European option pricing with transaction costs and stochastic volatility: an asymptotic analysis

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1. Introduction

The intrinsic limitations of the Black–Scholes model in describing real markets behaviour are very well known. Among the main assumptions underlying this model, the most relevant ones are probably constant volatility and no transaction costs. In this paper, we are going to consider the pricing problem of a European option in a model in which both proportional transaction costs are taken into account and the volatility is assumed to evolve according to a stochastic process of the Ornstein–Uhlenbeck type. To analyse this situation we shall follow a utility maximization procedure, following the seminal paper of Davis *et al.* (1993).

If one uses the following utility function $\mathcal{U}$:

$$\mathcal{U}(x) = 1 - \exp{(-\gamma x)},$$

where γ expresses the risk aversion of the investor, one gets, as the result of this analysis, a non-linear partial differential equation (PDE) for the expected value of the utility-maximized wealth held in the underlying asset of the option.

The pricing of a European option in the presence of small transaction costs was considered in Whalley & Wilmott (1997), where a correction term to the Black and Scholes pricing formula was

derived. This correction term was found to be $O(\varepsilon^{2/3})$, where ε denotes the asymptotic expansion param-eter. Moreover in Whalley & Wilmott (1997) it was found that the optimal hedging strategy consists in not transacting when the process driving the stock price is in a strip around the classical Black and Scholes *delta*—hedging formula, and in rebalancing the portfolio (selling or buying stocks) to keep the process inside the strip of no transaction. The width of the no transaction strip was found to be $O(\varepsilon^{2/3})$. In Whalley & Wilmott (1997), the volatility has been supposed to be constant.

A wide literature is available on the optimal consumption-investment problem for models with trans-action costs: we just mention two pioneer contributions on the subject, the paper by Davis & Norman (1990), and the paper by Cvitanic & Karatzas (1996). More recently, ⊘ksendal & Sulem (2002) inves-tigated the optimal consumption problem in a model including both fixed and proportional transac-tion costs, while Muthuraman & Kumar (2006) studied the optimal investment problem in a multi-dimensional setting. The problem of hedging contingent claims for models with transaction costs has been investigated in several papers: Albanese & Tompaidis (2008), Clewlow & Hodges (1997), Kallsen & Muhle-Karbe (2013) and Zakamouline (2006a,b).

The pricing of a European option with fast mean-reverting stochastic volatility was considered in a series of papers (see e.g. Fouque *et al.*, 2000, 2003, 2001, 2011; Fouque & Lorig, 2011). In the above-mentioned papers, the authors found the pricing formula whose leading order term is the classical Black and Scholes formula with averaged volatility. The correction term was order the square root of the characteristic time scale of the process driving the volatility. An optimal consumption-investment problem has been investigated in a paper by Bardi *et al.* (2010) where a rigorous asymptotic analysis is performed and where the solution is characterized in the limit of fast volatility dynamics.

In a recent paper, Mariani *et al.* (2012) proposed a numerical approximation scheme for European option prices in stochastic volatility models including transaction costs based on a finite-difference method. The stochastic volatility dynamics assumed there is a slight generalization of that proposed by Hull & White (1987), since they consider a drift coefficient which is a general (regular) deterministic function of both the time and the underlying asset price, while their diffusion coefficient is linear in the instantaneous volatility.

In the present paper, we propose a different approximation method for European option pricing in stochastic volatility models with transaction costs, based on an asymptotic analysis which follows the approach pioneered by Fouque *et al.* (2001). The model we consider for the stochastic volatility dynam-ics is of Ornstein–Uhlenbeck type. This model has been originally proposed by Stein & Stein (1991). We provide closed-formulas for the first terms appearing in the asymptotical expansion for European option prices in the limit of fast volatility and small transaction costs. In the present paper, we are not going to provide existence and uniqueness results for the stochastic optimal control problem arising in our pricing methodology, neither we shall discuss rigorously the questions related to asymptotic expansion convergence to the solution required.

The goal of the present investigation is to show how the approach pioneered by Davis *et al.* (1993) can be extended to stochastic volatility models. To this end we combine with a suitable scaling the asymptotic technique introduced by Whalley & Wilmott (1997) with that proposed by Fouque *et al.* (2001) in order to obtain quite explicit results. The existence and uniqueness of the solution for the singular control problem arising in American option pricing and in the same modelling framework considered in the present paper has been proved by Cosso & Sgarra (2013). We point out that a recent paper by Bichuch (2011) deals with contingent claim pricing in models with (proportional) transaction costs via an asymptotic analysis based on the pioneer work by Whalley & Wilmott (1997) by providing a rigorous derivation of the asymptotic expansion together with lower and upper bounds for the value function involved. In a previous paper, the same author presented a rigorous asymptotic analysis for

an optimal investment problem, in the case of a power utility function, in finite time for a model with proportional transaction costs (Bichuch, 2012). An approximate hedging strategy construction has been also proposed quite recently in a paper by Lepinette & Quoc (2012) for a local volatility model with proportional transaction costs.

The plan of the paper is the following: in Section 2 we introduce the model considered. In Section 3 we formulate the European option pricing problem as a stochastic control problem, and present the associated HJB equation. In Section 4, the asymptotic analysis is performed, by assuming small transaction costs and fast mean-reverting volatility. In Section 5, the price of the option is computed and in Section 6, the numerical results and the conclusions are provided. For the reader's convenience, in Appendix A the source term of the equation obtained through the asymptotic analysis at $O(\varepsilon)$ is calculated; in Appendix B the averages with respect to the Ornstein–Uhlenbeck invariant measure using the stochastic volatility proposed by Chesney & Scott (1989) are presented; finally, in Appendix C the derivatives, which appears in the obtained corrected pricing formula, with respect to the stock price of the classical Black and Scholes solution are recalled and collected.

2. A stochastic volatility model with transaction costs

We suppose to have the following multi-dimensional stochastic process:

$$dB = rB\,dt - (1+\lambda)S\,dL + (1-\mu)S\,dM, \tag{2.1}$$

$$dy = dL - dM, \tag{2.2}$$

$$dS = S(\alpha\,dt + f(z)\,dW), \tag{2.3}$$

$$dz = \xi(m-z)\,dt + \beta(\rho\,dW + \sqrt{1-\rho^2}\,dZ). \tag{2.4}$$

In the above equations B and S are the risk-free (the 'Bond') and the risky asset (the 'Stock'), respectively, r is the risk-free interest rate, α is the drift rate of the stock, λ and μ are the (proportional) cost of buying and selling a stock, f is the volatility function, that we shall suppose to depend on the stochastic variable z, which is sometimes called the volatility driving process. $L(t)$ and $M(t)$ are the cumulative number of shares bought or sold, respectively, up to time t. Both $L(t)$ and $M(t)$ are assumed to be right continuous with left-hand limits, non-negative and non-decreasing $\mathcal{F}_t$-adapted processes, where $\mathcal{F}_t$ denotes the filtration generated by (W_t, Z_t). Moreover, by convention, we shall assume $L(0) = M(0) = 0$. We keep the notations introduced in Davis *et al.* (1993) and Whalley & Wilmott (1997), where the reader can find a detailed justification for the transaction costs model just introduced. The set $\mathcal{T}$ of trading strategies $\pi(t)$, in the present setting, consists of all the two-dimensional, right-continuous, measurable processes $(B^\pi(t), y^\pi(t))$ which are the solution of (2.2) and for which some pair of right-continuous, measurable, $\mathcal{F}_t$-adapted, increasing processes $L(t)$, $M(t)$ exist, such that $(B^\pi(t), y^\pi(t), S(t)) \in \mathcal{E}_\Gamma \ \forall t \in [0,T]$, where $\mathcal{E}_\Gamma = (B, y, S) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ : B + c(y,S) > -\Gamma$, with Γ positive constant and $c(y,S)$ the liquidated cash value of the position y, i.e. the residual cash value when a long position $(y > 0)$ is sold or a short position $(y < 0)$ is closed. Conventionally, we assume $c(0, S) = 0$. In what follows we shall always suppose $f(z)$ to be a function bounded away from 0:

$$0 < m_1 \leqslant f(z) \leqslant m_2 < \infty, \quad \forall z.$$

The process followed by the stochastic variable z is a Ornstein–Uhlenbeck process with average m. The parameter ξ is the rate of mean-reversion volatility.

The Brownian motions W and Z are uncorrelated and ρ is the instantaneous correlation coefficient between the asset price and the volatility shocks. Usually one considers $\rho < 0$, i.e. the two processes are anti-correlated (e.g. when the prices go down the investors tend to be nervous and the volatility raises). For more details see Fouque *et al.* (2001) and Jonsson & Sircar (2002).

If we suppose to deal with trading strategies absolutely continuous with respect to time, we can write our stochastic dynamics in the following compact form:

$$L = \int_0^t l \, \mathrm{d}s, \quad M = \int_0^t m \, \mathrm{d}s.$$

Therefore the process we are dealing with can be written in the following form:

$$\mathrm{d}B = [rB - (1 + \lambda)Sl + (1 - \mu)Sm] \, \mathrm{d}t, \tag{2.5}$$

$$\mathrm{d}y = (l - m) \, \mathrm{d}t, \tag{2.6}$$

$$\mathrm{d}S = S(\alpha \, \mathrm{d}t + f(z) \, \mathrm{d}W), \tag{2.7}$$

$$\mathrm{d}z = \xi(m - z) \, \mathrm{d}t + \beta(\rho \, \mathrm{d}W + \sqrt{1 - \rho^2} \, \mathrm{d}Z). \tag{2.8}$$

REMARK 2.1 The assumption of absolute continuity with respect to time on the processes $L(t), M(t)$ was made in order to write our dynamics in a simpler form, but it can be relaxed. Following the treatment presented in Davis *et al.* (1993), all the results we are going to mention hold true for $L(t)$ and $M(t)$ right continuous with left-hand limits, non-negative, non-decreasing and $\mathcal{F}_t$-adapted.

If we denote the value at time t of a portfolio of the writer of a European option with strike price K, after following the strategy π, by $\Phi_w(t, B^\pi(t), y^\pi(t), S(t), z(t))$, its final value is given by:

$$\Phi_w(T, B^\pi(T), y^\pi(T), S(T), z(T)) = B^\pi(T) + I_{(S(T)<K)}c(y^\pi(T), S(T))$$
$$+ I_{(S(T)>K)}[c(y^\pi(T) - 1, S(T)) + K]. \tag{2.9}$$

On the other hand, the final value of a portfolio which does not include the option (denoted by Φ_1) is simply

$$\Phi_1(T, B^\pi(T), y^\pi(T), S(T), z(T)) = B^\pi(T) + c(y^\pi(T), S(T)). \tag{2.10}$$

3. The optimal control problem for European option pricing

The purpose of this section is to briefly recall the basic ideas of *Utility Indifference Pricing* and to resume, in a synthetic way, the framework proposed by Davis *et al.* (1993) for European option pricing in market models with transaction costs. We define the following value functions:

$$V_j(B) = \sup_{\pi \in \mathcal{T}} \mathbb{E}(\mathcal{U}(\Phi_j(T, B^\pi(T), y^\pi(T), S(T), z(T)))) \tag{3.1}$$

for $j = 1, w$. $\mathcal{U} : \mathbb{R} \to \mathbb{R}$ is the utility function, which is required to be concave and increasing and satisfying $\mathcal{U}(0) = 0$. Note how these value functions depend on the initial endowment $\mathfrak{B}$.

Following Davis *et al.* (1993) we now define:

$$\mathfrak{B}_j = \inf\{\mathfrak{B} : V_j(\mathfrak{B}) \geqslant 0\}.$$

In the present setting the price of the option C, i.e. the amount of money that the writer has to receive to accept the obligation implicit in writing the option, will therefore be:

$$C = \mathfrak{B}_w - \mathfrak{B}_1. \tag{3.2}$$

For this price the investor would in fact be indifferent between the two possibilities of going into the market to hedge the option, or of going into the market without the option. We point out that the present approach, named *Utility Indifference Pricing*, provides just a criterion to select one price, among the infinitely many compatible with the no-arbitrage requirement, since the present market model is incomplete.

We can define the following function that will be useful in the sequel

$$\Psi_j(T, B^\pi(T), y^\pi(T), S(T), z(T)) = \Phi_j(T, B^\pi(T), y^\pi(T), S(T), z(T)) - B^\pi(T). \tag{3.3}$$

We have to find an equation for V_j. In what follows we shall suppress the index j and denote V_j with V. The problem we are dealing with is a stochastic control problem, where the control is the trading strategy m and l.

By following step by step the procedure illustrated in Davis *et al.* (1993, pp. 476–477), including the dependence of just one more state variable, it is possible to show that V must satisfy an HJB equation. A rigorous derivation of the (HJB) equation and the existence and uniqueness proof of a solution in the same modelling context, but for the more complex case of American option pricing, can be found in Cosso & Sgarra (2013).

PROPOSITION 3.1 Let the stochastic process be described by (2.5–2.8), then the value function V defined before must satisfy the following (HJB) equation:

$$\max\{(\partial_y V_j - (1 + \lambda)S\partial_B V_j), -(\partial_y V_j - (1 - \mu)S\partial_B V_j),$$
$$\partial_t V_j + rB\partial_B V_j + \alpha S\partial_S V_j + \xi(m - z)\partial_z V_j$$
$$+ \tfrac{1}{2}[f(z)]^2 S^2 \partial_{SS} V_j + \frac{1}{2}\beta^2 \partial_{zz} V_j + \beta f S \rho \partial_{Sz} V_j\} = 0. \tag{3.4}$$

We now consider the case of the exponential utility function $\mathcal{U}(x) = 1 - \exp(-\gamma x)$. We note, just in passing by, that this gives for V_j the following expression:

$$V_j = 1 - \inf\{\mathbb{E}[\exp(-\gamma B(T))\exp(-\gamma \Psi_j)]\},$$

where Ψ_j has been previously introduced.

In the above maximization problem let us change the variables passing $V_j \longrightarrow Q_j$:

$$V_j = 1 - \exp\left(-\frac{\gamma}{\delta}(B + Q_j)\right),$$

where

$$\delta \equiv \exp[-r(T - t)].$$

Note that with the above expression for V_j, the price of the option C, as given in (3.2), now becomes $C = Q_1 - Q_w$ (3.5)

and we can state the auxiliary result:

PROPOSITION 3.2 The maximization problem for V_j is equivalent to the following minimization problem for Q_j:

$$\min \left\{ (\partial_y Q_j - (1+\lambda)S), (-\partial_y Q_j + (1-\mu)S), \ \partial_t Q_j - rQ_j + \alpha S \partial_S Q_j + \xi(m-z)\partial_z Q_j \right.$$
$$+ \frac{1}{2}[f(z)]^2 S^2 \left[\partial_{SS} Q_j - \frac{\gamma}{\delta}(\partial_S Q_j)^2 \right] + \frac{1}{2}\beta^2 \left[\partial_{zz} Q_j - \frac{\gamma}{\delta}(\partial_z Q_j)^2 \right]$$
$$\left. + \beta f S \rho \left[\partial_{Sz} Q_j - \frac{\gamma}{\delta}\partial_S Q_j \partial_z Q_j \right] \right\} = 0. \tag{3.6}$$

As shown in the papers by Davis *et al.* (1993) and Whalley & Wilmott (1997), the European option pricing problem is a free boundary problem and the (S, y, z) space divides into three regions, the Buy region, the Sell region and the No Transaction (NT from now on) region. The NT region is separated from the other two regions by two (unknown) boundaries and the optimal policy of the option writer consists in maintain his portfolio in the NT region; when an eventual movement of the asset price forces the portfolio to hit one of the two boundaries, he must trade so as to stay inside the NT region. In the Buy region the following condition must hold

$$\min(\partial_y Q_j - (1+\lambda)S) = 0, \tag{3.7}$$

while in the Sell region the following holds:

$$\min(\partial_y Q_j - (1-\mu)S) = 0. \tag{3.8}$$

The NT region is characterized by the following equation:

$$\partial_t Q_j - rQ_j + \alpha S \partial_S Q_j + \frac{1}{\varepsilon}(m-z)\partial_z Q_j + \frac{1}{2}[f(z)]^2 S^2 \left[\partial_{SS} Q_j - \frac{\gamma}{\delta}(\partial_S Q_j)^2 \right] + \frac{1}{\varepsilon}v^2 \left[\partial_{zz} Q_j - \frac{\gamma}{\delta}(\partial_z Q_j)^2 \right]$$
$$+ \frac{1}{\sqrt{\varepsilon}}v\sqrt{2}fS\rho \left[\partial_{Sz} Q_j - \frac{\gamma}{\delta}\partial_S Q_j \partial_z Q_j \right] = 0. \tag{3.9}$$

Some continuity conditions for the unknown function Q and its first and second derivatives across the free boundaries must be imposed in order to solve the free boundary problem. We shall present them in next section.

4. Small transaction costs and fast mean-reverting volatility: the asymptotic analysis

We now suppose small transaction costs and fast mean-reverting volatility. Moreover, we will assume that the transaction costs are much smaller than the rate of mean reversion:

$$\lambda = \bar{\lambda}\varepsilon^2, \quad \mu = \bar{\mu}\varepsilon^2, \quad \xi = \frac{1}{\varepsilon}, \quad \beta = \frac{\sqrt{2}v}{\sqrt{\varepsilon}}.$$

Buying and selling costs are assumed to be the same for simplicity in the following, whenever anything different will be specified.

We believe that our asymptotic assumptions are consistent with a situation where a large investor, facing very small transaction costs, is involved. In fact, in the empirical study (Fouque *et al.*, 2000), it is found that $\varepsilon \sim .005$. In the literature (Whalley & Wilmott, 1997; Davis *et al.*, 1993; Zakamouline, 2006a), typically it is assumed $0.2\% \geqslant \lambda \geqslant .01\%$.

In the absence of transaction costs and with a deterministic volatility $\varepsilon = 0$, the investor would continuously trade and get a perfect hedge staying at $y = y^*$, the 'B&S' hedging strategy. When transaction costs are present there is a strip of small thickness around $y = y^*$ where he does not transact. To resolve this strip we introduce the inner scaled coordinate Y

$$y = y^* + \varepsilon^a Y \quad \text{and} \quad \partial_y \longrightarrow \varepsilon^{-a} \partial_Y. \tag{4.1}$$

The unknown boundaries between the NT region and the buy and sell regions are located at:

$$y = y^* + \varepsilon^a Y^+ \quad \text{and} \quad y = y^* - \varepsilon^a Y^-.$$

It is very important from the practical hedger point of view to determine Y^+ and Y^-.
We impose the following matching conditions (see e.g. Whalley & Wilmott, 1997)

$$
\begin{array}{ll}
Q_{\mathrm{NT}}(Y = Y^\pm) = Q(y = y^* \pm \varepsilon^a Y^\pm) & \text{continuity,} \\
\partial_Y Q_{\mathrm{NT}}(Y = Y^\pm) = \varepsilon^a \partial_y Q(y = y^* \pm \varepsilon^a Y^\pm) & \text{continuity of the first derivative,} \\
\partial_{YY} Q_{\mathrm{NT}}(Y = Y^\pm) = \varepsilon^{2a} \partial_{yy} Q(y = y^* \pm \varepsilon^a Y^\pm) & \text{smooth pasting boundary condition.}
\end{array}
$$

These boundary conditions will force, in the asymptotic analysis below, $a = \frac{1}{3}$. Therefore the NT strip will have a thickness $O(\varepsilon^{1/3})$.

In the buy region $(Y < Y^-)$ we have (3.7) and a solution is

$$Q \quad = \quad (1 \quad + \quad \lambda)Sy \quad + \quad H^-(t, \quad S, \quad \lambda). \tag{4.2}$$

In the sell region $(Y > Y^+)$ we have (3.8), which solves to

$$Q = (1 - \mu)Sy + H^+(t, S, \lambda). \tag{4.3}$$

Here $H^+(t, S, \lambda)$ and $H^-(t, S, \lambda)$ are arbitrary functions not depending on y.

In the NT region we have (3.9), whose solution will be obtained and illustrated in the following sections.

4.1   *The solution in the NT region*

As we mentioned before, in the NT region we use the rescaled variable Y defined by (4.1). The change of variable leads to the following transformation rules for the derivatives:

$$
\begin{aligned}
\partial_y &\longrightarrow \varepsilon^{-1/3} \partial_Y, \\
\partial_S &\longrightarrow \partial_S - \varepsilon^{-1/3} y_S^* \partial_Y, \\
\partial_t &\longrightarrow \partial_t - \varepsilon^{-1/3} y_t^* \partial_Y, \\
\partial_z &\longrightarrow \partial_z - \varepsilon^{-1/3} y_z^* \partial_Y.
\end{aligned}
$$

By writing the solution in the NT region in the following form:

$$Q_{\mathrm{NT}} = S(y^* + \varepsilon^{1/3}Y) + U_0(S,t,z) + \sum_{i=1}^{13} \varepsilon^{i/6} U_i(S,t,z) + \varepsilon^{14/6} U_{14}(S,t,z,Y) + \cdots, \qquad (4.4)$$

we can easily obtain an expression for all the derivatives of Q_{NT} with respect to the variables t, S, z. By adopting an obvious notation they can be written as follows:

$$\partial_t Q_{\mathrm{NT}} = U_{0t} + \sum_{i=1}^{11} \varepsilon^{i/6} U_{it} + \varepsilon^{12/6}(U_{12t} - y_t^* U_{14Y}) + \cdots,$$

$$\partial_S Q_{\mathrm{NT}} = y^* + U_{0S} + \varepsilon^{1/6} U_{1S} + \varepsilon^{2/6}(Y + U_{2S}) + \sum_{i=3}^{11} \varepsilon^{i/6} U_{iS} + \varepsilon^{12/6}\left(U_{12S} - y_S^* U_{14Y}\right) + \cdots,$$

$$\partial_{SS} Q_{\mathrm{NT}} = U_{0SS} + \sum_{i=1}^{9} \varepsilon^{i/6} U_{iSS} + \varepsilon^{10/6}(U_{10S} + (y_S^*)^2 U_{14YY}) + \cdots,$$

$$\partial_z Q_{\mathrm{NT}} = U_{0z} + \sum_{i=1}^{11} \varepsilon^{i/6} U_{iz} + \varepsilon^{12/6}(U_{12z} - y_z^* U_{14Y}) + \cdots, \qquad (4.5)$$

$$\partial_{zz} Q_{\mathrm{NT}} = U_{0zz} + \sum_{i=1}^{9} \varepsilon^{i/6} U_{izz} + \varepsilon^{10/6}(U_{10zz} + (y_z^*)^2 U_{14YY}) + \cdots,$$

$$\partial_{Sz} Q_{\mathrm{NT}} = U_{0Sz} + \sum_{i=2}^{9} \varepsilon^{i/6} U_{iSz} + \varepsilon^{10/6}(U_{10Sz} + y_S^* y_z^* U_{14YY}) + \cdots$$

The introduction of the new scaled variables allows to split the description of the Black–Scholes delta-hedging strategy from the effects due to transaction costs and stochastic volatility and to consider separately their contribution.

To calculate the price of the option we shall use (3.5). The price will have the same asymptotic expansion as Q_j with $j = 1, w$, namely

$$C = C_0 + \varepsilon^{1/6} C_1 + \varepsilon^{1/3} C_2 + \sqrt{\varepsilon} C_3 + \varepsilon^{2/3} C_4 + \varepsilon^{5/6} C_5 + \varepsilon C_6 + \cdots. \qquad (4.6)$$

Each C_i is given by:

$$C_i = U_i^1 - U_i^w.$$

To find the appropriate final conditions for the C_i, we write the final conditions for Q^1 and Q^w. They are, respectively:

$$Q^1(T) = y(T)S(T) \qquad (4.7)$$

and

$$Q^w(T) = y(T)S(T) - \max(S(T) - K, 0). \qquad (4.8)$$

Given the expression (4.4) one has that the final conditions for the U_i are the following:

$$U_i^1(T) = 0 \quad \text{for } i = 0, \ldots, 6. \tag{4.9}$$

$$U_0^w(T) = -\max(S(T) - K, 0), \tag{4.10}$$

$$U_i^w(T) = 0 \quad \text{for } i = 1, \ldots, 6. $$

REMARK 4.1 We point out that the final conditions of our original problem have been imposed on the leading order term, while the corresponding conditions imposed for lower-order terms are homogeneous. This choice seems to be quite natural in the present setting and is also common for asymptotic expansions.

4.2 The $O(\varepsilon^{-1})$ up to $O(\varepsilon^{-1/6})$ order equations

To simplify the notation, and following the use in Fouque *et al.* (2003) and Jonsson & Sircar (2002), we define the linear operators $\mathcal{L}_i$ and the non-linear operator $\mathcal{N}_\mathcal{L}$:

$$\mathcal{L}_0 U = (m - z)U_z + v^2 U_{zz}, \tag{4.11}$$

$$\mathcal{L}_1 U = -v\sqrt{2}\rho \frac{(\alpha - r)}{f} U_z + v\sqrt{2}fS\rho U_{Sz}, \tag{4.12}$$

$$\mathcal{L}_2 U = U_t + \frac{1}{2}f^2 S^2 U_{SS} - rU + rSU_S, \tag{4.13}$$

$$\mathcal{N}_\mathcal{L} U = -v^2 \frac{\gamma}{\delta}(U_z)^2. \tag{4.14}$$

The $O(\varepsilon^{-1})$ equation is simply

$$\mathcal{L}_0 U_0 + \mathcal{N}_\mathcal{L} U_0 = 0. \tag{4.15}$$

The above equation can be considered an ordinary differential equation (ODE) in z for U_0:

$$v^2 U_{0zz} + (m - z)U_{0z} - v^2 \frac{\gamma}{\delta}(U_{0z})^2 = 0. \tag{4.16}$$

In Jonsson & Sircar (2002), it is proved that the only solution of an equation of this form is a U which does not depend on z. The conclusion we therefore draw is that

$$U_0 \text{ does not depend on } z.$$

Analogously, the $O(\varepsilon^{-i/6})$ equations, for $i = 1, \ldots, 5$ are

$$\mathcal{L}_0 U_i = 0.$$

The conclusion is

$$U_i \quad i = 1, \ldots, 5 \text{ do not depend on } z.$$

4.3 The O(1) equation

The $O(1)$ equation writes

$$\mathcal{L}_0 U_6 + \mathcal{L}_2 U_0 + S(\alpha - r)(U_{0S} + y^*S) - \frac{1}{2}[f(z)]^2 S^2 \frac{\gamma}{\delta}(y^* + U_{0S})^2 = 0. \tag{4.17}$$

The above equation will be analysed in Section 4.5.

4.4 The $O(\varepsilon^{1/6})$ equation

The $O(\varepsilon^{1/6})$ equation is

$$\mathcal{L}_0 U_7 + \mathcal{L}_2 U_1 + S(\alpha - r)U_{1S} - [f(z)]^2 S^2 \frac{\gamma}{\delta} U_{1S}(y^* + U_{0S}) = 0. \tag{4.18}$$

Also (4.18) will be analysed in Section 4.5.

4.5 The $O(\varepsilon^{2/6})$ equation

The $O(\varepsilon^{2/6})$ equation is

$$\mathcal{L}_0 U_8 + U_{2t} - r(SY + U_2) + \alpha S(Y + U_{2S}) + \frac{1}{2}[f(z)]^2 S^2 \left[U_{2SS} - 2\frac{\gamma}{\delta}(y^* + U_{0S})(Y + U_{2S}) \right] = 0. \tag{4.19}$$

In the above equation there are terms that do not depend on Y, and terms linear in Y. They must be equal to zero separately. From the terms linear in Y one gets:

$$y^* = -U_{0S} + \frac{(\alpha - r)\delta}{f^2 S\gamma}. \tag{4.20}$$

The above expression gives the leading order (in the absence of transaction costs) optimal hedging strategy. One recognizes the Black and Scholes Delta-hedging strategy. The same result appears in the paper by Whalley & Wilmott (1997, formula (3.6)).

By inserting the above expression into the $O(1)$ (4.17), one gets:

$$\mathcal{L}_0 U_6 + \partial_t U_0 + \frac{1}{2} f^2 S^2 \partial_{SS} U_0 + rS \partial_S U_0 - rU_0 + \frac{1}{2} \frac{\delta}{\gamma} \frac{(\alpha - r)^2}{f^2} = 0. \tag{4.21}$$

The above equation, considered as an ODE for U_6, is of the form:

$$\mathcal{L}_0 U = \chi. \tag{4.22}$$

In Fouque et al. (2003) it is shown that the solvability condition for (4.22) is

$$\langle \chi \rangle = 0, \tag{4.23}$$

where the average $\langle \cdot \rangle$ is taken with respect to the Ornstein–Uhlenbeck process invariant measure (see Klebaner, 1998 and Fouque *et al.*, 2003, formula (3.3), p. 1652)

$$\langle \chi \rangle = \frac{1}{v\sqrt{2\pi}} \int_{\mathbb{R}} \chi(z)\, e^{-(m-z)^2/2v^2}\, dz. \tag{4.24}$$

Therefore the solvability condition for (4.21) is

$$\partial_t U_0 + \frac{1}{2}\bar{\sigma}^2 S^2 \partial_{SS} U_0 - rU_0 + rS\partial_S U_0 = -\frac{\delta(\alpha-r)^2}{2\gamma}\frac{1}{\bar{\tau}^2}, \tag{4.25}$$

where $\bar{\sigma}$ is the effective constant volatility

$$\bar{\sigma}^2 = \langle f^2 \rangle,$$

and $\bar{\tau}$ is defined as

$$\frac{1}{\bar{\tau}^2} = \left\langle \frac{1}{f^2} \right\rangle.$$

Once one imposes to (4.25) the appropriate final condition, which will be different for the investor with option liability and the investor without it, then U_0 is determined. One can go back to equation (4.21) and solve it for U_6. Once (4.21) has been solved, and the proper boundary conditions are taken into account, it is easy to recognize that the solution for U_6 can be written as a superposition of two terms, one independent on z (the differential operator $\mathcal{L}_2$ contains only terms proportional to the first derivatives with respect to z and this term satisfies the homogeneous equation $\mathcal{L}_2 U = 0$) and one depend-ing on (S, z, t), in such a way that we can write

$$U_6 = U_6^{(z)}(S, z, t) + \tilde{U}_6(S, t), \tag{4.26}$$

where, $U_6^{(z)}(S, z, t)$, the part of U_6 which depends on z, has the following expression

$$U_6^{(z)}(S, z, t) = -\mathcal{L}_0^{-1}\left[\frac{1}{2}S^2(f^2 - \bar{\sigma}^2)U_{0SS} + \frac{1}{2}\frac{\delta}{\gamma}(\alpha - r)^2\left(\frac{1}{f^2} - \frac{1}{\bar{\tau}^2}\right)\right];$$

on the other hand, $\tilde{U}_6$ is a function that does not depend on z and that will be determined by the $O(\varepsilon)$ equation in the asymptotic procedure.

We can get a more explicit representation for $U_6^{(z)}$, that will be useful in the next subsection. We first define the functions $\varphi(z)$ and $\psi(z)$ as the solutions of the following problems:

$$\mathcal{L}_0\varphi = f^2 - \langle f^2 \rangle, \tag{4.27}$$

$$\mathcal{L}_0\psi = \frac{1}{f^2} - \left\langle \frac{1}{f^2} \right\rangle. \tag{4.28}$$

Therefore the above expression for $U_6^{(z)}$ can be written as:

$$U_6^{(z)} = -\left[\frac{1}{2}S^2 U_{0SS}\varphi + \frac{1}{2}\frac{\delta}{\gamma}(\alpha - r)^2\psi\right]. \tag{4.29}$$

Let us come back to (4.18). Once substituted the expression (4.20) for y^*, (4.18) reduces to

$$\mathcal{L}_0 U_7 + \mathcal{L}_2 U_1 = 0. \tag{4.30}$$

Equation (4.30) is a Poisson problem for U_7, whose solvability condition reads:

$$\langle \mathcal{L}_2 \rangle U_1 = 0, \tag{4.31}$$

which is a homogeneous Black–Scholes equation for U_1. Given that the final condition is zero, we get the conclusions

$$U_1 \equiv 0,$$

$$U_7 = \tilde{U}_7(S, t) \text{ does not dependent on } z.$$

One can now go back to (4.19), collect all terms independent of Y and get the following equation:

$$\mathcal{L}_0 U_8 + \mathcal{L}_2 U_2 = 0.$$

The above equation is a Poisson problem of the type (4.22). The solvability condition is:

$$\langle \mathcal{L}_2 \rangle U_2 = 0. \tag{4.32}$$

Note that the above equation is homogeneous in U_2. Given that the final condition, both for the investor with option liability and for the investor without it, is 0, one gets the following conclusions

$$U_2 \equiv 0,$$

$$U_8 = \tilde{U}_8(S, t) \text{ is independent of } z.$$

Therefore, the first significant correction to the Black and Scholes value is the $O(\varepsilon^{1/2})$ contribution.

4.6 The $O(\varepsilon^{3/6})$ equation

Using the expression (4.20) for y^*, the $O(\varepsilon^{3/6})$ equation can be written as

$$\mathcal{L}_2 U_3 + \mathcal{L}_1 U_6 + \mathcal{L}_0 U_9 = 0. \tag{4.33}$$

Note that in the above equation appears U_6, which until now we have derived only up to the function $\tilde{\zeta}(S, t)$, to be determined by a higher order asymptotic. However, in (4.33) U_6 is hit by the operator $\mathcal{L}_1$, which cancels $\tilde{U}_6(S, t)$.

Therefore, one can consider (4.33) as a Poisson problem for U_9, whose solvability condition is

$$\langle \mathcal{L}_2 \rangle U_3 = -\langle \mathcal{L}_1 U_6 \rangle. \tag{4.34}$$

The above equation is a Black and Scholes equation for U_3 with 0 final condition and with a source term. We now want to rewrite the source term.

By using for U_6 the expression (4.26) the operator $\mathcal{L}_1$ cancels the part not depending on z, and taking into account the expression (4.29), one can express the source term in (4.34) as

$$-\langle \mathcal{L}_1 U_6 \rangle = \left\langle \mathcal{L}_1 \left[\frac{1}{2} S^2 U_{0SS} \varphi + \frac{1}{2} \frac{\delta}{\gamma} (\alpha - r)^2 \psi \right] \right\rangle$$

$$= \frac{\nu\rho}{\sqrt{2}} \left[\langle f\varphi' \rangle (S^3 U_{0SSS} + 2S^2 U_{0SS}) - (\alpha - r)S^2 U_{0SS} \left\langle \frac{\varphi'}{f} \right\rangle - \frac{\delta}{\gamma}(\alpha - r)^3 \left\langle \frac{\psi'}{f} \right\rangle \right].$$

Therefore, U_3 solves the following Black and Scholes equation

$$U_{3t} + \frac{1}{2}\bar{\sigma}^2 S^2 U_{3SS} - rU_3 + rSU_{3S}$$

$$= \frac{\nu\rho}{\sqrt{2}} \left[\langle f\varphi' \rangle (S^3 U_{0SSS} + 2S^2 U_{0SS}) - (\alpha - r)S^2 U_{0SS} \left\langle \frac{\varphi'}{f} \right\rangle - \frac{\delta}{\gamma}(\alpha - r)^3 \left\langle \frac{\psi'}{f} \right\rangle \right] \tag{4.35}$$

with zero final data.

4.7 The $O(\varepsilon^{2/3})$ equation

Since U_7 does not depend on z, the $O(\varepsilon^{2/3})$ equation can be written as

$$\mathcal{L}_2 U_4 + \mathcal{L}_0 U_{10} + v^2(y_z^*)^2 U_{14YY} - \frac{\gamma}{\delta} f^2 S^2 Y^2 = 0. \tag{4.36}$$

Equation (4.36) can be considered as an ODE in Y for U_{14}. It writes as

$$U_{14YY} = AY^2 + B, \tag{4.37}$$

where we have defined the following quantities

$$A = \frac{\gamma}{\delta} \frac{f^2 S^2}{v^2 (y_z^*)^2},$$

$$B = -\frac{\mathcal{L}_2 U_4 + \mathcal{L}_0 U_{10}}{v^2 (y_z^*)^2}.$$

Equation (4.37) solves to

$$U_{14} = \frac{A}{12} Y^4 + \frac{1}{2} BY^2 + CY + D, \tag{4.38}$$

with C and D independent of Y. Now we have to impose the matching conditions.

Being

$$\begin{aligned} Q_{\mathrm{BUY}} &= (1 + \varepsilon^2 \bar{\lambda})Sy + H^-(S, z, t) \text{ in the outer buy region} \\ Q_{\mathrm{SELL}} &= (1 - \varepsilon^2 \bar{\mu})Sy + H^+(S, z, t) \text{ in the outer sell region} \end{aligned} \tag{4.39}$$

and imposing the continuity of the gradient at the two boundaries:

$$\partial_Y Q_{\mathrm{NT}}(Y = -Y^-) = \varepsilon^{1/3} \partial_y Q_{\mathrm{BUY}}(y = y^* - \varepsilon^{1/3} Y^-),$$

$$\partial_Y Q_{\mathrm{NT}}(Y = Y^+) = \varepsilon^{1/3} \partial_y Q_{\mathrm{SELL}}(y = y^* + \varepsilon^{1/3} Y^+),$$

one gets, at $O(\varepsilon^{14/6})$

$$\partial_Y U_{14}(Y=-Y^-) = \bar{\lambda}S, \tag{4.40}$$

$$\partial_Y U_{14}(Y=Y^+) = -\bar{\mu}S. \tag{4.41}$$

Therefore, using (4.38), one gets

$$-\frac{A}{3}(Y^-)^3 - BY^- + C = \bar{\lambda}S, \tag{4.42}$$

$$\frac{A}{3}(Y^+)^3 + BY^+ + C = -\bar{\mu}S. \tag{4.43}$$

Moreover, being W in the outer regions linear in y, one imposes the continuity of the second derivative as follows

$$\partial_{YY}Q_{NT}(Y=\pm Y^\pm) = 0,$$

i.e.

$$A(Y^+)^2 + B = 0,$$

$$A(Y^-)^2 + B = 0.$$

From these equations one sees that, at this order, the bandwidth about the Black and Scholes strategy is symmetric, i.e.

$$Y^+ = Y^- = \left(-\frac{B}{A}\right)^{1/2}. \tag{4.44}$$

Subtracting the two equations (4.42) and (4.43) to eliminate C, and using the above expressions for $Y^\pm$, one gets

$$\frac{4}{3}(-B)^{3/2}A^{-1/2} = (\bar{\lambda} + \bar{\mu})S.$$

After some manipulations, and using the expressions for A and B, (4.37) leads to the following equation:

$$\mathcal{L}_0 U_{10} + \mathcal{L}_2 U_4 = \left[\frac{3}{4}(\bar{\lambda} + \bar{\mu})fS^2\sqrt{\frac{\gamma}{\delta}}v^2(y_z^*)^2\right]^{2/3}. \tag{4.45}$$

One can also find an expression for the amplitude of the NT region

$$Y^+ = Y^- = \left[\frac{3}{2}\frac{1}{f^2 S}\frac{\delta}{\gamma}v^2(y_z^*)^2\right]^{1/3}. \tag{4.46}$$

Equation (4.45) is a Poisson problem of the type of (4.22). The solvability condition gives an equation for U_4:

$$\langle \mathcal{L}_2 \rangle U_4 = \left\langle \left[\frac{3}{4}(\bar{\lambda} + \bar{\mu})fS^2\sqrt{\frac{\gamma}{\delta}}v^2(y_z^*)^2\right]^{2/3}\right\rangle. \tag{4.47}$$

Note also that, adding the two equations (4.42) and (4.43), one gets that $C = 0$. Therefore

$$U_{14} = \frac{A}{12}Y^4 + \frac{1}{2}BY^2 + D,$$

(4.48)

which will be useful in Section 4.9.

4.8 The $O(\varepsilon^{5/6})$ equation

The $O(\varepsilon^{5/6})$ equation writes as:

$$\mathcal{L}_0 U_{11} + \mathcal{L}_2 U_5 - \sqrt{2}\frac{\partial U_6}{\partial z}\frac{\gamma}{\delta}v\rho f(z)SY - \frac{\partial U_3}{\partial S}\frac{\gamma}{\delta}f(z)^2 S^2 Y + \frac{\partial^2 U_{15}}{\partial Y^2}\left(\frac{\partial y^*}{\partial z}\right)^2 v^2 = 0.$$

(4.49)

This equation can be considered as an ODE for U_{15}:

$$\frac{\partial^2 U_{15}}{\partial Y^2} = \bar{A}Y + \bar{B},$$

where we have denoted

$$\bar{A} = \left(\sqrt{2}\frac{\partial U_6}{\partial z}\frac{\gamma}{\delta}v\rho f(z)S - \frac{\partial U_3}{\partial S}\frac{\gamma}{\delta}f(z)^2 S^2\right)\Big/ (v^2 y_z^{*2}),$$

$$\bar{B} = -\frac{\mathcal{L}_0 U_{11} + \mathcal{L}_2 U_5}{v^2 y_z^{*2}}.$$

Integrating (4.49) with respect to Y and using the boundary conditions:

$$U_{15Y}(Y^+) = U_{15Y}(-Y^-) = 0$$

which are needed to ensure the continuity of the gradient, one gets

$$\mathcal{L}_0 U_{11} + \mathcal{L}_2 U_5 = 0.$$

(4.50)

The above equation is a Poisson problem for U_{11}, whose solvability condition reads

$$\langle \mathcal{L}_2 \rangle U_5 = 0.$$

(4.51)

This is a homogeneous Black–Scholes equation for U_5. Given that the final condition is zero, we get the conclusions:

$$U_5 \equiv 0,$$

$$U_{11} = \tilde{U}_{11}(S, t) \text{ does not depend on } z.$$

4.9   The $O(\varepsilon)$ equation

Collecting the $O(\varepsilon)$ terms one gets

$$\mathcal{L}_2 U_6 + \mathcal{L}_1 U_9 + \mathcal{L}_0 U_{12} - \frac{1}{2} f^2 S^2 \frac{\gamma}{\delta} (U_{3S})^2 - \nu \sqrt{2} f S \rho \frac{\gamma}{\delta} U_{3S} U_{6z} - y_z^* (m-z) U_{14Y}$$

$$- \frac{\gamma}{\delta} f^2 Y S^2 U_{4S} + \nu^2 \left[-y_{zz}^* U_{14Y} - 2 y_z^* U_{14Yz} + (y_z^*)^2 U_{16YY} - \frac{\gamma}{\delta} (U_{6z})^2 \right] = 0. \tag{4.52}$$

The above equation can be considered as an ODE in Y for U_{16}:

$$\frac{\partial^2 U_{16}}{\partial Y^2} = \tilde{A} Y + \tilde{B},$$

where we have defined

$$\tilde{A} = \frac{\gamma}{\delta} \frac{f^2 S^2 U_{4S}}{\nu^2 (y_z^*)^2},$$

$$\tilde{B} = - \left[\mathcal{L}_0 U_{12} + \mathcal{L}_1 U_9 + \mathcal{L}_2 U_6 - \frac{1}{2} f^2 S^2 \frac{\gamma}{\delta} U_{3S}^2 - \nu \sqrt{2} f S \rho \frac{\gamma}{\delta} U_{3S} U_{6z} \right.$$

$$\left. - \nu^2 \left(y_{zz}^* U_{14Y} + 2 y_z^* U_{14Yz} + \frac{\gamma}{\delta} U_{6z}^2 \right) + U_{14Y} y_z^* (m-z) \right] \Big/ (\nu^2 (y_z^*)^2).$$

Note that $\tilde{A}$ does not depend on Y and in $\tilde{B}$ the Y-dependent terms appear only with their derivatives in Y.

We integrate (4.52) from $-Y^-$ to Y^+.
Let us use the boundary conditions:

$$U_{16Y}(Y^+) = U_{16Y}(-Y^-) = 0.$$

Moreover, being $Y^+ = Y^-$ and from the expression (4.48) it follows that

$$\int_{-Y^-}^{Y^+} U_{14Y} \, dY = 0.$$

By integrating (4.52) we get:

$$\mathcal{L}_0 U_{12} + \mathcal{L}_1 U_9 + \mathcal{L}_2 U_6 = \frac{1}{2} f^2 S^2 \frac{\gamma}{\delta} (U_{3S})^2 + \nu \sqrt{2} f S \rho \frac{\gamma}{\delta} U_{3S} U_{6z} + \nu^2 \frac{\gamma}{\delta} (U_{6z})^2.$$

The solvability condition for U_{12} gives the following equation for U_6:

$$\langle \mathcal{L}_2 \rangle U_6 = - \langle \mathcal{L}_1 U_9 \rangle + \frac{1}{2} \bar{\sigma}^2 S^2 \frac{\gamma}{\delta} (U_{3S})^2 + \nu \sqrt{2} S \rho \frac{\gamma}{\delta} U_{3S} \langle f U_{6z}^{(z)} \rangle + \nu^2 \frac{\gamma}{\delta} \langle (U_{6z}^{(z)})^2 \rangle. \tag{4.53}$$

The main results of this section are the following:

1. Equation (4.25) for U_0;

2. Equation (4.31) for U_1 which led us to $U_1 \equiv 0$;

3. Equation (4.32) for U_2 which led us to $U_2 \equiv 0$;

4. Equation (4.35) for U_3;

5. Equation (4.47) for U_4;

6. Equation (4.51) for U_5 which led us to $U_5 \equiv 0$;

7. Equation (4.53) for U_6;

8. The expression (4.20) for y^*, the centre of the NT region.

9. The expression (4.46) for the boundaries of the NT region.

5. The option pricing

To calculate the price of the option we now use (3.5), and the asymptotic expansion (4.6) together with the appropriate final conditions for the C_i, which, as we discussed in the previous section, can be obtained by the final conditions for the U_i, (4.9–4.11). Here we can state our main result, which will be proved in detail in the following subsections:

PROPOSITION 5.1 The first seven terms of the asymptotic expansion (4.6) in powers of the adopted parameter ε for the European Call option price in the *Utility Indifference* framework introduced by M.A.H. Davis, V.G. Panas and T. Zariphopoulou are provided by formulas (5.3), (5.4), (5.5), (5.8), (5.9), (5.10) and (5.15) written below.

5.1 *The leading order price*

In order to compute the leading order price we have to calculate U_0^1 and U_0^w where they both satisfy (4.25). Given the respective final conditions (4.9) and (4.10) one has that

$$U_0^1 = (T - t)\frac{\delta(\alpha - r)^2}{2\gamma}\frac{1}{\bar{\tau}^2}, \tag{5.1}$$

$$U_0^w = (T - t)\frac{\delta(\alpha - r)^2}{2\gamma}\frac{1}{\bar{\tau}^2} - C^{\mathrm{BS}}, \tag{5.2}$$

where C^{BS} is the classical pricing formula for a European call option, i.e.

$$C^{\mathrm{BS}}(S, t) = SN(d_1) - K\,\mathrm{e}^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(S/K) + (r + (1/2)\bar{\sigma}^2)(T - t)}{\bar{\sigma}\sqrt{T - t}}, \quad d_2 = d_1 - \bar{\sigma}\sqrt{T - t},$$

and $N(z)$ is the normal cumulative distribution function.

From the above expressions for U_0^j one obtains

$$C_0(S, t) = C^{\mathrm{BS}}(S, t). \tag{5.3}$$

5.2 The $O(\varepsilon^{1/6})$ correction

The equation for U_1^1 and U_1^w is (4.31), a homogeneous Black and Scholes equation. In both cases, the final condition is homogeneous. Therefore $U_1^1 \equiv 0$ and $U_1^w \equiv 0$ and

$$C_1(S,t) = 0. \tag{5.4}$$

5.3 The $O(\varepsilon^{1/3})$ correction

As for the $O(\varepsilon^{1/6})$ terms, the equation for U_2^1 and U_2^w is the homogeneous Black and Scholes equation (4.32), with homogeneous final condition, therefore both U_2^1 and U_2^w are zero and

$$C_2(S,t) = 0. \tag{5.5}$$

5.4 The $O(\varepsilon^{1/2})$ correction

The equation for U_3^1 and U_3^w is (4.35), in both cases with homogeneous final condition. Using, respectively, the expressions (5.1) and (5.2) in (4.35), one has that

$$U_3^1 = (T-t)\frac{\nu\rho}{\sqrt{2}}\frac{\delta}{\gamma}(\alpha-r)^3\left\langle\frac{\psi'}{f}\right\rangle, \tag{5.6}$$

$$U_3^w = -(T-t)\frac{\nu\rho}{\sqrt{2}}\left[-\frac{\delta}{\gamma}(\alpha-r)^3\left\langle\frac{\psi'}{f}\right\rangle - \langle f\varphi'\rangle(S^3 C_{3S}^{BS} + 2S^2 C_{SS}^{BS}) \right.$$

$$\left. + (\alpha-r)S^2\left\langle\frac{\varphi'}{f}\right\rangle C_{SS}^{BS}\right]. \tag{5.7}$$

Therefore

$$C_3(S,t) = -(T-t)\frac{\nu\rho}{\sqrt{2}}\left[\langle f\varphi'\rangle(S^3\partial_S^3 C^{BS} + 2S^2\partial_{SS}C^{BS}) - (\alpha-r)S^2\partial_{SS}C^{BS}\left\langle\frac{\varphi'}{f}\right\rangle\right]. \tag{5.8}$$

5.5 The $O(\varepsilon^{2/3})$ correction

The equation for U_4^1 and U_4^w is (4.47), in both cases with homogeneous final condition. The source term for the two problems is the same: in fact y_z^* has the same expression for both problems. Therefore $U_4^w = U_4^1$ and

$$C_4(S,t) = 0. \tag{5.9}$$

5.6 The $O(\varepsilon^{5/6})$ correction

The equation for U_5^1 and U_5^w is the homogeneous Black and Scholes equation (4.51) with zero final condition in both cases. Then

$$C_5(S,t) = 0. \tag{5.10}$$

5.7 The $O(\varepsilon)$ correction

To compute the $O(\varepsilon)$ correction we have to solve (4.53). It is a Black and Scholes equation with a source term. We know that U_6 is decomposed in a part dependent on z and a part that does not depend on z,

see (4.26). The same decomposition holds also for C_6:

$$C_6 = C_6^{(z)} + \tilde{C}_6,$$

where

$$C_6^{(z)} = U_6^{(z)1} - U_6^{(z)w},$$

and

$$\tilde{C}_6 = \tilde{U}_6^{(z)1} - \tilde{U}_6^w.$$

We have already computed $U_6^{(z)j}$, as given in (4.29), we can therefore calculate $C_6^{(z)}$. In fact, using (4.29) and the expressions (5.1) and (5.2), one gets

$$U_6^{(z)1} = -\frac{1}{2}\frac{\delta}{\gamma}(\alpha - r)^2\psi, \tag{5.11}$$

$$U_6^{(z)w} = \frac{1}{2}S^2 C_{SS}^{BS}\varphi - \frac{1}{2}\frac{\delta}{\gamma}(\alpha - r)^2\psi, \tag{5.12}$$

which gives

$$C_6^{(z)} = -\frac{1}{2}S^2 C_{SS}^{BS}\varphi.$$

We are now left with the task of computing $\tilde{C}_6$.

The equation for C_6 can be derived using (4.53). Subtracting the two equations relative to U_6^1 and U_6^w one gets

$$\langle\mathcal{L}_2\rangle C_6 = -[\langle\mathcal{L}_1 U_9^1\rangle - \langle\mathcal{L}_1 U_9^w\rangle] + \frac{1}{2}S^2\bar{\sigma}^2\frac{\gamma}{\delta}\left[(U_{3S}^1)^2 - (U_{3S}^w)^2\right]$$

$$+ v\sqrt{2}S\rho\frac{\gamma}{\delta}\left[U_{3S}^1\langle fU_{6z}^{(z)1}\rangle - U_{3S}^w\langle fU_{6z}^{(z)w}\rangle\right] + v^2\frac{\gamma}{\delta}\left[\langle(U_{6z}^{(z)1})^2\rangle - \langle(U_{6z}^{(z)w})^2\rangle\right]. \tag{5.13}$$

This equation is a Black and Scholes equation for C_6 with source term. In Appendix A, this source term is explicitly computed and (5.13) writes as

$$\langle\mathcal{L}_2\rangle C_6 = (T - t)^2\hat{A} + (T - t)\hat{B} + \hat{C}, \tag{5.14}$$

where

$$\hat{A} = -\frac{v^2\rho^2}{4}\frac{\gamma}{\delta}S^2\bar{\sigma}^2\left[\langle f\varphi'\rangle(S^3 C_{4S}^{BS} + 5S^2 C_{SSS}^{BS} + 4SC_{SS}^{BS}) - (\alpha - r)\left\langle\frac{\varphi'}{f}\right\rangle(S^2 C_{SSS}^{BS} + 2SC_{SS}^{BS})\right]^2,$$

$$\hat{B} = -v^2\rho^2 S^2\left(\langle\varphi'f\rangle - \frac{1}{2}(\alpha - r)\left\langle\frac{\varphi'}{f}\right\rangle\right)\left[\langle f\varphi'\rangle(S^3 C_{5S}^{BS} + 8S^2 C_{4S}^{BS}\right.$$

$$\left. + 14SC_{SSS}^{BS} + 4C_{SS}^{BS}) - (\alpha - r)\left\langle\frac{\varphi'}{f}\right\rangle(S^2 C_{4S}^{BS} + 4SC_{SSS}^{BS} + 2C_{SS}^{BS})\right]$$

$$- \frac{v^2 \rho^2}{2} S^3 \langle f \varphi' \rangle \left[(S^3 C_{6S}^{BS} + 11 S^2 C_{5S}^{BS} + 30 S C_{4S}^{BS} + 18 C_{SSS}^{BS}) \langle f \varphi' \rangle \right.$$

$$\left. - (\alpha - r)(S^2 C_{5S}^{BS} + 6 S C_{4S}^{BS} + 6 C_{SSS}^{BS}) \left\langle \frac{\varphi'}{f} \right\rangle \right]$$

$$- \frac{v^2 \rho^2}{2} \frac{\gamma}{\delta} S \left[\langle f \varphi' \rangle (S^3 C_{4S}^{BS} + 5 S^2 C_{3S}^{BS} + 4 S C_{SS}^{BS})(\alpha - r) \left\langle \frac{\varphi'}{f} \right\rangle (2 S C_{SS}^{BS} + S^2 C_{3S}^{BS}) \right]$$

$$\times \left(S^2 C_{SS}^{BS} \langle \varphi' f \rangle - \frac{\delta}{\gamma} (\alpha - r)^2 \langle \psi' f \rangle \right),$$

$$\hat{C} = v^2 \frac{\gamma}{\delta} \left[-\frac{1}{4} S^4 (C_{SS}^{BS})^2 \langle \varphi'^2 \rangle + \frac{1}{2} \frac{\delta}{\gamma} (\alpha - r)^2 S^2 C_{SS}^{BS} \langle \varphi' \psi' \rangle \right]$$

$$- v^2 \rho^2 (\alpha - r) \left[(\alpha - r) S^2 C_{SS}^{BS} \langle \frac{G'}{f} \rangle - (S^3 C_{SSS}^{BS} + 2 S^2 C_{SS}^{BS}) \left\langle \frac{F'}{f} \right\rangle \right]$$

$$- v^2 \rho^2 \left[(S^4 C_{4S}^{BS} + 5 S^3 C_{4S}^{BS} + 5 S^3 C_{SSS}^{BS} + 4 S^2 C_{SS}^{BS}) \langle F' f \rangle - (\alpha - r)(S^3 C_{SSS}^{BS} + 2 S^2 C_{SS}^{BS}) \langle G' f \rangle \right].$$

Given the homogeneous final condition, the solution of (5.14) writes as

$$C_6 = \frac{(T - t)^3}{3} \hat{A} + \frac{(T - t)^2}{2} \hat{B} + (T - t) \hat{C}. \tag{5.15}$$

We have used the fact that

$$\mathcal{L}_2 \left(\frac{(T - t)^3}{3} A + \frac{(T - t)^2}{2} B + (T - t) C \right)$$

$$= (T - t)^2 A + (T - t) B + C + \frac{(T - t)^3}{3} \mathcal{L}_2 A + \frac{(T - t)^2}{2} \mathcal{L}_2 B + (T - t) \mathcal{L}_2 C$$

and the last three terms are zero

$$\mathcal{L}_2 \left(S^n \frac{\partial^n C^{BS}}{\partial S^n} \right) = S^n \frac{\partial^n}{\partial S^n} \mathcal{L}_2 C^{BS} = 0.$$

In Appendix C, the reader can find the derivatives of C^{BS} with respect to S up to the sixth order. In Appendix B, the averages $\langle \cdot \rangle$ with respect to the Ornstein–Uhlenbeck process invariant measure are explicitly computed using Scott's model.

6. Numerical illustration of the results

In this section, we present the main results obtained via the asymptotic method. At first, we plot the NT region for different values of the volatility, ranging from 5 to 60%, both in the case which does not include the option, denoted by the index 1, and in the case which includes the option, denoted by the index w. The volatility is chosen as in the Scott model, $f(z) = e^z$. InFig. 1, the curves representing the Black and Scholes strategy y^* in the absence of transaction costs and the hedging boundaries, $y = y^* \pm \varepsilon^{1/3} Y^+$, are plotted versus S for the first problem. From the expressions (4.20) and (4.46) it follows
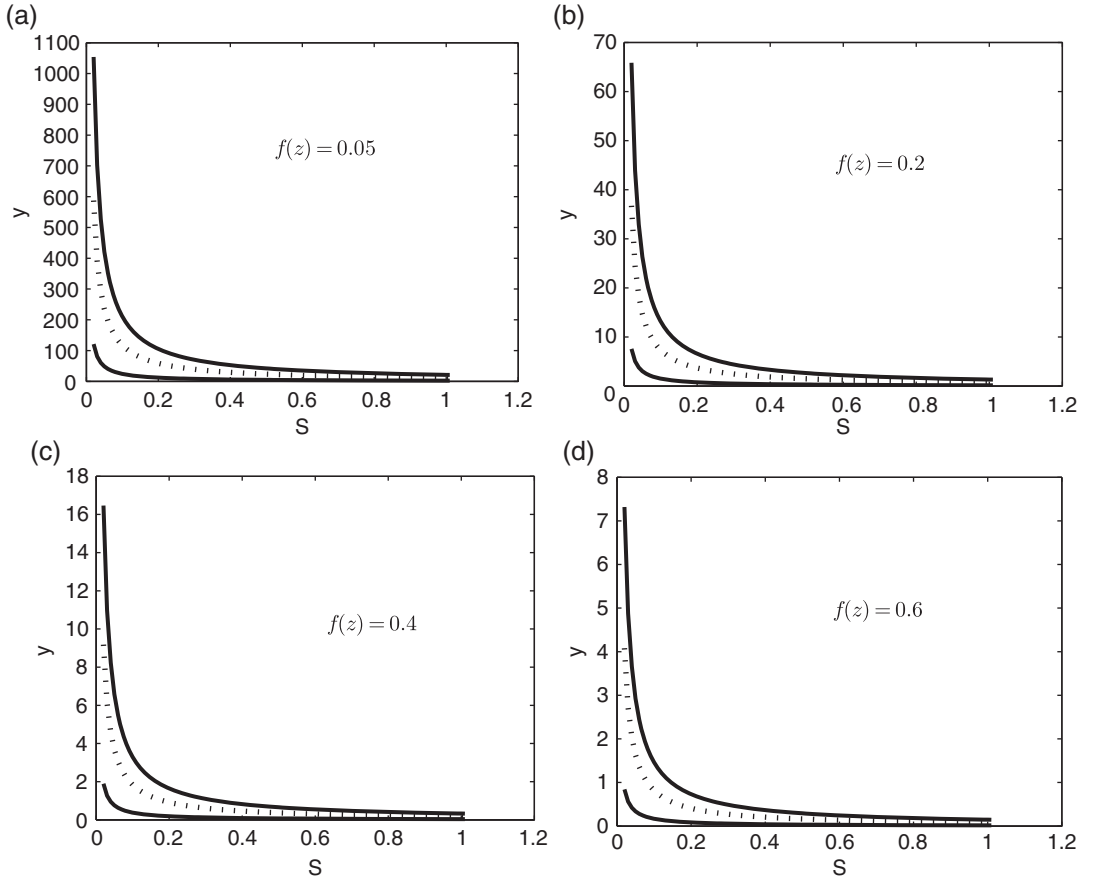
FIG. 1. The NT region in the case which does not include the option. The dotted curve represents the Black and Scholes strategy y^* in the absence of transaction costs, the other two curves represent the hedging boundaries. See the text for the choice of parameters. Parts (a,b,c,d) refer to different numerical values of the initial volatility.

that these curves are, respectively, given by

$$y^* = \frac{(\alpha - r)\delta}{e^{2z}S\gamma},$$ (6.1)

$$y = \frac{(\alpha - r)\delta}{e^{2z}S\gamma} \pm \varepsilon^{1/3} \left[\frac{3(\bar{\lambda} + \bar{\mu})(\alpha - r)^2 v^2 \delta^3}{e^{6z}S^3\gamma^3} \right]^{1/3}.$$ (6.2)

The corresponding curves in the second case are plotted in Fig. 2 and their equations are:

$$y^* = C_S^{\text{BS}} + \frac{(\alpha - r)\delta}{e^{2z}S\gamma},$$ (6.3)
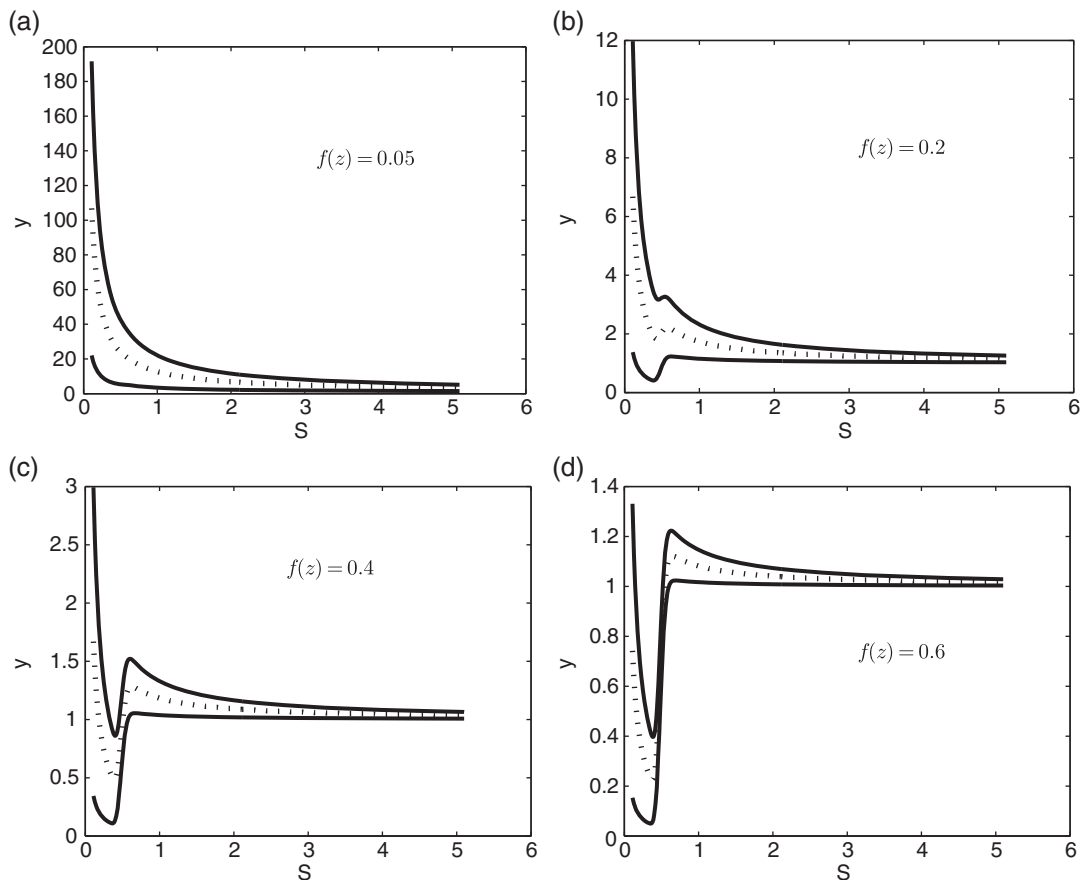
FIG. 2. The NT region in the case which includes the option. The dotted curve represents the Black and Scholes strategy y^* in the absence of transaction costs, the other two curves represent the hedging boundaries. See the text for the choice of parameters. Parts (a,b,c,d) refer to different numerical values of the initial volatility.

$$y = C_S^{BS} + \frac{(\alpha - r)\delta}{e^{2z}S\gamma} \pm \varepsilon^{1/3} \left[\frac{3(\bar{\lambda} + \bar{\mu})(\alpha - r)^2 v^2 \delta^3}{e^{6z}S^3\gamma^3} \right]^{1/3}. \tag{6.4}$$

Both in Figs 1 and 2, the strike price is $K = 0.5$, the risk-free interest rate is $r = 0.07$, the drift rate of the stock is $\alpha = 0.1$, the risk aversion is $\gamma = 1$, the mean volatility $\bar{\sigma} = 0.2$, the time to expiry is 0.3, $\bar{\lambda} = \bar{\mu} = 1$ and $\varepsilon = \frac{1}{200}$.

Finally, in Fig. 3 it is shown the curve representing the classical Black and Scholes price of a European call option with the first correction obtained at $O(\varepsilon^{1/2})$ and the second correction obtained at $O(\varepsilon)$. Here the parameters are chosen as $K = 100$, $r = 0.04$, $\alpha = 0.1$, $\gamma = 1$, the time to expiry is 0.25 and $\varepsilon = \frac{1}{200}$. In Fig. 3(a,b) the correlation coefficient is $\rho = 0$, therefore the Black and Scholes price and the corrected price at $O(\varepsilon^{1/2})$ coincide as follows from the expression (5.8). The solutions are computed at two levels of the current volatility $\sigma^2 = 0.165$ and $\sigma^2 = 0.66$ and in the range around the money $0.9 \leqslant S/K \leqslant 1.1$ the maximum deviation of the asymptotic approximation (at $O(\varepsilon)$) from the price with
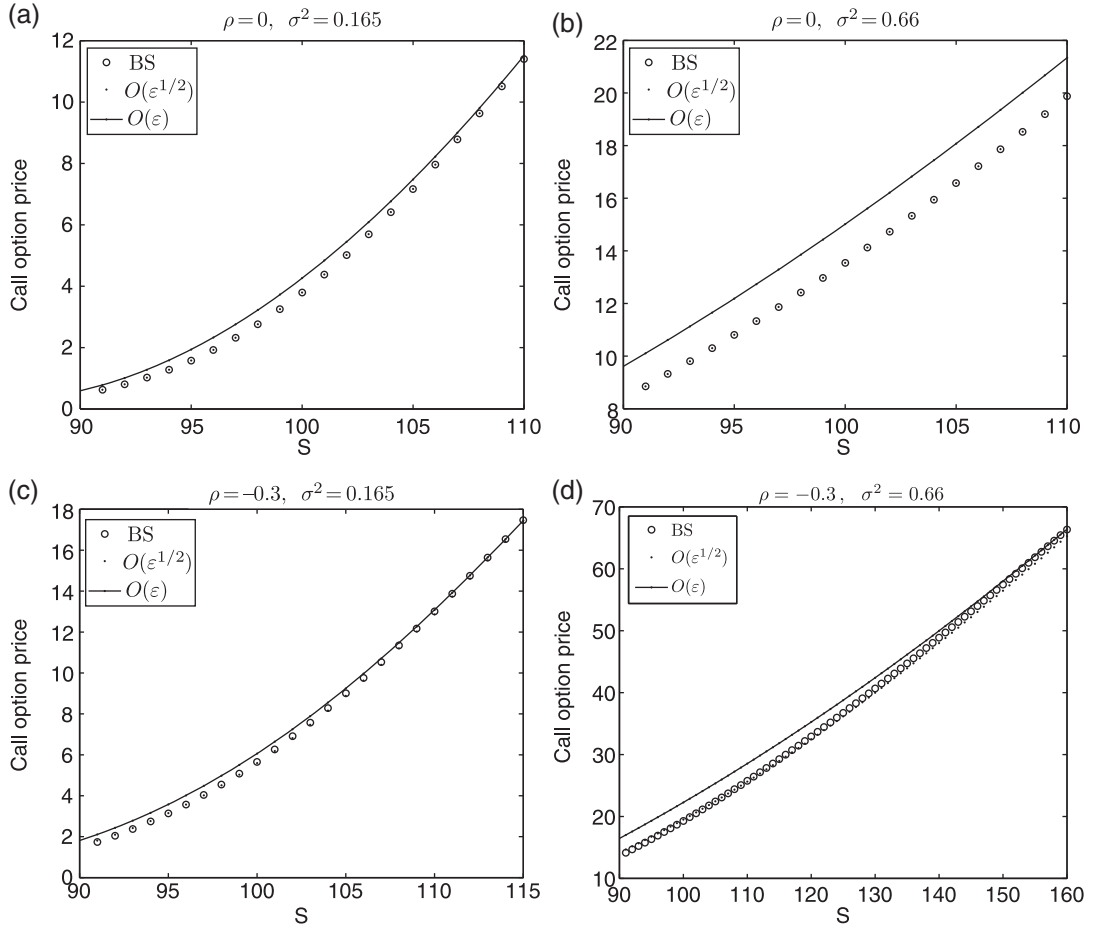
FIG. 3. Comparison between the classical price for a European call option and the price including the correction up to $O(\varepsilon^{1/2})$ and up to $O(\varepsilon)$, at two levels of the current volatility. (a, b) The region around the money, with $\rho = 0$ and (c, d) the region out of the money, with $\rho = -0.3$.

the lower volatility is 2.1% of this price, with the higher volatility is 7.5%. In Fig. 3(c,d) the correlation coefficient is $\rho = -0.3$. We note that in the region out of the money the second correction of the price obtained at $O(\varepsilon)$ improves the approximation obtained at $O(\varepsilon^{1/2})$. We want to remark that the oscillatory behaviour exhibited in Fig. 2(b–d) by y for small values of S has been already observed by Whalley & Wilmott (1997) also in models with constant volatility, although this feature seems to be more pro-nounced in the present context. Moreover, the thickness of the NT region in the presence of stochastic volatility seems to be bigger than in model with constant volatility. The $(\varepsilon)^{1/3}$ scaling was also a relevant feature already established in Whalley & Wilmott (1997) which is exhibited also by the present model.

Acknowledgements

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Appendix A. The source term in the equation for C_6

We separately compute the four terms appearing into the source of (5.13) for C_6.

A.1 *The term* $\langle \mathcal{L}_1 U_9^1 \rangle - \langle \mathcal{L}_1 U_9^w \rangle$

In the source term we have $[\langle \mathcal{L}_1 U_9^1 \rangle - \langle \mathcal{L}_1 U_9^w \rangle]$. We now write $\langle \mathcal{L}_1 U_9^j \rangle$ for $j = 1, w$, and for notational simplicity we omit the index j.

Consider (4.33) and solve it for U_9:

$$U_9 = -\mathcal{L}_0^{-1}(\mathcal{L}_2 U_3 + \mathcal{L}_1 U_6). \tag{A.1}$$

One can go back to (4.35) for the term $\mathcal{L}_2 U_3$:

$$\mathcal{L}_2 U_3 = \frac{1}{2} S^2 U_{3SS} \mathcal{L}_0 \varphi + \frac{\nu \rho}{\sqrt{2}} \left[\langle f \varphi' \rangle (S^3 U_{0SSS} + 2 S^2 U_{0SS}) \right.$$
$$\left. - (\alpha - r) S^2 U_{0SS} \left\langle \frac{\varphi'}{f} \right\rangle - \frac{\delta}{\gamma} (\alpha - r)^3 \left\langle \frac{\psi'}{f} \right\rangle \right].$$

Being $\mathcal{L}_1 U_6 = \mathcal{L}_1 U_6^{(z)}$, from (4.26) it follows that

$$\mathcal{L}_1 U_6^{(z)} = \frac{\nu \rho}{\sqrt{2}} \left[-f \varphi' (2 S^2 U_{0SS} + S^3 U_{0SSS}) + (\alpha - r) S^2 U_{0SS} \frac{\varphi'}{f} + \frac{\delta}{\gamma} (\alpha - r)^3 \frac{\psi'}{f} \right].$$

Defining F, G and H as the solutions of the following Poisson equations:

$$\mathcal{L}_0 F(z) = f \varphi' - \langle f \varphi' \rangle,$$
$$\mathcal{L}_0 G(z) = \frac{\varphi'}{f} - \left\langle \frac{\varphi'}{f} \right\rangle,$$
$$\mathcal{L}_0 H(z) = \frac{\psi'}{f} - \left\langle \frac{\psi'}{f} \right\rangle,$$

the formula (A.1) writes as

$$U_9 = -\frac{1}{2} S^2 U_{3SS} \varphi + \frac{\nu \rho}{\sqrt{2}} \left[(S^3 U_{0SSS} + 2 S^2 U_{0SS}) F(z) - (\alpha - r) S^2 U_{0SS} G(z) - \frac{\delta}{\gamma} (\alpha - r)^3 H(z) \right].$$

The first term in the right-hand side of (5.13) is

$$\langle \mathcal{L}_1 U_9^1 \rangle - \langle \mathcal{L}_1 U_9^w \rangle = v^2 \rho^2 S^2 (T-t)\left(\langle \varphi' f \rangle - \frac{1}{2}(\alpha - r)\left\langle \frac{\varphi'}{f} \right\rangle \right)\left[\langle f \varphi' \rangle (S^3 C_{5S}^{BS} + 8 S^2 C_{4S}^{BS} \right.$$

$$+ 14 S C_{SSS}^{BS} + 4 C_{SS}^{BS}) - (\alpha - r)\left\langle \frac{\varphi'}{f} \right\rangle (S^2 C_{4S}^{BS} + 4 S C_{SSS}^{BS} + 2 C_{SS}^{BS}) \bigg]$$

$$+ v^2 \rho^2 (\alpha - r)\left[(\alpha - r) S^2 C_{SS}^{BS}\left\langle \frac{G'}{f} \right\rangle - (S^3 C_{SSS}^{BS} + 2 S^2 C_{SS}^{BS})\left\langle \frac{F'}{f} \right\rangle \right]$$

$$+ \frac{v^2 \rho^2}{2} S^3 (T-t)\langle f\varphi' \rangle \left[(S^3 C_{6S}^{BS} + 11 S^2 C_{5S}^{BS} + 30 S C_{4S}^{BS} + 18 C_{SSS}^{BS})\langle f\varphi' \rangle \right.$$

$$- (\alpha - r)(S^2 C_{5S}^{BS} + 6 S C_{5S}^{BS} + 6 S C_{4S}^{BS} + 6 C_{SSS}^{BS})\left\langle \frac{\varphi'}{f} \right\rangle \bigg]$$

$$+ v^2 \rho^2 \left[(S^4 C_{4S}^{BS} + 5 S^3 C_{4S}^{BS} + 5 S^3 C_{SSS}^{BS} + 4 S^2 C_{SS}^{BS})\langle F' f \rangle \right.$$

$$- (\alpha - r)(S^3 C_{SSS}^{BS} + 2 S^2 C_{SS}^{BS})\langle G' f \rangle \big].$$

A.2   The term $(U_{3S}^1)^2 - (U_{3S}^w)^2$

Using the expressions (5.6) and (5.7), respectively, for U_3^1 and U_3^w, it follows that

$$(U_{3S}^1)^2 - (U_{3S}^w)^2 = -\frac{v^2 \rho^2}{2}(T-t)^2 \left[\langle f\varphi' \rangle (S^3 C_{4S}^{BS} + 5 S^2 C_{SSS}^{BS} + 4 S C_{SS}^{BS}) \right.$$

$$- (\alpha - r)\left\langle \frac{\varphi'}{f} \right\rangle (S^2 C_{SSS}^{BS} + 2 S C_{SS}^{BS}) \bigg]^2.$$

A.3   The term $U_{3S}^1 \langle f U_{6z}^{(z)1} \rangle - U_{3S}^w \langle f U_{6z}^{(z)w} \rangle$

Since U_3^1 is independent of S:

$$U_{3S}^1 \langle f U_{6z}^{(z)1} \rangle = 0.$$

Using the expression (4.29), we have

$$U_{3S}^w \langle f U_{6z}^{(z)w} \rangle = \frac{v\rho}{2\sqrt{2}}(T-t)\left[\langle f\varphi' \rangle (S^3 C_{4S}^{BS} + 5 S^2 C_{3S}^{BS} + 4 S C_{SS}^{BS}) - (\alpha - r)\left\langle \frac{\varphi'}{f} \right\rangle (2 S C_{SS}^{BS} + S^2 C_{3S}^{BS}) \right]$$

$$\times \left(S^2 C_{SS}^{BS}\langle \varphi' f \rangle - \frac{\delta}{\gamma}(\alpha - r)^2 \langle \psi' f \rangle \right).$$

A.4 The term $\langle (U_{6z}^{(z)1})^2 - (U_{6z}^{(z)w})^2 \rangle$

Using once again the formula (4.29) it follows

$$\langle (U_{6z}^{(z)1})^2 - (U_{6z}^{(z)w})^2 \rangle = -\frac{1}{4} S^4 (C_{SS}^{BS})^2 \langle \varphi'^2 \rangle + \frac{1}{2}\frac{\delta}{\gamma}(\alpha - r)^2 S^2 C_{SS}^{BS}\langle \varphi' \psi' \rangle.$$

Appendix B. Calculation of the averages $\langle \cdot \rangle$ for Scott's model

If one uses, for the volatility $f(z)$ the model of Scott, i.e.

$$f(z) = e^z,$$

then one can compute explicitly

$$\langle f^2 \rangle = e^{m+v^2},$$

$$\left\langle \frac{1}{f^2} \right\rangle = e^{m-v^2},$$

$$\left\langle \frac{\varphi'}{f} \right\rangle = \frac{1}{v^2}(e^{m+(1/2)v^2} - e^{m+(5/2)v^2}),$$

$$\left\langle \frac{\psi'}{f} \right\rangle = \frac{1}{v^2}(e^{(3/2)(-2m+3v^2)} - e^{m-(3/2)v^2}),$$

$$\langle \psi'f \rangle = \frac{1}{v^2}(e^{3(m-(1/2)v^2)} - e^{(1/2)(v^2-2m)}),$$

$$\langle f\varphi' \rangle = \frac{1}{v^2}(e^{3m+(5/2)v^2} - e^{3m+(9/2)v^2}),$$

$$\langle f \rangle = e^{m+(1/2)v^2},$$

$$\left\langle \frac{1}{f} \right\rangle = e^{-m+(1/2)v^2},$$

$$\langle f^2\varphi' \rangle = \frac{1}{2v^2}(e^{4(m+v^2)} - e^{4(m+2v^2)}),$$

$$\left\langle \frac{\varphi'}{f^2} \right\rangle = \frac{1}{2v^2}(1 - e^{4v^2}),$$

$$\langle \varphi' \rangle = -2\,e^{2(m+v^2)},$$

$$\langle fF' \rangle = \frac{1}{v^4}\left(e^{4m+3v^2} - e^{4m+5v^2} + \frac{1}{2}e^{4(m+2v^2)} - \frac{1}{2}e^{4(m+v^2)}\right),$$

$$\langle fG' \rangle = \frac{1}{v^4}(2v^2\,e^{2(m+v^2)} + e^{2m+v^2} - e^{2m+3v^2}),$$

$$\left\langle \frac{F'}{f} \right\rangle = \frac{1}{v^4}(-2v^2\,e^{2(m+v^2)} + e^{2m+5v^2} - e^{2m+3v^2}),$$

$$\left\langle \frac{G'}{f} \right\rangle = \frac{1}{v^4}\left(\frac{1}{2} - \frac{1}{2}e^{4v^2} + e^{3v^2} - e^{v^2}\right),$$

$$\left\langle \frac{H'}{f} \right\rangle = \frac{1}{v^4}\left(-\frac{1}{2} + \frac{1}{2}e^{4(2v^2-m)} - e^{-4m+5v^2} + e^{-v^2}\right).$$

Finally, the term $\langle \varphi' \psi' \rangle$ can be written as

$$\langle \varphi' \psi' \rangle = \frac{2\,e^{4v^2}}{\sqrt{\pi}\,v^2} \int_{-\infty}^{+\infty} e^{u^2}\,(\mathrm{erf}(u - \sqrt{2}v) - \mathrm{erf}(u))(\mathrm{erf}(u + \sqrt{2}v) - \mathrm{erf}(u))\,du.$$

The value of this integral is numerically calculated. In particular, for $v = 1/\sqrt{2}$ we obtain $\langle \varphi' \psi' \rangle = -24.8229$.

Analogously, we rewrite $\langle \varphi'^2 \rangle$ in the following form

$$\langle \varphi'^2 \rangle = \frac{2\,e^{4(m+v^2)}}{\sqrt{\pi}\,v^2} \int_{-\infty}^{+\infty} e^{u^2}\,(\mathrm{erf}(u - \sqrt{2}v) - \mathrm{erf}(u))^2\,du$$

and we numerically compute it. For $v = 1/\sqrt{2}$, it is $\langle \varphi'^2 \rangle = 17.329\,e^{2(2m+1)}$.

Appendix C. The derivatives of C^{BS}

The derivatives of C^{BS} with respect to S, up to the sixth order, are explicitly calculated as follows

$$\partial_S C^{\mathrm{BS}} = N(d_1),$$

$$\partial_{SS} C^{\mathrm{BS}} = \frac{e^{-d_1^2/2}}{S\bar{\sigma}\sqrt{2\pi(T-t)}},$$

$$\partial_S^3 C^{\mathrm{BS}} = \frac{-e^{-d_1^2/2}}{S^2\bar{\sigma}\sqrt{2\pi(T-t)}}\left(1 + \frac{d_1}{\bar{\sigma}\sqrt{T-t}}\right),$$

$$\partial_S^4 C^{\mathrm{BS}} = \frac{e^{-d_1^2/2}}{S^3\bar{\sigma}\sqrt{2\pi(T-t)}}\left(\frac{d_1^2 - 1}{\bar{\sigma}^2(T-t)} + \frac{3d_1}{\bar{\sigma}\sqrt{T-t}} + 2\right),$$

$$\partial_S^5 C^{\mathrm{BS}} = \frac{e^{-d_1^2/2}}{S^4\bar{\sigma}\sqrt{2\pi(T-t)}}\left[-\left(\frac{d_1}{\bar{\sigma}\sqrt{T-t}} + 3\right)\left(\frac{d_1^2 - 1}{\bar{\sigma}^2(T-t)} + \frac{3d_1}{\bar{\sigma}\sqrt{T-t}} + 2\right)\right.$$
$$\left. + \frac{1}{\bar{\sigma}^2(T-t)}\left(\frac{2d_1}{\bar{\sigma}\sqrt{T-t}} + 3\right)\right],$$

$$\partial_S^6 C^{\mathrm{BS}} = \frac{e^{-d_1^2/2}}{S^5\bar{\sigma}\sqrt{2\pi(T-t)}}\left\{\left(\frac{d_1^2 - 1}{\bar{\sigma}^2(T-t)} + \frac{3d_1}{\bar{\sigma}\sqrt{T-t}} + 2\right)\left[\left(\frac{d_1}{\bar{\sigma}\sqrt{T-t}} + 3\right)\right.\right.$$
$$\left.\times \left(\frac{d_1}{\bar{\sigma}\sqrt{T-t}} + 4\right) - \frac{1}{\bar{\sigma}(T-t)}\right] + \frac{2}{\bar{\sigma}^4(T-t)^2}$$
$$\left. - \frac{1}{\bar{\sigma}^2(T-t)}\left(\frac{2d_1}{\bar{\sigma}\sqrt{T-t}} + 3\right)\left(\frac{2d_1}{\bar{\sigma}\sqrt{T-t}} + 7\right)\right\}.$$