \setminus \bar{B} \text{ or } x \in B \,, \\ \phi^+(x) = \phi^-(x) & x \in \partial B \,, \\ \dfrac{\partial \phi^+}{\partial n}(x) - \kappa \dfrac{\partial \phi^-}{\partial n}(x) = -\kappa n(x) & x \in \partial B \,, \\ \lim_{|x| \to \infty} \phi(x) = 0 \,. \end{cases} \qquad (48)$$

Here n is the exterior unit normal vector to the boundary ∂B. Deriving an explicit formula for ϕ and evaluating the integral in (47) we obtain that

$$\mathcal{M}_B = \pi ab \begin{bmatrix} \dfrac{a+b}{a+b\kappa} & 0 \\ 0 & \dfrac{a+b}{a\kappa+b} \end{bmatrix} \,.$$

We remark that a similar asymptotic expansion has been derived by Amstutz [2] in the context of not self-adjoint problems (see [2], the expansion (6.1) with the formula (6.4)). In particular, for a rotated ellipse B with the semi-major axis of length a oriented in the direction of vector τ, the formula for the polarization tensor is

$$\mathcal{M}_B = \pi ab \left[\frac{a+b}{a\kappa+b} n \otimes n + \frac{a+b}{a+b\kappa} \tau \otimes \tau \right] \,,$$

where $n = \tau^\perp$. Using simple analysis, we get that, for fixed $y \in \Omega$, the right hand side in (46) is minimal for τ equal to the unit vector perpendicular to $\nabla u(y)$, which provides a criterion for finding the orientation of edges in an image. We also observe that taking now $a = 1$ and $b = \varepsilon$ and next passing formally to the limit as $\varepsilon \to 0$ we get

$$\lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^3} \left(\mathcal{J}(u_\varepsilon, v_\varepsilon) - \mathcal{J}(u, v) \right) = \frac{\alpha(\kappa - 1)}{2} \left[\frac{1}{\kappa} n \otimes n + \tau \otimes \tau \right] \nabla u(y) \cdot \nabla u(y),$$

which is the same result as in Theorem 2.1. However, we remark that we could not follow directly the analysis in [2] and [12] assuming from the beginning that the inhomogeneity is an ellipse B_ε with semi-axes equal to ε and ε^2 because of the impossibility of uniform scaling. On the other hand, we note the general approach of Capdeboscq & Vogelius [6, 7], which we use in this paper, allows to overcome this difficulty. We expect that the asymptotic expansion in Theorem 2.1 is also true in this case and can be rigorously obtained by a slight modification of the proof of Proposition 1.

We also remark that another way for obtaining information about position and orientation of edges in an image is the consideration of an asymptotic expansion with respect to the perturbation of a domain by a small crack. In the paper of Belaid et al. [3], the authors introduced the idea of constructing an algorithm for image segmentation based on such an asymptotic expansion but for not self–adjoint problems, which has been derived by Amstutz et al. [1]. It is interesting to consider this issue in the context of approximation of the Mumford–Shah functional, and this will be the subject of the forthcoming paper.

3. **Numerical Implementation.** We now propose two variants of an algorithm to edges detection that is based on the expansion given in Theorem 2.1, which states that

$$\mathcal{J}_\varepsilon(u_\varepsilon, v_{K \cup \omega_\varepsilon}) - \mathcal{J}_\varepsilon(u, v_K) \approx 2\beta\varepsilon - 2\alpha(1 - \kappa)\varepsilon^3 \mathcal{M} \nabla u(y) \cdot \nabla u(y) \qquad (49)$$

with $\mathcal{M} = \frac{1}{\kappa} n \otimes n + \tau \otimes \tau$. For fixed $y \in \Omega$, the right hand side in (49) is minimal for τ equal to the unit vector perpendicular to $\nabla u(y)$ in which case

$$\mathcal{M} \nabla u(y) \cdot \nabla u(y) = \frac{1}{\kappa} |\nabla u(y)|^2$$

and

$$\mathcal{J}_\varepsilon(u_\varepsilon, v_{K \cup \omega_\varepsilon}) - \mathcal{J}_\varepsilon(u, v_K) \approx 2\beta\varepsilon - 2\alpha\varepsilon^3 \frac{1 - \kappa}{\kappa} |\nabla u(y)|^2.$$

As a consequence, we can expect a decrease of the function $\mathcal{J}_\varepsilon$ in case

$$|\nabla u(y)|^2 \geq \frac{\beta\kappa}{\alpha\varepsilon^2(1 - \kappa)}$$

and the decrease is maximal at points y where the gradient of u is maximal.

1. Our first algorithm computes a smoothed version u_s of the input image f
   a-priori, and then finds, using only the smoothed image u_s, a sequence of edge indicators $K^{(k)}$, where $K^{(k+1)}$ is formed from $K^{(k)}$ by the addition of a strip $\sigma_\varepsilon(x^{(k)}, \tau^{(k)})$ for which the expected decrease in the approximated functional $\mathcal{J}_\varepsilon$ from (5) is maximal. Theorem 2.1 indicates that, as long as we only add strips that are away from $K^{(k)}$, this is the case if $x^{(k)}$ is chosen such that
   b-$|\nabla u(x)|$ is maximal and $\tau^{(k)} = (\nabla u(x^{(k)}))^\perp$. However, because the asymptotic expansion of Theorem 2.1 is only valid away from $K^{(k)}$, we have to restrict the search for a maximum of $|\nabla u|$ to some set $L^{(k)}$ which is compactly

contained in $\Omega \setminus K^{(k)}$. For instance, one can set $L^{(k)} := \Omega \setminus (\partial\Omega \cup K^{(k)} + B_\delta)$ for some $\delta > 0$; a different construction, which we have used in the numerical examples, is described below. The iteration is stopped when the expected decrease of the gradient term in the functional is compensated by the increase in the edge term. This is the case when $|\nabla u(x^{(k)})|^2 < \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}$. This method is summarized in Algorithm 1.

Data: input image $f \colon \Omega \to \mathbb{R}$, parameters α, $\beta > 0$, $\varepsilon > 0$, $0 < \kappa < 1$;
Result: edge indicator set K;

Initialization: set $K = \emptyset$ and $L = \Omega \setminus (\partial\Omega + B_\delta)$

compute the solution u of

$$\begin{cases} u - \alpha \operatorname{div}(\nabla u) = f & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

repeat
    find $x^* \in L$ with $|\nabla u(x^*)|$ maximal;
    compute a strip σ_ε of size ε centered at x^* with normal $\nabla u(x^*)$;
    set $K \leftarrow K \cup \sigma_\varepsilon$;
    compute an enlargement S of σ_ε;
    set $L \leftarrow L \setminus S$;
until $|\nabla u(x^*)|^2 < \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}$;

Algorithm 1: Implementation without updates of the smoothed function

---

    In fact this algorithm is an anisotropic edge detector, which take into account edge magnitudes and local edge orientations.

2. In our second algorithm, we combine updates of the edge indicator with updates of the smoothed function u: After adding a fixed number $n_{\max}$ of strips to the edge set K, we define the new diffusivity v by

$$v(x) := \begin{cases} \kappa & \text{if } x \in K, \\ 1 & \text{if } x \notin K, \end{cases}$$

and then compute a corresponding smoothed function u, which is then used for selecting the next at most $n_{\max}$ strips in the edge set. The process of alternating between the addition of strips and updates of the smoothed function u is repeated until no more admissible points $x \in L$ exist for which $|\nabla u(x)|^2 \geq \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}$.

    The rationale behind this idea is the fact that the expansion derived above, though still valid, becomes increasingly inaccurate as the number of added strips becomes larger. Therefore, at some point some reinitialization is necessary. Note, however, that the number $n_{\max}$ of strips that are added in each iteration mainly determines the computation time, as the computation of u is the most costly part of the algorithm. Thus the number $n_{\max}$ should not be chosen too small. In the numerical implementations, we chose $n_{\max}$ in such a way that approximately 10 computations of u were needed.

    The resulting method is described in Algorithm 2.

Data: input image $f\colon \Omega \to \mathbb{R}$, parameters α, $\beta > 0$, $\varepsilon > 0$, $0 < \kappa < 1$,
$\quad n_{\max} \in \mathbb{N}$;

Result: edge indicator function v, smoothed image u;

Initialization: set $v(x) = 1$ for $x \in \Omega$, $K = \emptyset$, and $L = \Omega \setminus (\partial\Omega + B_\delta)$;

compute the solution u of
$$\begin{cases} u - \alpha \operatorname{div}(\nabla u) = f & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega; \end{cases}$$

repeat
$\quad$ set $n = 1$;
$\quad$ **repeat**
$\quad\quad$ find $x^* \in L$ with $|\nabla u(x^*)|$ maximal;
$\quad\quad$ compute a strip σ_ε of size ε centered at x^* with normal $\nabla u(x^*)$;
$\quad\quad$ set $K \leftarrow K \cup \sigma_\varepsilon$;
$\quad\quad$ compute an enlargement S of σ_ε;
$\quad\quad$ set $L \leftarrow L \setminus S$;
$\quad\quad$ set $n \leftarrow n + 1$;
$\quad$ **until** $n > n_{\max}$ *or* $|\nabla u(x^*)|^2 < \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}$;
$\quad$ set $v(x) = \kappa$ for $x \in K$;
$\quad$ compute the solution u of
$$\begin{cases} u - \alpha \operatorname{div}(v\nabla u) = f & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega; \end{cases}$$

until $\max_{x^* \in L} |\nabla u(x^*)|^2 < \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}$;

$\qquad\qquad$ **Algorithm 2:** Implementation with updates of u

Solution of the PDE. For the numerical solution of the equation
$$\begin{cases} u - \alpha \operatorname{div}(v\nabla u) = f & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$
we have implemented a finite element method using bilinear ansatz functions on a rectangular grid for u and piecewise constant ansatz functions on the same grid for the diffusivity v. The solution of the resulting linear equation was computed with the CG method.

Update of the Edge Indicator. For updating the edge indicator set K (and the function v), we have to find maximizers of $|\nabla u|$. We restrict the search to midpoints of the rectangular elements E_k of the finite elements and evaluate the gradient on the elements analytically.

Assume now that the maximum of $|\nabla u|$ is attained at y^*. For the update of the set L we define the enlargement S of Σ_ε as a rectangle of side-lengths 2ε and 2δ for some $0 < \delta < \varepsilon$ around the center-line of the strip. That is, (see Figure 1)
$$S := \{y : \operatorname{dist}(y_k, \sigma_\varepsilon) \le \delta \text{ and } |(y - y_k) \cdot \nabla u(y_k)^\perp| \le \varepsilon |\nabla u(y_k)|\}$$
for some $\delta > 0$. In the numerical experiments below we have chosen $\varepsilon = 3h$ and $\delta = 2h$ with h being the pixel distance.
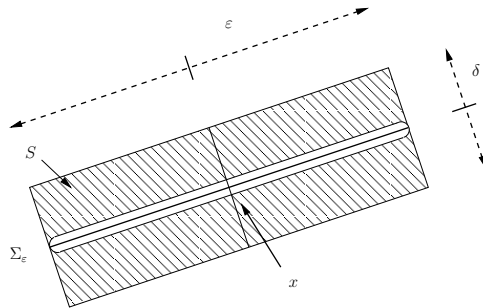
FIGURE 1. Sketch of the construction of the set σ_ε and the corresponding enlarged set S
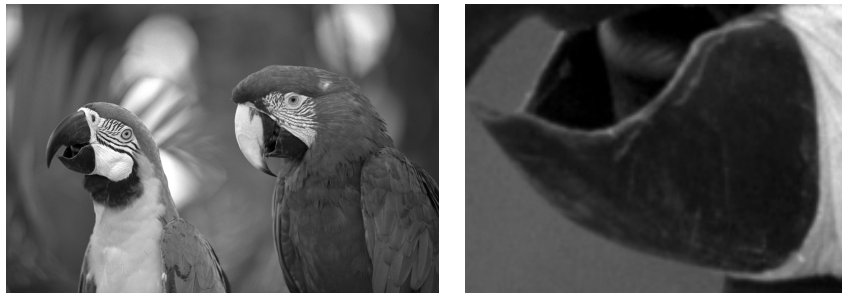

FIGURE 2. *Left:* Original Image. *Right:* Close up view of the lower part of the beak of the first parrot.

Numerical Experiments. We have tested the two algorithms proposed above using the *Parrots* image (see Figure 2). In all examples, the parameters were $\alpha = 8$, $\beta = 150$, and $\kappa = 0.05$. The length of the strips was set at 6 times the pixel distance and the value $n_{\max}$ was set to 1000 in Algorithm 2.

In addition, we provide a comparison with the results of the algorithm proposed in [12], where balls instead of strips are used for covering the edge set, and with the results obtained by the Canny edge detector. To perform tests with the latter one, we have used the Matlab function *edge* with the option *'canny'*, the threshold equal to 0.2 and default values for the remaining parameters involved in the algorithm. This parameter setting yields a segmentation of a similar detail as our algorithm. We note, however, that the Canny edge detector only yields a segmentation, while our algorithm at the same time segments and denoises the image.

While, generally speaking, the positions of the detected edges do approximately agree for the different algorithms, the actual form of the edges may significantly differ. Thus the algorithm of [12] results in thick edges, which do not appear in the results from the strip based methods and in the results obtained by the Canny edge detector. This difference is due to the fact that the directional information present in the strips allows the exclusion of laterally neighboring pixels from further considerations. In contrast, the balls that are used for edge covering in [12] do not allow a similar exclusion of pixels. In the case of the Canny algorithm, thin edges
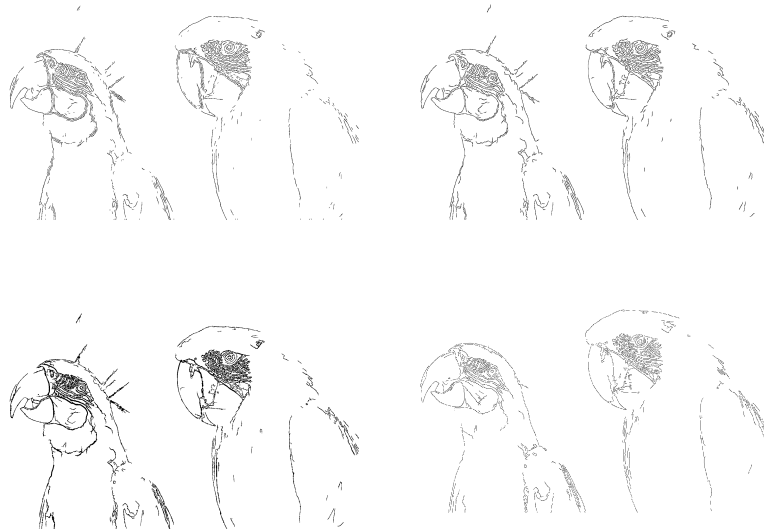
FIGURE 3.   *Upper left:* Result with Algorithm 1.   *Upper right:*
Result with Algorithm 2.   *Lower left:* Result using Algorithm
from [12]. *Lower right:* Result with the Canny edge detector.

FIGURE 4.   Close up view on a detail of Figure 2. *Upper left:*
Result with Algorithm 1. *Upper right:* Result with Algorithm 2.
Lower left: Result using Algorithm from [12] *Lower right:* Result
with the Canny edge detector. Note in particular the thick edges
in the third image.
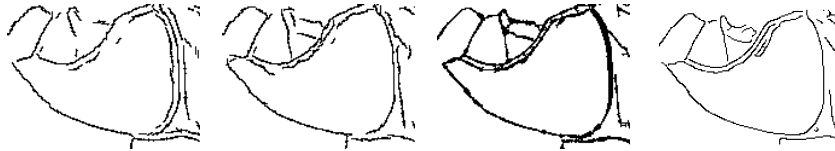
FIGURE 5. Close up view of the lower part of the beak of the first parrot in Figure 2. *First:* Result with Algorithm 1 *Second:* Result with Algorithm 2. *Third:* Result using Algorithm from [12] *Fourth:* Result with the Canny edge detector. One clearly sees the spurious edges in the first segmentation and the thick edge in the third segmentation, which is partly resolved in the second one.

are obtained thanks to application of the non-max suppression as an intermediate step.

Concerning computation times, Algorithm 1 is clearly faster than Algorithm 2, as the main computational effort of the methods lies in the solution of the PDE, which has to be computed several times in the case of Algorithm 2. We do note, however, that Algorithm 1 introduces artifacts in the form of parallel edges, which can be clearly seen in the close up view of the parrot's head in Figure 4 and 5. These false edge detections can be attributed to the smearing out of edges occurring in the first solution of the PDE.

Appendix — Γ-convergence. In the following we show, similarly as in [12], that the functional $\mathcal{J}_\varepsilon$ defined in (5) Γ-converges as $\varepsilon \to 0$ and $\kappa \to 0$ to the Mumford-Shah functional $\mathcal{F} \colon L^2(\Omega) \times L^2(\Omega) \to [0, +\infty]$, defined by

$$
\mathcal{F}(u,v) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega (u-f)^2\, dx + \dfrac{\alpha}{2} \int_{\Omega \setminus S_u} |\nabla u|^2\, dx + \beta \mathcal{H}^1(S_u) & \text{if } v \equiv 1\,, \\ +\infty & \text{else.} \end{cases} \tag{50}
$$

Here S_u denotes the discontinuity set of the function u (see [5]).

We do stress that, in contrast to the rest of the paper, where κ was constant, it is necessary for obtaining any non-trivial Γ-convergence result that this parameter tends to zero much faster than the size ε of the covering strips. Thus the following theorem can be interpreted as saying that the minimizers of $\mathcal{J}_\varepsilon$ are close to minimizers of the Mumford–Shah function, if both parameters ε and κ are close to zero. The asymptotic expansion derived in Theorem 2.1, however, relies on κ being bounded away from zero; the constant in the $O(\varepsilon^3)$ expansion tends to $+\infty$ as $\kappa \to 0$.

Theorem. *Assume that $\kappa(\varepsilon) = o(\varepsilon^2)$ as $\varepsilon \to 0$. Then,*

$$\mathcal{F} = \Gamma\text{-}\lim_{\varepsilon \to 0} \mathcal{J}_\varepsilon \ .$$

Proof. In order to prove the Γ-convergence result, we have to show that

$$\Gamma\text{-}\limsup_{\varepsilon \to 0} \mathcal{J}_\varepsilon \leq \mathcal{F} \leq \Gamma\text{-}\liminf_{\varepsilon \to 0} \mathcal{J}_\varepsilon.$$

The proof of the lim inf-inequality is along the lines of [12]. Therefore we only prove the lim sup-inequality.

Following [12], we introduce the set $\mathcal{W}(\Omega)$ consisting of all functions $u \in \mathrm{SBV}(\Omega)$ for which the following hold:

1. $\mathcal{H}^1(\overline{S}_u \setminus S_u) = 0$.
2. The set $\overline{S}_u$ is the union of a finite number of almost disjoint line segments contained in Ω, that is, their pairwise intersections are either empty or contain a single point.
3. $u|_{\Omega \setminus \overline{S}_u} \in W^{1,\infty}(\Omega \setminus S_u)$.

This set has been shown to be dense in $\mathrm{SBV}(\Omega)$ in the sense that, for every $u \in \mathrm{SBV}$, there exists a sequence $(u_j)_{j \in \mathbb{N}} \in \mathcal{W}(\Omega)$ such that $\|u_j - u\|_{L^2} \to 0$ and $\mathcal{F}(u_j) \to \mathcal{F}(u)$ (see [8, 9]).

Now assume that $u \in \mathcal{W}(\Omega)$ and $\varepsilon > 0$. In the following, we will construct sequences $u^\varepsilon \to u$ and $v^\varepsilon \to 1$ such that

$$\mathcal{J}_\varepsilon(u^\varepsilon, v^\varepsilon) \to \mathcal{F}(u, 1).$$

Because of the aforementioned density of $\mathcal{W}(\Omega)$ and the fact that $\mathcal{F}(u, v) = +\infty$ for $v \neq 1$, this will prove the lim sup-part.

Because $u \in \mathcal{W}(\Omega)$, there exists a finite number k of almost disjoint line segments $[a_i, b_i] \subset \mathbb{R}^2$ such that $S_u = \bigcup_{i=1}^k [a_i, b_i]$. Moreover,

$$\mathcal{H}^1(S_u) = \mathcal{H}^1(\overline{S}_u) = \sum_{i=1}^k |b_i - a_i| \ .$$

Now choose a minimal number of points $y_j^{(i)} \in [a_i, b_i]$, $j = 1, \ldots, l_i$, in such a way that the union of the strips $\omega_\varepsilon\big(y_j^{(i)}, \frac{b_i - a_i}{|b_i - a_i|}\big)$ covers the set

$$K_i^\varepsilon := \{x \in \mathbb{R}^2 \ : \ \mathrm{dist}(x, [a_i, b_i]) < \varepsilon^2\} \ .$$

This can be achieved with at most $1 + \frac{|b_i - a_i|}{2\varepsilon}$ points. Define

$$S_\varepsilon := \bigcup_{i=1}^k \bigcup_{j=1}^{l_i} \omega_\varepsilon\big(y_j^{(i)}, \frac{b_i - a_i}{|b_i - a_i|}\big)$$

and let $v^\varepsilon := v_{S_\varepsilon}$. Noting that

$$\mathcal{L}^2(S_\varepsilon) \leq \varepsilon^2 \big((2 + 2k\varepsilon)\mathcal{H}^1(S_u) + k\pi\varepsilon^2\big),$$

we see that $v^\varepsilon \to 1$ as $\varepsilon \to 0$. Moreover

$$m_\varepsilon(v^\varepsilon) \leq \sum_{i=1}^k \Big(1 + \frac{|b_i - a_i|}{2\varepsilon}\Big) \leq k + \frac{\mathcal{H}^1(S_u)}{2\varepsilon} \ ,$$

showing that

$$\limsup_{\varepsilon \to 0} 2\beta\varepsilon m_\varepsilon(v^\varepsilon) \leq \beta\mathcal{H}^1(S_u) \ .$$

Define moreover
$$u^\varepsilon(x) := u(x) \min\Big(\frac{\mathrm{dist}(x, S_u)}{\varepsilon^2}, 1\Big) \,.$$
Then Lebesgue's theorem of dominated convergence implies that $u^\varepsilon \to u$ in $L^2(\Omega)$, and therefore
$$\int_\Omega (u^\varepsilon - f)^2 \, dx \to \int_\Omega (u - f)^2 \, dx \text{ as } \varepsilon \to 0 \,.$$
Moreover, $\nabla u^\varepsilon(x) = \nabla u(x)$ for $x \notin K^\varepsilon := \{x \in \mathbb{R}^2 \, : \, \mathrm{dist}(x, S_u) < \varepsilon^2\}$, and
$$|\nabla u^\varepsilon(x)| \le |\nabla u(x)| + \frac{\|u\|_{L^\infty}}{\varepsilon^2} \text{ for almost every } x \in K^\varepsilon \,.$$
This implies that
$$\int_\Omega v^\varepsilon |\nabla u^\varepsilon|^2 \, dx \le \int_{\Omega \setminus K^\varepsilon} |\nabla u|^2 \, dx + 2\kappa(\varepsilon) \int_{K^\varepsilon \setminus S_u} |\nabla u|^2 + \frac{\|u\|_{L^\infty}^2}{\varepsilon^4} \, dx \,.$$
Because
$$\mathcal{L}^2(K^\varepsilon) \le \varepsilon^2 \big(2\mathcal{H}^1(S_u) + k\pi\varepsilon^2\big)$$
and $\kappa(\varepsilon) = o(\varepsilon^2)$ as $\varepsilon \to 0$, this shows that
$$\limsup_{\varepsilon \to 0} \int_\Omega v^\varepsilon |\nabla u^\varepsilon|^2 \, dx \le \int_{\Omega \setminus S_u} |\nabla u^\varepsilon|^2 \, dx \,.$$
Together, these estimates imply that
$$\limsup_{\varepsilon \to 0} \mathcal{J}_\varepsilon(u^\varepsilon, v^\varepsilon) \le \mathcal{F}(u, 1) \,,$$
which, because of the density of $\mathcal{W}(\Omega)$, in turn proves that
$$\mathcal{F} \ge \Gamma\text{-}\limsup_{\varepsilon \to 0} \mathcal{J}_\varepsilon \,.$$

REFERENCES

[1] S. Amstutz, I. Horchani, M. Masmoudi. Crack detection by the topological gradient method. *Control and Cybernetics*, 35(1), 2005.

[2] S. Amstutz. Sensitivity analysis with respect to a local perturbation of the material property. *Asymptotic Analysis*, 49:87-108, 2006.

[3] L. Belaid, M. Jaoua, M. Masmoudi, L. Siala. Image restoration and edge detection by topological asymptotic expansion, *C. R. Acad. Sci. Paris, Ser.* I 342:313–318, 2006.

[4] E. Beretta, Y. Capdeboscq, F. de Gournay, and E. Francini. Thin cylindrical conductivity inclusions in a three-dimensional domain: a polarization tensor and unique determination from boundary data. *Inverse Probl.*, 25(6):065004, 22, 2009.

[5] A. Braides. *Approximation of free-discontinuity problems*, volume 1694 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.

[6] Y. Capdeboscq and M.S. Vogelius. A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction. *M2AN Math. Model. Numer. Anal.*, 37(1):159–173, 2003.

[7] Y. Capdeboscq and M.S. Vogelius. Pointwise polarization tensor bounds, and applications to voltage perturbations caused by thin inhomogeneities. *Asymptot. Anal.*, 50(3-4):175–204, 2006.

[8] G. Cortesani. Strong approximation of GSBV functions by piecewise smooth functions. *Ann. Univ. Ferrara Sez. VII (N.S.)*, 43:27–49 (1998), 1997.

[9] G. Cortesani and R. Toader. A density result in SBV with respect to non-isotropic energies. *Nonlinear Anal.*, 38(5, Ser. B: Real World Appl.):585–604, 1999.

[10] G. Dong, M. Grasmair, S. H. Kang, and O. Scherzer. Scale and edge detection with topological derivatives of the Mumford-Shah functional. In A. Kuijper, editor, *SSVM'13: Proceedings of the fourth International Conference on Scale Space and Variational Methods in Computer Vision*, volume 7893 of *Lecture Notes in Computer Science*, pages 404–415, Berlin, Heidelberg, 2013. Springer-Verlag.

[11] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Reprint of the 2nd ed. Springer, 2001.

[12] M. Grasmair, M. Muszkieta, and O. Scherzer. An approach to the minimization of the Mumford–Shah functional using Γ-convergence and topological asymptotic expansion. To appear in *Interfaces Free Bound.*, 2013.

[13] Y. M. Jung, S. H. Kang, and J. Shen. Multiphase image segmentation via Modica-Mortola phase transition. *SIAM J. Appl. Math.*, 67(5):1213–1232, 2007.

[14] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York, 1968.

[15] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.*, 42(5):577–685, 1989.

[16] M. S. Vogelius and D. Volkov. Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter. *M2AN Math. Model. Numer. Anal.*, 34(4):723–748, 2000.