A new integral formula for the inverse Fueter mapping theorem

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Abstract

In this paper we provide an alternative method to construct the Fueter primitive of an axial monogenic function of degree k, which is complementary to the one used in [4]. As a byproduct, we obtain an explicit description of the kernel of the Fueter mapping. We also apply our method to obtain the Fueter primitives of the Cauchy kernels with singularities on the unit sphere.

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1 Introduction

Let us denote by $\mathbb{R}_{0,m}$ ($m \in \mathbb{N}$) the real Clifford algebra generated by the standard basis $\{e_1, \ldots, e_m\}$ of the Euclidean space $\mathbb{R}^m$ (see [2]). The multiplication in $\mathbb{R}_{0,m}$ is determined by the relations

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k = 1, \ldots, m$$

and a general element a of $\mathbb{R}_{0,m}$ may be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

in terms of the basis elements $e_A = e_{j_1} \dots e_{j_k}$, defined for every subset $A = \{j_1, \dots, j_k\}$ of $\{1, \dots, m\}$ with $j_1 < \cdots < j_k$. For the empty set, one puts $e_\emptyset = 1$, the latter being the identity element. Conjugation in $\mathbb{R}_{0,m}$ is given by $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e}_A = \overline{e}_{j_k} \cdots \overline{e}_{j_1}$ with $\overline{e}_j = -e_j$, $j = 1, \dots, m$.

Observe that $\mathbb{R}^{m+1}$ may be naturally embedded in the real Clifford algebra $\mathbb{R}_{0,m}$ by associating to any element $(x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$ the paravector given by

$$x_0 + \underline{x} = x_0 + \sum_{j=1}^m x_j e_j.$$

A function $f : \Omega \to \mathbb{R}_{0,m}$ defined and continuously differentiable in an open set Ω in $\mathbb{R}^{m+1}$ (resp. $\mathbb{R}^m$), is said to be monogenic if

$$(\partial_{x_0} + \partial_{\underline{x}})f = 0 \quad (\text{resp. } \partial_{\underline{x}} f = 0) \text{ in } \Omega,$$

where $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$ is the Dirac operator in $\mathbb{R}^m$ (see e.g. [1, 7, 9]). The differential operator $\partial_{x_0} + \partial_{\underline{x}}$, called generalized Cauchy-Riemann operator, gives a factorization of the Laplacian, i.e.

$$\Delta = \sum_{j=0}^m \partial_{x_j}^2 = (\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} - \partial_{\underline{x}}).$$

Thus the monogenic functions can be considered a subclass of the class of harmonic functions in $m + 1$ variables.

Throughout the paper we assume $h(z) = u(x, y) + iv(x, y)$ to be a holomorphic function in some open subset Ξ of the upper half of the complex plane $\mathbb{C}$ and $P_k(\underline{x})$ shall denote a homogeneous monogenic polynomial of degree k in $\mathbb{R}^m$. Let us recall the following generalization of Fueter's theorem obtained in [18]:

Theorem 1 *Put $\underline{\omega} = \underline{x}/r$, with $r = |\underline{x}|$, $\underline{x} \in \mathbb{R}^m$. If m is odd, then the function*

$$\mathsf{Ft}\,[h(z), P_k(\underline{x})]\,(x_0, \underline{x}) = \Delta^{k+\frac{m-1}{2}}\Big[\big(u(x_0, r) + \underline{\omega}\,v(x_0, r)\big)P_k(\underline{x})\Big]$$

is monogenic in $\Omega = \{(x_0, \underline{x}) \in \mathbb{R}^{m+1} : (x_0, r) \in \Xi\}$.

In other words, this result provides a way to generate monogenic functions starting from a holomorphic function in the upper half of the complex plane. It was originally formulated by R. Fueter in the setting of quaternionic analysis in [8] and later extended to the case of Clifford algebra-valued functions in [15, 17] (see also [10, 12, 14, 16]).

Remark 1 It easily seen that $\mathsf{Ft}\,[h(z), P_k(\underline{x})]$ defines an $\mathbb{R}$-linear operator between holomorphic functions and monogenic functions considered as real vector spaces, i.e.

$$\mathsf{Ft}\,[c_1 h_1(z) + c_2 h_2(z), P_k(\underline{x})] = c_1 \mathsf{Ft}\,[h_1(z), P_k(\underline{x})] + c_2 \mathsf{Ft}\,[h_2(z), P_k(\underline{x})]\,,$$

for all $c_1, c_2 \in \mathbb{R}$.

The functions generated by this technique are monogenic functions of the form

$$\big(A(x_0, r) + \underline{\omega}\, B(x_0, r)\big) P_k(\underline{x}), \tag{1}$$

where A and B are $\mathbb{R}$-valued continuously differentiable functions in $\mathbb{R}^2$ which satisfy the following Vekua-type system

$$\begin{cases} \partial_{x_0} A - \partial_r B &= \dfrac{2k + m - 1}{r}\, B \\ \partial_{x_0} B + \partial_r A &= 0. \end{cases} \tag{2}$$

Monogenic functions of the form (1) are called axial monogenic of degree k and represent an important class of functions in Clifford analysis (see [7]).

It is quite natural to ask whether given an axial monogenic function of degree k

$$H(x_0, \underline{x}) = \big(A(x_0, r) + \underline{\omega}\, B(x_0, r)\big) P_k(\underline{x}),$$

one can find a holomorphic function $h(z)$ such that

$$\mathsf{Ft}\,[h(z), P_k(\underline{x})]\,(x_0, \underline{x}) = H(x_0, \underline{x}).$$

The function h is called the Fueter primitive of H. This problem has been recently studied in [3, 4] and the Fueter primitive h has been explicitly constructed. To this purpose, it was necessary to determine the Fueter primitives $\mathcal{W}_{k,m}^{\pm}$ of the functions

$$\mathcal{F}_{k,m}^{+}(x_0, \underline{x}) = \int_{\mathbb{S}^{m-1}} \mathcal{G}(x_0 + \underline{x} - \underline{\omega})\, P_k(\underline{\omega})\, dS(\underline{\omega}),$$

$$\mathcal{F}_{k,m}^{-}(x_0, \underline{x}) = \int_{\mathbb{S}^{m-1}} \mathcal{G}(x_0 + \underline{x} - \underline{\omega})\, \underline{\omega}\, P_k(\underline{\omega}) dS(\underline{\omega}),$$

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where $dS(\underline{\omega})$ is the scalar element of surface area of $\mathbb{S}^{m-1}$, and

$$\mathcal{G}(x_0 + \underline{x}) = \frac{1}{A_{m+1}} \frac{\overline{x_0 + \underline{x}}}{|x_0 + \underline{x}|^{m+1}}$$

is the monogenic Cauchy kernel. Then, it is possible to express the Fueter primitive of H in terms of a suitable integral involving $\mathcal{W}_{k,m}^{\pm}$, A, B and P_k (see [4]). This method can be used on any axially symmetric open set of $\mathbb{R}^{m+1}$, i.e. on every open set which is invariant under rotations that fix the real axis x_0.

Every monogenic function f defined on an axially symmetric open set of $\mathbb{R}^{m+1}$ can be written as $f = \sum_{k=0}^{\infty} f_k$ where f_k are axial monogenic functions of degree k. Hence for each term in the series we can provide a Fueter primitive as described above. We would like to note that the problem of inverting the Fueter mapping theorem has been recently tackled in the case of bi-axial monogenic functions (see [5]).

The aim of this paper is to present an alternative proof of the fact that the Fueter mapping is surjective on the set of axial monogenic functions of degree k and to explicitly provide their Fueter primitives. The method we present here is complementary to the one presented in [4] in the sense that we here integrate with respect to the radius r instead of the axial coordinate x_0. For the sake of simplicity the method is developed on a rectangle; of course it remains applicable on more general axially symmetric domains.

As a byproduct of this method we describe the kernel of the Fueter mapping. We also compute with this method an exact formula for the Fueter primitives of the Cauchy kernels with singularities on the unit sphere. This corresponds to the integrals of the standard Cauchy kernel over the unit sphere.

2   Some preliminary results

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. From the Cauchy formula for repeated integration, we know that an n-th antiderivative of f is given by

$$f^{(-n)}(x) = \frac{1}{(n-1)!} \int_a^x (x - t)^{n-1} f(t) dt.$$

Inspired by this formula, we wish to find the solutions of the equations

$$\left(x^{-1} \frac{d}{dx}\right)^n g(x) = f(x) \quad \text{and} \quad \left(\frac{d}{dx} x^{-1}\right)^n g(x) = f(x), \qquad (3)$$

where $f : [a, b] \to \mathbb{R}$ is a given continuous function.

Remark 2 It is worth noting that the following identities hold (see [6, 13])

$$\left(x^{-1}\frac{d}{dx}\right)^n g(x) = \sum_{j=1}^{n}(-1)^{n+j}a_{j,n}x^{j-2n}\frac{d^j g}{dx^j}(x)$$

with

$$a_{j,n} = \frac{(2n-j-1)!}{2^{n-j}(n-j)!(j-1)!}$$

and

$$\left(\frac{d}{dx}x^{-1}\right)^n g(x) = \sum_{j=0}^{n}(-1)^{n+j}a_{j+1,n+1}x^{j-2n}\frac{d^j g}{dx^j}(x).$$

Moreover, the integers

$$a_{j+1,n+1} = \frac{(2n-j)!}{2^{n-j}(n-j)!j!}$$

are the coefficients of the Bessel polynomial of degree n (see [11]).

In order to find the solutions of (3), we define the following functions:

$$\phi_n(x) = \int_a^x t\phi_{n-1}(t)dt, \quad \psi_n(x) = x\int_a^x \psi_{n-1}(t)dt, \quad x \in [a, b], \; n \geq 1,$$

with $\phi_0(x) = \psi_0(x) = f(x)$. Obviously, ϕ_n and ψ_n satisfy

$$\frac{\phi_n'(x)}{x} = \phi_{n-1}(x), \quad \left(\frac{\psi_n(x)}{x}\right)' = \psi_{n-1}(x), \quad \phi_n(a) = \psi_n(a) = 0, \; n \geq 1. \quad (4)$$

Lemma 1 *The functions ϕ_n and ψ_n are given by*

$$\phi_n(x) = \frac{1}{(2n-2)!!}\int_a^x t(x^2-t^2)^{n-1}f(t)dt, \quad (5)$$

$$\psi_n(x) = \frac{x}{(2n-2)!!}\int_a^x (x^2-t^2)^{n-1}f(t)dt, \quad (6)$$

where $n!!$ denotes the double factorial of n.

Proof. Using integration by parts we obtain

$$\int_a^x t(x^2-t^2)^{n-1}f(t)dt = \int_a^x (x^2-t^2)^{n-1}\phi_1'(t)dt$$

$$= \left((x^2-t^2)^{n-1}\phi_1(t)\right)\big|_{t=a}^{t=x} + 2(n-1)\int_a^x t(x^2-t^2)^{n-2}\phi_1(t)dt.$$

Then

$$\int_a^x t(x^2 - t^2)^{n-1} f(t) dt = 2(n-1) \int_a^x t(x^2 - t^2)^{n-2} \phi_1(t) dt.$$

We iterate this procedure until the $(x^2 - t^2)$ term vanishes. Thus after $n-1$ steps we have

$$\int_a^x t(x^2 - t^2)^{n-1} f(t) dt = 2^{n-1}(n-1)! \int_a^x t\phi_{n-1}(t) dt = 2^{n-1}(n-1)!\phi_n(x),$$

which proves (5). Formula (6) may be proved in a similar way. $\qquad\square$

As an immediate consequence of Lemma 1 we obtain:

Theorem 2 *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The general solution of the equation*

$$\left(x^{-1} \frac{d}{dx} \right)^n g(x) = f(x)$$

is

$$\phi_n(x) + \sum_{j=0}^{n-1} C_j x^{2j}$$

while the general solution of

$$\left(\frac{d}{dx} x^{-1} \right)^n g(x) = f(x)$$

is

$$\psi_n(x) + \sum_{j=0}^{n-1} \tilde{C}_j x^{2j+1},$$

where C_j and $\tilde{C}_j$, $j = 0, \ldots, n-1$, are arbitrary real constants.

3   The inverse Fueter mapping theorem revisited

Here plays an essential role the explicit form of $\mathsf{Ft}\,[h(z), P_k(\underline{x})]$ determined in [12], namely:

$$\mathsf{Ft}\,[h(z), P_k(\underline{x})]\,(x_0, \underline{x}) = (2k + m - 1)!!$$
$$\times \left(\left(r^{-1} \partial_r \right)^{k + \frac{m-1}{2}} u(x_0, r) + \underline{\omega} \left(\partial_r\, r^{-1} \right)^{k + \frac{m-1}{2}} v(x_0, r) \right) P_k(\underline{x}). \quad (7)$$

Theorem 3 (The inverse Fueter mapping theorem) *Let*

$$H(x_0, \underline{x}) = \big(A(x_0, r) + \underline{\omega}\, B(x_0, r)\big) P_k(\underline{x}),$$

be a given arbitrary axial monogenic function of degree k in

$$\Omega = \big\{ (x_0, \underline{x}) \in \mathbb{R}^{m+1} : \ (x_0, r) \in [a, b] \times [c, d] \subset \mathbb{R}^2, \ c > 0 \big\}.$$

The Fueter primitives of $H(x_0, \underline{x})$ exist and are given by

$$u(x_0, r) = K_N \int_c^r t(r^2 - t^2)^{N-1} A(x_0, t)dt + \sum_{j=0}^{N-1} \alpha_j(x_0) r^{2j}, \qquad (8)$$

$$v(x_0, r) = K_N\, r \int_c^r (r^2 - t^2)^{N-1} B(x_0, t)dt + \sum_{j=0}^{N-1} \beta_j(x_0) r^{2j+1}, \qquad (9)$$

where

$$K_N = \frac{1}{2N\left((2N-2)!!\right)^2}, \qquad N = k + \frac{m-1}{2}.$$

Moreover, the $\mathbb{R}$-valued functions $\alpha_j(x_0)$ and $\beta_j(x_0)$ satisfy the following differential equations

$$\alpha_j'(x_0) - (2j+1)\beta_j(x_0)$$
$$= (-1)^{N-j-1} K_N \binom{N-1}{j} c^{2(N-j)-1} B(x_0, c), \ j = 0, \ldots, N-1, \quad (10)$$

$$\beta_j'(x_0) + 2(j+1)\alpha_{j+1}(x_0)$$
$$= (-1)^{N-j} K_N \binom{N-1}{j} c^{2(N-j-1)} A(x_0, c), \ j = 0, \ldots, N-2, \quad (11)$$

$$\beta_{N-1}'(x_0) = -K_N A(x_0, c). \qquad (12)$$

Proof. Let $h(z) = u(x, y) + iv(x, y)$ be a Fueter primitive of H, i.e.

$$\mathsf{Ft}\,[h(z), P_k(\underline{x})]\,(x_0, \underline{x}) = H(x_0, \underline{x}),$$

then from (7) we obtain

$$\left(r^{-1}\partial_r\right)^{k+\frac{m-1}{2}} u(x_0, r) = \frac{A(x_0, r)}{(2k+m-1)!!},$$

$$\left(\partial_r\, r^{-1}\right)^{k+\frac{m-1}{2}} v(x_0, r) = \frac{B(x_0, r)}{(2k+m-1)!!}.$$

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Theorem 2 now yields (8) and (9). We must now investigate when u and v given by these formulae satisfy the Cauchy-Riemann equations, taking into account that A and B fulfill the Vekua-type system (2).

Let us define

$$I_1(x_0, r) = \int_c^r t(r^2 - t^2)^{N-1} A(x_0, t) dt, \tag{13}$$

$$I_2(x_0, r) = r \int_c^r (r^2 - t^2)^{N-1} B(x_0, t) dt. \tag{14}$$

It follows from (2) and (4) that

$$\partial_{x_0} I_1(x_0, r) = \int_c^r t(r^2 - t^2)^{N-1} \left(\partial_t B(x_0, t) + \frac{2N}{t} B(x_0, t) \right) dt$$

$$= -c(r^2 - c^2)^{N-1} B(x_0, c) + (2N - 1) \int_c^r (r^2 - t^2)^{N-1} B(x_0, t) dt$$

$$+ 2(N - 1) \int_c^r t^2 (r^2 - t^2)^{N-2} B(x_0, t) dt,$$

$$\partial_r I_2(x_0, r) =$$

$$\int_c^r (r^2 - t^2)^{N-1} B(x_0, t) dt + 2(N - 1) r^2 \int_c^r (r^2 - t^2)^{N-2} B(x_0, t) dt =$$

$$(2N - 1) \int_c^r (r^2 - t^2)^{N-1} B(x_0, t) dt + 2(N - 1) \int_c^r t^2 (r^2 - t^2)^{N-2} B(x_0, t) dt,$$

which implies that

$$\partial_{x_0} I_1(x_0, r) - \partial_r I_2(x_0, r) = -c(r^2 - c^2)^{N-1} B(x_0, c).$$

Similarly, we may verify that

$$\partial_r I_1(x_0, r) + \partial_{x_0} I_2(x_0, r) = r(r^2 - c^2)^{N-1} A(x_0, c).$$

We thus have

$$\partial_{x_0} u(x_0, r) - \partial_r v(x_0, r) = \sum_{j=0}^{N-1} \left(\alpha'_j(x_0) - (2j + 1)\beta_j(x_0) \right) r^{2j}$$

$$- K_N c \left(r^2 - c^2 \right)^{N-1} B(x_0, c),$$

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$$\partial_r u(x_0, r) + \partial_{x_0} v(x_0, r) = \sum_{j=0}^{N-2} \left(\beta_j'(x_0) + 2(j+1)\alpha_{j+1}(x_0) \right) r^{2j+1}$$
$$+ \beta_{N-1}'(x_0) r^{2N-1} + K_N \, r \, (r^2 - c^2)^{N-1} A(x_0, c).$$

Therefore u and v satisfy the Cauchy-Riemann equations if and only if (10), (11) and (12) are fulfilled. $\qquad \square$

This theorem thus asserts that $\mathsf{Ft}\,[h(z), P_k(\underline{x})]$ is surjective on the space of axial monogenic functions of degree k. Furthermore, we note that this operator is not injective since $\mathsf{Ft}\,[z^n, P_k(\underline{x})]\,(x_0, \underline{x}) = 0$ for $0 \le n \le 2k+m-2$, as it was observed in [6, 13].

In the next result, we show that the set of all real linear combinations of z^n, $0 \le n \le 2k + m - 2$, is indeed the kernel of $\mathsf{Ft}\,[h(z), P_k(\underline{x})]$.

Corollary 1 *Let $\mathbb{R}_{2k+m-2}[z]$ be the vector space of all polynomials with real coefficients in z of degree at most $2k + m - 2$. Then*

$$ker\,(\mathsf{Ft}\,[h(z), P_k(\underline{x})]) = \mathbb{R}_{2k+m-2}[z].$$

Proof. We only have to prove that

$$ker\,(\mathsf{Ft}\,[h(z), P_k(\underline{x})]) \subset \mathbb{R}_{2k+m-2}[z].$$

If $\mathsf{Ft}\,[h(z), P_k(\underline{x})]\,(x_0, \underline{x}) = 0$, then from (8) and (9) we obtain

$$u(x, y) = \sum_{j=0}^{N-1} \alpha_j(x) y^{2j}, \quad v(x, y) = \sum_{j=0}^{N-1} \beta_j(x) y^{2j+1}.$$

The differential equations (10), (11) and (12) now tell us that $\alpha_j(x)$ (resp. $\beta_j(x)$) are polynomials of degree at most $2(N - j) - 1$ (resp. $2(N - j - 1)$). Therefore

$$u(x, 0) = C_0 + C_1 x + \ldots C_{2N-1} x^{2N-1}, \quad v(x, 0) = 0,$$

for certain real constants $C_0, \ldots, C_{2N-1}$. Then clearly $h(z) \in \mathbb{R}_{2k+m-2}[z]$. $\square$

Remark 3 As $\mathsf{Ft}\,[h(z), P_k(\underline{x})]$ is an $\mathbb{R}$-linear operator, it is clear that

$$\mathsf{Ft}\,[h_1(z), P_k(\underline{x})] = \mathsf{Ft}\,[h_2(z), P_k(\underline{x})] \Leftrightarrow h_1(z) - h_2(z) \in \mathbb{R}_{2k+m-2}[z].$$

We end the paper with two examples.

Example 1 From [12, 13] we have that

$$\mathsf{Ft}\left[1/z, P_k(\underline{x})\right](x_0, \underline{x}) = (-1)^{k+\frac{m-1}{2}}((2k+m-1)!!)^2 \left(\frac{x_0 - \underline{x}}{|x_0 + \underline{x}|^{2k+m+1}}\right) P_k(\underline{x}).$$

Thus if we apply (8) and (9) to this monogenic function we should be able to obtain the Cauchy kernel in the plane. Let us illustrate this for the case $k = 0$, $m = 5$. For this case $N = 2$ and

$$A(x_0, r) = \frac{x_0}{(x_0^2 + r^2)^3}, \quad B(x_0, r) = -\frac{r}{(x_0^2 + r^2)^3}.$$

It easily follows that

$$\beta_0(x_0) = -\frac{x_0^2 + 2c^2}{64(x_0^2 + c^2)^2}, \quad \alpha_0(x_0) = -x_0\beta_0(x_0),$$

$$\beta_1(x_0) = \frac{1}{64(x_0^2 + c^2)^2}, \quad \alpha_1(x_0) = -x_0\beta_1(x_0),$$

$$I_1(x_0, r) = \frac{(r^2 - c^2)^2 x_0}{4(x_0^2 + r^2)(x_0^2 + c^2)^2}, \quad I_2(x_0, r) = -\frac{(r^2 - c^2)^2 r}{4(x_0^2 + r^2)(x_0^2 + c^2)^2},$$

where I_1 and I_2 denote the functions defined in (13) and (14), respectively.

Using (8) and (9) we obtain

$$u(x_0, r) = \frac{x_0}{64(x_0^2 + r^2)}, \quad v(x_0, r) = -\frac{r}{64(x_0^2 + r^2)}.$$

That is $h(z)$ equals (up to a multiplicative constant) the Cauchy kernel in the plane.

Example 2 In [3] we have considered the functions

$$\mathcal{N}^+(q) = \int_{\mathbb{S}^2} \mathcal{G}(q - \underline{\omega})\, dS(\underline{\omega}),$$

and

$$\mathcal{N}^-(q) = \int_{\mathbb{S}^2} \mathcal{G}(q - \underline{\omega})\, \underline{\omega}\, dS(\underline{\omega}), \quad q = x_0 + r\underline{\omega},$$

and their Fueter primitives $\mathcal{W}^\pm$ in order to provide the Fueter inverse of a regular function of a quaternionic variable. This corresponds to what we have discussed in the introduction, i.e. the integrals $\mathcal{F}_{k,m}^\pm$, in the particular case $k = 0$, $m = 3$.

In a closed form, these two functions can be written as

$$\mathcal{N}^+(q) = \frac{\underline{\omega}}{\pi r}\left(\frac{1}{1+q^2} - \frac{1}{r}\mathrm{Im}(\arctan q)\right),$$

and

$$\mathcal{N}^-(q) = \frac{\underline{\omega}}{\pi r}\left(\arctan q + \frac{q}{1+q^2} - \frac{1}{r}\mathrm{Im}(q\arctan q)\right).$$

We shall use formulae (8) and (9) to retrieve the fact that the Fueter primitive of $\mathcal{N}^+$ is the function $\mathcal{W}^+(z) = \frac{1}{2\pi}\arctan z$. Note that the function $\mathcal{N}^+$ is regular of axial type and it can thus be written as $\mathcal{N}^+(q) = A(x_0, r) + \underline{\omega}B(x_0, r)$ where

$$A(x_0, r) = \frac{1}{\pi}\frac{2x_0}{(1+x_0^2 - r^2)^2 + 4x_0^2 r^2}$$

and

$$B(x_0, r) = \frac{1}{2\pi r}\left(\frac{2\left(1 + x_0^2 - r^2\right)}{(1+x_0^2 - r^2)^2 + 4x_0^2 r^2} - \frac{1}{2r}\ln\left(\frac{x_0^2 + (r+1)^2}{x_0^2 + (r-1)^2}\right)\right).$$

We now compute I_1 and I_2 defined in (13) and (14) with $N = 1$. We get that

$$2\pi I_1(x_0, r) = 2\pi\int_c^r tA(x_0, t)dt = \arctan\left(\frac{2x_0}{1 - x_0^2 - t^2}\right)\Big|_{t=c}^{t=r}$$
$$= 2\mathrm{Re}(\arctan z) - \arctan\left(\frac{2x_0}{1 - x_0^2 - c^2}\right)$$
$$= 2\mathrm{Re}(\arctan z) - 4\pi\alpha_0(x_0),$$

$$2\pi I_2(x_0, r) = 2\pi r\int_c^r B(x_0, t)dt = \frac{r}{2t}\ln\left(\frac{x_0^2 + (t+1)^2}{x_0^2 + (t-1)^2}\right)\Big|_{t=c}^{t=r}$$
$$= 2\mathrm{Im}(\arctan z) - \frac{r}{2c}\ln\left(\frac{x_0^2 + (c+1)^2}{x_0^2 + (c-1)^2}\right)$$
$$= 2\mathrm{Im}(\arctan z) - 4\pi r\beta_0(x_0).$$

On account of (8) and (9), we conclude that a Fueter primitive of $\mathcal{N}^+(q)$ is the function $\mathcal{W}^+(z) = \frac{1}{2\pi}\arctan z$ as it was computed in [3].

The function $\mathcal{N}^-(q)$ is also of axial type and so $\mathcal{N}^-(q) = A(x_0, r) + \underline{\omega}B(x_0, r)$ where

$$A(x_0, r) = \frac{1}{2\pi}\left(\frac{1}{2r}\ln\left(\frac{x_0^2 + (r-1)^2}{x_0^2 + (r+1)^2}\right) + \frac{2(x_0^2 + r^2 - 1)}{(1+x_0^2 - r^2)^2 + 4x_0^2 r^2}\right)$$

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and

$$B(x_0, r) = \frac{x_0}{2\pi r} \left(\frac{1}{2r} \ln \left(\frac{x_0^2 + (r-1)^2}{x_0^2 + (r+1)^2} \right) + \frac{2(1 + x_0^2 + r^2)}{(1 + x_0^2 - r^2)^2 + 4x_0^2 r^2} \right).$$

For this case we compute I_1 and I_2 with $N = 1$ and obtain

$$2\pi I_1(x_0, r) = 2\pi \int_c^r t A(x_0, t) dt$$
$$= \left(x_0 \arctan \left(\frac{2x_0}{1 - x_0^2 - t^2} \right) - \frac{t}{2} \ln \left(\frac{x_0^2 + (t+1)^2}{x_0^2 + (t-1)^2} \right) \right) \Big|_{t=c}^{t=r}$$
$$= 2\mathrm{Re}(z \arctan z) - 4\pi \alpha_0(x_0),$$

$$2\pi I_2(x_0, r) = 2\pi r \int_c^r B(x_0, t) dt$$
$$= r \left(\frac{x_0}{2t} \ln \left(\frac{x_0^2 + (t+1)^2}{x_0^2 + (t-1)^2} \right) + \arctan \left(\frac{2x_0}{1 - x_0^2 - t^2} \right) \right) \Big|_{t=c}^{t=r}$$
$$= 2\mathrm{Im}(z \arctan z) - 4\pi r \beta_0(x_0).$$

Thus, using formulae (8), (9) we obtain that the Fueter primitive of $\mathcal{N}^-(q)$ is $\mathcal{W}^-(z) = \frac{1}{2\pi} z \arctan z$, as shown in [3].

Acknowledgments

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