Discrete Tomography determination of bounded sets in $\mathbb{Z}^n$

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1. Introduction

Discrete Tomography is concerned with the inversion of the so-called Discrete Radon Transform to reconstruct discrete images from the collected counting data. The original motivation came from High-Resolution Transmission Electron Microscopy (see [25, 26, 28])

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which is able to obtain two-dimensional images with atomic resolution and provides quantitative information on the number of atoms that lie in single atomic columns in crystals choosing main X-ray directions to be resolvable by the microscopy. The ultimate goal is to achieve $3D$ electron tomography with atomic resolution. Although this is not yet possible for all structures, significant progress has recently been achieved using different approaches [30]. In the applications, due to the high energies required to produce the X-rays of a crystal before it is damaged, only a small number of X-rays can be taken. The aim is the reconstruction of images with few different grey levels, and, in particular, of binary images which resembles the original image as closely as possible.

In this paper we deal with the companion problem of uniqueness which consists in deciding whether an object accessible only via its X-rays is uniquely determined by the data or instead there are other objects *tomographically equivalent*, i.e., having the same X-rays. In the classical mathematical formulation, an object is a *lattice set*, that is a finite subset of points in the lattice $\mathbb{Z}^n$, and its X-rays count the number of its points computed along a prescribed set of m lattice directions. Uniqueness issues have been intensively studied in Discrete Tomography from different viewpoints (see, for instance, [4, 8, 11, 14, 16, 17, 19, 31]). In this setting with $n = 2$, the uniqueness problem can be solved in polynomial time for $m = 2$, whereas it is NP-complete for $m > 2$.

By increasing the dimension n, the problem can be extended in several ways. It leads for $n = 3$ to two natural generalizations which differ in the dimension of the linear spaces used for the X-rays: the first one considers one-dimensional X-rays according to the lines parallel to the three coordinate directions (see [15, 24]); the second one considers two-dimensional X-rays according to the planes orthogonal to the three axes (see [1, 18]). The complexity status of these problems is completely settled, and it turns out that they are NP-complete together with their generalizations to n-dimensional lattice sets and k-dimensional X-rays.

In this paper we focus on one-dimensional X-rays. In general, we note that uniqueness is not a property of the set of X-ray directions, since for any finite set S of lattice directions there are two different lattice sets tomographically equivalent w.r.t. S (see [14, Theorem 4.3.1]). Therefore, in the literature special classes of geometric objects are considered such as convex [13], Q-convex [5] and additive [10, 31] lattice sets. For the class of convex sets, uniqueness can always be achieved by means of X-rays in a set S of four lattice directions, apart from special cases which are related to the so-called S-polygons. These can be roughly defined as "switching-components" (see Section 2 for all terminology) with the extra property of convexity (detailed descriptions of such structures can be found for instance in [6, 7, 27]). A different restriction, in the same spirit, consists in considering subsets of a given rectangular grid $\mathcal{A} = [m_1] \times [m_2] \subset \mathbb{Z}^2$, where $[m_i] = \{0, 1, \ldots, m_i - 1\}$ and $i = 1, 2$. These sets have been introduced in [19] and are called *bounded sets.*

Overview of our results. We aim to extend some uniqueness results for bounded sets, obtained in [3] and [4], to higher dimensions by using an algebraic approach. This approach has been introduced in [20] and then used in [2] to prove some preliminary results for the successive papers. A trivial extension to $\mathbb{Z}^n$ is obtained as follows: if H is a two-dimensional subspace in $\mathbb{Z}^n$ and four directions lying in H are chosen according to [3], then the (one-dimensional) X-rays in these directions uniquely determine all the two-dimensional sections parallel to H of a bounded set E, and therefore distinguish E itself among all subsets of a given grid. However, in view of the possible applications to $3D$ electron tomography it is of interest to consider X-rays of bounded sets in noncoplanar sets of directions. In particular, we address the question of which sets of directions in general positions (i.e. with no three coplanar) are such that X-rays in these directions distinguish all the bounded sets in an n-dimensional grid $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n]$, where, for $p \in \mathbb{N}$, $[p] = \{0, 1, ..., p-1\}$. This issue is equivalently related to that of which sets of directions in general positions do not admit switching-components contained in $\mathcal{A}$. From the algebraic point of view, any finite set $S \subseteq \mathbb{Z}^n$ of lattice directions can be associated to a basic polynomial $F_S(x_1, \cdots, x_n)$ and switching-components can then be interpreted as multiples of $F_S(x_1, \cdots, x_n)$ having only coefficients in the set $\{-1, 0, 1\}$.

We point out that our results rely on the definition of (rectangular) grid and on the algebraic theory of switching component. Variations of the concept of grid can change this theory significantly as shown in [29].

In Section 2 we first show that $d+1$ represents the minimal number of directions we need in order to avoid switching-components, under the requirement that such directions span a d-dimensional subspace of $\mathbb{Z}^n$, where $n \geq d \geq 3$ (see Lemma 1). We then characterize all such sets S (see Theorem 6).

In Section 3 we use the previous results to characterize those sets S which distinguish all the subsets of $\mathcal{A}$ by their X-rays in the directions in S (see Theorem 12). Indeed we provide a necessary and sufficient uniqueness condition, for any fixed dimension, showing that, with few exceptions, as the dimension n grows, the grid $\mathcal{A}$ can be chosen arbitrary large in some coordinate directions. This means that working in $\mathcal{A}$ is a weak restriction, and there are no a priori limitations for the sizes of the bounded sets.

Note that as a consequence of Theorem 12, some results in [4] can be easily extended to the n-dimensional case (see also Remark 4 after Theorem 12 and the discussion concerning the links with [4] in the section of Concluding remarks).

2. Minimal sets of directions providing a weakly bad configuration

We denote the standard orthonormal coordinates for $\mathbb{Z}^n$ by $x_1, \ldots, x_n$. A vector $u = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, where $a_1 \geq 0$, is said to be a *lattice direction*, if $\gcd(a_1, ..., a_n) = 1$. We refer to a finite subset E of $\mathbb{Z}^n$ as a *lattice set*, and we denote its cardinality by $|E|$. Given a lattice direction u, the *X-ray* of a lattice set E in the direction u is the function giving the number of points in E on each line parallel to u. Let S be a prescribed finite set of lattice directions. Any two lattice sets E and F are *tomographically equivalent* if they have the same X-rays along the directions in S. Conversely, a lattice set E is said to be S-*unique* if there is no lattice set F different from but tomographically equivalent to E.

An S-*weakly bad configuration* is a pair of lists (Z, W) consisting of k lattice points not necessarily distinct (counted with multiplicity), $z_1, ..., z_k \in Z$ and $w_1, ..., w_k \in W$ such that for each direction $u \in S$, and for each $z_r \in Z$, the line through z_r in direction u contains a point $w_r \in W$ (see Figure 1(a)). If all the points in each set Z, W are distinct (multiplicity 1), then (Z, W) is called an S-*bad configuration* (see Figure 1(b)). If for some $k \geq 2$ an S-(weakly) bad configuration (Z, W) exists such that $Z \subseteq E$, $W \subseteq \mathbb{Z}^2 \setminus E$, we then say that a lattice set E has an S-(weakly) bad configuration. This notion plays a crucial role in investigating uniqueness problems, since a lattice set E is S-unique if and only if E has no S-bad configurations [12].

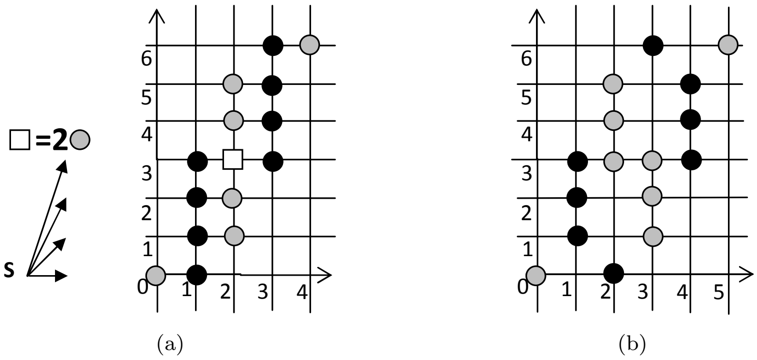

Figure 1: (a) An S-weakly bad configuration associated to $S = \{(1,0),(1,1),(1,3),(1,2)\}$, where Z consists of the grey points and the squared white point (counted twice), while W is the set of black points. (b) An S-bad configuration associated to the same set $S = \{(1,0),(1,1),(1,3),(1,2)\}$, with Z, W formed by grey and black points respectively.

To each finite set S of lattice directions we can associate an S-weakly bad configuration as follows. For each $I \subseteq S$, let $u(I) = \sum_{u \in I} u$, with $u(\emptyset) = 0 \in \mathbb{Z}^n$, and we define the multisets

$$A_S = \{u(I) : I \subseteq S, |I| \text{ even}\}, \quad B_S = \{u(I) : I \subseteq S, |I| \text{ odd}\}, \tag{2.1}$$

where the points $u(I)$ are counted with the appropriate multiplicities, whenever there exist distinct subsets I, J such that $u(I) = u(J)$. For every $v \in S$, each point $u(I)$ such that $v \in I$ can be paired to the point $u(I) - v$. This shows that the pair (A_S, B_S) is an S-weakly bad configuration, and it is an S-bad configuration if all the points $u(I)$, $I \subseteq S$, are distinct.

For $p \in \mathbb{N}$, denote $\{0, 1, ..., p-1\}$ by $[p]$. Let $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n]$ be a fixed lattice grid in $\mathbb{Z}^n$. We shall restrict our considerations to S-weakly bad configurations contained

The terminology is not uniform in the literature (see [22, 23] for an overview). For clarity, we adopt the terms "bad configuration" and "weakly bad configuration" to underlay whether all the points in each of the sets Z and W are distinct or not. Historically, Ryser used the name of "interchange" for "bad configuration", many authors wrote "switching-component" for "bad configuration" and others adopted the same term also for the broader concept of weakly bad configuration in the cases where the difference is not relevant.

in $\mathcal{A}$. We say that a set S is a *valid* set of directions for $\mathcal{A}$, if for all $i \in \{1, \dots, n\}$, the sum h_i of the absolute values of the i-th coordinates of the directions in S satisfies the condition $h_i < m_i$. Notice that the S-weakly bad configuration (A_S, B_S), associated to a set S of directions which is valid for $\mathcal{A}$, may not be contained in $\mathcal{A}$. This happens when some direction $u \in S$ has a negative coordinate. However, by applying a suitable translation we can get an S-weakly bad configuration contained in $\mathcal{A}$ (see Remark 1 below).

In order to study the S-weakly bad configurations contained in a finite grid $\mathcal{A}$, we shall adopt an algebraic approach introduced by Hajdu and Tijdeman in [20]. For a vector $u = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we simply write x^u in place of the monomial $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$. Consider now any lattice vector $u \in \mathbb{Z}^n$, where $u \neq 0$. Let $u_- \in \mathbb{Z}^n$ be the vector whose entries equal the corresponding entries of u if negative, and are 0 otherwise. Analogously, let $u_+ \in \mathbb{Z}^n$ be the vector whose entries equal the corresponding entries of u if positive, and are 0 otherwise.

For any finite set S of lattice directions in $\mathbb{Z}^n$, we define the rational function

$$R_S(x_1, \dots, x_n) = \prod_{u \in S} (x^u - 1) = \sum_{A \subset S} (-1)^{|S|-|A|} x^{u(A)}, \tag{2.2}$$

and denote by $F_S(x_1, \dots, x_n)$ the polynomial obtained from $R_S(x_1, \dots, x_n)$ as follows

$$F_S(x_1, \dots, x_n) = R_S(x_1, \dots, x_n) \prod_{u \in S} x^{-u_-} = \prod_{u \in S} x^{-u_-} \prod_{u \in S} (x^{(u_+)+(u_-)} - 1) = \prod_{u \in S} \left(x^{u_+} - x^{-u_-} \right). \tag{2.3}$$

For any function $f : \mathcal{A} \to \mathbb{Z}$, its *generating function* is the polynomial defined by

$$G_f(x_1, \dots, x_n) = \sum_{(a_1, \dots, a_n) \in \mathcal{A}} f(a_1, \dots, a_n) x_1^{a_1} \dots x_n^{a_n}.$$

Conversely, we say that the function f is *generated* by a polynomial $P(x_1, \dots, x_n)$ if $P(x_1, \dots, x_n) = G_f(x_1, \dots, x_n)$. Notice that the function f generated by the polynomial $F_S(x_1, \dots, x_n)$ vanishes outside $\mathcal{A}$ if and only if the set S is valid for $\mathcal{A}$.

The *line sum* of a function $f : \mathcal{A} \to \mathbb{Z}$ along the line $x = x_0 + tu$, passing through the point $x_0 \in \mathbb{Z}^n$ and with direction u, is the sum $\sum_{x = x_0 + tu, x \in \mathcal{A}} f(x)$. Further, we denote $||f|| = \max_{x \in \mathcal{A}} \{|f(x)|\}$. We can easily check that the function f generated by $F_S(x_1, \dots, x_n)$ has zero line sums along the lines taken in the directions belonging to S. Moreover, if $g : \mathcal{A} \to \mathbb{Z}$ has zero line sums along the lines taken in the directions of S, then $F_S(x_1, \dots, x_n)$ divides $G_g(x_1, \dots, x_n)$ over $\mathbb{Z}$ (see Lemma 3.1 in [20] and [21]). We recall that two functions $f, g : \mathcal{A} \subset \mathbb{Z}^n \to \{0, 1\}$ are *tomographically equivalent* with respect to a given finite set S of lattice directions if they have equal line sums along the lines corresponding to the directions in S. Note that two non-trivial functions $f, g : \mathcal{A} \to \{0, 1\}$ which are tomographically equivalent can be interpreted as characteristic functions of two lattice sets which are tomographically equivalent. Further, the difference $h = f - g$ of f and g has zero line sums. Hence there is a one-to-one correspondence between S-bad configurations contained in $\mathcal{A}$ and non-trivial functions $h : \mathcal{A} \to \mathbb{Z}$ having zero line sums along the lines corresponding to the directions in S and $||h|| \leq 1$.

To a monomial $kx_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ we associate the lattice point $z = (a_1, \dots, a_n) \in \mathbb{Z}^n$, together with its *weight* k. We say that a point $(a_1, \dots, a_n) \in \mathcal{A}$ is a *multiple positive point* for f (or G_f) if $f(a_1, \dots, a_n) > 1$. Analogously, $(a_1, \dots, a_n) \in \mathcal{A}$ is said to be a *multiple negative point* for f if $f(a_1, \dots, a_n) < -1$. Such points are simply referred to as *multiple points* when the signs are not relevant. For a polynomial $P(x_1, \dots, x_n)$ we denote by P^+ (resp. P^-) the set of lattice points corresponding to the monomials of $P(x_1, \dots, x_n)$ having positive (resp. negative) sign, referred to as positive (resp. negative) lattice points.

Remark 1. If S is a given valid finite set of directions for $\mathcal{A}$, then $F_S \subset \mathcal{A}$ and the pair of multisets (F_S^+, F_S^-) is an S-weakly bad configuration contained in $\mathcal{A}$. Moreover, it is an S-bad configuration if and only if the polynomial $F_S(x_1, \dots, x_n)$ has all the coefficients in the set $\{-1, 0, 1\}$. If each direction $u \in S$ has non-negative coordinates, then $u \in \mathcal{A}$, so that $(A_S, B_S) = (F_S^+, F_S^-)$. Otherwise, (F_S^+, F_S^-) can be obtained by translating (A_S, B_S) by the vector $\sum_{u\in S} -u_-$. From the algebraic point of view this is equivalent to multiplying the rational function $R_S(x_1, \dots, x_n)$ by $\prod_{u\in S} x^{-u_-}$. Therefore, when we address problems involving just the coefficients of the polynomial $F_S(x_1, \dots, x_n)$, we can replace this polynomial by the rational function $R_S(x_1, \dots, x_n)$.

For a finite set $S = \{u_1, u_2, ..., u_m\}$ of directions in $\mathbb{Z}^n$, the *dimension* of S, denoted by $\dim S$, is the dimension of the vector space generated by the vectors $u_1, u_2, ..., u_m$. We now show that a set of lattice directions in $\mathbb{Z}^n$ which contains $d > 2$ linearly independent directions and whose associated pair (F_S^+, F_S^-) is not an S-bad configuration must contain at least $d + 1$ directions, independently of any grid $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n]$.

Lemma 1. *Let $S = \{u_1, u_2, ..., u_m\}$ a set of distinct lattice directions in $\mathbb{Z}^n$, where $3 \le \dim S = d \le n$. If $F_S(x_1, \dots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$, then $m \ge d+1$.*

Proof. By Remark 1, if $F_S(x_1, \dots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$ then the rational function $R_S(x_1, \dots, x_n) = \prod_{u\in S}(x^u - 1) = \sum_{A\subseteq S}(-1)^{m-|A|}x^{u(A)}$ has a coefficient outside the set $\{-1, 0, 1\}$, as well. This means that when we expand the product which factorizes $R_S(x_1, \dots, x_n)$, we get some monomials which are repeated. Therefore, there are two distinct subsets $I, J \subset S$ such that $x^{u(I)} = x^{u(J)}$ and $(-1)^{m-|I|} = (-1)^{m-|J|}$, so that $u(I) = u(J)$ and $|I| \equiv |J| \pmod 2$ (see also [19, Lemma 3.2]). By removing common elements in I, J, we may assume that I, J are disjoint. The relation $u(I) = u(J)$ gives a linear dependence among vectors in $I \cup J \subseteq S$, so that the vectors in S are not linearly independent. This implies $m \ge d + 1$, as required.

We now exhibit a family of sets $S \subset \mathbb{Z}^n$ with $d + 1$ lattice directions and $\dim S = d \ge 3$, whose associated polynomials $F_S(x_1, \dots, x_n)$ have a coefficient outside the set $\{-1, 0, 1\}$.

Proposition 2. *Let $B = \{u_1, u_2, ..., u_d\}$ be a set of linearly independent directions in $\mathbb{Z}^n$, where $n \ge d \ge 3$. Let I, J be disjoint subsets of $\{u_1, ..., u_d\}$ such that $|I| \not\equiv |J| \pmod 2$, and $w = u(I)-u(J)$ is a lattice direction with $w \notin B$. If $S = \{u_1, ..., u_d, w\}$, then $R_S(x_1, \dots, x_n)$ and $F_S(x_1, \dots, x_n)$ have a coefficient outside the set $\{-1, 0, 1\}$, corresponding to the term $x^{u(I)}$ and $x^{u(I)-\sum_{u\in S} u_-}$, respectively. Moreover, if $|I| + |J| = d$, there exists exactly one coefficient outside $\{-1, 0, 1\}$.*

Proof. Since w is a lattice direction, its first coordinate cannot be negative, so that $I \neq \emptyset$. Further if $J = \emptyset$, then $|I| > 1$, since $w \notin B$. Moreover, we have $w + u(J) = u(I)$, so that $x^{w+u(J)} = x^{u(I)}$. Since $|I| \not\equiv |J| (\mathrm{mod} 2)$, we have $|I| \equiv |\{w\} \cup J| (\mathrm{mod} 2)$, hence $(-1)^{d+1-|I|} = (-1)^{d+1-|\{w\} \cup J|}$. Therefore, in the expansion of the product $R_S(x_1, \ldots, x_n) = \prod_{u \in S}(x^u - 1)$, we have at least two occurrences of the term $(-1)^{d+1-|I|}x^{u(I)}$ (see relation (2.2)).

We now show that there are no other occurrences of such terms, so that the term $x^{u(I)}$ in $R_S(x_1, \ldots, x_n)$ has just one coefficient outside the set $\{-1, 0, 1\}$ (equal to ± 2). Suppose that there exists a set $H \subset S$ such that $u(H) = u(I)$, where $H \neq \{w\} \cup J$, and $|H| \equiv |I| \, (\mathrm{mod}\, 2)$. If $w \notin H$ then the relation $u(H) = u(I)$ provides a linear dependence among the vectors of the set $I \cup H \subseteq \{u_1, ..., u_d\}$, which contradicts the assumption that $u_1, ..., u_d$ are linearly independent. If $w \in H$, then we have $u(H) = w + u(\widehat{H})$, where $\widehat{H} = H \setminus \{w\}$ with $\widehat{H} \neq J$. Thus, we get $w = u(I) - u(J) = u(H) - u(\widehat{H}) = u(I) - u(\widehat{H})$, which again provides a linear dependence among the vectors $u_1, ..., u_d$. This proves that $R_S(x_1, \ldots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$, corresponding to the term $x^{u(I)}$. By Remark 1 and (2.3) the polynomial $F_S(x_1, \ldots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$ corresponding to the monomial $x^{u(I)} \cdot \prod_{u \in S} x^{-u_-} = x^{u(I) - \sum_{u \in S} u_-}$.

Suppose now $|I| + |J| = d$, so that $B = I \cup J$. We show that $R_S(x_1, \ldots, x_n)$ does not contain other terms with coefficients distinct from $\{-1, 0, 1\}$, apart from $x^{u(I)}$. Suppose, on the contrary, that there exist two terms $x^{u(H)} = x^{u(K)} \neq x^{u(I)}$, where $H, K \subset S$ are disjoint sets such that $|H| \equiv |K| \, (\mathrm{mod}\, 2)$. Equality $u(H) = u(K)$ implies that the vectors of the set $H \cup K \subseteq S$ are linearly dependent. Since $S = \{w\} \cup I \cup J$, where the vectors in $B = I \cup J$ are linearly independent, $w \in H \cup K$. Assume, without loss of generality, $w \in H$, so that $w \notin K$, since H, K are disjoint. Then $u(H) = u(K)$ implies $w = u(K) - u(H')$, where $H' = H \setminus \{w\}$. Therefore, we get $u(K) - u(H') = u(I) - u(J)$ which gives a linear dependence among the vectors in $I \cup J$. Again this contradicts the assumption that the vectors in $I \cup J$ are linearly independent. Hence, we can conclude that $F_S(x_1, \ldots, x_n)$ has exactly one coefficient outside the set $\{-1, 0, 1\}$.

Remark 2. If d is even, then $|I| \not\equiv |J| \, (\mathrm{mod}\, 2)$ implies $|I| + |J| \neq d$, so that w must be a linear combination of vectors contained in a proper subset of $\{u_1, ..., u_d\}$ (see Example 5 below). In particular, for $d = 2$, no set S consisting of three distinct lattice directions u_1, u_2, w can satisfy the condition $w = u(I) - u(J)$ with $|I| \neq |J| \, (\mathrm{mod}\, 2)$, and this explains the assumption $d \geq 3$ in Proposition 2. A detailed description of the case $d = n = 2$ can be found in [2] and [3].

For the sake of simplicity, in the following examples we shall take $n = d$.

Example 3. *Consider the set $S \subset \mathbb{Z}^3$ consisting of the directions $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$, $u_3 = (1, 2, 1)$, $w = (2, 3, 1)$, where $w = u_1 + u_2 + u_3$. Then*

$$\begin{array}{l} F_S(x_1,x_2,x_3) = \\ (x_1-1)(x_2-1)(x_1x_2^2x_3-1)(x_1^2x_2^3x_3-1) = \\ x_1^4x_2^6x_3^2 - x_1^4x_2^5x_3^2 - x_1^3x_2^6x_3^2 + x_1^3x_2^5x_3^2 - x_1^3x_2^4x_3 + \\ +x_1^3x_2^3x_3 + x_1^2x_2^4x_3 - 2x_1^2x_2^3x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 - \\ -x_1x_2^2x_3 + x_1x_2 - x_1 - x_2 + 1. \end{array}$$

According to (the proof of) Proposition 2 with $I = \{u_1, u_2, u_3\}$ *and* $J = \emptyset$, $F_S(x_1, x_2, x_3)$ *contains a unique monomial with coefficient outside* $\{-1, 0, 1\}$, *given by* $-2x^w = -2x_1^2x_2^3x_3$.

Example 4. *Consider the set* $S \subset \mathbb{Z}^3$ *of directions* $u_1 = (1,0,0)$, $u_2 = (0,1,0)$, $u_3 = (0,0,1)$, $w = (1,1,-1)$, *where* $w = u_1 + u_2 - u_3$. *Then*

$$\begin{array}{l} F_S(x_1,x_2,x_3) = \\ (x_1-1)(x_2-1)(x_3-1)(x_1x_2-x_3) = \\ -x_1x_2x_3^2 + x_2x_3^2 + x_1x_3^2 - x_3^2 + x_1^2x_2^2x_3 - \\ -x_1x_2^2x_3 - x_1^2x_2x_3 + 2x_1x_2x_3 - x_2x_3 - x_1x_3 + \\ +x_3 - x_1^2x_2^2 + x_1x_2^2 + x_1^2x_2 - x_1x_2. \end{array}$$

According to (the proof of) Proposition 2 with $I = \{u_1, u_2\}$ *and* $J = \{u_3\}$, $F_S(x_1, x_2, x_3)$ *contains a unique monomial with coefficient outside* $\{-1, 0, 1\}$, *given by* $2x^{u_1+u_2-w_-} = 2x_1x_2x_3$.

The following example shows that if $|I| + |J| \neq d$, then $F_S(x_1, \dots, x_n)$ may have more than one coefficient outside the set $\{-1, 0, 1\}$.

Example 5. *Consider the following set of linearly independent directions of* $\mathbb{Z}^4$:

$$\{u_1 = (1,0,0,0), u_2 = (0,1,0,0), u_3 = (0,1,1,0), u_4 = (0,1,2,1)\}.$$

Let us choose one of the following sets

$$\begin{array}{ll} S_1 & = \{u_1, u_2, u_3, u_4, w = u_1 + u_2 + u_4\} = \\ & = \{(1,0,0,0), (0,1,0,0), (0,1,1,0), (0,1,2,1), (1,2,2,1)\} \\ & (I = \{u_1, u_2, u_4\}, J = \emptyset) \\ & \\ S_2 & = \{u_1, u_2, u_3, u_4, w = u_1 + u_2 - u_4\} = \\ & = \{(1,0,0,0), (0,1,0,0), (0,1,1,0), (0,1,2,1), (1,0,-2,-1)\} \\ & (I = \{u_1, u_2\}, J = \{u_4\}). \end{array}$$

According to (the proof of) Proposition 2, $F_{S_1}(x_1, x_2, x_3, x_4)$ *contains the monomials* $2x^w = 2x_1x_2^2x_3^2x_4$ *and* $-2x^{w+u_3} = -2x_1x_2^3x_3^3x_4$ *with coefficient outside* $\{-1, 0, 1\}$, *and* $F_{S_2}(x_1, x_2, x_3, x_4)$ *contains the monomials* $-2x^{u_1+u_2-w_-} = -2x_1x_2x_3^2x_4$ *and* $2x^{u_1+u_2-w_-+u_3} = 2x_1x_2^2x_3^3x_4$ *with coefficient outside* $\{-1, 0, 1\}$.

We now provide a complete characterization of the minimal sets S of lattice directions, containing $d > 2$ linearly independent directions, for which $F_S(x_1, \ldots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$. The following theorem shows that all such sets must belong to the family presented in Proposition 2.

Theorem 6. *Let $S \subset \mathbb{Z}^n$ be a set of distinct lattice directions such that $|S| = d + 1$ and $\dim S = d \geq 3$. Then $F_S(x_1, \ldots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$ if and only if S is of the form*

$$S = \{u_1, ..., u_d, w = u(I) - u(J)\} \tag{2.4}$$

where the vectors $u_1, ..., u_d$ are linearly independent, and I, J are disjoint subsets of $\{u_1, ..., u_d\}$ such that $|I| \equiv |\{w\} \cup J| \pmod 2$.

Proof. Let S be of the form specified in (2.4), where the vectors $u_1, ..., u_d$ are linearly independent, and I, J are disjoint subsets of $\{u_1, ..., u_d\}$ such that $|I| \equiv |\{w\} \cup J| \pmod 2$. Then $F_S(x_1, \ldots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$ by Proposition 2.

Vice versa, let $S \subset \mathbb{Z}^n$ be a set of $d+1$ distinct lattice directions such that $\dim S = d \geq 3$ and $F_S(x_1, \ldots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$. By Remark 1 the rational function $R_S(x_1, \ldots, x_n) = \prod_{u \in S}(x^u - 1)$ has a coefficient outside the set $\{-1, 0, 1\}$, as well. This means that there exist two disjoint subsets $S_1, S_2 \subset S$ such that $u(S_1) = u(S_2)$ and $|S_1| \equiv |S_1| \pmod 2$. Let $S_1 = \{v_1, ..., v_h\}$ and $S_2 = \{z_1, \ldots, z_k\}$. Then $v_1 = z_1 + \cdots + z_k - (v_2 + \cdots + v_h)$. Let us define $w = v_1$, $I = \{z_1, ..., z_k\}$ and $J = \{v_2 + \cdots + v_h\}$ (notice that we may have $J = \emptyset$). Then we have $S = \{u_1, ..., u_d, w = u(I) - u(J)\}$, where $\{u_1, ..., u_d\} = S \setminus \{w\}$. The vectors $\{u_1, ..., u_d\}$ are linearly independent since $\dim S = d$. □

According to Proposition 2, in Examples 3 and 4 we have just one monomial with coefficients different from $\{-1, 0, 1\}$, and in these cases all the vectors u_1, u_2, u_3, which are linearly independent, appear in the expression giving the fourth vector w. In Example 5 the linear combination defining w does not contain the vector u_3, and in this case we have an additional monomial with coefficients outside the set $\{-1, 0, 1\}$ whose exponent is obtained by adding the vector u_3 to $u(I) - \sum_{u \in S} u_-$. This is true in general, as it is shown by the following theorem.

Theorem 7. *Let $S = \{u_1, ..., u_d, w = u(I) - u(J)\}$ be a set of distinct lattice directions in $\mathbb{Z}^n$, where the vectors $u_1, ..., u_d$ are linearly independent, and I, J are disjoint subsets of $\{u_1, ..., u_d\}$ such that $|I| \equiv |\{w\} \cup J| \pmod 2$. Then the number η of monomials in $F_S(x_1, \ldots, x_n)$ with coefficients outside the set $\{-1, 0, 1\}$ is given by*

$$\eta = 2^{d-|I|-|J|}. \tag{2.5}$$

The terms in $R_S(x_1, \ldots, x_n) = \prod_{u \in S}(x^u - 1)$ with coefficients outside the set $\{-1, 0, 1\}$ are given by

$$(-1)^{d+1-|I|-|H|} 2x^{u(I)+u(H)}, \tag{2.6}$$

where $H \subseteq S \setminus (I \cup J \cup \{w\})$, and to distinct sets H correspond distinct terms.

PROOF. If $|I|+|J|=d$, then the result follows from Proposition 2. Suppose $|I|+|J|<d$, and denote $T=I\cup J\cup\{w\}$. We have

$$\begin{aligned} R_S(x_1,\ldots,x_n) &= R_T(x_1,\ldots,x_n)\prod_{u\in S\setminus T}(x^u-1)= \\ &= R_T(x_1,\ldots,x_n)\sum_{H\subseteq S\setminus T}(-1)^{|S\setminus T|-|H|}x^{u(H)}. \end{aligned}$$

By Proposition 2, applied to the set T of directions, the expansion of $R_T(x_1,\ldots,x_n)$ contains a unique term with coefficient outside $\{-1,0,1\}$, given by $(-1)^{|T|-|I|}2x^{u(I)}$. Since the vectors $u_1,\ldots,u_d$ are linearly independent, all the terms obtained by multiplying $(-1)^{|T|-|I|}2x^{u(I)}$ by $(-1)^{|S\setminus T|-|H|}x^{u(H)}$, where $H\subseteq S\setminus T$, are distinct and none of them can be equal to a term in $R_T(x_1,\ldots,x_n)$. Therefore, the number of terms in $R_S(x_1,\ldots,x_n)$ with coefficients outside the set $\{-1,0,1\}$ equals the number of subsets of $S\setminus T$ (including the empty set which corresponds to the term $(-1)^{d+1-|I|}2x^{u(I)}$), so that we have $\eta=2^{|S\setminus T|}=2^{d-|I|-|J|}$, as required.

The previous argument also shows that all the terms in $R_S(x_1,\ldots,x_n)$ with coefficients outside the set $\{-1,0,1\}$ are given by $(-1)^{d+1-|I|-|H|}2x^{u(I)+u(H)}$, for $H\subseteq S\setminus(I\cup J\cup\{w\})$. All these terms are distinct, since $S\setminus T$ consists of linearly independent vectors. □

From the geometric point of view, Theorem 7 says that the multiple points in the S-weakly bad configuration (A_S,B_S) are obtained by translating the multiple point $u(I)$ by the vectors $u\in S$ which do not belong to the vector space generated by $I\cup J\cup\{w\}$.

Remark 3. Note that if S contains all the n coordinate directions, then $d=n=|I|+|J|$, so that the weakly bad-configuration associated to F_S has a unique multiple point.

3. Minimal sets of directions providing uniqueness in a finite grid

In this section we characterize the minimal sets of lattice directions that ensure uniqueness inside a finite grid $\mathcal{A}=[m_1]\times[m_2]\times\cdots\times[m_n]$ in $\mathbb{Z}^n$, where, for $p\in\mathbb{N}$, $[p]=\{0,1,...,p-1\}$. We refer to all the subsets of $\mathcal{A}$ as *bounded sets*. We say that a finite set $S\subseteq\mathbb{Z}^n$ of lattice directions is a *set of uniqueness* for a finite grid $\mathcal{A}$ if no distinct lattice sets $E,F\subseteq\mathcal{A}$ exist which are tomographically equivalent. In this case we also say that each set $E\subset\mathcal{A}$ is *uniquely determined* in $\mathcal{A}$. Notice that this does not imply that E is S-unique, as there may exist $F\nsubseteq\mathcal{A}$ different from but tomographically equivalent to E. Therefore, looking for sets S which are sets of uniqueness for $\mathcal{A}$ is equivalent to add a priori knowledge on the class of sets to be reconstructed, namely to restrict to the class of bounded sets.

By means of Theorem 6 we can state a necessary condition on a minimal set S of lattice directions to be a set of uniqueness for $\mathcal{A}$.

Proposition 8. *Let $S\subset\mathbb{Z}^n$ be a set of distinct lattice directions such that $|S|=d+1$ and $\dim S=d\geq 3$. Suppose that S is a valid set of uniqueness for a finite grid $\mathcal{A}=[m_1]\times[m_2]\times\cdots\times[m_n]\subset\mathbb{Z}^n$. Then S is of the form*

$$S=\{u_1,...,u_d,w=u(I)-u(J)\}$$

where the vectors $u_1, ..., u_d$ *are linearly independent, and* I, J *are disjoint subsets of* $\{u_1, ..., u_d\}$ *such that* $|I| \equiv |\{w\} \cup J| \pmod 2$.

Proof. Since S is a valid set of directions for $\mathcal{A}$, the S-weakly bad configuration (F_S^+, F_S^-) is contained in $\mathcal{A}$. Moreover, since S is a set of uniqueness for $\mathcal{A}$, (F_S^+, F_S^-) cannot be an S-bad configuration, otherwise the sets F_S^+, and F_S^- would be tomographically equivalent. Therefore the polynomial $F_S(x_1, \ldots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$. The result then follows from Theorem 6. $\square$

To get a sufficient condition we now study all the S-weakly bad configurations contained in $\mathcal{A}$, or equivalently the functions $g : \mathcal{A} \to \mathbb{Z}$ which have zero line sums along the lines corresponding to the directions in S. In [20], Hajdu and Tijdeman showed that all these functions can be obtained as a linear combination of shifts of a basic one corresponding to the function f (where f denotes the function generated by the basic polynomial $F_S(x_1, \ldots, x_n)$). In view of this result we first consider a shift by a single vector $v \in \mathbb{Z}^n$.

For any $v \in \mathbb{Z}^n$ we define the polynomials $\phi_v^-(x_1, \ldots, x_n) = x^{v+} - x^{-v-}$, and $\phi_v^+(x_1, \ldots, x_n) = x^{v+} + x^{-v-}$. Given a finite set S of lattice directions and $v \in \mathbb{Z}^n$, we define the functions $f_-^v : \mathbb{Z}^n \to \mathbb{Z}$, and $f_+^v : \mathbb{Z}^n \to \mathbb{Z}$ to be the maps whose generating functions are $G_{f_-^v}(x_1, \ldots, x_n)) = \phi_v^-(x_1, \ldots, x_n) F_S(x_1, \ldots, x_n)$ and $G_{f_+^v}(x_1, \ldots, x_n) = \phi_v^+(x_1, \ldots, x_n) F_S(x_1, \ldots, x_n)$, respectively.

Lemma 9. *Let* $S = \{u_1, ..., u_d, w = u(I) - u(J)\}$ *be a set of distinct lattice directions in* $\mathbb{Z}^n$*, where the vectors* $u_1, ..., u_d$ *are linearly independent, and* I, J *are disjoint subsets of* $\{u_1, ..., u_d\}$ *such that* $|I| \equiv |\{w\} \cup J| \pmod 2$. *Then the following holds*

1. $||f_-^v|| \leq 1$ *if and only if* $v = u(X) - u(I)$*, where* $X \subseteq T = I \cup J \cup \{w\}$ *and* $|X| \equiv |I| \pmod 2$.
2. $||f_+^v|| \leq 1$ *if and only if* $v = u(X) - u(I)$*, where* $X \subseteq T = I \cup J \cup \{w\}$ *and* $|X| \not\equiv |I| \pmod 2$.

Proof. We first note that $||f_-^v|| \leq 1$ ($||f_+^v|| \leq 1$) if and only if its generating function $G_{f_-^v}(x_1, \ldots, x_n)$ ($G_{f_+^v}(x_1, \ldots, x_n)$, respectively) has all the coefficients in the set $\{-1, 0, 1\}$. Moreover, the coefficients of these polynomials equal the coefficients of the rational functions $(x^v \pm 1) R_S(x_1, \ldots, x_n)$ (see Remark 1).

We shall prove the theorem for the function f_-^v (case (1)), as the other case is similar. Suppose $v = u(X) - u(I)$, where $X \subseteq I \cup J \cup \{w\}$ and $|X| \equiv |I| \pmod 2$. If $X = I$ then $v = 0$ and $f_-^v = 0$. Suppose $X \neq I$, and denote $T = I \cup J \cup \{w\}$. We have $x^{u(I)+v} = x^{u(X)}$ and $(-1)^{|T|-|I|} = (-1)^{|T|-|X|}$. Consider

$$(x^v - 1) R_S(x_1, \ldots, x_n) = (x^v - 1) R_T(x_1, \ldots, x_n) \prod_{u \in S \setminus T} (x^u - 1)\,.$$

By Proposition 2 the rational function $R_T(x_1, \ldots, x_n)$ contains a unique term with coefficient outside $\{-1, 0, 1\}$, given by $(-1)^{|T|-|I|} 2x^{u(I)}$. Thus, in the expansion of the product $x^v R_T(x_1, \ldots, x_n)$ we have the term $(-1)^{|T|-|I|} 2x^{u(I)+v}$ and in $-R_T(x_1, \ldots, x_n)$ we have the term $-(-1)^{|T|-|X|}x^{u(X)} = -(-1)^{|T|-|I|}x^{u(I)+v}$. Therefore, the term $x^{u(I)+v}$ in $(x^v - 1)R_T(x_1, \ldots, x_n)$ has coefficient $(-1)^{|T|-|I|} \in \{-1, 0, 1\}$. Since the vectors in $I \cup J$ are linearly independent, all the other terms in $(x^v - 1)R_T(x_1, \ldots, x_n)$ have coefficients in $\{-1, 0, 1\}$. Further, the linear independence of the vectors in $S \setminus T$ implies that distinct terms in $(x^v - 1)R_T(x_1, \ldots, x_n)$ remain distinct when they are multiplied by $\prod_{u \in S \setminus T}(x^u - 1)$.

Vice versa, suppose $||f_-^v|| \leq 1$. Then the rational function $(x^v - 1)R_S(x_1, \ldots, x_n)$ has all the coefficients in $\{-1, 0, 1\}$. By Theorem 7, the rational function $R_S(x_1, \ldots, x_n)$ contains the terms $(-1)^{d+1-|I|-|H|}2x^{u(I)+u(H)}$, where $H \subset S \setminus T$. The absolute values of the coefficients of these terms must decrease by multiplying with $(x^v - 1)$. Thus, for each $H \subset S \setminus T$ there exists a set $\widehat{H} \subseteq S$ such that $v + u(I) + u(H) = u(\widehat{H})$, and $(-1)^{d+1-|I|-|H|} = (-1)^{d+1-|\widehat{H}|}$. In particular, for $H = \emptyset$, we denote $X = \widehat{H}$, so that we have $v + u(I) = u(X)$, where $X \subseteq S$ and $(-1)^{d+1-|I|} = (-1)^{d+1-|X|}$. If $S = I \cup J \cup \{w\}$ then $X \subseteq I \cup J \cup \{w\}$, with $|X| \equiv |I| \pmod 2$, as required.

Let us suppose $S \neq I \cup J \cup \{w\}$ and X not contained in $I \cup J \cup \{w\}$. Let $z \in X \setminus (I \cup J \cup \{w\})$ and denote $Y = X \setminus \{z\}$. We then have $v + u(I) = u(X) = z + u(Y)$ (notice that $u(\{z\}) = z$). We now consider the set $H = \{z\}$. Then there exists $\widehat{H} \subseteq S$ such that $v + u(I) + z = u(\widehat{H})$. Thus, we have $z + u(Y) + z = u(\widehat{H})$. Since $z \notin I \cup J \cup \{w\}$, the conditions $w = u(I) - u(J)$ and $z + u(Y) + z = u(\widehat{H})$ provide two distinct linear dependence relations among the vectors in S, which contradicts the assumption $\dim S = d$. This proves that $X \subseteq I \cup J \cup \{w\}$, with $(-1)^{d+1-|I|} = (-1)^{d+1-|X|}$, so that $|X| \equiv |I| \pmod 2$, as required. □

Example 10. *Consider the set S as in Example 5:*

$$\begin{aligned} S &= \{u_1, u_2, u_3, u_4, w = u_1 + u_2 + u_4\} \\ &= \{(1,0,0,0), (0,1,0,0), (0,1,1,0), (0,1,2,1), (1,2,2,1)\} \end{aligned}$$

where $I = \{u_1, u_2, u_4\}, J = \emptyset$.
We have $T = \{u_1, u_2, u_4, w = u_1 + u_2 + u_4\}$, and $u(I) = w = u_1 + u_2 + u_4$. According to Lemma 9, $\phi_v^-(x_1, \ldots, x_n)F_S((x_1, \ldots, x_4)$ has all the coefficients in $\{-1, 0, 1\}$ if

$$v \in \{0, \quad \pm(u_1 + u_2), \quad \pm(u_1 + u_4), \quad \pm(u_2 + u_4)\},$$

as we can easily verify.
Analogously, $\phi_v^+(x_1, \ldots, x_n)F_S((x_1, \ldots, x_4)$ has all the coefficients in $\{-1, 0, 1\}$ if

$$v \in \{\pm u_1, \quad \pm u_2, \quad \pm u_4, \quad \pm w\}.$$

We now consider an arbitrary function $g : \mathcal{A} \subset \mathbb{Z}^n \to \mathbb{Z}$, which has zero line sums along the lines corresponding to a finite set of directions S, which is valid for $\mathcal{A}$. The following result represents a higher-dimensional generalization of Theorem 3 in [2] (see also [3, Theorem 2]), and the proof is similar. However, for the convenience of the reader, we prefer to give the proof explicitly. For any pair of vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ we denote $\min(a, b) = (\min(a_1, b_1), \ldots, \min(a_n, b_n))$.

Theorem 11. *Let $S = \{u_1, ..., u_d, w = u(I) - u(J)\}$ be a valid set of distinct lattice directions for a given grid $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n] \subset \mathbb{Z}^n$, where the vectors $u_1, ..., u_d$ are linearly independent, and I, J are disjoint subsets of $\{u_1, ..., u_d\}$ such that $|I| \equiv |\{w\} \cup J| \pmod 2$. Let $g : \mathcal{A} \to \mathbb{Z}$ be a non-trivial function which has zero line sums along the lines corresponding to the directions in S. If $||g|| \leq 1$ then there exists $r \in \mathbb{N}$ such that*

$$G_g(x_1, \ldots, x_n) = \sum_{t=1}^{r} \delta(t) x^{v(t)} F_S(x_1, \ldots, x_n), \tag{3.1}$$

where $\delta(t) = \pm 1$, and for each $t \in \{1, ..., r\}$, there exists $t' \in \{1, ..., r\}$ such that the vector $u(t) = v(t) - v(t')$ satisfies the following conditions

1. *$u(t) = u(X) - u(I)$, where $X \subseteq T = I \cup J \cup \{w\}$ and $|X| \equiv |I| \pmod 2$, if $\delta(t) \neq \delta(t')$.*
2. *$u(t) = u(X) - u(I)$, where $X \subseteq T = I \cup J \cup \{w\}$ and $|X| \not\equiv |I| \pmod 2$, if $\delta(t) = \delta(t')$.*

Proof. By the higher dimensional version of [20, Theorem 1] (see also [21]), we have $G_g(x_1, \ldots, x_n) = P(x_1, \ldots, x_n) F_S(x_1, \ldots, x_n)$ for some polynomial $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$. Each monomial of $P(x_1, \ldots, x_n)$ can be split into a sum of monomials with coefficients ± 1, so that

$$P(x_1, \ldots, x_n) = \sum_{t=1}^{r} \delta(t) x^{s(t)},$$

for suitable vectors $s(t) \in \mathbb{Z}^n$. Therefore we get

$$G_g(x_1, \ldots, x_n) = \sum_{t=1}^{r} \delta(t) x^{s(t)} F_S(x_1, \ldots, x_n),$$

where $\delta(t) = \pm 1$. Let f be the function generated by $F_S(x_1, \ldots, x_n)$. By Theorem 7 there exists $q \in \mathbb{Z}^n$ such that $|f(q)| = 2$. By multiplying $F_S(x_1, \ldots, x_n)$ by $x^{s(t)}$, the value $f(q) = \pm 2$ is attached to the translated point $q + s(t)$. Since $||g|| \leq 1$, such value must be reduced in g. This occurs by adding to the corresponding monomial a monomial with the same degree and a coefficient with opposite sign. Thus, for each $t \in \{1, ..., r\}$, some point $z_t \in \mathbb{Z}^n$ must exist, and some monomial $\delta(t') x^{s(t')} \neq \delta(t) x^{s(t)}$ of $P(x_1, \ldots, x_n)$, such that $f(z_t) \neq 0$, and $z_t + s(t') = q + s(t)$. It follows that the function

$$\left(\delta(t) x^{s(t)} + \delta(t') x^{s(t')}\right) F_S(x_1, \ldots, x_n)$$

has no multiple points. Let us define $u(t) = s(t) - s(t') = z_t - q$. Then we have

$$\begin{gathered}\left(\delta(t) x^{s(t)} + \delta(t') x^{s(t')}\right) F_S(x_1, \ldots, x_n) = \\ x^{\min(s(t), s(t'))} \left(\delta(t) x^{u(t)_+} + \delta(t') x^{-u(t)_-}\right) F_S(x_1, \ldots, x_n).\end{gathered}$$

By Lemma 9 applied to the vector $u(t)$ and to the function

$$\left(\delta(t) x^{u(t)_+} + \delta(t') x^{-u(t)_-}\right) F_S(x_1, \ldots, x_n),$$

we get either 1. or 2., depending on the signs of $\delta(t)$ and $\delta(t')$, which provide a necessary condition for $||g|| \leq 1$.

By Proposition 8, if a set $S \subset \mathbb{Z}^n$ of distinct lattice directions with $|S| = d + 1$ and $\dim S = d \geq 3$ is a valid set of uniqueness for a rectangular grid $\mathcal{A} \subset \mathbb{Z}^n$, then it must be of the form $S = \{u_1, ..., u_d, w = u(I) - u(J)\}$, where the vectors $u_1, ..., u_d$ are linearly independent, and I, J are disjoint subsets of $\{u_1, ..., u_d\}$ such that $|I| \equiv |\{w\} \cup J| \pmod 2$. Among the sets S having this form we can now specify which are sets of uniqueness for $\mathcal{A}$, thanks to Theorem 11. For a given set $S = \{u_1, ..., u_d, w = u(I) - u(J)\}$, where I, J are disjoint subsets of $\{u_1, ..., u_d\}$ we define $D = \{\pm v : v = u(X) - u(I) \neq 0, X \subseteq I \cup J \cup \{w\}\}$.

Theorem 12. *Let $S = \{u_r = (a_{r1}, \dots, a_{rn}) : r = 1, \dots, d+1\}$, where $u_1, ..., u_d$ are linearly independent, $u_{d+1} = u(I) - u(J)$, and I, J are disjoint subsets of $\{u_1, ..., u_d\}$ such that $|I| \equiv |\{w\} \cup J| \pmod 2$. Suppose S is valid for the grid $\mathcal{A} = [m_1] \times [m_2] \times \cdots \times [m_n]$. Denote $\sum_{r=1}^{d+1} |a_{ri}| = h_i$, for each $i \in \{1, \dots, n\}$. Suppose that $g : \mathcal{A} \to \mathbb{Z}$ has zero line sums along the lines in the directions in S, and $||g|| \leq 1$. Then g is identically zero if and only if for each $v = (v_1, \dots, v_n) \in D$, there exists $s \in \{1, \dots, n\}$ such that $|v_s| \geq m_s - h_s$.*

Proof. We first show that if there exists a vector $v = (v_1, \dots, v_n) \in D$ such that $|v_i| < m_i - h_i$ for all $i \in \{1, \dots, n\}$, then we can find a non-trivial function $g : \mathcal{A} \to \mathbb{Z}$ such that $||g|| \leq 1$.
Since $v \in D$, by Lemma 9, either $||f_v^-|| \leq 1$ or $||f_v^+|| \leq 1$. Moreover, we have

$$\deg_{x_i} G_{f_v^-}(x_1, \dots, x_n) = \deg_{x_i} G_{f_v^+}(x_1, \dots, x_n) = |v_i| + h_i < m_i,$$

for all $i \in \{1, \dots, n\}$. Therefore, we can choose $g = f_v^-$, or $g = f_v^+$, depending on which one satisfies the condition $||g|| \leq 1$.

Vice versa, let us suppose that for each $v = (v_1, \dots, v_n) \in D$ there exists $s \in \{1, \dots, n\}$ such that $|v_s| \geq m_s - h_s$. Consider a non-trivial function $g : \mathcal{A} \to \mathbb{Z}$ such that $||g|| \leq 1$. By Theorem 11, its generating function $G_g(x_1, \dots, x_n)$ is a polynomial of the form (3.1), where $\delta(t) = \pm 1$, and for each $t \in \{1, ..., r\}$, there exists $t' \in \{1, ..., r\}$ such that the vector $u(t) = v(t) - v(t') \in D$.

We must prove that no such polynomial $G_g(x_1, \dots, x_n)$ exists, with degree less than m_i in x_i, for all $i \in \{1, ..., n\}$. Suppose the converse. Then $G_g(x_1, \dots, x_n)$ contains the expression

$$\left(\delta(t')x^{v(t')} + \delta(t)x^{v(t)}\right) F_S(x_1, \dots, x_n)$$

which, up to a sign, can be written in one of the following forms

$$\begin{aligned} &x^{\min\{v(t), v(t')\}} \phi^+_{u(t)}(x_1, \dots, x_n) \cdot F_S(x_1, \dots, x_n) = \\ &= x^{\min\{v(t), v(t')\}} \cdot G_{f^+_{u(t)}}(x_1, \dots, x_n), \end{aligned}$$

if $\delta(t) = \delta(t')$, or

$$x^{\min\{v(t),v(t')\}}\phi^{-}_{u(t)}(x_1,\ldots,x_n)\cdot F_S(x_1,\ldots,x_n)=$$
$$=x^{\min\{v(t),v(t')\}}\cdot G_{f^{-}_{u(t)}}(x_1,\ldots,x_n),$$

if $\delta(t)\neq\delta(t')$, where $u(t)\in D$.
Therefore, we have

$$\deg_{x_i}G_g(x_1,\ldots,x_n)\geq\deg_{x_i}G_{f^{+}_{u(t)}}(x_1,\ldots,x_n)=$$
$$=\deg_{x_i}G_{f^{-}_{u(t)}}(x_1,\ldots,x_n).$$

Assume $u(t)=(u_1(t),\ldots,u_n(t))$, so that we have

$$\deg_{x_i}G_{f^{-}_{u(t)}}(x_1,\ldots,x_n)=\deg_{x_i}G_{f^{+}_{u(t)}}(x_1,\ldots,x_n)=|u_i(t)|+h_i, \tag{3.2}$$

for all $i\in\{1,\ldots,n\}$. Since $u(t)\in D$, there exists $s\in\{1,\ldots,n\}$, so that $|u_s(t)|\geq m_s-h_s$. Therefore the degree in x_s of the polynomial $G_g(x_1,\ldots,x_n)$ is at least equal to m_s, so contradicting our assumption. Therefore, being $||g||\leq 1$, we get that g is identically zero.
□

Remark 4. Let us also recall that a lattice set E is *additive* if and only if E has no weakly-bad configurations (see [10]). In [4] we get an algorithm which constructs all the non-additive lattice sets which can be uniquely reconstructed inside a given rectangular two-dimensional grid by their X-rays taken in a prescribed set of four lattice directions. Such an algorithm is based on a uniqueness result [4, Theorem 1] which enables us to select such suitable 4-tuples of uniqueness. Now, Theorem 12 extends this result to any dimension $n\geq 2$, and consequently the algorithm in [4] can be adapted and employed even in higher dimensions.

Furthermore, the same argument as in [4] can be applied to show that Theorem 2 in [4] (which is free-dimensional) recognizes all the non-additive sets of uniqueness. Also, in the case when S contains all the n coordinate directions, by Remark 3, we have that Theorems 4 and 5 in [4] can be easily extended to the n-dimensional case. This allows to count the number of bounded additive and bounded non-additive sets of uniqueness.

From the geometrical point of view, Theorem 12 can then be rephrased as follows: *If S and $\mathcal{A}$ are chosen according to Theorem 12, and D is such that its members satisfy the conditions in Theorem 12, then every subset of $\mathcal{A}$ is uniquely determined in $\mathcal{A}$ by its X-rays in the directions in S.*

Example 13. *Consider the set $S\subset\mathbb{Z}^4$ defined by*

$$\begin{aligned} S &= \{u_1,u_2,u_3,w=u_1+u_2+u_3\}\\ &= \{(1,0,0,0),(1,1,0,0),(1,1,2,1),(3,2,2,1)\}\end{aligned}$$

where $I=\{u_1,u_2,u_3\}$, $J=\emptyset$.

We have $h_1 = 6$, $h_2 = 4$, $h_3 = 4$, $h_4 = 2$, so that S is a valid set of directions for every grid $\mathcal{A} = [m_1] \times [m_2] \times [m_3] \times [m_4]$, such that $m_1 \geq 7$, $m_2 \geq 5$, $m_3 \geq 5$ and $m_4 \geq 3$. Since

$$D = \{\pm u_1,\ \pm u_2,\ \pm u_3,\ \pm w,\ \pm(u_1 + u_2),\ \pm(u_1 + u_3),\ \pm(u_2 + u_3)\},$$

the conditions of Theorem 12 cause the restriction $m_1 = 7$. Therefore, the set S is a valid set of uniqueness for every grid $\mathcal{A} = [7] \times [m_2] \times [m_3] \times [m_4]$, such that $m_2 \geq 5$, $m_3 \geq 5$ and $m_4 \geq 3$.
We also note that by adding a further direction to the set S we can enlarge the size of the grid $\mathcal{A}$. For example, let $\widehat{S} \subset \mathbb{Z}^4$ be defined by

$$\begin{aligned} \widehat{S} &= \{u_1, u_2, u_3, u_4, w = u_1 + u_2 + u_3\} \\ &= \{(1,0,0,0), (1,1,0,0), (1,1,2,1), (k,0,1,0), (3,2,2,1)\} \end{aligned}$$

where $I = \{u_1, u_2, u_3\}$, $J = \emptyset$, and $k \neq 0$. Then $\widehat{S}$ is a valid set of uniqueness for every grid $\mathcal{A} = [7 + |k|] \times [m_2] \times [m_3] \times [m_4]$, such that $m_2 \geq 5$, $m_3 \geq 6$ and $m_4 \geq 3$.

4. Concluding remarks

In this paper we focused on the uniqueness of reconstruction of bounded sets in Discrete Tomography. Following the algebraic approach introduced in [20] and [21] we considered sets S of lattice directions in $\mathbb{Z}^n$ which contain $d > 2$ linearly independent directions, and we proved that $d + 1$ represents the minimal number of directions we need to avoid switching-components (Lemma 1). We then characterized all the sets consisting of $d + 1$ lattice directions in $\mathbb{Z}^n$ spanning a d-dimensional subspace of $\mathbb{Z}^n$, for which the associated polynomial $F_S(x_1, \ldots, x_n)$ has a coefficient outside the set $\{-1, 0, 1\}$ (see Theorem 6). These results were then applied to bounded sets contained in a rectangular grid $\mathcal{A} = [m_1] \times \ldots \times [m_n]$ in order to characterize those sets S of directions which are sets of uniqueness for $\mathcal{A}$. Thanks to Proposition 8 we know the structure of such sets, while Theorem 12 specifies the size of the corresponding grid $\mathcal{A}$. Since $n \geq d \geq 3$, a set of uniqueness could be constructed just with 4 directions (obtained by selecting the minimum value $d = 3$). However, for reconstructing purposes, it could happen that $\mathcal{A}$ is too much "narrow" in some coordinate direction. In this case a selection of $d > 3$ could then be useful in order to try to increase the value of some m_s. This allows to enlarge the grid of uniqueness in the corresponding coordinate direction. See also Example 13 for more details.

Finally, we wish to discuss the choice we made of the grid $\mathcal{A}$. We note that the reconstruction task consists in recovering a lattice set consistent with the data, if one exists. The given data identify a finite grid $\mathcal{G}$ (i.e. the set of lattice points which are intersections of lines corresponding to nonzero X-rays), called the *tomographic grid* of the unknown object. Feasible solutions of the reconstruction problem are subsets of $\mathcal{G}$ with same X-rays taken along lines in the given directions. Our results show that the reconstruction must be unique when $\mathcal{G} \subseteq \mathcal{A}$. This happens, for instance, when the set S of directions contains the coordinate directions. Note that in this case some results (Theorems 4 and 5) of [4] immediately extend to n-dimension. Uniqueness might be lost whenever $\mathcal{G} \nsubseteq \mathcal{A}$, and in this case uniqueness may be achieved by a priori knowledge that the unknown lattice set is contained in a given n-dimensional grid $\mathcal{A}$. Moreover, in the framework of verification, where directions can be chosen depending on the object, for instance, in order to check a given shape (as we perceive by [9]), the coordinate directions can be used to find the minimal grid $\mathcal{A}(\equiv \mathcal{G})$ containing all feasible solutions. In this view, our a priori requirement is weaker than the concept of verification, and in general, stronger than that of determination. Therefore our problem can be placed in between so providing a hierarchy of uniqueness issues which depends on the choice of the set of directions.

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