Maximal subgroups of amalgams of finite inverse semigroups

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1 Introduction

If S_1 and S_2 are semigroups (groups) such that $S_1 \cap S_2 = U$ is a non-empty subsemigroup (subgroup) of both S_1 and S_2, then $[S_1, S_2; U]$ is called an amalgam of

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semigroups (groups). The amalgamated free product $S_1 *_U S_2$ associated with this amalgam in the category of semigroups (groups) is defined by the usual universal diagram.

The amalgam $[S_1, S_2; U]$ is said to be strongly embeddable in a semigroup (group) S if there are injective homomorphisms $\phi_i : S_i \rightarrow S$ such that $\phi_1|_U = \phi_2|_U$ and $S_1\phi_1 \cap S_2\phi_2 = U\phi_1 = U\phi_2$. It is well known that every amalgam of groups embeds in a group while semigroup amalgams do not necessarily embed in any semigroup [14]. On the other hand, every amalgam of inverse semigroups (in the category of inverse semigroups) embeds in an inverse semigroup, and hence in the corresponding amalgamated free product in the category of inverse semigroups [11].

An inverse semigroup is a semigroup S with the property that for each element $a \in S$ there is a unique element $a^{-1} \in S$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$; the element a^{-1} is called the inverse of a. As a consequence of the definition, the set of the idempotents $E(S)$ is a semilattice. One may also define a natural partial order on S by setting $a \leq b$ if and only if $a = eb$ for some $e \in E(S)$.

Inverse semigroups may be regarded as semigroups of partial one-to-one transformations, so they arise very naturally in several areas of mathematics and more recently also in computer science, mainly since the inverse of an element can be seen as an "undo with a trace" of the action represented by that element. We refer the reader to the book of Petrich [18] for basic results and notation about inverse semigroups and to the more recent books of Lawson [15] and Paterson [17] for many references to the connections between inverse semigroups and other branches of mathematics.

The free object on a set X in the category of inverse semigroups is denoted by $FIS(X)$. It is the quotient of the free semigroup $(X \cup X^{-1})^+$ by the least congruence ν that makes the resulting quotient semigroup inverse (see [18]). The inverse semigroup S presented by a set X of generators and a set T of relations is denoted by $S = Inv\langle X; T\rangle$. This is the quotient of the free semigroup $(X \cup X^{-1})^+$ by the least congruence τ that contains ν and the relations in T.

The structure of $FIS(X)$ was studied via graphical methods by Munn [16]. Munn's work was greatly extended by Stephen [25] who introduced the notion of Schützenberger graphs associated with presentations of inverse semigroups. These graphs were widely used in the study of algorithmic and structural questions for several classes of inverse semigroups (see, for instance [3–8, 10, 12, 13, 20–22, 24, 26]). In particular Haataja, Margolis and Meakin were the first to show that Bass–Serre theory may be applied to study the structure of maximal subgroups. They obtained results for amalgams of inverse semigroups where U contained all idempotents of S_1 and S_2. Their construction was extended by Bennett [3] and Jajcayová [12] to respectively study the maximal subgroups of a special class of amalgams and HNN-extensions of inverse semigroups. A topological approach using a standard 2-complex associated to an inverse semigroup presentation has been used by Steinberg [24] who generalized the results of Haataja et al. [10] and overlapped with some of the results of Bennett [3].

In [6], the word problem for amalgams of finite inverse semigroups was shown to be decidable by constructing an automaton that is a good approximation of the Schützenberger automaton. Here, we make use of this construction to study the structure of maximal subgroups in such amalgams. This is done along the lines of Bennett's study of maximal subgroups of lower bounded amalgams, but as amalgams of finite inverse semigroups are not necessarily lower bounded, Schützenberger automata of amalgams of finite inverse semigroups differ from Bennett's automata mainly by the fact the Schützenberger graphs of the two original semigroups do not appear as subgraphs of the resulting Schützenberger graph of the amalgam. This leads to several important technical differences in the treatment and in the results.

Our paper is organized as follows: In Sect. 2, we recall basic definitions and relevant results concerning Schützenberger automata of inverse semigroups, and the structure and properties of Schützenberger automata of amalgams of finite inverse semigroups in particular. In Sect. 3, we introduce the concept of a host which is a fundamental notion. In Sect. 4, we prove that the automorphism groups of the Schützenberger graphs of amalgams (which are isomorphic to the maximal subgroups) are isomorphic to the automorphism groups of particular subgraphs formed by hosts. We study these subgraphs and their properties in Sect. 5. Finally, in Sect. 6, merging the theory of Bass–Serre with the previous results, we give a complete description of the maximal subgroups in an amalgam of finite inverse semigroups.

2 Preliminaries

In this section we review definitions and results concerning Schützenberger automata of inverse semigroups, and briefly describe the construction of Schützenberger graphs of amalgams of finite inverse semigroups. We refer the reader to [2,6,18,25] for more details.

An *inverse word graph* over an alphabet X is a strongly connected labelled digraph whose edges are labelled over $X \cup X^{-1}$, where X^{-1} is the set of formal inverses of elements in X, so that for each edge e labelled by $x \in X$ there is an edge labelled by x^{-1} in the reverse direction. A finite sequence of edges $e_i = (\alpha_i, a_i, \beta_i)$, $1 \leq i \leq n, a_i \in X \cup X^{-1}$ with $\beta_i = \alpha_{i+1}$ for all i with $1 \leq i < n$, is an $\alpha_1 - \beta_n$ path of Γ labelled by $a_1 a_2 \ldots a_n \in (X \cup X^{-1})^+$. An *inverse automaton* over X is a triple $\mathcal{A} = (\alpha, \Gamma, \beta)$ where Γ is an inverse word graph over X with the set of vertices $V(\Gamma)$, set of edges $Ed(\Gamma)$, and $\alpha, \beta \in V(\Gamma)$ are two special vertices called the initial and final state of $\mathcal{A}$. The language $L[\mathcal{A}]$ recognized by $\mathcal{A}$ is the set of labels of all $\alpha - \beta$ paths of Γ. The inverse word graph Γ over X is *deterministic* if for each $\nu \in V(\Gamma)$, $a \in X \cup X^{-1}$, $(\nu, a, \nu_1), (\nu, a, \nu_2) \in Ed(\Gamma)$ implies $\nu_1 = \nu_2$.

Morphisms between inverse word graphs are graph morphisms that preserve labeling of edges and are referred to as *V-homomorphisms* in [25]. In this paper we simply refer to them as homomorphisms, and in the case a morphism is surjective, injective or bijective, we refer to it as an epimorphism, monomorphism or isomorphism, respectively. The group of all automorphisms of a graph Γ is denoted by $Aut(\Gamma)$. If Γ is an inverse word graph over X and ρ is an equivalence relation on the set of vertices of Γ, the corresponding quotient graph Γ/ρ is called a *V-quotient* of Γ (see [25] for details). There is a least equivalence relation on the vertices of an inverse automaton Γ such that the corresponding V-quotient is deterministic. A deterministic V-quotient of Γ is called a *DV-quotient*. There is a natural homomorphism from Γ onto a V-quotient of Γ. The notions of morphism, V-quotient and DV-quotient of inverse graphs extend analogously to inverse automata (see [25]).

Let $S = Inv\langle X; T\rangle \simeq (X \cup X^{-1})^+/\tau$ be an inverse semigroup. The Schützenberger graph $S\Gamma(X, T; w)$ for a word $w \in (X \cup X^{-1})^+$ relative to the presentation $\langle X|T\rangle$ has the $\mathcal{R}$-class of $w\tau$ in S for its set of vertices and its set of edges consists of all the triples (s, x, t) with $x \in X \cup X^{-1}$, and $s \cdot x\tau = t$. We view the edge (s, x, t) as being directed from s to t. The graph $S\Gamma(X, T; w)$ is a deterministic inverse word graph over X. The structure of Schützenberger graphs is closely connected with the Green's relations on S. In particular the following results by Stephen will be important for our purposes.

Proposition 1 *Let $S = Inv\langle X; T\rangle$ be an inverse semigroup and let $e, f \in E(S)$. Then*

1. *$e\mathcal{D}f$ if and only if there exists a V-isomorphism $\phi : S\Gamma(X, T; e) \to S\Gamma(X, T; f)$ [25, Theorem 3.4 (a)].*
2. *The $\mathcal{H}$-class of e and $Aut(S\Gamma(X, T; e))$ are isomorphic groups [25, Theorem 3.5].*

The second statement identifies the group of symmetries of $S\Gamma(X, T; e)$ with the maximal subgroup of S having e as the unity. This fact is fundamental since it gives a geometric interpretation of maximal subgroups. It will be implicitly used throughout the paper. The automaton $\mathcal{A}(X, T; w)$ whose underlying graph is $S\Gamma(X, T; w)$ with the vertex $ww^{-1}\tau$ as the initial state and the vertex $w\tau$ as the terminal state, is called the Schützenberger automaton of $w \in (X \cup X^{-1})^+$ relative to the presentation $\langle X|T\rangle$.

In [25] Stephen provides an iterative (but in general not effective) procedure to build $\mathcal{A}(X, T; w)$ via two operations, *the elementary determination* and *the elementary expansion*. An inverse word graph is called *closed* with respect to $\langle Y|T\rangle$ if it is a deterministic word graph where no expansion relative to $\langle Y|T\rangle$ can be performed. An inverse automaton is *closed* with respect to $\langle Y|T\rangle$ if its underlying graph is closed.

Let $S_i = Inv\langle X_i; R_i\rangle = (X_i \cup X_i^{-1})^+/\eta_i$, $i = 1, 2$, where the X_i are disjoint alphabets. For an amalgam $[S_1, S_2; U]$, we view the natural image of $u \in U$ in S_i under the embedding of U as a word in the alphabet X_i, and $\langle X_1 \cup X_2|R_1 \cup R_2 \cup W\rangle$ with $W = \{(\phi_1(u), \phi_2(u))|u \in U\}$ is a presentation of $S_1 *_U S_2$. We put $X = X_1 \cup X_2$ and $R = R_1 \cup R_2$ and we call $\langle X|R \cup W\rangle$ the *standard presentation of $S_1 *_U S_2$ with respect to the presentations of S_1 and S_2*. For short, the standard presentation of $S_1 *_U S_2 \simeq (X \cup X^{-1})^+/\tau$. In the sequel, we will use the superscript notations $\mathcal{D}^U$, $\mathcal{D}^{S_i}$, $\mathcal{D}^{S_1 *_U S_2}$ to discriminate the $\mathcal{D}$-classes in U, S_i, $S_1 *_U S_2$, respectively. We keep this convention for all the Green's relations as well as for their classes. For instance, for the maximal subgroup in S_1 of an idempotent e in S_1 we use the symbol $H_e^{S_1}$, but if we consider the maximal subgroup in the free product with amalgamation we use the notation $H_e^{S_1 *_U S_2}$. We adhere to this notation throughout the paper, and we always assume that S_1 and S_2 are finite inverse semigroups.

Let Γ be an inverse word graph labeled over $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$, an edge of Γ that is labeled from $X_i \cup X_i^{-1}$, $i \in \{1, 2\}$, is sai d to be *colored* i. A subgraph of Γ is called *monochromatic* if all of its edges have the same color. A *lobe* of Γ is defined to be a maximal monochromatic connected subgraph of Γ. The coloring of edges extends to a coloring of lobes. Two lobes are said to be *adjacent* if they share common vertices, called *intersection vertices*. If $\nu \in V(\Gamma)$ is an intersection vertex, then it is common to two unique lobes, which we denote by $\Delta_1(\nu)$ and $\Delta_2(\nu)$, colored

1 and 2, respectively. We define the *lobe graph* of Γ to be the graph whose vertices are the lobes of Γ and whose edges correspond to the adjacency of lobes. We remark that a nontrivial inverse word graph Δ colored i and closed relative to $\langle X_i | R_i \rangle$ contains all the paths (ν_1, v', ν_2) with $v' \in (X_i \cup X_i^{-1})^+$ such that $v'\eta_i = v\eta_i$, provided that (ν_1, v, ν_2) is a path of Δ. Hence we often say that there is a path (ν_1, s, ν_2) with $s \in S_i$ in Δ whenever $\{(\nu_1, v, \nu_2) | v\eta_i = s\} \neq \emptyset$. Similarly, we say that (ν_1, u, ν_2) with $u \in U$ is a path of Δ to mean that $(\nu_1, \phi_i(u), \nu_2)$ is a path of Δ. For all $\nu \in V(\Delta)$ we denote by $\mathcal{L}_U(\nu, \Delta)$ the set of all the elements $u \in U$ such that (ν, u, ν) is a loop based at ν in Δ. In [6], an algorithm for the word problem for $S_1 *_U S_2$ is described. The algorithm is based on five constructions whose iterative applications starting from the linear automaton of $(\alpha, lin(w), \beta)$ lead to the Schützenberger automaton of w. We use these constructions without describing them in detail, and we refer the reader to [6] for relevant definitions and notions. It is important to emphasize that the first four constructions are applied only finitely many times, and after their application a finite automaton called $Core(w)$ is produced. Automaton $Core(w)$ has a particularly shaped underlying inverse graph, called opuntoid graph. The notion of an opuntoid graph is crucial for the geometric insight in what follows. We recall that an inverse graph Γ is an *opuntoid* if the following conditions are satisfied:

- Γ is a deterministic inverse word graph;
- the lobe graph is a tree, denoted $T(\Gamma)$;
- each lobe of Γ is a finite closed DV-quotient of a Schützenberger graph relative to $\langle X_i | R_i \rangle$, $i \in \{1, 2\}$;
- for each intersection vertex ν of Γ, the loop equality property holds, i.e. $\mathcal{L}_U(\nu, \Delta_1(\nu)) = \mathcal{L}_U(\nu, \Delta_2(\nu))$;
- for each intersection vertex ν of Γ, and for each $\nu' \in V(\Delta_1(\nu)) \cap V(\Delta_2(\nu))$ with $\nu' \neq \nu$ there is $u \in U$ such that (ν, u, ν') is a path in both $\Delta_1(\nu)$ and $\Delta_2(\nu)$. Moreover (ν, u', ν') with $u' \in U$ is path in $\Delta_1(\nu)$ if and only if (ν, u', ν') is a path in $\Delta_2(\nu)$.

A *subopuntoid subgraph* Θ is an inverse word subgraph of an opuntoid graph Γ such that if Δ is a lobe of Θ, then Δ is also a lobe of Γ. If $\phi : \Gamma \to \Gamma'$ is a homomorphism between two opuntoid graphs Γ, Γ' and Θ is a subopuntoid subgraph of Γ, we denote the restriction of ϕ to Θ by $\phi|_\Theta$. Opuntoid automata are automata whose underlying graphs are opuntoid graphs. For a vertex $\nu \in V(\Gamma)$, $e_i(\nu)$ is the minimal idempotent in S_i labeling a loop in ν, while $f(e_i(\nu))$ is the minimal idempotent in U labeling a loop in ν.

The last construction from [6], Construction 5, can be, in general, applied infinitely many times. Each time it is applied to an opuntoid graph Γ, it produces an opuntoid graph Γ' which has one more lobe and into which Γ embeds. Thus, we obtain a directed system of opuntoid automata. Its direct limit is $\mathcal{A}(X, R \cup W; w)$. Since Construction 5 is instrumental in understanding the notion of a host introduced in the next section, we recall it here. The construction is applied to particular vertices of Γ called buds. A vertex ν is a *bud* if it is not an intersection vertex and there is a loop based in ν labelled by an element of U, i.e. $\mathcal{L}_U(\nu, \Delta) \neq \emptyset$, for the (unique) lobe Δ containing ν.

Construction 5 (see [6])

– Let Γ be an opuntoid graph and let $v \in V(\Delta)$ be a bud belonging to a lobe Δ colored by $i \in \{1, 2\}$. Put $f = f(e_i(v))$ and let $(x, \Lambda, x) = \mathcal{A}(X_{3-i}, R_{3-i}; f)$. Consider the smallest equivalence relation $\rho \subseteq V(\Lambda) \times V(\Lambda)$ which identifies all the vertices of Λ connected to x by some word of $\mathcal{L}_U(v, \Delta)$ and such that Λ/ρ is deterministic. Let $\Delta' = \Lambda/\rho$.
– Consider the automaton $\mathcal{B} = (v, \Gamma, v) \times (x\rho, \Delta', x\rho)$. For all $u \in U$ such that (v, u, y) and (v, u, y') are paths of Δ and Δ' respectively, consider the equivalence relation κ on $V(\mathcal{B})$ that identifies y and y' and call $\overline{\Gamma}$ the underlying graph of $\mathcal{B}/\kappa$.

Lemma 10 of [6] states that $\overline{\Gamma}$ is an opuntoid graph with one more lobe than Γ and the graph Γ embeds into $\overline{\Gamma}$.

3 Host of an opuntoid automaton

Construction 5 depends on local information contained in the lobe with the bud at which the construction is applied. Thus, the information for building the entire Schützenberger automaton is already contained in the core, the automaton $Core(w)$. In this section, we generalize this concept by introducing a *host*—a smallest subopuntoid subgraph from which it is possible to retrieve the entire Schützenberger graph.

Given two adjacent lobes Δ and Δ' of an opuntoid graph Γ, following [2,19], we say that Δ' *directly feeds off* Δ, in symbols $\Delta \mapsto \Delta'$, if Δ' can be obtained from Δ by applying Construction 5 at some intersection vertex $v \in V(\Delta) \cap V(\Delta')$. Moreover for each pair of lobes Δ, Δ' of Γ we say that Δ' *feeds off* Δ, and we write $\Delta \mapsto^* \Delta'$, if Δ and Δ' are related in the transitive closure of $\mapsto$. We remark that the definition of feeding off in our case differs from the one given by Bennett in [2]. In Bennett's case all the lobes are Schützenberger graphs, and no quotients, like those in Construction 5, are needed.

For each finite opuntoid graph Γ consider a sequence

$$\Gamma = \Gamma_1 \hookrightarrow \Gamma_2 \hookrightarrow \ldots \Gamma_m \hookrightarrow \ldots$$

where Γ_{m+1} is obtained from Γ_m by applying Construction 5 at some bud of $V(\Gamma_m)$, and consider the directed limit $\varinjlim \Gamma_i = \bigcup_{k>0} \Gamma_k$. Then $\varinjlim \Gamma_i$ is a closed automaton with respect to $\langle X | R \cup W \rangle$ (see [6]); called in the sequel the closure of Γ and denoted by $cl_{R \cup W}(\Gamma)$. The uniqueness of the closure of an opuntoid graph follows from the work of Stephen [25,26]. Note that the closure of the underlying graph of $Core(w)$ is $S\Gamma(X, R \cup W; w)$.

A lobe Δ' of an opuntoid graph Γ that is adjacent to precisely one other lobe Δ of Γ is called a *parasite* (see also [3]) of Γ if Δ' feeds off Δ. Note that if we delete a parasite from Γ, we get an opuntoid graph Γ' whose closure is the same as the closure of Γ. Thus, parasites do not contain any 'essential' information for building closures of opuntoid graphs, and we have the following definition:

Definition 1 A subopuntoid subgraph Γ' of an opuntoid graph Γ is called a *host* of Γ if:

- its lobe tree is finite,
- it is parasite-free,
- every lobe of Γ not belonging to Γ' feeds off some lobe of Γ'.

A host of an opuntoid automaton is a host of its underlying graph. We remark that not all opuntoid graphs possess a host, and even when a host exists, it is not in general unique. The following proposition describes the case when there is more than one host. We present it without a proof, since it essentially follows along the lines of the proof given in [2, Lemma 6.2].

Proposition 2 *Let Γ be an opuntoid graph possessing a host. If Γ has more than one host, then every host is a lobe and the unique reduced lobe path connecting any two hosts consists entirely of lobes that are hosts.*

Obviously, an opuntoid graph with a finite lobe graph always has a host. Since the underlying graph Γ_0 of $Core(w)$ is finite and $S\Gamma(X, R; w) = cl_{R\cup W}(\Gamma_0)$, by Lemma 6.1 of [2], we have that the Schützenberger graph $S\Gamma(X, R; w)$ always possesses a host contained in Γ_0. Moreover, by Proposition 2, the union of all hosts of an opuntoid graph Γ that possesses a host, is a subopuntoid subgraph of Γ. We denote this subgraph by $Host(\Gamma)$.

The closure of an opuntoid automaton (α, Γ, β) is $(\alpha', cl_{R\cup W}(\Gamma), \beta')$ where α', β' are the natural images of α and β in $cl_{R\cup W}(\Gamma)$. An opuntoid automaton (α, Γ, β) is called *complete* if $(\alpha, \Gamma, \beta) = (\alpha', cl_{R\cup W}(\Gamma), \beta')$, or equivalently, if Γ has no buds. This yields the following description of Schützenberger automata which slightly extends Theorem 3 in [6] (see [19]).

Proposition 3 *Let $S = S_1 *_U S_2$ be an amalgamated free product of finite inverse semigroups S_1 and S_2 amalgamating a common inverse subsemigroup U, where $\langle X_i | R_i\rangle$ are presentations of S_i for $i = 1, 2$. Let $X = X_1 \cup X_2$, $R = R_1 \cup R_2$ and W be the set of all pairs $(\phi_1(u), \phi_2(u))$ for $u \in U$. Then the Schützenberger automata relative to $\langle X | R \cup W\rangle$ are complete opuntoid automata possessing a host.*

It is important for the sequel to point out the differences between the opuntoid graphs, feeding off relations, and the Schützenberger automata given in Bennett's papers [2,3] and the corresponding concepts presented here. According to Bennett, lobes of opuntoid graphs are always Schützenberger graphs relative to $\langle X_i | R_i\rangle$ for some $i \in \{1, 2\}$, while our lobes are in general only DV-quotients of Schützenberger graphs. The lower bound equality property of opuntoid graphs of Bennett is here replaced by the loop equality property that, as remarked in [6] p. 10, coincides with the lower bound equality property in the case when lobes are Schützenberger graphs. In [2] a lobe Δ' colored i directly feeds off Δ in a vertex ν if $\Delta' = S\Gamma(X_i, R_i; f(e_{3-i}(\nu)))$, while in our case, some DV-quotient is generally needed to guarantee the loop equality property in ν. Finally, while Schützenberger automata of lower bounded amalgams are *exactly* the complete opuntoid automata with hosts, in the case of amalgams of finite inverse semigroups not all the complete opuntoid automata with hosts are Schützenberger automata.

4 Automorphisms of opuntoid graphs

The main result proved in this section shows that there is a group isomorphism between the automorphism group of a complete opuntoid graph Γ and the automorphism group of its subopuntoid $Host(\Gamma)$.

The lemmas stated in this section are analogous to Lemmas 6 and 7 in Bennett [3]. In particular, the proofs of our lemmas follow along the same lines as those in [3], since the proofs in Bennett's case of lower bounded amalgams as well as in our case of amalgams of finite inverse semigroups use only the facts that the lobe graphs are trees, homomorphisms of graphs preserve labels (colors) of edges, and automorphisms of deterministic inverse word graphs that agree on a vertex are equal.

Lemma 1 *Let $\Gamma,\ \Gamma'$ be opuntoid graphs and let $\varphi : \Gamma \to \Gamma'$ be a homomorphism. Then φ is an isomorphism if and only if it induces an isomorphism of the lobe trees of Γ and Γ', and also maps the lobes of Γ isomorphically onto the corresponding lobes of Γ'.*

Let Γ be an opuntoid automaton with finitely many lobes. It is straightforward to check that the automorphism group of Γ embeds into the automorphism group of some lobe of Γ by using the fact that the automorphism group of a finite tree fixes a vertex or an edge (see [1, Subsection 27.1.3]).

Let Γ be an opuntoid graph and let Θ be a subopuntoid subgraph of Γ. For any lobe Δ not belonging to Θ the notation $\Theta \cup \Delta$ will denote the least subopuntoid subgraph of Γ which contains Θ and Δ. We have the following lemma.

Lemma 2 *Let Γ, Γ' be two complete opuntoid graphs and let Θ, Θ' be subopuntoid subgraphs containing a host of Γ and Γ' respectively. Let ν be a bud of Θ and let $\Delta_i(\nu)$ be a lobe of Θ for some $i \in \{1, 2\}$. Let $\varphi : \Theta \to \Theta'$ be an isomorphism and let $\nu' = \varphi(\nu)$. Then φ can be extended to an isomorphism from $\Theta \cup \Delta_{3-i}(\nu)$ onto $\Theta' \cup \Delta_{3-i}(\nu')$.*

Proof Since ν is a bud of Θ and $\Delta_i(\nu)$ is a lobe of Θ, the lobe $\Delta_{3-i}(\nu)$ of Γ does not belong to Θ. Moreover, since Θ contains a host of Γ, then $\Delta_{3-i}(\nu)$ feeds off $\Delta_i(\nu)$, $\Delta_i(\nu) \mapsto \Delta_{3-i}(\nu)$. Let $f = f(e_i(\nu))$ be the minimum idempotent in U labeling a loop based at ν in $\Delta_i(\nu)$. Since φ preserves labels, $f = f(e_i(\varphi(\nu)))$. As Γ' is complete, $\varphi(\nu)$ is not a bud, hence it is an intersection vertex. Moreover, by Lemma 1, $\Delta_i(\varphi(\nu))$ is a lobe of Θ'. If $\Delta_{3-i}(\varphi(\nu))$ is also a lobe of Θ' then again by Lemma 1, $\Delta_i(\nu) = \varphi^{-1}(\Delta_i(\varphi(\nu)))$ and $\varphi^{-1}(\Delta_{3-i}(\varphi(\nu)))$ are adjacent lobes in Θ with intersection vertex ν, hence ν is not a bud of Θ, and we have a contradiction. So $\Delta_{3-i}(\varphi(\nu))$ is not a lobe of Θ' and $\Delta_i(\varphi(\nu)) \mapsto \Delta_{3-i}(\varphi(\nu))$ because Θ' contains a host. Since $f(e_i(\varphi(\nu))) = f(e_i(\nu)) = f$ and $\mathcal{L}_U(\nu, \Delta_i(\nu)) = \mathcal{L}_U(\varphi(\nu), \Delta_i(\varphi(\nu)))$, by the definition of direct feed off, $(\nu, \Delta_{3-i}(\nu), \nu)$ and $(\varphi(\nu), \Delta_{3-i}(\varphi(\nu)), \varphi(\nu))$ are isomorphic to the same DV-quotient of $\mathcal{A}(X_{3-i}, R_{3-i}; f)$ and so the lobes $\Delta_{3-i}(\nu)$ and $\Delta_{3-i}(\varphi(\nu))$ are isomorphic under an isomorphism ψ such that $\psi(\nu) = \varphi(\nu)$. It is now straightforward to see that φ can be extended to an isomorphism from $\Theta \cup \Delta_{3-i}(\nu)$ onto $\Theta' \cup \Delta_{3-i}(\varphi(\nu))$ whose restriction to $\Delta_{3-i}(\nu)$ is ψ. □

The above lemma yields the following:

Proposition 4 *Let* Γ, Γ' *be two complete opuntoid graphs and let* Θ, Θ' *be subopuntoid subgraphs containing a host of* Γ, Γ' *respectively. Let* $\varphi : \Theta \to \Theta'$ *be an isomorphism. Then* φ *can be extended to an isomorphism* $\varphi^* : \Gamma \to \Gamma'$.

Proof Let $\mathcal{P}$ be the set of the pairs $(\phi, \overline{\Gamma})$ where $\overline{\Gamma}$ is a subopuntoid subgraph of Γ containing Θ and ϕ is a graph monomorphism of $\overline{\Gamma}$ into Γ' such that $\phi|_{\Theta} = \varphi$. Obviously $(\varphi, \Theta) \in \mathcal{P}$. Let $\leq$ be the natural partial order on $\mathcal{P}$ defined by $(\phi_1, \Gamma_1) \leq (\phi_2, \Gamma_2)$ if Γ_1 is a subopuntoid subgraph of Γ_2 and $\phi_2|_{\Gamma_1} = \phi_1$. By Hausdorff maximality lemma there is a maximal chain $\Omega = \{(\phi_\alpha, \Gamma_\alpha)\}_{\alpha \in I}$. Consider the pair $(\widehat{\phi}, \widehat{\Gamma})$ where $\widehat{\Gamma} = \cup_\alpha \Gamma_\alpha$ and $\widehat{\phi}(\nu) = \phi_\alpha(\nu)$ for $\nu \in V(\Gamma_\alpha)$. It is easy to show that the element $(\widehat{\phi}, \widehat{\Gamma})$ belongs to $\mathcal{P}$, and in particular it is a maximal element of the chain Ω.
We claim $\widehat{\Gamma} = \Gamma$. Suppose that, contrary to our claim, $\widehat{\Gamma} \neq \Gamma$. Then $\widehat{\Gamma}$ has a bud ν, and so only one of the two lobes $\Delta_1(\nu), \Delta_2(\nu)$ of Γ is in $\widehat{\Gamma}$. Assume without loss of generality that it is $\Delta_1(\nu)$. Then by Lemma 2 the monomorphism $\widehat{\phi}$ can be extended to an isomorphism from $\widehat{\Gamma} \cup \Delta_2(\nu)$ onto a subopuntoid graph of Γ', which contradicts the maximality of $(\widehat{\phi}, \widehat{\Gamma})$. Whence $\widehat{\Gamma} = \Gamma$ and $\widehat{\phi}$ is an isomorphism between the opuntoid graph Γ and the subopuntoid subgraph $\widehat{\phi}(\Gamma) \subseteq \Gamma'$. Suppose $\widehat{\phi}(\Gamma) \neq \Gamma'$, then $\widehat{\phi}(\Gamma)$ has a bud μ and again only one of the two lobes $\Delta_1(\mu), \Delta_2(\mu)$ of Γ' belongs to $\widehat{\phi}(\Gamma)$. Repeating the above argument for the isomorphism $\widehat{\varphi}^{-1} : \widehat{\phi}(\Gamma) \to \Gamma$, we get $\widehat{\phi}(\Gamma) = \Gamma'$.

We are now ready to prove the following proposition.

Proposition 5 *Let* Γ *be a complete opuntoid graph which possesses a host. Then the automorphism group of* Γ *is isomorphic to the automorphism group of the union of all hosts of* Γ, $Host(\Gamma)$.

Proof From Proposition 4 we know that each $\phi \in Aut(Host(\Gamma))$ can be extended to an automorphism $\varphi \in Aut(\Gamma)$. We prove that φ preserves the feed off relation. Assume that $\Delta \mapsto \Delta'$, and let $\nu \in V(\Delta) \cap V(\Delta')$. Then obviously $\varphi(\nu) \in V(\varphi(\Delta)) \cap V(\varphi(\Delta'))$. Moreover, if $i \in \{1, 2\}$ is the color of Δ, then $f(e_i(\nu)) = f(e_i(\varphi(\nu)))$ and $\mathcal{L}_U(\nu, \Delta) = \mathcal{L}_U(\varphi(\nu), \varphi(\Delta))$. Let $f = f(e_i(\nu))$, since $\Delta \mapsto \Delta'$ then

$$(\nu, \Delta', \nu) \simeq \mathcal{A}(X_{3-i}, R_{3-i}; f)/\rho$$

where ρ is the least equivalence relation that identifies the initial vertex ν of $\mathcal{A}(X_{3-i}, R_{3-i}; f(e_i(\nu)))$ with the final vertices y of all the paths (ν, u, y) with $u \in \mathcal{L}_U(\nu, \Delta)$ and makes the quotient deterministic. Thus,

$$(\varphi(\nu), \varphi(\Delta'), \varphi(\nu)) \simeq \mathcal{A}(X_{3-i}, R_{3-i}; f)/\rho$$

and so by the definition of direct feed off, $\varphi(\Delta) \mapsto \varphi(\Delta')$. Therefore φ sends hosts into hosts and $\varphi|_{Host(\Gamma)}$ belongs to $Aut(Host(\Gamma))$. It is straightforward to check that the map χ defined by $\chi(\varphi) = \varphi|_{Host(\Gamma)}$ is a group isomorphism from $Aut(\Gamma)$ onto $Aut(Host(\Gamma))$. □

5 Union of hosts of Schützenberger graphs

In this section, we study the union of the hosts of a Schützenberger graph of the amalgamated free product $S_1 *_U S_2$ of finite inverse semigroups S_1, S_2. We start with a characterization of the Schützenberger graphs with more than one host. We need to recall some results from [20]:

Proposition 6 [20, Proposition 10]
Let Γ be an opuntoid graph. Let Δ, Δ' be two lobes of Γ colored respectively by $i, 3-i$ for some $i = 1, 2$ with $\Delta \mapsto \Delta'$. Let $\nu \in V(\Delta) \cap V(\Delta')$ be an intersection vertex of Γ. Then $f = f(e_i(\nu)) = e_{3-i}(\nu) \in E(U)$. Conversely if $\Delta \simeq S\Gamma(X_i, R_i; f)$ is a lobe of Γ and $f \in E(U)$, then $\Delta' \simeq S\Gamma(X_{3-i}, R_{3-i}; f)$ is a lobe of Γ and $\Delta' \mapsto \Delta$.

The following proposition is the key to our extension of Bennett's results. While in the lower bounded case all the lobes are Schützenberger automata, in our case this does not hold in general. The following proposition is fundamental in the proof of Theorem 1 in order to show that in the case when there is more than one host, all the hosts are Schützenberger automata.

Proposition 7 [20, Theorem 23 and Proposition 18]
Let Δ, Δ' be two lobes of $S\Gamma(X, R \cup W; w)$ colored respectively by $i, 3-i$ for some $i = 1, 2$ with $\Delta \mapsto \Delta'$. Let $\nu \in V(\Delta) \cap V(\Delta')$, $f \in E(U)$ be such that $(\nu, \Delta', \nu) \simeq (x\rho, S\Gamma(X_{3-i}, R_{3-i}; f)/\rho, x\rho)$, where ρ is the least equivalence relation on $S\Gamma(X_{3-i}, R_{3-i}; f)$ which identifies the net

$$N(x, S\Gamma(X_{3-i}, R_{3-i}; f)) = \{v : \exists u \in \mathcal{L}_U(\nu, \Delta) \text{ such that } (x, u, v) \text{ is a path in } S\Gamma(X_{3-i}, R_{3-i}; f)\}$$

and makes $S\Gamma(X_{3-i}, R_{3-i}; f)/\rho$ deterministic. Then

$$(\nu, S\Gamma(X, R \cup W; f), \nu) \simeq (x\Xi, S\Gamma(X, R \cup W; f)/\Xi, x\Xi)$$

where the relation

$$\Xi \subseteq V(S\Gamma(X, R \cup W; f)) \times V(S\Gamma(X, R \cup W; f))$$

is defined by: $q\Xi q'$ if there are $y, y' \in N(x, S\Gamma(X_{3\_i}, R_{3\_i}; f))$ and $t \in (X \cup X^{-1})^*$ such that (y, t, q) and (y', t, q') are paths in $S\Gamma(X, R \cup W; f)$.
Moreover the following lifting property for Ξ holds: if $(p\Xi, h, q\Xi)$ is a path in $S\Gamma(X, R \cup W; f)/\Xi$ then for each $p \in p\Xi$ there is a path (p, h, q') in $S\Gamma(X, R \cup W; f)$ with $q' \in q\Xi$.

Finally, we are ready to present the following characterization of the Schützenberger graphs with more than one host:

Theorem 1 *Let $[S_1, S_2; U]$ be an amalgam of finite inverse semigroups, let $w \in (X \cup X^{-1})^+$. The following are equivalent:*

1. *$S\Gamma(X, R \cup W; w)$ has more than one host.*
2. *Each host of $S\Gamma(X, R \cup W; w)$ is the Schützenberger graph of some idempotent of U relative to the presentation $\langle X_i | R_i \rangle$ of S_i for some $i \in \{1, 2\}$.*
3. *$ww^{-1} \mathcal{D}^{S_1 *_U S_2} f$, for some idempotent $f \in E(U)$.*

Proof 1) $\Rightarrow$ 3). Assume that $S\Gamma(X, R \cup W; w)$ has more than one host. Then by Proposition 2, there are (at least) two adjacent lobes of $S\Gamma(X, R \cup W; w)$ which are hosts. Let ν be an intersection vertex between these adjacent hosts $\Delta_i = \Delta_i(\nu)$, $i \in \{1, 2\}$ and let $f = f(e_1(\nu))$. By the definition of host, we have $cl_{R \cup W}(\Delta_2) = S\Gamma(X, R \cup W; w)$. Moreover, $\Delta_1 \mapsto \Delta_2$, so by Proposition 7, $(\nu, S\Gamma(X, R \cup W; w), \nu) \simeq (x\Xi, S\Gamma(X, R \cup W; f)/\Xi, x\Xi)$. Let e be an idempotent labeling a loop based at ν in $S\Gamma(X, R \cup W; w)$ then again by Proposition 7, e also labels a loop based at x in $S\Gamma(X, R \cup W; f)$, hence $e \geq f$. Thus f is the minimum idempotent labeling a loop based at ν in $S\Gamma(X, R \cup W; w)$. Whence $S\Gamma(X, R \cup W; w) = S\Gamma(X, R \cup W; ww^{-1}) \simeq S\Gamma(X, R \cup W; f)$, and by Proposition 1, $ww^{-1}\mathcal{D}^{S_1 *_U S_2} f$.
3) $\Rightarrow$ 2). Set $\Delta_i = S\Gamma(X_i, R_i; f)$, for $i = 1, 2$. By Proposition 1, $S\Gamma(X, R \cup W; ww^{-1}) \simeq S\Gamma(X, R \cup W; f)$. Obviously, $S\Gamma(X, R \cup W; f)$ is obtained by iterated applications of Construction 5 to Δ_1, and Δ_1 is a host of $S\Gamma(X, R \cup W; w)$. Let now Δ be any host of $S\Gamma(X, R \cup W; w)$ and assume that it is colored $j \in \{1, 2\}$. We will prove by induction on the length n of the reduced lobe path connecting Δ_1 to Δ, that Δ is a Schützenberger graph of some idempotent of U. If $n = 0$, the statement is trivially true. So let $P : \Delta_1, \Delta_2, \ldots, \Delta_n = \Delta$ be the reduced lobe path connecting Δ_1 with Δ. Since Δ_1 and $\Delta_n = \Delta$ are hosts, by Proposition 2, Δ_{n-1} is a host, and by induction hypothesis it is a Schützenberger graph of some idempotent of U. Let $\nu \in V(\Delta_{n-1}) \cap V(\Delta_n)$. Since $\Delta_{n-1} \mapsto \Delta_n$, by Proposition 6, $e_{3-j}(\nu) = f(e_j(\nu)) \in E(U)$. Since Δ_{n-1} is a Schützenberger graph, this implies that $(\nu, \Delta_{n-1}, \nu) \simeq \mathcal{A}(X_{3-j}, R_{3-j}; f(e_j(\nu)))$. Now, by Proposition 6, $\Delta_n \simeq S\Gamma(X_j, R_j; f(e_j(\nu)))$.
2) $\Rightarrow$ 1). Let Δ be a host of $S\Gamma(X, R \cup W; w)$ colored i. Then $\Delta \simeq S\Gamma(X_i, R_i; f)$ for some $f \in E(U)$. Then $f = e_i(\nu) = f(e_i(\nu))$ for some $\nu \in V(\Delta)$. Applying Construction 5 at ν, we get a new lobe Δ' such that $\Delta' \mapsto \Delta$ by Proposition 6. Let Λ be any lobe of $S\Gamma(X, R \cup W; w)$. Since Δ is a host, then Λ feeds off Δ that in turns directly feeds off Δ'. So Λ feeds off Δ', and Δ' is a host.

From now on, we will denote by $Host(S\Gamma(w))$ the union of all hosts of a Schützenberger graph $S\Gamma(X, R \cup W; w)$. We will characterize Schützenberger graphs $S\Gamma(X, R \cup W; e)$ with infinite $Host(S\Gamma(e))$. Since a host has finitely many finite lobes, infinite $Host(S\Gamma(e))$ are opuntoids with infinitely many hosts. Hence by Theorem 1, we have $e\mathcal{D}^{S_1 *_U S_2} f$ for some idempotent $f \in E(U)$, and all hosts are lobes which are Schützenberger graphs of some idempotents of U relative to the presentation $\langle X_i | R_i \rangle$ of S_i for some $i \in \{1, 2\}$.

Definition 2 Let Δ, Δ' be two lobes of an opuntoid graph such that $\Delta' = \phi(\Delta)$ for some isomorphism ϕ. Let $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ be the reduced lobe path connecting Δ to Δ' and let $\nu_1 \in V(\Delta_0) \cap V(\Delta_1)$. The isomorphism ϕ is called a shift-isomorphism (and Δ, Δ' are called shift-isomorphic by ϕ) if $\phi(\nu_1) \notin V(\Delta_{n-1}) \cap V(\Delta_n)$.

The lobes Δ, Δ' are called successive isomorphic lobes if no Δ_i $(0 < i < n)$ is isomorphic to Δ_0.

We have the following lemma.

Lemma 3 *Let $e \in E(S_1 *_U S_2)$ with S_1, S_2 finite inverse semigroups and let $e\mathcal{D}^{S_1 *_U S_2} f$ for some idempotent $f \in E(U)$. Let Δ, Δ' be two distinct lobes of $Host(S\Gamma(e))$ colored by i, such that $\Delta' = \phi(\Delta)$ for some $\phi \in Aut(Host(S\Gamma(e)))$. Let $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ be the reduced lobe path connecting Δ to Δ'. Then either for all j with $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$, $\phi(\Delta_j) = \Delta_{n-j}$, or for some j with $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$, $\phi|_{\Delta_j}$ is a shift-isomorphism.*

Proof By Proposition 2 and Theorem 1, each Δ_h, $0 \leq h \leq n$, is a host that is the Schützenberger graph of some idempotent of U relative to the presentation $\langle X_i | R_i \rangle$ for some $i = 1, 2$. We prove the statement of the lemma by induction on $n \geq 2$. The base step $n = 2$ is trivial. If $n > 2$ and $\phi|_{\Delta_0}$ is a shift-isomorphism of Δ_0 onto Δ_n, the statement is trivially true again. So assume that $\phi|_{\Delta_0}$ is not a shift-isomorphism. Let $\nu_1 \in V(\Delta_0) \cap V(\Delta_1)$, then $\phi(\nu_1) \in V(\Delta_{n-1}) \cap V(\Delta_n)$. Moreover, by Proposition 6, we get $f_1 = e_i(\nu_1) \in E(U)$. Hence, $e_i(\phi(\nu_1)) = f_1 \in E(U)$, and so $(\phi(\nu_1), \Delta_{n-1}, \phi(\nu_1)) \simeq \mathcal{A}(X_{3-i}, R_{3-i}; f_1) \simeq (\nu_1, \Delta_1, \nu_1)$, and $\phi(\Delta_1) = \Delta_{n-1}$. Since the reduced lobe path from Δ_1 to Δ_{n-1} has length $n - 1$, the statement holds by the induction hypothesis. □

Proposition 8 *Let $e \in E(S_1 *_U S_2)$ with S_1, S_2 finite inverse semigroups, and let $e\mathcal{D}^{S_1 *_U S_2} f$ for some idempotent $f \in E(U)$. Then the following are equivalent*

1. *$Host(S\Gamma(e))$ is infinite;*
2. *$Host(S\Gamma(e))$ has infinitely many lobes;*
3. *There are two isomorphic hosts of $S\Gamma(e)$ which are not successive isomorphic lobes;*
4. *There are two hosts of $S\Gamma(e)$ with a shift-isomorphism between them.*

Proof The equivalence between (1) and (2) is obvious.
(1) $\Rightarrow$ (3) By Theorem 1, each lobe of $Host(S\Gamma(e))$ is a Schützenberger graph of some idempotent of U relative to the presentation $\langle X_i | R_i \rangle$, for some $i \in \{1, 2\}$. Since S_1, S_2 are finite, there are finitely many Schützenberger graphs of idempotents of U relative to the presentations $\langle X_i | R_i \rangle$ with $i \in \{1, 2\}$. Since all the lobes are finite, each lobe has finitely many adjacent lobes and so the degree of each vertex of the lobe tree $T(Host(S\Gamma(e)))$ is finite. Therefore, there is an infinite reduced lobe path in $T(Host(S\Gamma(e)))$ in which there are at least three isomorphic lobes, whence there are two isomorphic hosts that are not successive isomorphic.
3) $\Rightarrow$ 4) Suppose that $S\Gamma(e)$ has two isomorphic hosts Δ and Δ' that are not successive isomorphic lobes. Then, in the reduced lobe path $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ connecting them, there is a lobe Δ_h, with $1 \leq h \leq n-1$, isomorphic to both Δ and Δ'. We can assume without loss of generality, that no other Δ_j, $1 \leq j \leq n-1$, $j \neq h$, is isomorphic to Δ. Since isomorphic lobes have the same color, n must be at least 4 and even. Let $t = n/2$, and let ϕ be the isomorphism sending Δ onto Δ'. By Propositions 4 and 5, ϕ can be extended to an automorphism $\overline{\phi} \in Aut(Host(S\Gamma(e)))$. Assume that

for no lobe Λ of $Host(S\Gamma(e))$ the restriction $\overline{\phi}|_{\Lambda}$ is a shift-isomorphism of Λ onto some host Λ'. Then, by Lemma 3, $\overline{\phi}(\Delta_j) = \Delta_{2t-j}$, for all j, $0 \leq j < t$. But then $t = h$, because otherwise both Δ_h and Δ_{n-h} would be isomorphic to Δ. That means that $\overline{\phi}|_{\Delta_t} \in Aut(\Delta_t)$. Let $\nu_t \in V(\Delta_{t-1}) \cap V(\Delta_t)$. Then $\overline{\phi}(\nu_t) \in V(\Delta_t) \cap V(\Delta_{t+1})$. Now let $\psi : \Delta \to \Delta_t$ be an isomorphism. If ψ is a shift-isomorphism, we are done. Otherwise, if $\nu_1 \in V(\Delta_0) \cap V(\Delta_1)$, then $\psi(\nu_1) \in V(\Delta_t) \cap V(\Delta_{t-1})$. Using the fact that ψ preserves labeling, it is easy to see that ψ is actually a bijection between the two sets $V(\Delta_0) \cap V(\Delta_1)$ and $V(\Delta_t) \cap V(\Delta_{t-1})$. Thus take ν' to be the vertex of $V(\Delta_0) \cap V(\Delta_1)$ for which $\psi(\nu') = \nu_t \in V(\Delta_{t-1}) \cap V(\Delta_t)$. Hence $\overline{\phi}(\psi(\nu')) = \overline{\phi}(\nu_t) \in V(\Delta_t) \cap V(\Delta_{t+1})$, i.e. $\overline{\phi}(\psi(\nu')) \notin V(\Delta_{t-1}) \cap V(\Delta_t)$. It follows that the map $\psi \cdot \overline{\phi} : \Delta \to \Delta_t$ (defined by $\psi \cdot \overline{\phi}(\nu) = \overline{\phi}(\psi(\nu))$) is a shift-isomorphism from Δ to Δ_t.
4) $\Rightarrow$ 2) Assume by contradiction, that $Host(S\Gamma(e))$ has two shift-isomorphic lobes and finitely many lobes. Let Δ, Δ' be two shift-isomorphic lobes in $Host(S\Gamma(e))$, such that the reduced lobe path $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ from Δ to Δ' is of maximal length. By Proposition 2 and Theorem 1, each Δ_j, $0 \leq j \leq n$, is a host and the Schützenberger graph of some idempotent of U. Moreover, by Proposition 6, $(\nu_j, \Delta_{j-1}, \nu_j) \simeq \mathcal{A}(X_i, R_i; f_j)$, and $(\nu_j, \Delta_j, \nu_j) \simeq \mathcal{A}(X_{3-i}, R_{3-i}, f_j)$, where $\nu_j \in V(\Delta_{j-1}) \cap V(\Delta_j)$, $f_j = e_i(\nu_j) = e_{3-i}(\nu_j) \in U$ and $i \in \{1, 2\}$ is the color of Δ_{j-1}. By Propositions 4 and 5, the isomorphism $\phi : \Delta \to \Delta'$ can be extended to an automorphism $\overline{\phi} \in Aut(Host(S\Gamma(e)))$. We will prove by induction on h, that for all h with $0 \leq h \leq n$, $\overline{\phi}$ maps the subopuntoid subgraph $\Theta_h = \bigcup_{0\leq j\leq h} \Delta_j$ of $Host(S\Gamma(e))$ onto a subopuntoid subgraph of $Host(S\Gamma(e))$ whose lobes, except eventually Δ_n, are all different from the lobes of Θ_h. We will also prove that $\overline{\phi}$ is a shift-isomorphism between Δ_h and $\overline{\phi}(\Delta_h)$. The base of the induction is trivial. So let $\Theta_{h-1} = \bigcup_{0\leq j\leq h-1} \Delta_j$ and put $\Delta_{n+j} = \overline{\phi}(\Delta_j)$ for all $0 \leq j \leq h-1$. By Lemma 1, $\overline{\phi}(\Theta_{h-1}) = \bigcup_{0\leq j\leq h-1} \Delta_{n+j}$ and for all j with $0 \leq j \leq h-2$, the lobe Δ_{n+j} is adjacent to Δ_{n+j+1}. Moreover, by the induction hypothesis, Θ_{h-1} and $\overline{\phi}(\Theta_{h-1})$ have disjoint sets of lobes, and $\overline{\phi}$ is a shift-isomorphism of Δ_{h-1} onto Δ_{n+h-1}. Let ν_h be an intersection vertex between Δ_{h-1} and Δ_h and let $i \in \{1, 2\}$ be the color of Δ_{h-1}. By Lemma 2, $\overline{\phi}$ maps $\Theta_h = \Theta_{h-1} \cup \Delta_h$, onto $\overline{\phi}(\Theta_{h-1}) \cup S\Gamma(X_{3-i}, R_{3-i}; f(e_i(\overline{\phi}(\nu_h))))$. Therefore, $S\Gamma(X_{3-i}, R_{3-i}; f(e_i(\overline{\phi}(\nu_h)))) = \Delta_{n+h}$ does not coincide with any lobe of Θ_h. Moreover, if ν_{h+1} is an intersection vertex between Δ_h and Δ_{h+1}, then $\overline{\phi}(\nu_{h+1}) \notin V(\Delta_{n+h-1}) \cap V(\Delta_{n+h})$, so $\overline{\phi}$ is a shift isomorphism between Δ_h and Δ_{n+h}. In particular, for $h = n$, $\overline{\phi}$ is a shift-isomorphism of Δ_n onto Δ_{2n} and $\overline{\phi}^2$ is a shift-isomorphism of Δ_0 onto Δ_{2n}, contradicting the assumption that the reduced lobe path connecting $\Delta = \Delta_0$ to $\Delta' = \Delta_n$ is a path of maximal length among the reduced lobe paths connecting two hosts which are isomorphic under a shift-isomorphism. Thus, $Host(S\Gamma(e))$ has infinitely many lobes.

From the above proposition we derive the following corollary:

Corollary 1 *Let $e \in E(S_1 *_U S_2)$, with S_1, S_2 finite inverse semigroups, and let $e\mathcal{D}^{S_1 *_U S_2} f$, for some idempotent $f \in E(U)$. Then $Host(S\Gamma(e))$ is infinite if and only* *if there is a reduced lobe path* $P : \Delta_0, \ldots, \Delta_{t-1}, \Delta_t, \Delta_{t+1}, \ldots \Delta_{2t}$ *in* $Host(S\Gamma(e))$ *with* $\Delta_0 \simeq \Delta_t \simeq \Delta_{2t}$.

Proof By Proposition 8, if $Host(S\Gamma(e))$ is infinite, then there are two lobes of $Host(S\Gamma(e))$ with a shift-isomorphism between them. Let Δ and Δ' be such lobes, and let $\phi : \Delta \to \Delta'$ be a shift-isomorphism. If P is the reduced lobe path from Δ to Δ', we can show, by the same argument as in the proof of implication 4) $\Rightarrow$ 2) of Proposition 8, that $P \cup \phi(P)$ is a reduced lobe path in $Host(S\Gamma(e))$ satisfying the statement. Conversely, let $P : \Delta_0, \ldots, \Delta_{t-1}, \Delta_t, \Delta_{t+1}, \ldots \Delta_{2t}$ with $\Delta_0 \simeq \Delta_t \simeq \Delta_{2t}$ be a reduced lobe path of $Host(S\Gamma(e))$. Then there are two non-successive isomorphic lobes in P, and hence, by Proposition 8, $Host(S\Gamma(e))$ is infinite. □

6 Maximal subgroups

In this section we make use of the Bass–Serre theory of groups acting on trees. We recall basic notation which we will use through the section, and we refer the reader interested in more details to [9,23]. Let G be a group acting (in the sense of Serre) on a graph $X = (Vert(X), Edge(X), \alpha, \omega)$ with maps $\alpha, \omega : Edge(x) \to Vert(X)$ mapping an edge to its initial and terminal vertex. We assume that G acts without inversion, i.e. $g \cdot y \neq \overline{y}$, ($\overline{y}$ denotes the opposite edge to y), for any $y \in Edge(X)$, and any $g \in G$. We denote the *quotient graph* of such action of G on X by $G\backslash X$. Let X be a connected non-empty graph and let $\mathcal{G}$ be a mapping assigning to each $v \in Vert(X)$ a group G_v, and to each $y \in Edge(X)$ a group G_y with the property that $G_y = G_{\overline{y}}$. Assume that for each $y \in Edge(X)$ there are two group monomorphisms $\sigma_y : G_y \to G_{\alpha(y)}$, and $\tau_y : G_y \to G_{\omega(y)}$ embedding the edge group into the corresponding initial, resp. terminal vertex group, such that $\sigma_y = \tau_{\overline{y}}$. Then X with the group assignment $\mathcal{G}$ is called a *graph of groups* $(\mathcal{G}(-), X)$. Let T be any maximal subtree of X, the *fundamental group* $\pi(\mathcal{G}(-), X, T)$ of $(\mathcal{G}(-), X)$ with respect to T is generated by the disjoint union of vertex groups $G_v,\ v \in Vert(X)$ and by the edges from $Edge(X)$, subject to the relations $\{\overline{y} = y^{-1}, y^{-1}\sigma_y(a)y = \tau_y(a) | y \in Edge(X), a \in G_y\} \cup \{y = 1 | y \in T\}$. The fundamental group of a graph of groups is, up to isomorphisms, independent of the choice of T and so it will be denoted by $\pi(\mathcal{G}(-), X)$. Two graphs of groups $(\mathcal{G}(-), X)$, $(\mathcal{H}(-), Y)$ are said to be *isomorphic* if there is a graph isomorphism $\Phi : X \to Y$ together with a collection of group isomorphisms $\Phi_v : G_v \to H_{\Phi(v)}$, $\Phi_y : G_y \to H_{\Phi(y)}$ satisfying the conditions $\Phi_y = \Phi_{\overline{y}}$, $\Phi_{\alpha(y)}\sigma_y = \sigma_{\Phi(y)}\Phi(y)$ and $\Phi_{\omega(y)}\tau_y = \tau_{\Phi(y)}\Phi(y)$. It is easy to see that isomorphic graphs of groups have isomorphic fundamental groups.
A graph of groups $(\mathcal{H}(-), X)$ is *conjugate* to $(\mathcal{G}(-), X)$ if it has the same group assignments as $(\mathcal{G}(-), X)$, the same embeddings σ_y and whose embeddings τ_y are the ones of $(\mathcal{G}(-), X)$ followed by a conjugation by an element of $G_{\omega(y)}$. A graph of groups $(\mathcal{H}(-), Y)$ is *conjugate isomorphic* to a graph of groups $(\mathcal{G}(-), X)$, if it is isomorphic to a conjugate of $(\mathcal{G}(-), X)$. Two conjugate isomorphic graphs of groups have the same fundamental group.
There is a standard way to construct a graph of groups $(\mathcal{G}(-), G\backslash X)$ starting from an action of a group G on a connected non-empty graph X. The relationship between this graph of groups and the original group G is explained in the following theorem [9,23]:

Theorem 2 *Let G be a group acting without inversions on a connected graph X. Then X is a tree if and only if $\Phi : \pi(\mathcal{G}(-), G\backslash X) \to G$ is an isomorphism of groups.*

If we fix an idempotent $e \in E(S_1 *_U S_2)$, the lobe graph of the union of the hosts $\mathcal{T}_e = \mathcal{T}(Host(S\Gamma(e)))$ can be seen as a directed tree (in the sense of Serre) $(Vert(\mathcal{T}_e), Edge(\mathcal{T}_e), \alpha, \omega)$, where $Vert(\mathcal{T}_e)$ is the set of lobes of the opuntoid graph $Host(S\Gamma(e))$, and $Edge(\mathcal{T}_e)$ is formed by the pairs of adjacent lobes of $Host(S\Gamma(e))$, oriented always "from color 1 to color 2", i.e. for each $y = (\Delta, \Delta') \in Edge(\mathcal{T}_e)$, $\alpha(y)$ is the lobe Δ colored 1 and $\omega(y)$ is the lobe Δ' colored 2. Lemma 1 and Proposition 5 prove that the group $H_e^{S_1 *_U S_2} \simeq Aut(S\Gamma(e))$ acts on $\mathcal{T}_e$ without inversions. Thus we can build the associated graph of groups $(\mathcal{G}(-), G\backslash\mathcal{T}_e)$ starting from the action of the group $G = H_e^{S_1 *_U S_2}$ on the connected non-empty graph $\mathcal{T}_e$. The quotient graph $G\backslash\mathcal{T}_e$ will be denoted by Y. Theorem 2 gives us immediately the following:

Corollary 2 *Let $e \in E(S_1 *_U S_2)$ be an idempotent in the amalgamated free product of two finite inverse semigroups S_1, S_2. Let $Y = H_e^{S_1 *_U S_2}\backslash\mathcal{T}_e$. Then*

$$H_e^{S_1 *_U S_2} \simeq \pi(\mathcal{G}(-), Y).$$

To study the structure of maximal subgroups in more detail, we analyze two different cases:

- **Case 1:** e is an "old" idempotent: e is $\mathcal{D}^{S_1 *_U S_2}$-related to some idempotent of S_1 or S_2.
- **Case 2:** e is a "new" idempotent: e is not $\mathcal{D}^{S_1 *_U S_2}$-related to any idempotent of S_1 or S_2.

6.1 Case 1: "Old" idempotent

Although in general the lobes of Schützenberger graphs of elements of amalgams of finite inverse semigroups are only DV-quotients, if we restrict our attention to the hosts of Schützenberger graphs of the original idempotents, the situation becomes nicer:

Theorem 3 *Let e be an idempotent in S_1 or S_2. Using the above notation, let $Y = H_e^{S_1 *_U S_2}\backslash\mathcal{T}_e$. Then*

1. *each $\Delta \in Vert(Y)$ is a Schützenberger graph $S\Gamma(X_i, R_i; e_i(\nu))$ for some $\nu \in Vert(\Delta)$;*
2. *Y is finite;*
3. *$(\Delta_1, \Delta_2) \in Edge(Y)$ if and only if the following conditions hold*
   - *e is $\mathcal{D}^{S_1 *_U S_2}$-related to some idempotent of U;*
   - *$\Delta_i(\nu) \simeq S\Gamma(X_i, R_i; f)$, for some $f \in E(U)$;*
   - *there is a lobe Δ_2' of $Host(S\Gamma(e))$ such that $(\Delta_1, \Delta_2') \in Edge(\mathcal{T}_e)$, $\psi(\Delta_2') = \Delta_2$, for some automorphism $\psi \in Aut(Host(S\Gamma(e)))$, and $e_i(\nu') = f$ for an intersection vertex ν' between Δ_1 and Δ_2'.*

Proof Assume that e is an idempotent of S_1. If we start from a word $u \in (X_1 \cup X_1^{-1})^+$ equivalent to e in S_1, then it is clear that the underlying graph Δ_0 of $Core(u)$ is isomorphic to $S\Gamma(X_1, R_1; u)$. Thus Δ_0 is a host of $\Gamma = S\Gamma(X, R \cup W; e)$. If e is not $\mathcal{D}^{S_1 *_U S_2}$-related to any idempotent of U then it is the unique host. Otherwise Γ has more than one host, and each host is a lobe that is a Schützenberger graph of some idempotent of U relative to the presentation $\langle X_i | R_i \rangle$ (Theorem 1). This proves Part 1. of the Theorem.
For Part 2., if e is not $\mathcal{D}^{S_1 *_U S_2}$-related to any idempotent of U then Y is trivially finite. If e is $\mathcal{D}^{S_1 *_U S_2}$-related to some idempotent of U, by Theorem 1, all hosts of $S\Gamma(e)$ are Schützenberger graphs of idempotents of U, and therefore there are at most $|U|$ different hosts up to isomorphisms. From Propositions 4 and 5, it follows that two isomorphic hosts lie in the same $H_e^{S_1 *_U S_2}$-orbit; Part 2. holds again.
In Part 3. of the theorem, the "if" direction is trivial. For the "only if" direction, let $(\Delta_1, \Delta_2) \in Edge(Y)$. Then Γ has more than one host, hence by Theorem 1, e is $\mathcal{D}^{S_1 *_U S_2}$-related to some idempotent of U. In addition, there is an edge $(\Delta_1', \Delta_2') \in Edge(\mathcal{T}_e)$ which lies in the same $H_e^{S_1 *_U S_2}$-orbit as (Δ_1, Δ_2). The lobes Δ_1', Δ_2' are adjacent, feed off each other and share at least one common vertex q. By Theorem 1, $\Delta_i' \simeq S\Gamma(X_i, R_i; e_i(q))$ with $e_i(q) \in E(U)$ for $i = 1, 2$. Moreover, there is an automorphism φ of $Host(S\Gamma(e))$ such that $\varphi(\Delta_1') = \Delta_1$. The lobes Δ_1 and $\varphi(\Delta_2')$ are therefore adjacent, and share the vertex $\varphi(q) = \nu'$. Hence, $e_i(\nu') = e_i(q) \in E(U)$, the lobe $\varphi(\Delta_2') \simeq S\Gamma(X_2, R_2; e_2(\nu'))$ is a host, and $(\Delta_1, \varphi(\Delta_2')) \in Edge(\mathcal{T}_e)$ lies in the same $H_e^{S_1 *_U S_2}$-orbit as (Δ_1, Δ_2). Again, there is an automorphism ψ of $Host(S\Gamma(e))$ such that $\psi(\varphi(\Delta_2')) = \Delta_2$, $\psi(\Delta_1) = \Delta_1$ and $\psi(\nu') = \nu$, hence $e_1(\nu) \in E(U)$ and $\Delta_i \simeq S\Gamma(X_i, R_i; f)$ with $f = e_1(\nu)$. □

As an immediate consequence of the previous proof, we conclude that in the case when e is an original idempotent that it is not $\mathcal{D}$-related in the amalgam to any idempotent in U, the graph of groups consists of just a single vertex. This vertex corresponds to the Schützenberger graph of some g in S_i, and by Proposition 1, we obtain the following description of the maximal subgroups.

Corollary 3 *Let e be $\mathcal{D}$-related in $S_1 *_U S_2$ to some idempotent g of $S_i \setminus U$, $i = 1, 2$, then the maximal subgroup $H_e^{S_1 *_U S_2}$ is isomorphic to the group $H_g^{S_i}$, $H_e^{S_1 *_U S_2} \simeq H_g^{S_i}$.*

Let us now look at the situation when e is $\mathcal{D}$-related in $S_1 *_U S_2$ to some idempotent of U. Then, by Theorem 1, the Schützenberger graph $S\Gamma(X, R \cup W; e)$ has more than one host, and each host is a lobe, and is a Schützenberger graph of some idempotent of U relative to the presentation $\langle X_i | R_i \rangle$ of S_i, for some $i \in \{1, 2\}$. By Lemma 3, the quotient graph Y is finite. Note that in this case, the graph of groups $(\mathcal{H}_e(-), Z_e)$ built according to Bennett's construction can be used to describe the structure of the maximal subgroup of $S_1 *_U S_2$ containing e. Namely, we could prove, along the same lines as in the proof of Theorem 2 in [3], that the graph of groups $(\mathcal{G}(-), Y)$ built from the action of the group $G = H_e^{S_1 *_U S_2}$ on the connected non-empty graph $\mathcal{T}_e$, is conjugate isomorphic to the graph of groups $(\mathcal{H}_e(-), Z_e)$ - the graph, which would provide some information about the maximal subgroup of $S_1 *_U S_2$ containing e. However, thanks to Theorem 3, the graph of groups $(\mathcal{G}(-), Y)$ provides more information, namely, we obtain a better description of the associated groups that are stabilizers of vertices and edges in $\mathcal{T}_e$. By Theorem 3, vertices of $\mathcal{T}_e$ are Schützenberger graphs of idempotents belonging to U. Thus, we immediately derive (Propositions 1, 4 and 5) that the stabilizers of the vertices appearing in the graph of groups $(\mathcal{G}(-), Y)$ are maximal subgroups of idempotents of U in the original semigroups S_1 or S_2. Proposition 9 below gives a description of the stabilizer of an edge in $\mathcal{T}_e$. First, we need the following technical lemma.

Lemma 4 *Let* $(\nu, \Delta, \nu) = \mathcal{A}(X_k, R_k, f)$ *for some* $k \in \{1, 2\}$ *and* $f \in U$. *Let* $I(\nu, \Delta) = \{y \in V(\Delta) : (\nu, u, y)$ *is a path in* Δ *for some* $u \in U\}$. *Then*

$$H_f^U \simeq \{\varphi \in Aut(\Delta) : \varphi(I(\nu, \Delta)) \subseteq I(\nu, \Delta)\}.$$

Proof Theorem 3.5 of [25] shows that $H_f^{S_k} \simeq Aut(\Delta)$ by the isomorphism $m \mapsto \phi_m$ defined by $\phi_m(v) = m^{-1}v$. Since ϕ_k is an embedding of H_f^U into $H_f^{S_k}$, then H_f^U also embeds into $Aut(\Delta)$. We will show that the map $u \mapsto \psi_{\phi_k(u)}$ defined by $\psi_{\phi_k(u)}(v) = \phi_k(u^{-1})v$ with $v \in V(\Delta)$, $u \in H_f^U$, is an isomorphism between H_f^U and $\{\varphi \in Aut(\Delta) : \varphi(I(\nu, \Delta)) \subseteq I(\nu, \Delta)\}$. To show that $\psi_{\phi_k(u)} \in \{\varphi \in Aut(\Delta) : \varphi(I(\nu, \Delta)) \subseteq I(\nu, \Delta)\}$, it is enough to prove that $\psi_{\phi_k(u)}(\nu) \in I(\nu, \Delta)$, since each element of $I(\nu, \Delta)$ is connected to ν by some element of U, and $\psi_{\phi_k(u)} \in Aut(\Delta)$. Since f is the unity of H_f^U, we get $u = fuf$. Moreover, since $\nu = \phi_k(f)$ we have:

$$\psi_{\phi_k(u)}(\nu) = \phi_k(fu^{-1}f)\phi_k(f) = \phi_k(f)\phi_k(fu^{-1}f) = \nu\phi_k(fu^{-1}f),$$

and, by [25], $fu^{-1}f$ labels a path connecting ν to $\psi_{\phi_k(u)}(\nu)$. Whence, we get $\psi_{\phi_k(u)}(\nu) \in I(\nu, \Delta)$. It remains to show that $u \mapsto \psi_{\phi_k(u)}$ is surjective. Let $\psi \in \{\varphi \in Aut(\Delta) : \varphi(I(\nu, \Delta)) \subseteq I(\nu, \Delta)\}$. Then there is some $u \in U$ which labels a path in Δ connecting ν to $\psi(\nu)$. Since ψ is an automorphism, fuf also labels a path connecting ν to $\psi(\nu)$. Note that the element $fu^{-1}f \in H_f^U$, and $\nu\phi_k(fuf) = \psi(\nu)$. Consider the automorphism $\psi_{\phi_k(fu^{-1}f)}$:

$$\psi_{\phi_k(fu^{-1}f)}(\nu) = \psi_{\phi_k(fu^{-1}f)}(\phi_k(f)) = \phi_k(f)\phi_k(fuf) = \nu\phi_k(fuf) = \psi(\nu),$$

and hence $\psi_{\phi_k(fu^{-1}f)} = \psi$, since they coincide on a vertex.

Proposition 9 *Let* Δ_1, Δ_2 *be two adjacent lobes of* $Host(S\Gamma(e))$, *where* e *is an idempotent of* $S_1 *_U S_2$ $\mathcal{D}$*-related to some idempotent of* U. *Let* $f = f(e_1(\nu)) = f(e_2(\nu))$, *for some intersection vertex* $\nu \in V(\Delta_1) \cap V(\Delta_2)$. *Then the stabilizer in* $G = H_e^{S_1 *_U S_2}$ *of the edge* (Δ_1, Δ_2), $Stab_G((\Delta_1, \Delta_2))$, *is isomorphic to* H_f^U:

$$Stab_G((\Delta_1, \Delta_2)) \simeq H_f^U.$$

Proof Using Propositions 4 and 5, it is straightforward to check

$$Stab_G((\Delta_1, \Delta_2)) \simeq \{\varphi \in Aut(\Delta_1) : \varphi(V(\Delta_1) \cap V(\Delta_2)) \subseteq V(\Delta_1) \cap V(\Delta_2)\}.$$

By the assimilation property we have

$$V(\Delta_1) \cap V(\Delta_2) = \{y \in V(\Delta_1) : (v, u, y) \text{ is a path in } \Delta_1 \text{ for some } u \in U\},$$

hence $Stab_G((\Delta_1, \Delta_2)) \simeq H_f^U$ by Lemma 4.

For future reference, we record the previous facts in the form of a theorem:

Theorem 4 *With the above notation, if e is $\mathcal{D}^{S_1 *_U S_2}$-related to some idempotent of U, then*

$$H_e^{S_1 *_U S_2} \simeq \pi(\mathcal{G}(-), Y)$$

where Y is finite. Moreover, the group G_v, $v \in Vert(Y)$, is a maximal subgroup in S_1 or S_2 of some idempotent of U, while G_y, $y \in Edge(Y)$, is a maximal subgroup in U.

Since Y is finite, from [9, p. 14] it follows that $H_e^{S_1 *_U S_2}$ is built by iteratively forming amalgamated free products for each of the edges of the maximal subtree T of Y, followed by forming HNN-extensions for each of the edges not in T. Therefore the next natural step is to ask whether Y is a tree or not. This will reveal in which cases the construction of $H_e^{S_1 *_U S_2}$ involves just iterated group amalgams, and in which it also involves HNN-extensions. First we characterize the case when $H_e^{S_1 *_U S_2}$ is finite.

Proposition 10 *Let $e \in E(S_1 *_U S_2)$ with $e\mathcal{D}^{S_1 *_U S_2} f$, for some $f \in E(U)$. Then $H_e^{S_1 *_U S_2}$ is finite if and only if $H_e^{S_1 *_U S_2} \simeq H_g^{S_k}$, for some $g \in E(U)$, $k \in \{1, 2\}$.*

Proof By Proposition 8 and Propositions 4 and 5, $H_e^{S_1 *_U S_2}$ is finite if and only if $Host(S\Gamma(f))$ is finite. Since the automorphism group of a finite tree fixes a vertex or an edge (see, for instance [1]), it is straightforward to check that for the finite $Host(S\Gamma(e))$ each automorphism φ of $Host(S\Gamma(e))$ has to fix a lobe $\Delta = S\Gamma(X_k, R_k; g)$, for some $k \in \{1, 2\}$, $g \in E(U)$. Thus $Aut(Host(S\Gamma(e))) \simeq Aut(\Delta)$, and whence $H_e^{S_1 *_U S_2} \simeq H_g^{S_k}$. The converse is trivial.

We have the following characterization of infinite maximal subgroups:

Theorem 5 *Let $e \in E(S_1 *_U S_2)$ with $e\mathcal{D}^{S_1 *_U S_2} f$, for some $f \in E(U)$. Then the following are equivalent:*

1. *$H_e^{S_1 *_U S_2}$ is infinite;*
2. *there is a sequence $f_1, f_2, ..., f_{2t-2}$ of idempotents of U for some $t > 1$ such that:*
   - *$f\mathcal{D}^{S_1 *_U S_2} f_1$,*
   - *f_{i-1} and f_i are not $\mathcal{D}^U$-related, for $1 < i \leq 2t - 2$,*
   - *there is an index $k \in \{1, 2\}$ such that $f_1\mathcal{D}^{S_k} f_t\mathcal{D}^{S_k} f_{2t-2}$, and for each even i with $1 < i < 2t - 2$, $f_{i-1}\mathcal{D}^{S_{3-k}} f_i\mathcal{D}^{S_k} f_{i+1}$, and $f_{2t-3}\mathcal{D}^{S_{3-k}} f_{2t-2}$.*
3. *$Y = H_e^{S_1 *_U S_2} \backslash \mathcal{T}_e$ is not a tree.*

Proof Again, by Proposition 8 and Propositions 4 and 5, $H_e^{S_1 *_U S_2}$ is infinite if and only if $Host(S\Gamma(f))$ is infinite. Moreover, by Corollary 1, $Host(S\Gamma(f))$ is infinite if and only if there is a reduced lobe path

$$P : \Delta_1, \ldots, \Delta_t, \ldots, \Delta_{2t-1}$$

with $\Delta_1 \simeq \Delta_t \simeq \Delta_{2t-1}$. We prove the equivalence 1) $\Leftrightarrow$ 2) by showing that this geometric characterization is equivalent to the algebraic conditions described in Statement 2. of the Theorem.

1) $\Rightarrow$ 2) Take any intersection vertex ν_i of $V(\Delta_i) \cap V(\Delta_{i+1})$ for $1 \le i \le 2t-2$ of P. Assume, without loss of generality, that the color of Δ_1 is 1. By Proposition 6, we have a sequence

$$e_1(\nu_1) = e_2(\nu_1), e_2(\nu_2) = e_1(\nu_2), \ldots, e_2(\nu_{2t-2}) = e_1(\nu_{2t-2})$$

of idempotents of U with $e_1(\nu_1)\mathcal{D}^{S_1 *_U S_2} f$. Set $f_i = e_1(\nu_i)$. Since $\Delta_1 \simeq \Delta_t \simeq \Delta_{2t-1}$, we have $f_1\mathcal{D}^1 f_t \mathcal{D}^1 f_{2t-2}$. It is straightforward to check that this sequence satisfies the remaining conditions of Statement 2. as well.

2) $\Rightarrow$ 1) If we assume, without loss of generality, $k = 1$, then $\Delta_1 = S\Gamma(X_1, R_1; f_1)$ is a host of $\Gamma = S\Gamma(X, R \cup W; f) \simeq S\Gamma(X, R \cup W; e)$. Let $\nu_1 \in V(\Delta_1)$ such that $e_1(\nu_1) = f_1$. Since Γ is complete, ν_1 is an intersection vertex, so let Δ_2 be the lobe of Γ that shares the vertex ν_1 with Δ_1. Then $\Delta_2 = S\Gamma(X_2, R_2; f_1)$ is a host by Proposition 6. Since $f_1\mathcal{D}^{S_2} f_2$ and f_2 is not $\mathcal{D}^U$ related to f_1, there is a vertex $\nu_2 \in V(\Delta_2)$ which is not connected to ν_1 by any path labeled by an element in U, and $e_2(\nu_2) = f_2$. Thus ν_2 is not among the intersection vertices of Δ_1, Δ_2, and so there is a lobe Δ_3, different from Δ_1, such that $\nu_2 \in V(\Delta_2) \cap V(\Delta_3)$. The lobe $\Delta_3 \simeq S\Gamma(X_1, R_1; f_2)$ is a host by Proposition 6. Using now $f_2\mathcal{D}^{S_1} f_3$, and the fact that f_2 and f_3 are not $\mathcal{D}^U$-related, we get that there is a vertex ν_3 with $e_1(\nu_3) = f_3$ that is not an intersection vertex between Δ_2, Δ_3. Continuing in this way, we build a reduced lobe path $P : \Delta_1, \ldots, \Delta_t, \ldots, \Delta_{2t-1}$ such that $\nu_i \in V(\Delta_1) \cap V(\Delta_{i+1})$ for $1 \le i \le 2t-2$, with $e_1(\nu_1) = f_1, e_1(\nu_t) = f_t, e_1(\nu_{2t-2}) = f_{2t-2}$. Finally, since $f_1\mathcal{D}^{S_1} f_t \mathcal{D}^{S_1} f_{2t-2}$, we get $\Delta_1 \simeq \Delta_t \simeq \Delta_{2t-1}$.

1) $\Rightarrow$ 3) Applying Proposition 8 yields the existence of two hosts with a shift-isomorphism φ between them. Hence, there is a reduced lobe path $P : \Delta_1, \ldots, \Delta_{2t-1}$ such that Δ_1 is sent onto Δ_{2t-1} by φ, and the automorphism of the lobe graph induced by φ does not map the edge (Δ_1, Δ_2) onto the edge $(\Delta_{2t-1}, \Delta_{2t-2})$. Therefore, these two edges do not belong to the same $H_e^{S_1 *_U S_2}$-orbit. Hence, Y contains a non-trivial loop P' induced by P.

3) $\Rightarrow$ 1) A reduced loop P' in Y lifts to a reduced lobe path $P : \Delta_1, \ldots, \Delta_{2t-1}$ in $Host(S\Gamma(f))$, for some $t > 1$, with Δ_1, Δ_{2t-1} belonging to the same $H_e^{S_1 *_U S_2}$-orbit. Hence, there is an automorphism $\varphi \in Aut(Host(S\Gamma(f)))$ which sends Δ_1 onto Δ_{2t-1}. Furthermore, no automorphism sends the edge (Δ_1, Δ_2) onto the edge $(\Delta_{2t-1}, \Delta_{2t-2})$, otherwise P' would not be reduced. Hence, $\varphi|_{\Delta_1} : \Delta_1 \to \Delta_{2t-1}$ is a shift-isomorphism. Therefore, by Proposition 8, $Host(S\Gamma(f))$ is infinite. □

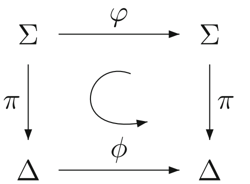
From the above theorem, we obtain that Y is a tree if and only if $H_e^{S_1 *_U S_2}$ is finite. Which means that the only case when $H_e^{S_1 *_U S_2}$ is isomorphic to iterated amalgams of groups is when $H_e^{S_1 *_U S_2}$ is finite.

Remark 1 We recall that an amalgam $[S_1, S_2; U]$ respects the $\mathcal{J}$-order if for each $e_1, e_2 \in E(U)$, $e_1 \mathcal{J}^{S_k} e_2$ implies $e_1 \mathcal{J}^{S_{3-k}} e_2$. In a $\mathcal{J}$-order respecting amalgam, if e is $\mathcal{J}^{S_1 *_U S_2}$-related to some idempotent $f \in U$, then (using arguments similar to those in the proof of Theorem 5), it is not hard to check that each host is isomorphic to either $S\Gamma(X_1, R_1; f)$ or $S\Gamma(X_2, R_2; f)$. Therefore, in this case $H_e^{S_1 *_U S_2}$ is finite, Y has exactly two vertices, and

$$H_e^{S_1 *_U S_2} \simeq H_f^{S_1} *_{H_f^U} H_f^{S_2}.$$

6.2 Case 2: "New" idempotent

Let e be an idempotent that is not $\mathcal{D}^{S_1 *_U S_2}$-related to any idempotent of S_1 or S_2. Then $S\Gamma(X, R \cup W; e)$ has a unique host that is a subopuntoid subgraph of the underlying graph of $Core(f)$. Thus $H_e^{S_1 *_U S_2}$ stabilizes some lobe Δ of the host. Since this lobe is finite, for any $\nu \in V(\Delta)$ there is a minimum idempotent, namely $e = e_k(\nu)$, labeling a loop based at ν. Thus, by [6, Lemma 2] (ν, Δ, ν) is a DV-quotient of the Schützenberger automaton $\mathcal{A}(X_k, R_k; e) = (\alpha, \Sigma, \alpha)$ called in [6] the maximum determinizing Schützenberger automaton of (ν, Δ, ν). If we denote the natural homomorphism induced by this quotient by $\pi : (\alpha, \Sigma, \alpha) \to (\nu, \Delta, \nu)$, we will show that we can lift an automorphism ϕ of Δ to an automorphism φ of Σ, such that the following diagram commutes:



Theorem 6 *Let (ν, Δ, ν) be a closed inverse automaton relative to the presentation $\langle X_k | R_k \rangle$ for some $k \in \{1, 2\}$. With the above notation let (α, Σ, α) be the maximum determinizing Schützenberger automaton of (ν, Δ, ν) with $\pi(\alpha) = \nu$. Then every automorphism $\phi \in Aut(\Delta)$ can be lifted to an automorphism $\varphi \in Aut(\Sigma)$ such that $\varphi \circ \pi = \pi \circ \phi$. Moreover, there is a group epimorphism from the subgroup $H := \{\varphi \in Aut(\Sigma) : \exists \phi \in Aut(\Delta), \varphi \circ \pi = \pi \circ \phi\}$ of $Aut(\Sigma)$ onto $Aut(\Delta)$ whose kernel is $N = H \cap S$, where $S = \{\varphi \in Aut(\Sigma) : \varphi(\pi^{-1}(\nu)) \subseteq \pi^{-1}(\nu)\}$.*

Proof Let $\phi \in Aut(\Delta)$, let $\nu' = \phi(\nu)$. Since ϕ preserves labeling, we have $e_k(\nu') = e_k(\nu) = e$. Thus, there is a word $w \in (X \cup X^{-1})^*$ labeling a path (ν, w, ν') in Δ such that $ww^{-1} = w^{-1}w = e$. Since $(\alpha, \Sigma, \alpha) = \mathcal{A}(X_k, R_k; e)$, there is also a path (α, w, α'), for some $\alpha' \in V(\Sigma)$. From the minimality of $e_k(\nu')$, we get $e_k(\alpha') = e$. Hence $(\alpha', \Sigma, \alpha')$ and (α, Σ, α) are Schützenberger automata that accept the same language, and by Proposition 1, there is an automorphism $\varphi \in Aut(\Sigma)$ such that

$\varphi(\alpha) = \alpha'$. We show that φ is an automorphism that satisfies the lifting property $\varphi \circ \pi = \pi \circ \phi$. For this purpose, let v be a vertex of Σ and let $r \in (X \cup X^{-1})^+$ be a word labeling a path (α, r, v). This path is mapped by the automorphism φ to (α', r, v'), with $v' = \varphi(v)$. Consider $\pi(v)$. Clearly, $(\nu, r, \pi(v))$ is a path in Δ, and the image of this path by ϕ is $(\nu', r, \phi(\pi(v)))$. Hence $(\nu, wr, \phi(\pi(v)))$ is also a path in Δ. Consider now $\pi(v')$. Since (α, wr, v') is a path in Σ, $(\nu, wr, \pi(v'))$ is also a path in Δ. By the determinism of Δ, we get $\phi(\pi(v)) = \pi(v') = \pi(\varphi(v))$. Let $H := \{\varphi \in Aut(\Sigma) : \exists \phi \in Aut(\Delta), \varphi \circ \pi = \pi \circ \phi\}$. It is straightforward to check that H is a subgroup of $Aut(\Delta)$.

For any $\varphi \in H$, the relation $\pi^{-1} \circ \varphi \circ \pi \subseteq V(\Delta) \times V(\Delta)$ is a function: By the definition of H there is a ϕ such that $\varphi \circ \pi = \pi \circ \phi$. Taking into account that π is surjective, for any left inverse π^{-1} holds

$$\pi^{-1} \circ (\varphi \circ \pi) = \pi^{-1} \circ (\pi \circ \phi) = (\pi^{-1} \circ \pi) \circ \phi = 1_\Delta \circ \phi = \phi.$$

Hence, there is a map $\lambda : H \to Aut(\Delta)$ defined by $\lambda(\varphi) = \pi^{-1} \circ \varphi \circ \pi$. Since we already proved that for any $\phi \in Aut(\Delta)$ there is a $\varphi \in Aut(\Sigma)$ such that $\pi \circ \phi = \varphi \circ \pi$, it follows that $\varphi \in H$ and $\lambda(\varphi) = \phi$, and therefore λ is surjective. It remains to show that λ is a homomorphism. Let $\varphi_i \in H$ and let $\phi_i \in Aut(\Delta)$ such that $\varphi_i \circ \pi = \pi \circ \phi_i$ for $i = 1, 2$. Using the definitions we obtain

$$\begin{aligned}\lambda(\varphi_1 \circ \varphi_2) &= \pi^{-1} \circ (\varphi_1 \circ \varphi_2) \circ \pi \\ &= (\pi^{-1} \circ \varphi_1) \circ (\pi \circ \phi_2) = (\pi^{-1} \circ \varphi_1 \circ \pi) \circ \phi_2 \\ &= \lambda(\varphi_1) \circ \phi_2 = \lambda(\varphi_1) \circ \lambda(\varphi_2).\end{aligned}$$

The proof of the last part of the Theorem follows from the definitions of H and S. □

Note that without the requirement for the finiteness of the inverse semigroup S, it is not in general possible to define the maximum determinizing Schützenberger automaton of a closed inverse word automaton relative to the presentation $\langle X|R\rangle$ of the inverse semigroup S. It is also not hard to produce an example where it is not possible to lift an automorphism of a closed DV-quotient Δ of a Schützenberger automaton Σ to an automorphism of Σ (see [19]). In addition the subgroup H used in the previous theorem is in general only a proper subgroup of $Aut(\Sigma)$. To illustrate this, consider the dihedral group $D_4 = Gp\langle r, s|r^2, s^2, (rs)^4\rangle$ which seen as a finite inverse semigroup has only one Schützenberger graph - the Cayley graph of D_4. In this Cayley graph, if we identify the identity e with the element s, and then we determinize, we obtain an inverse word graph Δ with $Aut(\Delta) \simeq Gp\langle\sigma|\sigma^2\rangle$. It is easy to show that σ can be lifted to the automorphism induced by the left multiplication by the element $(sr)^2$. However, it is also not difficult to check that the automorphism φ induced by left multiplication by (sr) is not in H, i.e. there is no $\phi \in Aut(\Delta)$ for which $\varphi \circ \pi = \pi \circ \phi$.

The next theorem describes maximal subgroups corresponding to idempotents that are not $\mathcal{D}$-related in the amalgam to any of the old idempotents of S_1 or S_2:

Theorem 7 *Let $e \in E(S_1 *_U S_2)$, S_1, S_2 finite inverse semigroups, and suppose that e is not $\mathcal{D}^{S_1 *_U S_2}$-related to any idempotent of S_1 or S_2. Then $H_e^{S_1 *_U S_2}$ is a homomorphic* *image of a subgroup of the maximal subgroup $H_g^{S_k}$, for some $g \in E(S_k)$ and some $k \in \{1, 2\}$.*

Proof We already noted that under the assumption of our theorem, $H_e^{S_1 *_U S_2}$ is isomorphic to the automorphism group of some lobe Δ of $Host(S\Gamma(e))$. By Theorem 6, $Aut(\Delta)$ is a homomorphic image of $Aut(\Sigma)$, where $\Sigma = S\Gamma(X_k, R_k; g)$, for some $g \in E(S_k)$. Therefore $H_e^{S_1 *_U S_2}$ is a homomorphic image of $Aut(\Sigma) \simeq H_g^{S_k}$. In particular, $H_e^{S_1 *_U S_2} \simeq H_g^{S_k}/N$, where N is the normal subgroup described in Theorem 6.

Remark 2 When S_1 and S_2 are E-unitary, then no quotient has to be performed in the construction of the Schützenberger graph of a word with respect to the standard presentation of the amalgam $S_1 *_U S_2$. Then, if e is not $\mathcal{D}^{S_1 *_U S_2}$-related to any idempotent of S_1 or S_2, the maximal subgroup $H_e^{S_1 *_U S_2}$ is isomorphic to the maximal subgroup $H_g^{S_k}$ of S_k, for some $k \in \{1, 2\}$ and $g \in E(S_k)$.

7 Presentation for $H_e^{S_1 *_U S_2}$

We have completely determined the structure of maximal subgroups of amalgamated free products of finite inverse semigroups. All these groups are finitely presented, and as a concluding remark, we note, that their presentations are effectively computable:

Theorem 8 *Let $e \in E(S_1 *_U S_2)$, S_1, S_2 finite inverse semigroups, then there is an algorithm to compute a presentation for $H_e^{S_1 *_U S_2}$.*

Proof We provide only a brief sketch of the idea of the proof. For more details see the final chapter of [19]. Note that checking whether a lobe Δ' feeds off another lobe Δ is a decidable task, since it is enough to apply Construction 5 to Δ to obtain a lobe Δ'', and then check whether Δ' and Δ'' are isomorphic. Using this fact, we can build a host for a Schützenberger graph: A host is always contained in the core - which can be built effectively. Having the core, removing the lobes which are parasites will eventually result in a host. Therefore, if $S\Gamma(e)$ has a unique host, then $H_e^{S_1 *_U S_2}$ is the automorphism group of a lobe Δ of the (unique) host contained in $Core(u)$, for some word u equivalent to e. This host is finite, the lobe Δ can be constructed, and a presentation for $Aut(\Delta)$ can be determined. In the case when $S\Gamma(e)$ has more than one host, the union of all hosts can be infinite, and in order to determine a presentation for $H_e^{S_1 *_U S_2}$, we need the information contained in the graph of groups $(\mathcal{G}(-), Y)$. This can be done effectively by starting from any host in $Core(u)$, building a maximal subtree T of Y by considering the subopuntoid graph of $Host(S\Gamma(e))$ built recursively by adding at each step adjacent hosts that are non-isomorphic to the hosts previously chosen. Since, up to isomorphism, there are only finitely many lobes, after finitely many steps we obtain an opuntoid graph Θ for which each adjacent host is isomorphic to a lobe already occurring in Θ. By Propositions 4 and 5, all the lobes of Θ are representatives of all the orbits, therefore $T = \mathcal{T}(\Theta)$. Furthermore, by Theorem 3, in order to build the fundamental $H_e^{S_1 *_U S_2}$-transversal Y, we need to add edges to T. The edges to be added are the edges $d = (\Delta, \Delta')$ for which there exists a host Δ'' adjacent in $\mathcal{T}_e$ to Δ for which Δ' is the (unique) host of T isomorphic to Δ''. By Propositions 4 and 5, the isomorphism $\Delta' \simeq \Delta''$ also defines the connecting element t_d. Using the above considerations in combination with the finiteness of Y and its lobes, we can effectively compute the graph of groups $(\mathcal{G}(-), Y)$, and therefore also the presentation of $\pi(\mathcal{G}(-), Y) \simeq H_e^{S_1 *_U S_2}$. $\square$

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