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Basic Properties of a Mean Field Laser Equation

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Abstract. We study the nonlinear quantum master equation describing a laser under the mean field approximation. The quantum system is formed by a single mode optical cavity and two level atoms, which interact with reservoirs. Namely, we establish the existence and uniqueness of the regular solution to the nonlinear operator equation under consideration, as well as we get a probabilistic representation for this solution in terms of a mean field stochastic Schrödinger equation. To this end, we find a regular solution for the nonautonomous linear quantum master equation in Gorini-Kossakowski-Sudarshan-Lindblad form, and we prove the uniqueness of the solution to the nonautonomous linear adjoint quantum master equation in Gorini-Kossakowski-Sudarshan-Lindblad form. Moreover, we obtain rigorously the Maxwell-Bloch equations from the mean field laser equation.

Keywords: Open quantum system, nonlinear quantum master equation, Maxwell-Bloch equations, quantum master equation in the Gorini-Kossakowski-Sudarshan-Lindblad form, existence and uniqueness, regular solution, Ehrenfest-type theorem, stochastic Schrödinger equation.

1. Introduction

This paper provides the mathematical foundation for the nonlinear laser equation

$$
\begin{aligned}
\frac{d}{dt}\rho_t \;=\;& -i\,\frac{\omega}{2}[2\,a^\dagger a + \sigma^3, \rho_t] \qquad (1)\\
&+ g\Big[\Big(\mathrm{Tr}(\sigma^-\rho_t)a^\dagger - \mathrm{Tr}(\sigma^+\rho_t)a\Big) + \Big(\mathrm{Tr}(a^\dagger\rho_t)\sigma^- - \mathrm{Tr}(a\,\rho_t)\sigma^+\Big), \rho_t\Big]\\
&+ \kappa_-\Big(\sigma^-\rho_t\,\sigma^+ - \frac{1}{2}\sigma^+\sigma^-\rho_t - \frac{1}{2}\rho_t\,\sigma^+\sigma^-\Big)\\
&+ \kappa_+\Big(\sigma^+\rho_t\,\sigma^- - \frac{1}{2}\sigma^-\sigma^+\rho_t - \frac{1}{2}\,\rho_t\,\sigma^-\sigma^+\Big)
\end{aligned}
$$

$$+ 2\kappa\Big(a\,\rho_t a^\dagger - \frac{1}{2}a^\dagger a\rho_t - \frac{1}{2}\rho_t a^\dagger a\Big)\,,$$

where $\omega \in \mathbb{R}$, g is a nonzero real number, $\kappa, \kappa_+, \kappa_- > 0$ and ρ_t is an unknown nonnegative trace-class operator on $\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$. As usual, $[\,\cdot\,,\,\cdot\,]$ stands for the commutator of two operators,

$$\sigma^+ \;=\; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \sigma^- \;=\; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \sigma^3 \;=\; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and a, $a^\dagger$ are the closed operators on $\ell^2(\mathbb{Z}_+)$ given by

$$a\,e_n \;=\; \begin{cases} \sqrt{n}\,e_{n-1} & \text{if} \quad n \in \mathbb{N} \\ \quad 0 & \text{if} \quad n = 0 \end{cases}$$

and $a^\dagger e_n = \sqrt{n+1}\,e_{n+1}$ for all $n \in \mathbb{Z}_+$. Here and subsequently, $(e_n)_{n\geq 0}$ denotes the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$.

Under the mean field approximation, (1) describes the dynamics of a laser consisting of a radiation field coupled to a set of identical noninteracting two-level systems (see, e.g., Section 3.7.3 of [8] and [27, 32, 40, 47] for more details on mean field quantum master equations). The first term of the right-hand side of (1) is determined by the free Hamiltonians of the field mode and the atoms, the second term governs the atom-field interaction, and the last three terms, i.e., the Gorini-Kossakowski-Sudarshan-Lindblad superoperators [24, 31], represent decay/pumping in the atoms and radiation losses. We are interested in establishing rigorously the well-posedness of (1), the equations of motion of the observables $a + a^\dagger$, $\sigma^- + \sigma^+$ and σ^3, and a probabilistic representation of ρ_t. This gives the mathematical basis to study, for instance, dynamical properties of (1) and the numerical solution of (1).

Our approach to the nonlinear quantum master equation (1) involves the study of nonautonomous linear quantum master equations in the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form [1, 8, 24, 31]. In the time-homogeneous setup, E. B. Davies and A. M. Chebotarev [12, 17] constructed the minimal solution of GKSL linear master equations with unbounded coefficients (see, e.g., [13, 18]). Using semigroup methods, [11, 14, 15, 18] prove that these equations have a unique solution under a quantum version of the Lyapunov condition for nonexplosion of classical Markov processes. Applying probabilistic techniques, one deduces that the GKSL quantum master equation preserves the regularity of the initial state (see, e.g., [37]), and one also obtains the well-posedness of the GKSL adjoint quantum master equation with an initial condition given by an unbounded operator (see, e.g., [36]). Using a limit procedure, one gets a conservative solution to a linear adjoint quantum master equation with time-dependent coefficients (see, e.g, [10]). In this article, we address a class of time-local linear master equations, which describes relevant physical systems (see, e.g., [7, 9, 16, 26, 46]). Namely, by extending some results given by [36, 37], we construct a regular solution for the nonautonomous linear quantum master equation

$$\frac{d}{dt}\rho_t = G(t)\rho_t + \rho_t G(t)^* + \sum_{k=1}^{\infty} L_k(t)\rho_t L_k(t)^* \qquad t \geq 0\,, \tag{2}$$

where ρ_t is a density operator in $\mathfrak{h}$, the initial datum ρ_0 is regular, and $G(t), L_1(t), L_2(t), \ldots$ are linear operators in $\mathfrak{h}$ satisfying (on appropriate domain)

$$G(t) = -i\,H(t) - \frac{1}{2}\sum_{\ell=1}^{\infty} L_\ell(t)^* L_\ell(t)$$

with $H(t)$ self-adjoint operator in $\mathfrak{h}$. Furthermore, we prove the uniqueness of the solution to the adjoint version of (2), which models the evolution of the quantum observables in the Heisenberg picture. This leads to prove the well-posedness of the GKSL quantum master equation resulting from replacing in (1) the unknown values of $\mathrm{Tr}(\sigma^- \rho_t)$ and $\mathrm{Tr}(a\,\rho_t)$ by known functions $\alpha(t)$ and $\beta(t)$.

Our main objective is to develop the mathematical theory for the nonlinear equation (1). First, we establish the existence and uniqueness of the regular solution to (1). In this direction, Belavkin [5, 6] treated a general class of nonlinear quantum master equations with bounded coefficients, and Kolokoltsov [30] obtained the well-posedness of nonlinear quantum dynamic semigroups having nonlinear Hamiltonians that are bounded perturbations of unbounded linear self-adjoint operators, together with nonlinear bounded Gorini-Kossakowski-Sudarshan-Lindblad superoperators. Arnold and Sparber [2] showed the existence and uniqueness of global solution to a nonlinear quantum master equation involving Hartree potential by means of semigroup techniques.

Moreover, we deal with the equations of motion for the mean values of a, σ^- and σ^3. It is well known that the following first-order differential equations is formally obtained from (1):

$$\begin{cases} \frac{d}{dt}\mathrm{Tr}(a\,\rho_t) = -(\kappa + i\,\omega)\,\mathrm{Tr}(a\,\rho_t) + g\ \mathrm{Tr}(\sigma^-\rho_t) \\ \frac{d}{dt}\mathrm{Tr}(\sigma^-\rho_t) = -(\gamma + i\,\omega)\,\mathrm{Tr}(\sigma^-\rho_t) + g\ \mathrm{Tr}(a\,\rho_t)\,\mathrm{Tr}(\sigma^3\rho_t) \\ \frac{d}{dt}\mathrm{Tr}(\sigma^3\rho_t) = -4g\,\mathrm{Re}\left(\mathrm{Tr}(a\,\rho_t)\,\overline{\mathrm{Tr}(\sigma^-\rho_t)}\right) - 2\gamma(\mathrm{Tr}(\sigma^3\rho_t) - d)\,, \end{cases} \tag{3}$$

where ≥ 0, $\gamma = (\kappa_+ + \kappa_-)/2$ and $d = (\kappa_+ - \kappa_-)/(\kappa_+ + \kappa_-)$ (see, e.g., [8]). Please verify!
In the semiclassical laser theory, the Maxwell-Bloch equations (3) describe the evolution of the field (i.e., $\mathrm{Tr}(a\,\rho_t)$), the polarization (i.e., $\mathrm{Tr}(\sigma^-\rho_t)$) and

the population inversion (i.e., $\mathrm{Tr}(\sigma^3\rho_t)$) of ring lasers like far-infrared NH_3 lasers (see, e.g., [25, 42, 48]). The system (3) has received much attention in the physical literature due to its important role in the description of laser dynamics (see, e.g., [8, 21, 41]). In this paper, we prove rigorously the validity of (3) whenever the initial state is regular enough, and thus we get an Ehrenfest theorem for (1) (see, e.g., [19, 22, 23]).

Finally, we obtain a probabilistic representation of (1). The solution of the linear quantum master equations in GKSL form is characterized as the mean value of random pure states given by the linear and nonlinear stochastic Schrödinger equations (see, e.g., [3, 4, 8, 37, 49]). This representation plays an important tool in the numerical simulation of open quantum systems (see, e.g., [8, 35, 33, 43, 45]), and it has also been used for proving theoretical properties of the GKSL quantum master equations (see, e.g., [20, 36, 37]). In this paper, we get a probabilistic representation of (1) in terms of a mean field version of the linear stochastic Schrödinger equation. To the best of our knowledge this is the first rigorously established result, at the level of infinite dimensional density matrices, with an unbounded nonlinear evolution operator, in the study of nonlinear mean field laser evolution equations

This paper is organized as follows. Section 2 presents the main results. Section 3 is devoted to general linear master equations. In Sect. 4 we study a linear quantum master equation associated with (1), moreover, for the sake of completeness, we recall the basic properties of the complex Lorenz equations. All proofs are deferred to Sect. 4.2.

1.1. Notation

In this paper, $(\mathfrak{h}, \langle \cdot, \cdot \rangle)$ is a separable complex Hilbert space, where the scalar product $\langle \cdot, \cdot \rangle$ is linear in the second variable and anti-linear in the first one. The standard basis of $\mathbb{C}^2$ is denoted by

$$e_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If A, B are linear operators in $\mathfrak{h}$, then $[A, B] = AB - BA$ and $\mathcal{D}(A)$ stands for the domain of A. We take $N = a^\dagger a$. In case $\mathfrak{X}$, $\mathfrak{Z}$ are normed spaces, we denote by $\mathfrak{L}(\mathfrak{X}, \mathfrak{Z})$ the set of all bounded operators from $\mathfrak{X}$ to $\mathfrak{Z}$ and we choose $\mathfrak{L}(\mathfrak{X}) = \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$. We write $\mathfrak{L}_1(\mathfrak{h})$ for the set of all trace-class operators on $\mathfrak{h}$ equipped with the trace norm. For simplicity of notation, generic nonnegative constants are denoted by K, as well as $K(\cdot)$ stands for different nondecreasing nonnegative functions on $[0, \infty[$.

Let C be a self-adjoint positive operator in $\mathfrak{h}$. Then, $\pi_C : \mathfrak{harrowh}$ is defined by $\pi_C(x) = x$ if $x \in \mathcal{D}(C)$ and $\pi_C(x) = 0$ if $x \notin \mathcal{D}(C)$, as well as $\|x\|_C = \sqrt{\langle x, x \rangle_C}$ with $\langle x, y \rangle_C = \langle x, y \rangle + \langle Cx, Cy \rangle$ for any $x, y \in \mathcal{D}(C)$. We write $L^2(\mathbb{P}, \mathfrak{h})$ for the set of all square integrable random variables from

$(\Omega, \mathfrak{F}, \mathbb{P})$ to $(\mathfrak{h}, \mathfrak{B}(\mathfrak{h}))$, where $\mathcal{B}(\mathfrak{Y})$ is the collection of all Borel set of the topological space $\mathfrak{Y}$. Finally, $L^2_C(\mathbb{P}, \mathfrak{h})$ denotes the set of all $\xi \in L^2(\mathbb{P}, \mathfrak{h})$ satisfying $\xi \in \mathcal{D}(C)$ a.s. and $\mathbb{E}(\|\xi\|^2_C) < \infty$.

2. Basic Properties of the Mean Field Laser Equation

This section presents the main results of the paper, which are summarized in Theorem 1 given below. We start by adapting the notion of regular weak solution — of a linear quantum master equation (see, e.g., [37] and Definition 5 given below) — to the mean field laser equation (1). To this end, we recall that a density operator ϱ is C-regular if, roughly speaking, $C\varrho C$ is a trace-class operator, where C is a suitable reference operator (see, e.g., [11, 37]).

DEFINITION 1 Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$. An operator $\varrho \in \mathfrak{L}_1(\mathfrak{h})$ is called density operator iff ϱ is a nonnegative operator with unit trace. The nonnegative operator $\varrho \in \mathfrak{L}(\mathfrak{h})$ is said to be C-regular iff $\varrho = \sum_{n\in\mathfrak{I}} \lambda_n |u_n\rangle\langle u_n|$ for some countable set $\mathfrak{I}$, summable nonnegative real numbers $(\lambda_n)_{n\in\mathfrak{I}}$ and collection $(u_n)_{n\in\mathfrak{I}}$ of elements of $\mathcal{D}(C)$, which together satisfy: $\sum_{n\in\mathfrak{I}} \lambda_n \|Cu_n\|^2 < \infty$. Let $\mathfrak{L}^+_{1,C}(\mathfrak{h})$ denote the set of all C-regular density operators in $\mathfrak{h}$.

DEFINITION 2 Let C be a self-adjoint positive operator in $\mathfrak{h}$. A family $(\rho_t)_{t\geq 0}$ of operators belonging to $\mathfrak{L}^+_{1,C}(\mathfrak{h})$ is called C-weak solution to (1) iff the function $t \mapsto \mathrm{Tr}(a\rho_t)$ is continuous and for all $t \geq 0$ we have

$$\frac{d}{dt}\mathrm{Tr}(A\rho_t) \;=\; \mathrm{Tr}(A\mathcal{L}_\star(\rho_t)\rho_t) \qquad \forall\, A \in \mathfrak{L}(\mathfrak{h})\,,$$

where

$$\begin{aligned}
\mathcal{L}_\star(\widetilde{\varrho})\varrho \;=\; & -\frac{i\,\omega}{2}[2a^\dagger a + \sigma^3, \varrho] + 2\kappa\Big(a\,\varrho a^\dagger - \frac{1}{2}a^\dagger a\varrho - \frac{1}{2}\varrho a^\dagger a\Big) \\
& + \kappa_-\Big(\sigma^-\varrho\,\sigma^+ - \frac{1}{2}\sigma^+\sigma^-\varrho - \frac{1}{2}\varrho\,\sigma^+\sigma^-\Big) \\
& + \kappa_+\Big(\sigma^+\varrho\,\sigma^- - \frac{1}{2}\sigma^-\sigma^+\varrho - \frac{1}{2}\varrho\,\sigma^-\sigma^+\Big) \\
& + g[\mathrm{Tr}(\sigma^-\widetilde{\varrho})a^\dagger - \mathrm{Tr}(\sigma^+\widetilde{\varrho})a, \varrho] + g[\mathrm{Tr}(a^\dagger\widetilde{\varrho})\sigma^- - \mathrm{Tr}(a\,\widetilde{\varrho})\sigma^+, \varrho]\,.
\end{aligned}$$

Similar to the linear case, (1) is strongly related with the following nonlinear stochastic evolution equation on $\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2$:

$$\begin{aligned}
Z_t(\xi) \;=\; & \xi + \int_0^t \Big(-\,i\,H(t, Z_t(\xi)) - \frac{1}{2}\sum_{\ell=1}^{3} L^*_\ell L_\ell\Big) Z_s(\xi)ds \qquad (4) \\
& + \sum_{\ell=1}^{3}\int_0^t L_\ell\, Z_s(\xi)dW^\ell_s\,,
\end{aligned}$$

where $L_1 = \sqrt{2\kappa}\, a$, $L_2 = \sqrt{\gamma(1-d)}\, \sigma^-$, $L_3 = \sqrt{\gamma(1+d)}\, \sigma^+$,

$$\begin{aligned} H(t, Z_t(\xi)) &= \frac{\omega}{2}(2a^\dagger a + \sigma^3) \\ &\quad + i\, g\Big(\mathbb{E}\langle Z_t(\xi), \sigma^- Z_t(\xi)\rangle a^\dagger - \mathbb{E}\langle Z_t(\xi), \sigma^+ Z_t(\xi)\rangle a\Big) \\ &\quad + i\, g\Big(\mathbb{E}\langle Z_t(\xi), a^\dagger Z_t(\xi)\rangle \sigma^- - \mathbb{E}\langle Z_t(\xi), a\, Z_t(\xi)\rangle \sigma^+\Big)\,, \end{aligned} \tag{5}$$

and W^1, W^2, W^3 are real valued independent Wiener processes on a filtered complete probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$. Next, we tailor the concept of regular weak solution — used in the framework of stochastic Schödinger equations (see, e.g., [19, 38, 39] and Definition 4 given below) — to suit (4).

DEFINITION 3 Let $p \in \mathbb{N}$. An $\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$-valued adapted process with continuous sample paths $(Z_t(\xi))_{t\in\mathbb{I}}$ is called strong N^p-solution of (4) if:

- For all $t \geq 0$: $\mathbb{E}\|Z_t(\xi)\|^2 \leq K(t)\mathbb{E}\|\xi\|^2$, $Z_t(\xi) \in \mathcal{D}(N^p)$ a.s., and
$$\sup_{s\in[0,t]} \mathbb{E}\|N^p X_s(\xi)\|^2 \;<\; \infty\,.$$
- The functions $t \mapsto \mathbb{E}\langle Z_t(\xi), \sigma^- Z_t(\xi)\rangle$ and $t \mapsto \mathbb{E}\langle Z_t(\xi), a\, Z_t(\xi)\rangle$ are continuous.
- a.s. for all $t \geq 0$:
$$\begin{aligned} Z_t(\xi) &= \xi + \int_0^t \Big(-i\, H(t) - \frac{1}{2}\sum_{\ell=1}^{3} L_\ell^* L_\ell\Big)\pi_{N^p}(Z_s(\xi))ds \\ &\quad + \sum_{\ell=1}^{3}\int_0^t L_\ell\, \pi_{N^p}(Z_s(\xi))dW_s^\ell \end{aligned}$$
with $H(t, Z_t(\xi))$ described by (5), and L_ℓ, W^ℓ as in (4).

Now, we establish the existence and uniqueness of the regular solution to (1), a Ehrenfest-type theorem describing the evolution of the mean values of the observables $a + a^\dagger$, $\sigma^- + \sigma^+$ and σ^3, and the probabilistic representation of (1).

THEOREM 1 *Suppose that $\varrho \in \mathfrak{L}^+_{1,N^p}(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$, with $p \in \mathbb{N}$. Then, there exists a unique N^p-weak solution $(\rho_t)_{t\geq 0}$ to (1) with initial datum ϱ. Moreover, the Maxwell-Bloch equations (3) hold, and*

$$\rho_t \;=\; \mathbb{E}|Z_t(\xi)\rangle\langle Z_t(\xi)| \quad \forall\, t \geq 0\,, \tag{6}$$

where $\xi \in L^2_{N^p}(\mathbb{P}, \mathfrak{h})$ satisfies $\varrho = \mathbb{E}|\xi\rangle\langle\xi|$, and $Z_t(\xi) \in \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$ is the strong N^p-solution of (4).

Proof. Deferred to Sect. A.7. □

Remark 1 If $g^2 d < \kappa\gamma$, then $(0, 0, d)$ is an asymptotically stable equilibrium point of (3). In fact, from (23) and (24), given below, it follows that $\mathrm{Tr}(a\,\rho_t)$, $\mathrm{Tr}(\sigma^- \rho_t)$ and $\mathrm{Tr}(\sigma^3 \rho_t) - d$ converge exponentially fast to 0 as t goes to $+\infty$.

3. General Linear Quantum Master Equations

3.1. Regular solution for the GKSL quantum master equation

This subsection provides a regular solution for the linear quantum master equation (2). By generalizing [37] to the nonautonomous framework, we will describe a solution of (2) with the help of the linear stochastic evolution equation in $\mathfrak{h}$:

$$X_t(\xi) \;=\; \xi + \int_0^t G(s) X_s(\xi) ds + \sum_{\ell=1}^{\infty} \int_0^t L_\ell(s) X_s(\xi) dW_s^\ell \,, \tag{7}$$

where $W^1, W^2, \ldots$ are real-valued independent Wiener processes on a filtered complete probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$.

Suppose that the density operator ρ_0 is C-regular. According to Theorem 3.1 of [37] we have $\rho_0 = \mathbb{E}|\xi\rangle\langle\xi|$ for certain $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$. We set

$$\rho_t \;:=\; \mathbb{E}|X_t(\xi)\rangle\langle X_t(\xi)| \,, \tag{8}$$

where we use Dirac notation, $X_t(\xi)$ is the unique strong C-solution of (7) (see Definition 4), and the mathematical expectation can be interpreted as a Bochner integral in both $\mathfrak{L}_1(\mathfrak{h})$ and $\mathfrak{L}(\mathfrak{h})$. Then, ρ_t is a C-regular density operator (see [37] for details).

HYPOTHESIS 1 *There exists a self-adjoint positive operator C in $\mathfrak{h}$ such that $\mathcal{D}(C) \subset \mathcal{D}(G(t))$ and $\mathcal{D}(C) \subset \mathcal{D}(L_\ell(t))$ for all $t \geq 0$, and $G(\cdot) \circ \pi_C$ and $L_\ell(\cdot) \circ \pi_C$ are measurable as functions from $([0,\infty[\times\mathfrak{h}, \mathcal{B}([0,\infty[\times\mathfrak{h}))$ to $(\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$.*

DEFINITION 4 Assume Hypothesis 1. Let $\mathbb{I}$ be either $[0,\infty[$ or $[0,T]$, with $T \in \mathbb{R}_+$. By strong C-solution of (7) with initial condition ξ, on the interval $\mathbb{I}$, we mean an $\mathfrak{h}$-valued adapted process $(X_t(\xi))_{t\in\mathbb{I}}$ with continuous sample paths such that for all $t \in \mathbb{I}$: $\mathbb{E}\|X_t(\xi)\|^2 \leq K(t)\mathbb{E}\|\xi\|^2$, $X_t(\xi) \in \mathcal{D}(C)$ a.s., $\sup_{s\in[0,t]} \mathbb{E}\|CX_s(\xi)\|^2 < \infty$, and

$$X_t(\xi) \;=\; \xi + \int_0^t G(s)\pi_C(X_s(\xi)) ds + \sum_{\ell=1}^{\infty} \int_0^t L_\ell(s)\pi_C(X_s(\xi)) dW_s^\ell \quad \text{a.s.}$$

The following theorem, which extends Theorem 4.4 of [37] to the nonautonomous context, asserts that ρ_t given by (8) is a regular solution to (2).

DEFINITION 5 Let C be a self-adjoint positive operator in $\mathfrak{h}$. A family $(\rho_t)_{t\geq 0}$ of C-regular density operators is called C-weak solution to (2) if and only if

$$\frac{d}{dt}\,\mathrm{Tr}(A\rho_t) \;=\; \mathrm{Tr}\left[A\Big(G(t)\rho_t \;+\rho_t G(t)^* + \sum_{\ell=1}^{\infty} L_\ell(t)\rho_t L_\ell(t)^*\Big)\right] \tag{9}$$

for all $A \in \mathfrak{L}(\mathfrak{h})$ and $t \geq 0$.

HYPOTHESIS 2 *Suppose that C satisfies Hypothesis 1, together with:*

(H2.1) *for any $t \geq 0$ and $x \in \mathcal{D}(C)$, $\|G(t)x\|^2 \leq K(t)\|x\|_C^2$,*

(H2.2) *for any $t \geq 0$ and $x \in \mathcal{D}(C)$, $2\mathrm{Re}\,\langle x, G(t)x\rangle + \sum_{\ell=1}^{\infty}\|L_\ell(t)x\|^2 = 0$,*

(H2.3) *for any initial datum $\xi \in L_C^2(\mathbb{P},\mathfrak{h})$, (7) has a unique strong C-solution on any bounded interval,*

(H2.4) *there exist functions $f_k : [0,\infty[\times[0,\infty[arrow[0,\infty[$ such that:*

(i) f_k is bounded on bounded subintervals of $[0,\infty[\times[0,\infty[$;

(ii) $\lim_{s\to t} f_k(s,t) = 0$ and

(iii) for all $s,t \geq 0$ and $x \in \mathcal{D}(C)$, we have

$$\|G(s)x - G(t)x\|^2 \;\leq\; f_0(s,t)\|x\|_C^2 \quad \text{and} \quad \|L_\ell(s)x - L_\ell(t)x\|^2 \;\leq\; f_\ell(s,t)\|x\|_C^2 .$$

THEOREM 2 *Let Hypotheses 1 and 2 hold. Assume that ϱ_0 be C-regular, and that $G(t), L_1(t),\ L_2(t),\ \ldots$ are closable for all $t \geq 0$. Then ρ_t given by (8) is a C-weak solution to (2). Moreover, for all $t \geq 0$ we have*

$$\rho_t \;=\; \rho_0 + \int_0^t \Big(G(s)\rho_s + \rho_s G(s)^* + \sum_{\ell=1}^{\infty} L_\ell(s)\rho_s L_\ell(s)^*\Big)ds\,, \tag{10}$$

where we understand the above integral in the sense of Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$.

Proof. Deferred to Sect. A.1 □

Remark 2 Sufficient conditions for the regularity of the solution to the linear stochastic Schödinger equation (7) (i.e., Hypothesis 2.3) are given, for instance, in [19, 34, 38].

3.2. Uniqueness of the solution to the adjoint quantum master equation in the GKSL form

The next theorem introduces the operator $\mathcal{T}_t(A)$ that describes the evolution of the observable A at time t in the Heisenberg picture. Roughly speaking, the maps $A \mapsto \mathcal{T}_t(A)$ is the adjoint operator of the application $\varrho \mapsto \rho_t$, where ρ_t is defined by (8).

HYPOTHESIS 3 *Let Hypothesis 1 hold together with (H2.1) and (H2.3). Suppose that*

(H3.1) *For all* $t \geq 0$ *and* $x \in \mathcal{D}(C)$,

$$2\mathrm{Re}\,\langle x, G(t)x\rangle + \sum_{\ell=1}^{\infty} \|L_\ell(t)x\|^2 \ \leq\ K(t)\|x\|^2 .$$

THEOREM 3 *Assume that Hypothesis 1 and (H2.1) and (H2.3) hold. Consider* $A \in \mathfrak{L}(\mathfrak{h})$. *Then, for every* $t \geq 0$ *there exists a unique* $\mathcal{T}_t(A) \in \mathfrak{L}(\mathfrak{h})$ *for which*

$$\langle x, \mathcal{T}_t(A)y\rangle \ =\ \mathbb{E}\langle X_t(x), AX_t(y)\rangle \quad \forall\, x, y \in \mathcal{D}(C)\,. \tag{11}$$

Moreover, $\sup_{t\in[0,T]} \|\mathcal{T}_t(A)\| < \infty$ *for all* $T \geq 0$.

Proof. Deferred to Sect. A.2 □

Theorem 4 below shows that $\mathcal{T}_t(A)$ is the unique possible solution of the adjoint quantum master equation

$$\frac{d}{dt}\mathcal{T}_t(A) \ =\ \mathcal{T}_t(A)G(t) + G(t)^*\mathcal{T}_t(A) + \sum_{k=1}^{\infty} L_k(t)^*\mathcal{T}_t(A)L_k(t)\,. \tag{12}$$

Thus, we generalize Theorem 2.2 of [37] to the nonautonomous framework.

THEOREM 4 *Let Hypothesis 3 holds, and let* $\mathcal{T}_t(A)$ *be as in Theorem 3 with* $A \in \mathfrak{L}(\mathfrak{h})$. *Assume that* $(\mathcal{A}_t)_{t\geq 0}$ *is a family of operators belonging to* $\mathfrak{L}(\mathfrak{h})$ *such that* $\mathcal{A}_0 = A$, $\sup_{s\in[0,t]} \|\mathcal{A}_s\|_{\mathfrak{L}(\mathfrak{h})} < \infty$, *and*

$$\frac{d}{dt}\langle x, \mathcal{A}_t y\rangle \ =\ \langle x, \mathcal{A}_t G(t)y\rangle + \langle G(t)x, \mathcal{A}_t y\rangle + \sum_{\ell=1}^{\infty}\langle L_\ell(t)x, \mathcal{A}_t L_\ell(t)y\rangle \tag{13}$$

for all $x, y \in \mathcal{D}(C)$. *Then* $\mathcal{A}_t = \mathcal{T}_t(A)$ *for all* $t \geq 0$.

Proof. Deferred to Sect. A.3 □

Remark 3 In the autonomous case, [36, 37] obtain sufficient conditions for $\mathcal{T}_t(A)$ defined by (11) to be solution of (12). Using semigroup methods, [11, 14, 15, 18] show the existence and uniqueness of solutions to (2) and (12), in the semigroup sense.

In order to check (H2.3) we establish the following extension of Theorem 2.4 of [19].

HYPOTHESIS 4 *Suppose that C satisfies Hypothesis 1, together with*

(H4.1) *for any $t \geq 0$ and $x \in \mathcal{D}(C)$, $\|G(t)x\|^2 \leq K(t)\|x\|_C^2$,*

(H4.2) *for every $\ell \in \mathbb{N}$, there exists a nondecreasing function $K_\ell : [0, \infty[\to [0, \infty[$ satisfying $\|L_\ell(t)x\|^2 \leq K_\ell(t)\|x\|_C^2$ for all $x \in \mathcal{D}(C)$ and $t \geq 0$,*

(H4.3) *there exists a nondecreasing function $\alpha : [0, \infty[\to [0, \infty[$ and a core $\mathfrak{D}_1$ of C^2 such that for any $x \in \mathfrak{D}_1$ we have*

$$2\mathrm{Re}\,\langle C^2 x, G(t)x\rangle + \sum_{\ell=1}^{\infty} \|CL_\ell(t)x\|^2 \ \leq\ \alpha(t)\|x\|_C^2 \quad \forall\, t \geq 0\,,$$

(H4.4) *there exists a nondecreasing function $\beta : [0, \infty[\to [0, \infty[$ and a core $\mathfrak{D}_2$ of C such that*

$$2\mathrm{Re}\,\langle x, G(t)x\rangle + \sum_{\ell=1}^{\infty} \|L_\ell(t)x\|^2 \ \leq\ \beta(t)\|x\|^2 \quad \forall\, t \geq 0 \quad \forall\, x \in \mathfrak{D}_2\,.$$

THEOREM 5 *In addition to Hypothesis 4, we assume that $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$ is $\mathfrak{F}_0$-measurable. Then (7) has a unique strong C-solution $(X_t(\xi))_{t\geq 0}$ with initial condition ξ. Moreover,*

$$\mathbb{E}\|CX_t(\xi)\|^2 \ \leq\ K(t)(\mathbb{E}\|C\xi\|^2 + \mathbb{E}\|\xi\|^2)\,.$$

Proof. Our assertion can be proved in much the same way as Theorem 2.4 of [19]. □

Remark 4 Theorem 5 given above asserts that Theorem 2.4 of [19] still holds if we replace the assumption (H2.4) of [19] by Hypothesis (H4.4). We will apply Theorem 4 to the case: $L_1 = \sqrt{2\kappa}a^\dagger$, $L_2 = \sqrt{\gamma(1-d)}\sigma^+$, $L_3 = \sqrt{\gamma(1+d)}\sigma^-$ and $G(t) = i\,H(t) - \frac{1}{2}\sum_{\ell=1}^3 L_\ell L_\ell^*$ with

$$H(t) \ =\ \frac{\omega}{2}(2a^\dagger a + \sigma^3) + i\,g(\alpha(t)a^\dagger - \overline{\alpha(t)}a) + i\,g(\overline{\beta(t)}\sigma^- - \beta(t)\sigma^+)\,.$$

Since

$$G(t) + G(t)^* + \sum_{\ell=1}^{3} L_\ell^* L_\ell \ =\ 4\kappa^2 I + 2\gamma^2(1+d^2)\sigma_3\,,$$

condition (H2.4) of Theorem 2.4 of [19] does not apply to our situation.

4. Auxiliary Equations

4.1. Auxiliary linear quantum master equation

This subsection deals with the linear evolution equation obtained by replacing in (1) the unknown functions $t \mapsto g\ \mathrm{Tr}(\sigma^- \rho_t)$ and $t \mapsto g\ \mathrm{Tr}(a\,\rho_t)$ by general functions $\alpha, \beta : [0, \infty[\to \mathbb{C}$. More precisely, we study the well-posedness of the linear quantum master equation

$$\frac{d}{dt}\rho_t \;=\; \mathcal{L}^h_\star\,\rho_t + [\alpha(t)a^\dagger - \overline{\alpha(t)}a + \overline{\beta(t)}\sigma^- - \beta(t)\sigma^+, \rho_t]\,, \qquad (14)$$

where $\rho_t \in \mathfrak{L}^+_1(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$,

$$\begin{aligned} \mathcal{L}^h_\star\,\varrho \;=\; & \Big[-\frac{i\,\omega}{2}(2a^\dagger a + \sigma^3), \varrho\Big] + 2\kappa\Big(a\,\varrho a^\dagger - \frac{1}{2}a^\dagger a\varrho - \frac{1}{2}\varrho a^\dagger a\Big) \qquad (15)\\ & + \gamma(1-d)\Big(\sigma^-\varrho\,\sigma^+ - \frac{1}{2}\sigma^+\sigma^-\varrho - \frac{1}{2}\varrho\,\sigma^+\sigma^-\Big)\\ & + \gamma(1+d)\Big(\sigma^+\varrho\,\sigma^- - \frac{1}{2}\sigma^-\sigma^+\varrho - \frac{1}{2}\varrho\,\sigma^-\sigma^+\Big)\,, \end{aligned}$$

where $d \in]-1, 1[$, $\omega \in \mathbb{R}$ and $\kappa, \gamma > 0$. Furthermore, we represent (14) by using

$$X_t(\xi) \;=\; \xi + \int_0^t G(s)X_s(\xi)ds + \sum_{\ell=1}^{3}\int_0^t L_\ell(s)X_s(\xi)dW^\ell_s\,, \qquad (16)$$

where $X_t(\xi) \in \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$, W^1, W^2, W^3 are real valued independent Wiener processes, $L_1 = \sqrt{2\kappa}\,a$, $L_2 = \sqrt{\gamma(1-d)}\,\sigma^-$, $L_3 = \sqrt{\gamma(1+d)}\,\sigma^+$ and $G(t) = -i\,H(t) - \frac{1}{2}\sum_{\ell=1}^{3} L^*_\ell L_\ell$ with

$$H(t) \;=\; \frac{\omega}{2}(2a^\dagger a + \sigma^3) + i\,(\alpha(t)a^\dagger - \overline{\alpha(t)}a) + i\,(\overline{\beta(t)}\sigma^- - \beta(t)\sigma^+)\,.$$

Though the open quantum system (14) deserves attention in its own right, our main objective is to develop key tools for proving the results of Sect. 2. First, combining Theorems 2, 4 and 5 we obtain the existence and uniqueness of the regular solution to (14).

THEOREM 6 *Consider (14) with $\alpha, \beta : [0, \infty[\to \mathbb{C}$ continuous. Let ϱ be N^p-regular, where $p \in \mathbb{N}$. Then, there exists a unique N^p-weak solution $(\rho_t)_{t\geq 0}$ to (14) with initial datum $\rho_0 = \varrho$. Moreover, for any $t \geq 0$ we have*

$$\rho_t \;=\; \rho_0 + \int_0^t \Big(\mathcal{L}^h_\star\,\rho_s + [\alpha(s)a^\dagger - \overline{\alpha(s)}a + \overline{\beta(s)}\sigma^- - \beta(s)\sigma^+, \rho_s]\Big)ds \qquad (17)$$

and

$$\rho_t \;=\; \mathbb{E}|X_t(\xi)\rangle\langle X_t(\xi)| \quad \forall\, t \geq 0\,, \tag{18}$$

where the integral of (17) is understood in the sense of Bochner integral in $\mathfrak{L}_1(\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2)$, $\xi \in L^2_{N^p}(\mathbb{P},\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2)$ *satisfies* $\varrho = \mathbb{E}|\xi\rangle\langle\xi|$ *and* $X_t(\xi)$ *is the unique strong* N^p*-solution of (16).*

Proof. Deferred to Sect. A.4. □

Remark 5 Assume the framework of Theorem 6. From the proof of Theorem 6 it follows that $\mathbb{E}\|X_t(\xi)\|^2_{N^p} \leq K(t)\mathbb{E}\|\xi\|^2_{N^p}$ for all $t \geq 0$. In the operator language we have $\mathrm{Tr}(N^p\,\rho_t\,N^p) \leq K(t)(1+\mathrm{Tr}(N^p\,\rho_0\,N^p))$ (see, e.g., [37]) since $\mathbb{E}\|X_t(\xi)\|^2 = \mathbb{E}\|\xi\|^2 = 1$ (see, e.g., [19]).

Using the Ehrenfest-type theorem given in [19] we get a system of ordinary differential equations that describes the evolution of $\mathrm{Tr}(\rho_t\,a)$, $\mathrm{Tr}(\rho_t\,\sigma^-)$ and $\mathrm{Tr}(\rho_t\,\sigma^3)$.

THEOREM 7 *Under the assumptions and notation of Theorem 6,*

$$\frac{d}{dt}\,\mathrm{Tr}(\rho_t\,a) \;=\; -(\kappa+i\,\omega)\,\mathrm{Tr}(\rho_t\,a)+\alpha(t)\,, \tag{19}$$

$$\frac{d}{dt}\,\mathrm{Tr}(\rho_t\,\sigma^-) \;=\; -(\gamma+i\,\omega)\,\mathrm{Tr}(\rho_t\,\sigma^-)+\beta(t)\,\mathrm{Tr}(\rho_t\,\sigma^3)\,, \tag{20}$$

$$\begin{aligned}\frac{d}{dt}\,\mathrm{Tr}(\rho_t\,\sigma^3) \;=\;& -2(\overline{\beta(t)}\,\mathrm{Tr}(\rho_t\sigma^-)+\beta(t)\overline{\mathrm{Tr}(\rho_t\sigma^-)}) \\ & -2\gamma(\mathrm{Tr}(\rho_t\sigma^3)-d)\,.\end{aligned} \tag{21}$$

Proof. Deferred to Sect. A.5 □

4.2. Complex Lorenz equations

Taking $A(t)=\mathrm{Tr}(a\,\rho_t)$, $S(t)=\mathrm{Tr}(\sigma^-\rho_t)$ and $D(t)=\mathrm{Tr}(\sigma^3\rho_t)$ we rewrite (3) as

$$\left\{\begin{aligned}\frac{d}{dt}A(t) &= -(\kappa+i\,\omega)A(t)+g\;S(t)\\ \frac{d}{dt}S(t) &= -(\gamma+i\,\omega)S(t)+g\;A(t)D(t)\\ \frac{d}{dt}D(t) &= -4g\,\mathrm{Re}\,(\overline{A(t)}S(t))-2\gamma(D(t)-d)\,,\end{aligned}\right. \tag{22}$$

where $t \geq 0$, $D(t)\in\mathbb{R}$ and $A(t), Y(t)\in\mathbb{C}$. The complex Lorenz equation (22) has received much attention in the physical literature (see, e.g., [21, 41]) due to its important role in the description of laser dynamics. Just for the sake of completeness, we next present relevant properties of (22), together with their mathematical proofs.

THEOREM 8 *Suppose that* $d \in]-1,1[$, $\omega \in \mathbb{R}$, $g \in \mathbb{R} \setminus \{0\}$ *and* $\kappa, \gamma > 0$. *Then, for every initial condition* $A(0) \in \mathbb{C}$, $S(0) \in \mathbb{C}$, $D(0) \in \mathbb{R}$ *there exists a unique solution defined on* $[0, +\infty[$ *to the system (22). Moreover, we have:*

- *If* $d < 0$, *then for all* $t \geq 0$,

$$4|d||A(t)|^2 + 4|S(t)|^2 + (D(t)-d)^2 \leq \mathrm{e}^{-2t\min\{\kappa,\gamma\}}\Big(4|d||A(0)|^2 + 4|S(0)|^2 + (D(0)-d)^2\Big). \tag{23}$$

- *If* $d \geq 0$, *then for any* $t \geq 0$,

$$|A(t)|^2 + \frac{g^2}{\gamma\kappa}|S(t)|^2 + \frac{g^2}{4\gamma\kappa}(D(t)-d)^2 \leq \mathrm{e}^{-t\min\{\kappa-\frac{g^2 d}{\gamma},\gamma-\frac{g^2 d}{\kappa}\}}\Big(|A(0)|^2 + \frac{g^2}{\gamma\kappa}|S(0)|^2 + \frac{g^2}{4\gamma\kappa}(D(0)-d)^2\Big). \tag{24}$$

Proof. Deferred to Sect. A.6 $\square$

Remark 6 According to $\gamma = (\kappa_+ + \kappa_-)/2$, $d = (\kappa_+ - \kappa_-)/(\kappa_+ + \kappa_-)$ we have $\kappa_- = \gamma(1-d)$ and $\kappa_+ = \gamma(1+d)$. Since $\kappa_+, \kappa_- > 0$, $\gamma > 0$ and $d \in]-1,1[$.

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Appendix A: Proofs

A.1. Proof of Theorem 2

The proof of Theorem 2 follows from combining Lemma A.2, given below, with the arguments used in the proof of Theorem 4.4 of [37]. First, we get the weak continuity of the map $t \mapsto AX_t(\xi)$ in case A is relatively bounded by C.

LEMMA A.1 *Let (H2.3) of Hypothesis 2 holds. Suppose that* $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$ *and* $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$. *Then, for any* $\psi \in L^2(\mathbb{P}, \mathfrak{h})$ *and* $t \geq 0$ *we have*

$$\lim_{s \to t} \mathbb{E}\langle \psi, AX_s(\xi)\rangle = \mathbb{E}\langle \psi, AX_t(\xi)\rangle. \tag{A.1}$$

Proof. Consider a sequence of nonnegative real numbers $(s_n)_n$ satisfying $s_n \to t$ as $n \to +\infty$. Since $((X_{s_n}(\xi), AX_{s_n}(\xi), CX_{s_n}(\xi)))_n$ is a bounded sequence in $L^2(\mathbb{P}, \mathfrak{h}^3)$, where $\mathfrak{h}^3 = \mathfrak{h} \times \mathfrak{h} \times \mathfrak{h}$, there exists a subsequence $(s_{n(k)})_k$ such that

$$(X_{s_{n(k)}}(\xi), AX_{s_{n(k)}}(\xi), CX_{s_{n(k)}}(\xi)) \overset{k\to\infty}{\longrightarrow} (Y, U, V) \tag{A.2}$$

weakly in $L^2(\mathbb{P}, \mathfrak{h}^3)$. Define $\mathfrak{M} = \{(\eta, A\eta, C\eta) : \eta \in L^2_C(\mathbb{P}, \mathfrak{h})\}$. Thus,

$$(X_{s_{n(k)}}(\xi), AX_{s_{n(k)}}(\xi), CX_{s_{n(k)}}(\xi)) \in \mathfrak{M} \quad \forall\, k \in \mathbb{N}\,.$$

Since $\mathfrak{M}$ is a linear manifold of $L^2(\mathbb{P}, \mathfrak{h}^3)$ closed with respect to the strong topology (see, e.g., proof of Lemma 7.15 of [37]), (A.2) implies $(Y, U, V) \in \mathfrak{M}$ (see, e.g., Section III.1.6 of [29]). Using $\mathbb{E}(\sup_{s\in[0,t+1]} \|X_s(\xi)\|^2) < \infty$, together with the dominated convergence theorem we obtain that

$$\mathbb{E}\|X_{s_{n(k)}}(\xi) - X_t(\xi)\|^2 \overset{k\to+\infty}{\longrightarrow} 0\,.$$

Hence $Y = X_t(\xi)$, and so $U = AX_t(\xi)$. Therefore, $AX_{s_{n(k)}}(\xi)$ converges to $AX_t(\xi)$ weakly in $L^2(\mathbb{P}, \mathfrak{h})$. □

LEMMA A.2 *Assume Hypothesis 2, together with* $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$ *and* $A \in \mathfrak{L}(\mathfrak{h})$. *Then,* $t \mapsto L_k(t)X_t(\xi)$ *is continuous as a map from* $[0, +\infty[$ *to* $L^2(\mathbb{P}, \mathfrak{h})$. *Moreover,*

$$\begin{aligned} t \quad \longmapsto \quad & \mathbb{E}\langle G(t)X_t(\xi), AX_t(\xi)\rangle + \mathbb{E}\langle X_t(\xi), AG(t)X_t(\xi)\rangle \\ & + \sum_{\ell=1}^{\infty} \mathbb{E}\langle L_\ell(t)X_t(\xi), AL_\ell(t)X_t(\xi)\rangle \end{aligned}$$

is a continuous function.

Proof. Suppose that $(t_n)_n$ is a sequence of nonnegative real numbers satisfying $t_n \to t$ as $n \to +\infty$. By $\mathbb{E}(\sup_{s\in[0,t+1]} \|X_s(\xi)\|^2) < \infty$ (see, e.g., Th. 4.2.5 of [44]), using the dominated convergence theorem gives

$$\mathbb{E}\|X_{t_n}(\xi) - X_t(\xi)\|^2 \overset{n\to\infty}{\longrightarrow} 0\,,$$

and hence $AX_{t_n}(\xi) \overset{n\to\infty}{\longrightarrow} AX_t(\xi)$ in $L^2(\mathbb{P}, \mathfrak{h})$. For any $\psi \in L^2(\mathbb{P}, \mathfrak{h})$,

$$\begin{aligned} &|\mathbb{E}\langle \psi, G(s)X_s(\xi)\rangle - \mathbb{E}\langle \psi, G(t)X_t(\xi)\rangle| \\ &\quad \leq \mathbb{E}\|\psi\|\|G(s)X_s(\xi) - G(t)X_s(\xi)\| + |\mathbb{E}\langle \psi, G(t)X_s(\xi)\rangle - \mathbb{E}\langle \psi, G(t)X_t(\xi)\rangle|\,, \end{aligned}$$

and so combining Lemma A.1 with

$$\mathbb{E}\|G(s)X_s(\xi) - G(t)X_s(\xi)\|^2 \ \leq \ f_0(s,t)\mathbb{E}\|X_s(\xi)\|_C^2$$

yields

$$\lim_{s\to t}\mathbb{E}\langle \psi, G(s)X_s(\xi)\rangle \ = \ \mathbb{E}\langle \psi, G(t)X_t(\xi)\rangle \,. \tag{A.3}$$

Therefore

$$\lim_{n\to\infty}\mathbb{E}\langle G(t_n)X_{t_n}(\xi), AX_{t_n}(\xi)\rangle \ = \ \mathbb{E}\langle G(t)X_t(\xi), AX_t(\xi)\rangle \tag{A.4}$$

(see, e.g., Section III.1.7 of [29]). Analysis similar to that in (A.3) shows

$$\lim_{s\to t}\mathbb{E}\langle \psi, L_\ell(s)X_s(\xi)\rangle \ = \ \mathbb{E}\langle \psi, L_\ell(t)X_t(\xi)\rangle \,,$$

and hence

$$L_\ell(t_n)X_{t_n}(\xi) \ \overset{n\to\infty}{\longrightarrow} \ L_\ell(t)X_t(\xi) \quad \text{weakly in} \quad L^2(\mathbb{P},\mathfrak{h})\,. \tag{A.5}$$

According to (A.4) with A replaced by A^* we have the continuity of $t \mapsto \mathbb{E}\langle A^*X_t(\xi), G(t)X_t(\xi)\rangle$, and so $t \mapsto \mathbb{E}\langle X_t(\xi), AG(t)X_t(\xi)\rangle$ is continuous. Moreover, taking $A = I$ in (A.4) we deduce that

$$\mathbb{E}\,\mathrm{Re}\,\langle X_{t_n}(\xi), G(t_n)X_{t_n}(\xi)\rangle \ \overset{n\to\infty}{\longrightarrow} \ \mathbb{E}\,\mathrm{Re}\,\langle X_t(\xi), G(t)X_t(\xi)\rangle\,.$$

Applying (H2.2) we now get

$$\sum_{\ell=1}^{\infty}\mathbb{E}\|L_\ell(t_n)X_{t_n}(\xi)\|^2 \ \overset{n\to\infty}{\longrightarrow} \ \sum_{\ell=1}^{\infty}\mathbb{E}\|L_\ell(t)X_t(\xi)\|^2. \tag{A.6}$$

Combining (A.5) and (A.6) yields

$$\limsup_{n\to\infty}\mathbb{E}\|L_\ell(t_n)X_{t_n}(\xi)\|^2 \ \leq \ \mathbb{E}\|L_\ell(t)X_t(\xi)\|^2$$

(see, e.g., proof of Lemma 7.16 of [37] for details) which, together with (A.5), implies that $L_\ell(t_n)X_{t_n}(\xi)$ converges strongly in $L^2(\mathbb{P},\mathfrak{h})$ to $L_\ell(t)X_t(\xi)$ as $n\to\infty$. Therefore, $t\mapsto L_\ell(t)X_t(\xi)$ is continuous as a function from $[0,+\infty[$ to $L^2(\mathbb{P},\mathfrak{h})$.

Using (H2.2) we obtain that $\sum_{\ell=1}^{n}\mathbb{E}\langle L_\ell(t)X_t(\xi), AL_\ell(t)X_t(\xi)\rangle$ converges to $\sum_{\ell=1}^{\infty}\mathbb{E}\langle L_\ell(t)X_t(\xi), AL_\ell(t)X_t(\xi)\rangle$ as $n\to\infty$ uniformly on any finite interval. Since

$$\mathbb{E}\langle L_\ell(t_n)X_{t_n}(\xi), AL_\ell(t_n)X_{t_n}(\xi)\rangle \ \overset{n\to\infty}{\longrightarrow} \ \mathbb{E}\langle L_\ell(t)X_t(\xi), AL_\ell(t)X_t(\xi)\rangle\,,$$

the map $t\mapsto \sum_{\ell=1}^{\infty}\mathbb{E}\langle L_\ell(t)X_t(\xi), AL_\ell(t)X_t(\xi)\rangle$ is continuous. $\square$

LEMMA A.3 *Let Hypothesis 2 hold, except (H2.4). For any* $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$, *we define*

$$\begin{aligned}\mathcal{L}_*(\xi, t) &= \mathbb{E}|G(t)X_t(\xi)\rangle\langle X_t(\xi)| + \mathbb{E}|X_t(\xi)\rangle\langle G(t)X_t(\xi)| \\ &\quad + \sum_{\ell=1}^{\infty} \mathbb{E}|L_\ell(t)X_t(\xi)\rangle\langle L_\ell(t)X_t(\xi)| \,.\end{aligned}$$

Then $\mathcal{L}_*(\xi, t)$ *is a trace-class operator on* $\mathfrak{h}$ *whose trace-norm is uniformly bounded with respect to* t *on bounded time intervals; the series involved in the definition of* $\mathcal{L}_*$ *converges in* $\mathfrak{L}_1(\mathfrak{h})$.

Proof. By condition (H2.2), using $\||x\rangle\langle y|\|_1 = \|x\|\|y\|$ and Lemma 7.3 of [37] we get

$$\begin{aligned}&\|\mathbb{E}|G(t)X_t(\xi)\rangle\langle X_t(\xi)|\|_1 + \|\mathbb{E}|X_t(\xi)\rangle\langle G(t)X_t(\xi)|\|_1 \\ &\qquad + \sum_{\ell=1}^{\infty} \|\mathbb{E}|L_\ell(t)X_t(\xi)\rangle\langle L_\ell(t)X_t(\xi)|\|_1 \\ &\quad \le \; 4\mathbb{E}(\|X_t(\xi)\|\|G(t)X_t(\xi)\|) \; \le \; K(t)\sqrt{\mathbb{E}\|\xi\|^2}\sqrt{\mathbb{E}\|X_t(\xi)\|_C^2}\,,\end{aligned}$$

where the last inequality follows from (H2.1). □

Applying Lemma 7.3 of [37] and Lemma A.2 we easily obtain Lemma A.4.

LEMMA A.4 *Suppose that Hypothesis 2 holds,* $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$, *and* $A \in \mathfrak{L}(\mathfrak{h})$. *Then* $t \mapsto \mathrm{Tr}(A\mathcal{L}_*(\xi, t))$ *is continuous as a function from* $[0, \infty[$ *to* $\mathbb{C}$, *and*

$$\begin{aligned}\mathrm{Tr}(A\mathcal{L}_*(\xi, t)) &= \mathbb{E}\langle X_t(\xi), AG(t)X_t(\xi)\rangle + \mathbb{E}\langle G(t)X_t(\xi), AX_t(\xi)\rangle \\ &\quad + \sum_{\ell=1}^{\infty} \mathbb{E}\langle L_\ell(t)X_t(\xi), AL_\ell(t)X_t(\xi)\rangle \,.\end{aligned}$$

Here, $\mathcal{L}_*(\xi, t)$ *is as in Lemma A.3.*

LEMMA A.5 *Adopt Hypothesis 2, together with* $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$. *Then*

$$\rho_t \;=\; \mathbb{E}|\xi\rangle\langle\xi| + \int_0^t \mathcal{L}_*(\xi, s)ds\,, \tag{A.7}$$

where $t \ge 0$ *and* $\mathcal{L}_*(\xi, s)$ *is as in Lemma A.3; we understand the above integral in the sense of Bochner integral in* $\mathfrak{L}_1(\mathfrak{h})$.

Proof. Fix $x \in \mathfrak{h}$ and choose $\tau_n = \inf\{s \geq 0 : \|X_s(\xi)\| > n\}$ with $n \in \mathbb{N}$. Applying the complex Itô formula we obtain that

$$\langle X_{t\wedge\tau_n}(\xi), x\rangle X_{t\wedge\tau_n}(\xi) \;=\; \langle \xi, x\rangle\xi + \mathbb{E}\int_0^{t\wedge\tau_n} L_x(X_s(\xi), s)ds + M_t\,, \qquad \text{(A.8)}$$

where

$$M_t \;=\; \sum_{\ell=1}^{\infty}\int_0^{t\wedge\tau_n} (\langle X_s(\xi), x\rangle L_\ell(s)X_s(\xi) + \langle L_\ell(s)X_s(\xi), x\rangle X_s(\xi))dW_s^\ell$$

and for any $z \in \mathcal{D}(C)$,

$$L_x(z, s) \;=\; \langle z, x\rangle G(s)z + \langle G(s)z, x\rangle z + \sum_{k=1}^{\infty}\langle L_k(s)z, x\rangle L_k(s)z\,.$$

According to (H2.2) we have

$$\begin{aligned}&\mathbb{E}\sum_{\ell=1}^{\infty}\int_0^{t\wedge\tau_n} \|\langle X_s(\xi), x\rangle L_\ell(s)X_s(\xi) + \langle L_\ell(s)X_s(\xi), x\rangle X_s(\xi)\|^2 ds\\ &\qquad\leq\; 4n^3\|x\|^2\mathbb{E}\int_0^{t\wedge\tau_n}\|G(s)X_s\|ds\,.\end{aligned}$$

Therefore, $\mathbb{E}M_t = 0$ by (H2.1), and so (A.8) yields

$$\mathbb{E}\langle X_{t\wedge\tau_n}(\xi), x\rangle X_{t\wedge\tau_n}(\xi) \;=\; \mathbb{E}\langle \xi, x\rangle\xi + \mathbb{E}\int_0^{t\wedge\tau_n} L_x(X_s(\xi), s)ds\,. \qquad \text{(A.9)}$$

We will take the limit as $n \to \infty$ in (A.9). Since $X(\xi)$ has continuous sample paths, $\tau_n \nearrow_{n\to\infty} \infty$. By (H2.1) and (H2.2), applying the dominated convergence yields

$$\lim_{n\to\infty}\mathbb{E}\int_0^{t\wedge\tau_n} L_x(X_s(\xi), s)ds \;=\; \mathbb{E}\int_0^t L_x(X_s(\xi), s)ds\,.$$

Combining $\mathbb{E}(\sup_{s\in[0,t+1]}\|X_s(\xi)\|^2) < \infty$ with the dominated convergence theorem gives

$$\lim_{n\to\infty}\mathbb{E}\langle X_{t\wedge\tau_n}(\xi), x\rangle X_{t\wedge\tau_n}(\xi) \;=\; \mathbb{E}\langle X_t(\xi), x\rangle X_t(\xi)\,.$$

Then, letting first $n \to \infty$ in (A.9) and then using Fubini's theorem we get

$$\mathbb{E}\langle X_t(\xi), x\rangle X_t(\xi) \;=\; \mathbb{E}\langle \xi, x\rangle \xi + \int_0^t \mathbb{E}L_x(X_s(\xi), s)\,. \tag{A.10}$$

By (H2.2), the dominated convergence theorem leads to

$$\mathbb{E}\sum_{k=1}^{\infty}\langle L_k(s)X_s(\xi), x\rangle L_k(s)X_s(\xi) \;=\; \sum_{k=1}^{\infty}\mathbb{E}\langle L_k(s)X_s(\xi), x\rangle L_k(s)X_s(\xi)\,,$$

and so Lemma 7.3 of [37] yields $\mathbb{E}L_x(X_s(\xi), s) = \mathcal{L}_*(\xi, s)x$, hence

$$\int_0^t \mathbb{E}L_x(X_s(\xi), s) \;=\; \int_0^t \mathcal{L}_*(\xi, s)x ds\,. \tag{A.11}$$

Since the dual of $\mathfrak{L}_1(\mathfrak{h})$ consists in all linear maps $\varrho \mapsto \operatorname{Tr}(A\varrho)$ with $A \in \mathfrak{L}(\mathfrak{h})$, Lemma A.4 implies that $t \mapsto \mathcal{L}_*(\xi, t)$ is measurable as a function from $[0, \infty[$ to $\mathfrak{L}_1(\mathfrak{h})$. Furthermore, using Lemma A.3 we get that $t \mapsto \mathcal{L}_*(\xi, t)$ is a Bochner integrable $\mathfrak{L}_1(\mathfrak{h})$-valued function on bounded intervals. Then (A.10), together with (A.11), gives (A.7). □

Proof of Theorem 2. According to Theorem 3.2 of [37] we have

$$AG(t)\rho_t \;=\; \mathbb{E}|AG(t)X_t(\xi)\rangle\langle X_t(\xi)|\,.$$

Since $G(t), L_1(t), L_2(t), \ldots$ are closable, $G(t)^*, L_1(t)^*, L_2(t)^*, \ldots$ are densely defined and $G(t)^{**}$, $L_1(t)^{**}, \ldots$ coincide with the closures of $G(t), L_1(t), \ldots$ respectively (see, e.g., Theorem III.5.29 of [29]). Now, Theorem 3.2 of [37] yields $A\rho_t G(t)^* = \mathbb{E}|AX_t(\xi)\rangle\langle G(t)X_t(\xi)|$ and

$$AL_k(t)\rho_t L_k(t)^* \;=\; \mathbb{E}|AL_k(t)X_t(\xi)\rangle\langle L_k(t)X_t(\xi)|\,.$$

Therefore,

$$\mathcal{L}_*(\xi, t) \;=\; G(t)\rho_t + \rho_t G(t)^* + \sum_{k=1}^{\infty} L_k(t)\rho_t L_k(t)^*, \tag{A.12}$$

where $\mathcal{L}_*(\xi, t)$ is as in Lemma A.3. Combining (A.12) with Lemma A.5 we get (10), and so

$$\operatorname{Tr}(A\rho_t) \;=\; \operatorname{Tr}(A\varrho) + \int_0^t \operatorname{Tr}(A\mathcal{L}_*(\xi, s))ds \quad \forall\, t \geq 0\,.$$

Using the continuity of $\mathcal{L}_*(\xi,\,\cdot\,)$ we obtain (9). □

A.2. Proof of Theorem 3

Proof. For any $x, y \in \mathcal{D}(C)$ we set $[x, y] = \mathbb{E}\langle X_t(x), AX_t(y)\rangle$. According to Definition 4 we have

$$|[x, y]| \;=\; |\mathbb{E}\langle X_t(x), AX_t(y)\rangle| \;\leq\; K(t)\|A\|\|x\|\|y\| \quad \forall\, x, y \in \mathcal{D}(C)\,.$$

Since $\mathcal{D}(C)$ is dense in $\mathfrak{h}$, $[\,\cdot\,,\,\cdot\,]$ can be extended uniquely to a sesquilinear form $[\,\cdot\,,\,\cdot\,]$ over $\mathfrak{h} \times \mathfrak{h}$ satisfying $|[x, y]| \leq K(t)\|A\|\|x\|\|y\|$ for any $x, y \in \mathfrak{h}$. Hence, there exists a unique bounded operator $\mathcal{T}_t(A)$ on $\mathfrak{h}$ such that $|[x, y]| = \langle x, \mathcal{T}_t(A)y\rangle$ for all x, y in $\mathfrak{h}$. Moreover, $\|\mathcal{T}_t(A)\| \leq K(t)\|A\|$. □

A.3. Proof of Theorem 4

Proof. Using Itô formula we will prove that for all $x, y \in \mathcal{D}(C)$,

$$\mathbb{E}\langle X_t(x), AX_t(y)\rangle \;=\; \langle x, \mathcal{A}_t y\rangle. \tag{A.13}$$

This, together with Theorem 3, implies $\mathcal{A}_t = \mathcal{T}_t(A)$.

Motivated by $\mathcal{A}_t$ is only a weak solution, we fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{h}$ and consider the function $F_n : [0, t] \times \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ defined by

$$F_n(s, u, v) \;=\; \langle R_n\overline{u}, \mathcal{A}_{t-s}R_n v\rangle\,,$$

where $R_n = n(n + C)^{-1}$ and $\bar{u} = \sum_{n\in\mathbb{N}} \overline{\langle e_n, u\rangle} e_n$. Since the range of R_n is contained in $\mathcal{D}(C)$,

$$\frac{d}{ds}F_n(s, u, v) \;=\; -g(s, R_n\overline{u}, R_n v)\,, \tag{A.14}$$

with $g(s, x, y) = \langle x, \mathcal{A}_{t-s}Gy\rangle + \langle Gx, \mathcal{A}_{t-s}y\rangle + \sum_{k=1}^{\infty}\langle L_k x, \mathcal{A}_{t-s}L_k y\rangle$. We have that $t \mapsto \langle u, \mathcal{A}_t v\rangle$ is continuous for all $u, v \in \mathfrak{h}$, and so combining $CR_n \in \mathfrak{L}(\mathfrak{h})$ with Hypothesis 3 we get the uniformly continuity of $(s, u, v) \mapsto g(s, R_n\overline{u}, R_n v)$ on bounded subsets of $[0, t] \times \mathfrak{h} \times \mathfrak{h}$. Then, we can apply Itô's formula to $F_n(s \wedge \tau_j, \overline{X_s^{\tau_j}(x)}, X_s^{\tau_j}(y))$, with

$$\tau_j \;=\; \inf\{t \geq 0 : \|X_t(x)\| + \|X_t(y)\| > j\}\,.$$

Fix $x, y \in \mathcal{D}(C)$. Combining Itô's formula with (A.14) we deduce that

$$F_n(t \wedge \tau_j, \overline{X_t^{\tau_j}(x)}, X_t^{\tau_j}(y)) \;=\; F_n(0, \overline{X_0(x)}, X_0(y)) + I^n_{t\wedge\tau_j} + M_t\,,$$

where for $s \in [0,t]$:

$$
\begin{aligned}
M_s &= \sum_{k=1}^{\infty} \int_0^{s\wedge\tau_j} \langle R_n X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n L_k X_r^{\tau_j}(y)\rangle dW_r^k \\
&\quad + \sum_{k=1}^{\infty} \int_0^{s\wedge\tau_j} \langle R_n L_k X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n X_r^{\tau_j}(y)\rangle dW_r^k
\end{aligned}
$$

and

$$
I_s^n = \int_0^s \Big(- g(r, R_n X_r(x), R_n X_r(y)) + g_n(r, X_r(x), X_r(y)) \Big) dr
$$

with

$$
g_n(r,u,v) = \langle R_n u, \mathcal{A}_{t-r} R_n G v\rangle + \langle R_n G u, \mathcal{A}_{t-r} R_n v\rangle + \sum_{k=1}^{\infty} \langle R_n L_k u, \mathcal{A}_{t-r} R_n L_k v\rangle \,.
$$

We next establish the martingale property of M_s. For all $r \in [0,t]$ we have

$$
\|R_n X_r^{\tau_j}(x)\|^2 \|\mathcal{A}_{t-r}\|^2 \|R_n L_k X_r^{\tau_j}(y)\|^2 \;\leq\; j^2 \sup_{s\in[0,t]} \|\mathcal{A}_s\|^2 \|L_k X_r^{\tau_j}(y)\|^2.
$$

By (H2.1) and (H3.1),

$$
\mathbb{E} \int_0^{t\wedge\tau_j} \sum_{k=1}^{\infty} |\langle R_n X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n L_k X_r^{\tau_j}(y)\rangle|^2 ds \;<\; \infty \,.
$$

Thus

$$
\Big(\sum_{k=1}^{\infty} \int_0^{s\wedge\tau_j} \langle R_n X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n L_k X_r^{\tau_j}(y)\rangle dW_r^k \Big)_{s\in[0,t]}
$$

is a martingale. The same conclusion can be drawn for

$$
\sum_{k=1}^{\infty} \int_0^{s\wedge\tau_j} \langle R_n L_k X_r^{\tau_j}(x), \mathcal{A}_{t-r} R_n X_r^{\tau_j}(y)\rangle dW_r^k,
$$

and so $(M_s)_{s\in[0,t]}$ is a martingale. Hence

$$
\mathbb{E}\langle R_n X_t^{\tau_j}(x), \mathcal{A}_{t-t\wedge\tau_j} R_n X_t^{\tau_j}(y)\rangle \;=\; \langle R_n x, \mathcal{A}_t R_n y\rangle + \mathbb{E} I_{t\wedge\tau_j}^n \,. \tag{A.15}
$$

We will take the limit as $j \to \infty$ in (A.15). Since $\mathbb{E}(\sup_{s\in[0,t]} \|X_s(\xi)\|^2) < \infty$ for $\xi = x, y$ (see, e.g., Th. 4.2.5 of [44]), using the dominated convergence theorem, together with the continuity of $t \mapsto \langle u, \mathcal{A}_t v\rangle$, we get

$$\mathbb{E}\langle R_n X_t^{\tau_j}(x), \mathcal{A}_{t-t\wedge\tau_j} R_n X_t^{\tau_j}(y)\rangle \xrightarrow{j\to\infty} \mathbb{E}\langle R_n X_t(x), A R_n X_t(y)\rangle \,.$$

Applying again the dominated convergence theorem yields $\mathbb{E}I^n_{t\wedge\tau_j} \xrightarrow{j\to\infty} \mathbb{E}I^n_t$, and hence letting $j \to \infty$ in (A.15) we deduce that

$$\begin{aligned} &\mathbb{E}\langle R_n X_t(x), A R_n X_t(y)\rangle - \langle R_n x, \mathcal{A}_t R_n y\rangle \\ &= \mathbb{E}\int_0^t \Big(-g(s, R_n X_s(x), R_n X_s(y)) + g_n(s, X_s(x), X_s(y))\Big) ds \,. \end{aligned} \tag{A.16}$$

Finally, we take the limit as $n \to \infty$ in (A.16). Since $\|R_n\| \le 1$ and R_n tends pointwise to I as $n \to \infty$, the dominated convergence theorem yields

$$\lim_{n\to\infty} \mathbb{E}\int_0^t g_n(s, X_s(x), X_s(y)) ds = \mathbb{E}\int_0^t g(s, X_s(x), X_s(y)) ds \,.$$

For any $x \in \mathcal{D}(C)$, $\lim_{n\to\infty} CR_n x = Cx$. By $\|CR_n x\| \le \|Cx\|$, using the dominated convergence theorem gives

$$\lim_{n\to\infty} \mathbb{E}\int_0^t g(s, R_n X_s(x), R_n X_s(y)) ds = \mathbb{E}\int_0^t g(s, X_s(x), X_s(y)) ds \,.$$

Thus, letting $n \to \infty$ in (A.16) we obtain (A.13). □

A.4. Proof of Theorem 6

Proof. First, we show that ρ_t given by (18) is a N^p-weak solution to (14). To this end, we will verify that $C = N^p$ satisfies Hypothesis 2, where, here and subsequently, $H(t)$, $G(t)$, L_1, L_2, L_3 are defined as in Theorem 6. Since $L_2, L_3 \in \mathfrak{L}(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$, $L_1, L_1^* L_1$ are relatively bounded with respect to N and

$$\|H(t)x\|^2 \le K \max(|\alpha(t)|, |\beta(t)|) \|x\|_N \quad \forall\, x \in \mathcal{D}(N) \,,$$

C fulfills (H2.1) of Hypothesis 2. By definition of $G(t)$ and L_ℓ,

$$2\mathrm{Re}\langle x, G(t)x\rangle + \sum_{\ell=1}^{3} \|L_\ell x\|^2 = 0 \quad \forall\, x \in \mathcal{D}(N) \,,$$

and hence (H2.2) holds. Condition (H2.4) follows from the continuity of α and β.

In order to check (H2.3), we denote by $\mathfrak{D}$ the set of all $x \in \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$ such that $x(n,\eta) := \langle e_n \otimes e_\eta, x\rangle$ is equal to 0 for all combinations of $n \in \mathbb{Z}_+$ and $\eta = \pm$ except a finite number. Consider $x \in \mathfrak{D}$. A careful computation yields

$$\begin{aligned} &2\mathrm{Re}\langle N^{2p}x, G(t)x\rangle + \sum_{\ell=1}^{3} \|N^p L_\ell x\|^2 \qquad \text{(A.17)} \\ &\quad = \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\,\mathrm{Re}\Big(\alpha(t)x(k,\eta)\overline{x(k+1,\eta)}\Big)\sqrt{k+1}\Big((k+1)^{2p} - k^{2p}\Big) \\ &\qquad + \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\kappa|x(k,\eta)|^2 k\Big((k-1)^{2p} - k^{2p}\Big)\,. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\,\mathrm{Re}\Big(\alpha(t)x(k,\eta)\overline{x(k+1,\eta)}\Big)\sqrt{k+1}\Big((k+1)^{2p} - k^{2p}\Big) \\ &\qquad \le 2|\alpha(t)| \sum_{k\in\mathbb{Z}_+,\eta=\pm} |x(k,\eta)||x(k+1,\eta)|\phi(k) \\ &\qquad \le 2|\alpha(t)| \sum_{k\in\mathbb{Z}_+,\eta=\pm} |x(k,\eta)|^2\phi(k) \end{aligned}$$

with

$$\phi(k) = \sqrt{k+1}((k+1)^{2p} - k^{2p}) = \sqrt{k+1}\sum_{j=0}^{2p-1}\binom{2p}{j}k^j\,,$$

$$\begin{aligned} &\sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\,\mathrm{Re}\Big(\alpha(t)x(k,\eta)\overline{x(k+1,\eta)}\Big)\sqrt{k+1}\Big((k+1)^{2p} - k^{2p}\Big) \qquad \text{(A.18)} \\ &\qquad \le |\alpha(t)|K \sum_{k\in\mathbb{Z}_+,\eta=\pm} |x(k,\eta)|^2(1 + k^{2p-1/2})\,. \end{aligned}$$

Combining (A.17) with (A.18) we get

$$2\mathrm{Re}\langle N^{2p}x, G(t)x\rangle + \sum_{\ell=1}^{3} \|N^p L_\ell x\|^2 \le K|\alpha(t)|\|x\|_{N^p}^2\,,$$

and so (H4.3) of Hypothesis 4 holds because $\mathfrak{D}$ is a core of N^p. Then, applying Theorem 2.4 of [19] (see also Theorem 5) we obtain that for any initial condition $\xi \in L^2_{N^p}(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$ there exists a unique strong N^p-solution of (16), together with

$$\mathbb{E}\|X_t(\xi)\|^2_{N^p} \ \leq \ K(t)\mathbb{E}\|\xi\|^2_{N^p} \,. \tag{A.19}$$

Therefore, (H2.3) holds and so we have checked Hypothesis 2 with $C = N^p$.

Applying Theorem 3.1 of [37] yields $\varrho = \mathbb{E}|\xi\rangle\langle\xi|$ for certain

$$\xi \in L^2_{N^p}(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)\,.$$

Using Theorem 2 we obtain that $\rho_t := \mathbb{E}|X_t(\xi)\rangle\langle X_t(\xi)|$ satisfies the relation (17) and

$$\begin{cases} \dfrac{d}{dt}\operatorname{Tr}(A\rho_t) & = & \operatorname{Tr}\left[A\Big(G(t)\rho_t \ + \rho_t G(t)^* + \sum_{\ell=1}^{3} L_\ell \rho_t L_\ell^*\Big)\right] \\ \qquad \rho_0 & = & \varrho \end{cases} \tag{A.20}$$

for all $A \in \mathfrak{L}(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$.

Second, we will prove that (14) has at most one N^p-weak solution provided that the initial condition is N^p-regular. Suppose that (A.20) holds. Taking $A = |y\rangle\langle x|$ in (A.20) we get

$$\frac{d}{dt}\langle x, \rho_t y\rangle \ = \ \langle G(t)^* x, \rho_t y\rangle + \langle x, \rho_t G(t)^* y\rangle + \sum_{\ell=1}^{3}\langle L_\ell^* x, \rho_t L_\ell^* y\rangle \tag{A.21}$$

for all $x, y \in \mathcal{D}(N^p)$. Relation (A.21) coincides with (13) with $\mathcal{A}_t$, $G(t)$, L_1, L_2 and L_3 replaced by ρ_t, $G(t)^*$, L_1^*, L_2^* and L_3^*. This suggests us to apply Theorem 4 to (A.21) in order to prove the uniqueness of the solution of (A.20). To this end, we next deduce that the linear stochastic Schrödinger equation

$$Y_t(\xi) \ = \ \xi + \int_0^t G(s)^* Y_s(\xi) ds + \sum_{\ell=1}^{3}\int_0^t L_\ell^* \, Y_s(\xi) dW_s^\ell \tag{A.22}$$

satisfies Hypothesis 4 with $C = N^p$.

Now, we check Hypothesis 4 with $G(t)$, L_1, L_2 and L_3 replaced by $G(t)^*$, L_1^*, L_2^* and L_3^*. Take $C = N^p$. Since $a^\dagger$ is relatively bounded with respect to N, using analysis similar to that in the second paragraph we can check that $G(t)^* = i\,H(t) - \frac{1}{2}\sum_{\ell=1}^{3} L_\ell^* L_\ell$ satisfies (H4.1) of Hypothesis 4 with $G(t)$ substituted by $G(t)^*$, as well as (H4.2) holds with $L_\ell(t)$ replaced by $L_1^* = \sqrt{2\kappa}a^\dagger$, $L_2^* = \sqrt{\gamma(1-d)}\sigma^+$, $L_3^* = \sqrt{\gamma(1+d)}\sigma^-$. On $\mathfrak{D}$ we have

$$\begin{aligned} G(t)^* + (G(t)^*)^* + \sum_{\ell=1}^{3}(L_\ell^*)^* L_\ell^* \ &= \ \sum_{\ell=1}^{3}(L_\ell L_\ell^* - L_\ell^* L_\ell) \\ &= \ 4\kappa^2 I + 2\gamma^2(1+d^2)\sigma_3\,, \end{aligned}$$

which gives (H4.4). For any $x \in \mathfrak{D}$,

$$
\begin{aligned}
&2\mathrm{Re}\langle N^{2p}x, i\,H(t)x\rangle && \text{(A.23)}\\
&\quad = \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\mathrm{Re}\Big(\alpha(t)x(k,\eta)\overline{x(k+1,\eta)}\Big)\sqrt{k+1}\Big((k+1)^{2p}-k^{2p}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\langle x, \Big(L_1N^{2p}L_1^* - \frac{1}{2}L_1^*L_1N^{2p} - \frac{1}{2}N^{2p}L_1^*L_1\Big)x\Big\rangle && \text{(A.24)}\\
&\qquad = \sum_{k\in\mathbb{Z}_+,\eta=\pm} 2\kappa|x(k,\eta)|^2((k+1)^{2p+1}-k^{2p+1})\,.
\end{aligned}
$$

Since L_2, L_3 are bounded operators with conmute with N^{2p}, using (A.23) and (A.24) yields

$$
2\mathrm{Re}\langle N^{2p}x, G(t)^*x\rangle + \sum_{\ell=1}^{3}\|N^pL_\ell^*x\|^2 \ \le\ K(t)\|N^px\|^2
$$

and hence (H4.3) holds. By Theorem 5, (A.22) has a unique strong N^p-solution whenever $\xi \in L^2_C(\mathbb{P},\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2)$. It follows from Theorem 4 that (A.21) has at most one solution $\varrho_t \in \mathfrak{L}(\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2)$ satisfying $\varrho_0 = \varrho$. Thus, (14) has a unique N^p-regular solution, which is equal to $\rho_t := \mathbb{E}|X_t(\xi)\rangle\langle X_t(\xi)|$. $\square$

A.5. Proof of Theorem 7

Proof. From Theorem 6 it follows that (16) has a unique strong N^p-solution $X_t(\xi)$ for any initial datum $\xi \in L^2_{N^p}(\mathbb{P},\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2)$. In order to establish (19) we apply Theorem 4.1 of [19] to obtain

$$
\begin{aligned}
\mathrm{Tr}(a\rho_t) &= \mathrm{Tr}(a\rho_0) + \sum_{\ell=1}^{3}\int_0^t \mathbb{E}\langle L_\ell X_s(\xi), aL_\ell X_s(\xi)\rangle ds && \text{(A.25)}\\
&\quad + \int_0^t \Big(\mathbb{E}\langle a^\dagger X_s(\xi), G(s)X_s(\xi)\rangle + \mathbb{E}\langle G(s)X_s(\xi), aX_s(\xi)\rangle\Big)ds\,,
\end{aligned}
$$

where, throughout the proof, $G(t)$, $H(t)$, L_1, L_2, L_3 are as in Theorem 6. Therefore, $t \mapsto \mathrm{Tr}(a\rho_t)$ is a continuous function.

Suppose that $x \in \mathfrak{D}$, where $\mathfrak{D}$ is the set of all $x \in \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2$ satisfying $\langle e_n \otimes e_\eta, x \rangle = 0$ for all combinations of $n \in \mathbb{Z}_+$ and $\eta = \pm$ except a finite number. Since a commutes with σ^3 and $\sigma^\pm$, using $[a, a^\dagger] = I$ we deduce that

$$\begin{aligned} \langle a^\dagger x, -i\, H(s) x \rangle + \langle -i\, H(s) x, a x \rangle &= \langle x, i\, [H(s), a] x \rangle \\ &= \langle x, [i\, \omega\, a^\dagger a - \alpha(t) a^\dagger + \overline{\alpha(t)} a, a] x \rangle \\ &= \langle x, (-i\, \omega\, a + \alpha(t)) x \rangle \end{aligned}$$

and

$$\begin{aligned} &\sum_{\ell=1}^{3} \left\langle x, \left(L_\ell^\star a L_\ell - \frac{1}{2} a L_\ell^\star L_\ell - \frac{1}{2} L_\ell^\star L_\ell a \right) x \right\rangle \\ &= \left\langle x, \left(L_1^\star a L_1 - \frac{1}{2} a L_1^\star L_1 - \frac{1}{2} L_1^\star L_1 a \right) x \right\rangle \ = \ -\kappa \langle x, a x \rangle \,. \end{aligned}$$

Because $\mathfrak{D}$ is a core for N, we obtain that for all $x \in \mathcal{D}(N)$,

$$\langle a^\dagger x, G(s) x \rangle + \langle G(s) x, a x \rangle + \sum_{\ell=1}^{3} \langle L_\ell x, a L_\ell x \rangle \ = \ \langle x, -(\kappa + i\, \omega) a x + \alpha(t) x \rangle \,.$$

Then, from (A.25) it follows that

$$\mathrm{Tr}(a \rho_t) \ = \ \mathrm{Tr}(a \rho_0) + \int_0^t \Big(- (\kappa + i\, \omega)\, \mathrm{Tr}(a \rho_s) + \alpha(s) \Big) ds \,,$$

which leads to (19).

Fix $\eta = -$ or $\eta = 3$. According to (A.20) we have

$$\frac{d}{dt} \mathrm{Tr}(\rho_t \sigma^\eta) \ = \ \mathrm{Tr} \left[\sigma^\eta \Big(G(t) \rho_t \ + \rho_t G(t)^* + \sum_{\ell=1}^{3} L_\ell \rho_t L_\ell^* \Big) \right] ,$$

and so applying Theorem 3.2 of [37] we deduce that

$$\begin{aligned} \frac{d}{dt} \mathrm{Tr}(\rho_t \sigma^\eta) \ &= \ \mathrm{Tr} \left[\rho_t \Big(\sigma^\eta G(t) + G(t)^* \sigma^\eta + \sum_{\ell=1}^{3} L_\ell^* \sigma^\eta L_\ell \Big) \right] \\ &= \ \mathrm{Tr} \left[\rho_t \Big(- i\, [\sigma^\eta, H(t)] + \sum_{\ell=1}^{3} \Big(L_\ell^* \sigma^\eta L_\ell - \frac{1}{2} \sigma^\eta L_\ell^* L_\ell - \frac{1}{2} L_\ell^* L_\ell \sigma^\eta \Big) \Big) \right] \\ &= \ \mathrm{Tr} \left(- i\, \rho_t \Big[\sigma^\eta, \frac{\omega}{2} \sigma^3 + i\, (\overline{\beta(t)} \sigma^- - \beta(t) \sigma^+) \Big] \right) \\ &\quad + \sum_{\ell=2}^{3} \mathrm{Tr} \left[\rho_t \Big(L_\ell^* \sigma^\eta L_\ell - \frac{1}{2} \sigma^\eta L_\ell^* L_\ell - \frac{1}{2} L_\ell^* L_\ell \sigma^\eta \Big) \right] . \end{aligned}$$

Now, we use the commutation relations

$$[\sigma^+, \sigma^-] \;=\; \sigma^3\,, \quad [\sigma^3, \sigma^+] \;=\; 2\sigma^+\,, \quad [\sigma^-, \sigma^3] \;=\; 2\sigma^-$$

to derive (20) and (21). □

A.6. Proof of Theorem 8

Proof. Fix $A(0) \in \mathbb{C}$, $S(0) \in \mathbb{C}$ and $D(0) \in \mathbb{R}$. Since (22) is an ordinary differential equation with locally Lipschitz coefficients, (22) has a unique solution defined on a maximal interval $[0, T[$ (see, e.g., [28]).

For all $t \in [0, T[$, we set $X(t) = \exp(i\omega t)A(t)$, $Y(t) = \exp(i\omega t)S(t)$ and $Z(t) = D(t) - d$. Thus, (22) becomes

$$\begin{cases} X'(t) &=& -\kappa\, X(t) + g\, Y(t) \\ Y'(t) &=& dg\, X(t) - \gamma\, Y(t) + g\, X(t)Z(t) \\ Z'(t) &=& -4g\, \mathrm{Re}(\overline{X(t)}\, Y(t)) - 2\gamma\, Z(t)\,. \end{cases}$$

Therefore,

$$\frac{d}{dt}|X(t)|^2 \;=\; 2\,\mathrm{Re}(X'(t)\overline{X(t)}) \;=\; -2\kappa|X(t)|^2 + 2g\,\mathrm{Re}(Y(t)\overline{X(t)})$$

and

$$\begin{cases} \dfrac{d}{dt}|Y(t)|^2 &=& 2dg\,\mathrm{Re}(X(t)\overline{Y(t)}) - 2\gamma|Y(t)|^2 + 2g\, Z(t)\mathrm{Re}(X(t)\overline{Y(t)}) \\ \dfrac{d}{dt}Z(t)^2 &=& -4\gamma Z(t)^2 - 8g\, Z(t)\mathrm{Re}(\overline{X(t)}Y(t))\,. \end{cases}$$

Hence,

$$4\,\frac{d}{dt}|Y(t)|^2 + \frac{d}{dt}Z(t)^2 \;=\; 8dg\,\mathrm{Re}(X(t)\overline{Y(t)}) - 8\gamma|Y(t)|^2 - 4\gamma Z(t)^2\,. \tag{A.26}$$

Suppose, for a moment, that $d < 0$. Then

$$-4d\frac{d}{dt}|X(t)|^2 + 4\frac{d}{dt}|Y(t)|^2 + \frac{d}{dt}Z(t)^2 = 8d\kappa|X(t)|^2 - 8\gamma|Y(t)|^2 - 4\gamma Z(t)^2.$$

This gives

$$\begin{aligned} &\frac{d}{dt}(-4d\,|X(t)|^2 + 4\,|Y(t)|^2 + (Z(t))^2) \\ &\qquad\qquad \le\; -\min\{2\kappa, 2\gamma\}(-4d\,|X(t)|^2 + 4\,|Y(t)|^2 + Z(t)^2)\,, \end{aligned}$$

which implies

$$4|d|\,|X(t)|^2 + 4\,|Y(t)|^2 + Z(t)^2 \le \exp(-2t\,\min\{\kappa,\gamma\})(4|d|\,|X(0)|^2 + 4\,|Y(0)|^2 + Z(0)^2) \tag{A.27}$$

for any $t \in [0,T[$.

On the other hand, assume that $d \geq 0$. Combining

$$\frac{d}{dt}|X(t)|^2 + \frac{g^2}{4\gamma\kappa}\Big(4\,\frac{d}{dt}|Y(t)|^2 + \frac{d}{dt}Z(t)^2\Big) = 2g\Big(1+\frac{g^2 d}{\gamma\kappa}\Big)\mathrm{Re}(X(t)\overline{Y(t)}) - 2\kappa|X(t)|^2 - 2\frac{g^2}{\kappa}|Y(t)|^2 - \frac{g^2}{\kappa}Z(t)^2$$

with $2\mathrm{Re}(X(t)\overline{\frac{g}{\kappa}Y(t)}) \leq |X(t)|^2 + \frac{g^2}{\kappa^2}|Y(t)|^2$ we obtain

$$\frac{d}{dt}\Big(|X(t)|^2 + \frac{g^2}{\gamma\kappa}|Y(t)|^2 + \frac{g^2}{4\gamma\kappa}Z(t)^2\Big) \le \Big(-\kappa + \frac{g^2 d}{\gamma}\Big)|X(t)|^2 + \Big(-\gamma + \frac{g^2 d}{\kappa}\Big)\frac{g^2}{\gamma\kappa}|Y(t)|^2 - 4\gamma\frac{g^2}{4\gamma\kappa}Z(t)^2.$$

Therefore, for all $t \in [0,T[$ we have

$$\frac{d}{dt}\Big(|X(t)|^2 + \frac{g^2}{\gamma\kappa}|Y(t)|^2 + \frac{g^2}{4\gamma\kappa}Z(t)^2\Big) \le -\min\Big\{\kappa - \frac{g^2 d}{\gamma}, \gamma - \frac{g^2 d}{\kappa}\Big\}\Big(|X(t)|^2 + \frac{g^2}{\gamma\kappa}|Y(t)|^2 + \frac{g^2}{4\gamma\kappa}Z(t)^2\Big)\,.$$

This yields

$$|X(t)|^2 + \frac{g^2}{\gamma\kappa}|Y(t)|^2 + \frac{g^2}{4\gamma\kappa}Z(t)^2 \le \mathrm{e}^{-t\min\{\kappa-\frac{g^2 d}{\gamma},\gamma-\frac{g^2 d}{\kappa}\}}\Big(|X(0)|^2 + \frac{g^2}{\gamma\kappa}|Y(0)|^2 + \frac{g^2}{4\gamma\kappa}Z(0)^2\Big)\,. \tag{A.28}$$

Suppose that $T < +\infty$. According to (A.27) and (A.28) we have that

$$\|(A(t), S(t), D(t))\| \ < \ K\,,$$

where $K > 0$ and $t \in [0,T[$. This contradicts the property

$$\lim_{t\to T}\|(A(t), S(t), D(t))\| \ = \ \infty\,.$$

Therefore, $T = +\infty$. Moreover, (A.27) and (A.28) lead to (23) and (24), respectively. $\square$

A.7. Proof of Theorem 1

Proof. Let $(A(t), S(t), D(t))$ be the unique global solution of (22) with $A(0) = \mathrm{Tr}(a\varrho)$, $S(0) = \mathrm{Tr}(\sigma^- \varrho)$ and $D(0) = \mathrm{Tr}(\sigma^3 \varrho)$. According to Theorem 6 we have that there exists a unique N^p-weak solution $(\rho_t)_{t\geq 0}$ to (14) with $\alpha(t) = g\, S(t)$, $\beta(t) = g\, A(t)$ and initial datum $\rho_0 = \varrho$. Moreover, Theorem 6 ensures that $\rho_t = \mathbb{E}|Z_t(\xi)\rangle\langle Z_t(\xi)|$, where $Z_t(\xi)$ is the strong N^p-solution of (16) with $\alpha(t) = g\, S(t)$, $\beta(t) = g\, A(t)$ and initial condition $\xi \in L^2_{N^p}(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$ such that $\varrho = \mathbb{E}|\xi\rangle\langle\xi|$. Applying Theorem 7 we deduce that the evolutions of $\mathrm{Tr}(a\, \rho_t)$, $\mathrm{Tr}(\sigma^- \rho_t)$ and $\mathrm{Tr}(\sigma^3 \rho_t)$ are governed by

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\mathrm{Tr}(a\, \rho_t) & = & -(\kappa + i\,\omega)\,\mathrm{Tr}(a\, \rho_t) + g\ S(t) \\ \dfrac{d}{dt}\mathrm{Tr}(\sigma^- \rho_t) & = & -(\gamma + i\,\omega)\,\mathrm{Tr}(\sigma^- \rho_t) + g\ A(t)\,\mathrm{Tr}(\sigma^3 \rho_t) \\ \dfrac{d}{dt}\mathrm{Tr}(\sigma^3 \rho_t) & = & -4g\ \mathrm{Re}(\overline{A(t)}\ \ \mathrm{Tr}(\sigma^- \rho_t)) - 2\gamma(\mathrm{Tr}(\sigma^3 \rho_t) - d)\,. \end{array} \right. \tag{A.29}$$

From the uniqueness of solution to (A.29) we find $\mathrm{Tr}(a\, \rho_t) = A(t)$, $\mathrm{Tr}(\sigma^- \rho_t) = S(t)$ and $\mathrm{Tr}(\sigma^3 \rho_t) = D(t)$. Hence

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\mathrm{Tr}(A\rho_t) & = & \mathrm{Tr}(A\mathcal{L}_\star(\rho_t)\rho_t) \quad \forall\, A \in \mathfrak{L}(\ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2) \\ \rho_0 & = & \varrho\,, \end{array} \right. \tag{A.30}$$

as well as $\alpha(t) = g\,\mathbb{E}\langle Z_t(\xi), \sigma^- Z_t(\xi)\rangle$ and $\beta(t) = g\,\mathbb{E}\langle Z_t(\xi), a\, Z_t(\xi)\rangle$ (see, e.g., [37]). Therefore, $Z_t(\xi)$ is a strong N^p-solution of (4).

Let $Z_t(\xi)$ and $\widetilde{Z}_t(\xi)$ be strong N^p-solutions of (4) with initial datum ξ belonging to $L^2_{N^p}(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$. Then, $Z_t(\xi)$ is the strong N^p-solution of (16) with initial datum $\xi \in L^2_{N^p}(\mathbb{P}, \ell^2(\mathbb{Z}_+) \otimes \mathbb{C}^2)$, $\alpha(t) = g\,\mathbb{E}\langle Z_t(\xi), \sigma^- Z_t(\xi)\rangle$ and $\beta(t) = g\,\mathbb{E}\langle Z_t(\xi), a\, Z_t(\xi)\rangle$. Since

$$t \longmapsto \mathbb{E}\langle Z_t(\xi), \sigma^- Z_t(\xi)\rangle \quad \text{and} \quad t \longmapsto \mathbb{E}\langle Z_t(\xi), a\, Z_t(\xi)\rangle$$

are continuous functions, applying Theorems 6 and 7, together with Theorem 3.2 of [37], we deduce that

$$\mathbb{E}\langle Z_t(\xi), \sigma^- Z_t(\xi)\rangle\,, \quad \mathbb{E}\langle Z_t(\xi), a\, Z_t(\xi)\rangle\,, \quad \mathbb{E}\langle Z_t(\xi), \sigma^3 Z_t(\xi)\rangle$$

is a solution of (22) with initial condition $A(0) = \mathrm{Tr}(a\,\varrho)$, $S(0) = \mathrm{Tr}(\sigma^- \varrho)$ and $D(0) = \mathrm{Tr}(\sigma^3 \varrho)$. The same is true for $\widetilde{Z}_t(\xi)$ in place of $Z_t(\xi)$, and so Theorem 8 leads to $\mathbb{E}\langle Z_t(\xi), \sigma^- Z_t(\xi)\rangle = \mathbb{E}\langle \widetilde{Z}_t(\xi), \sigma^- \widetilde{Z}_t(\xi)\rangle$ and

$$\mathbb{E}\langle Z_t(\xi), a\, Z_t(\xi)\rangle \;=\; \mathbb{E}\langle \widetilde{Z}_t(\xi), a\, \widetilde{Z}_t(\xi)\rangle$$

for all $t \geq 0$. Now, the uniqueness of the strong N^p-solution of (16) implies $Z = \widetilde{Z}$.

On the other hand, suppose that $(\rho_t)_{t\geq 0}$ and $(\widetilde{\rho}_t)_{t\geq 0}$ are families of N^p-regular operators satisfying (A.30) such that $\rho_0 = \widetilde{\rho}_0 = \varrho$ and $t \mapsto \mathrm{Tr}(a\,\rho_t)$, $t \mapsto \mathrm{Tr}(a\,\widetilde{\rho}_t)$ are continuous. Then, $(\rho_t)_{t\geq 0}$ is a N^p-weak solution to (14) with $\alpha(t) = g\ \mathrm{Tr}(\sigma^- \rho_t)$ and $\beta(t) = g\ \mathrm{Tr}(a\,\rho_t)$, as well as $(\widetilde{\rho}_t)_{t\geq 0}$ is a N^p-weak solution to (14) with $\alpha(t) = g\ \mathrm{Tr}(\sigma^- \widetilde{\rho}_t)$ and $\beta(t) = g\ \mathrm{Tr}(a\widetilde{\rho}_t)$. Using Theorem 7 we get that $(\mathrm{Tr}(a\,\rho_t), \mathrm{Tr}(\sigma^- \rho_t), \mathrm{Tr}(\sigma^3 \rho_t))$ and

$$(\mathrm{Tr}(a\,\widetilde{\rho}_t), \mathrm{Tr}(\sigma^- \widetilde{\rho}_t), \mathrm{Tr}(\sigma^3 \widetilde{\rho}_t))$$

are solutions of (22) with initial condition $A(0) = \mathrm{Tr}(a\,\varrho)$, $S(0) = \mathrm{Tr}(\sigma^- \varrho)$ and $D(0) = \mathrm{Tr}(\sigma^3 \varrho)$. Since the solution of (22) is unique (see, e.g., Theorem 8), $\mathrm{Tr}(a\,\rho_t) = \mathrm{Tr}(a\,\widetilde{\rho}_t)$, $\mathrm{Tr}(\sigma^- \rho_t) = \mathrm{Tr}(\sigma^- \widetilde{\rho}_t)$ and $\mathrm{Tr}(\sigma^3 \rho_t) = \mathrm{Tr}(\sigma^3 \widetilde{\rho}_t)$. Therefore, $(\rho_t)_{t\geq 0}$ and $(\widetilde{\rho}_t)_{t\geq 0}$ are N^p-weak solution to (14) with the same $\alpha(t)$ and $\beta(t)$, and hence using Theorem 6 yields $\rho_t = \widetilde{\rho}_t$ for all $t \geq 0$. $\square$

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