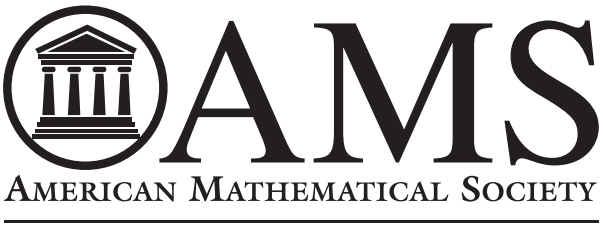

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A THEORETICAL STUDY OF COMPRESSED SOLVING FOR ADVECTION-DIFFUSION-REACTION PROBLEMS

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Abstract. We present a theoretical analysis of the CORSING (COmpRessed SolvING) method for the numerical approximation of partial differential equations based on compressed sensing. In particular, we show that the best s-term approximation of the weak solution of a PDE with respect to a system of N trial functions, can be recovered via a Petrov-Galerkin approach using $m \ll N$ test functions. This recovery is guaranteed if the local a-coherence associated with the bilinear form and the selected trial and test bases fulfills suitable decay properties. The fundamental tool of this analysis is the restricted inf-sup property, i.e., a combination of the classical inf-sup condition and the well-known restricted isometry property of compressed sensing.

1. Introduction

Compressed Sensing (CS) is an extremely powerful tool of signal processing employed to recover a sparse signal using far fewer measurements than those required by the Nyquist-Shannon sampling theorem. In particular, expanding the signal with respect to a basis of N vectors, it is possible to recover the best s-term approximation to the signal, with $s \ll N$, by means of m random measurements, with $s < m \ll N$ [23, 12, 27].

In [11], we introduced an application of CS to the numerical approximation of Partial Differential Equations (PDEs). For this purpose, we rely on an analogy between the sampling process of a signal and the evaluation of the bilinear form associated with a Petrov-Galerkin discretization ([6, 25, 37]) of the PDE against randomly chosen test functions. We named the resulting numerical method CORSING, acronym for COmpRessed SolvING.

Comparison with other techniques. The CORSING method aims at computing the best s-term approximation to the solution to a PDE. Therefore, it can be classified

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among nonlinear approximation methods ([22, 46]) for PDEs. Although the framework for CORSING is very general and can accommodate many different choices of trial and test spaces, when considering hierarchical piecewise polynomials over an initial coarse triangulation as trial basis functions, a possible competitor approach is the Adaptive Finite Element Method (AFEM) (see, e.g., [33] and the references therein). AFEM and CORSING are, however, thoroughly different: in AFEM, the solution is iteratively computed according to the loop

$$\text{SOLVE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE},$$

and exploiting suitable *a posteriori* error estimators. On the contrary, with CORSING we employ a reduced Petrov-Galerkin discretization, using a fixed trial space of dimension N (which corresponds ideally to a very fine uniform refinement, expressed in a hierarchical basis) and performing a fixed number of random measurements in the test space. In particular:

(1) the trial space is not iteratively enlarged, but fixed initially;
(2) the measurements in the test space are performed non-adaptively;
(3) no *a posteriori* error estimators/indicators are needed.

The CORSING procedure then recovers an s-sparse solution (with $s \ll N$), which can be compared with the AFEM solution on the same ground. We consider (1) as a possible drawback of CORSING, whereas (2) and (3) are upsides. In principle (1) requires a higher computational cost in the recovery phase, whereas (2) allows for full parallelization and (3) significantly reduces the implementation complexity.

From a different perspective, CORSING can be considered as a variant of the infinite-dimensional CS, where CS is applied to infinite-dimensional Hilbert spaces [3, 4]. This is achieved by subsampling a given isometry of the Hilbert space, usually associated with an inner product and a change of basis (e.g., from a wavelet basis to the Fourier basis). The main idea behind CORSING is different, since it deals with the bilinear form arising from the weak formulation, that can be even nonsymmetric. Nevertheless, we think that the theory developed in [3, 4] could play a significant role for a deeper understanding of the CORSING technique and this will be a subject of future investigation.

The present work and the proposed theoretical analysis have relations with the techniques presented in [40, 41, 1, 2], where polynomial approximation and function interpolation are addressed, by employing principles from CS, such as (weighted) ℓ^1-minimization.

Finally, it is worth mentioning that CS has already been used in combination with PDEs, however in the different context of approximating stochastic/parametric PDEs, of particular interest in *Uncertainty Quantification* [24, 50, 45, 35, 7, 39]. The goal is to approximate some statistical information of the random solution to the PDE through CS-based sampling strategies in the space of parameters.

Main contributions of the paper. The goal of this paper is to set up a theoretical analysis of CORSING, providing sufficient conditions for convergence, and formalizing the empirical recipes given in [11]. With this aim, we introduce a novel variant of the classical inf-sup condition [9], where the infimum is considered among the sparse elements of the trial space and the supremum over a small test space. We refer to this condition as *Restricted Inf-Sup Property* (RISP), since it combines the inf-sup condition and the Restricted Isometry Property (RIP), a well-known tool in the CS literature. Another important tool of the analysis is the concept of *local*

a-coherence, a generalization of the *local coherence* to bilinear forms on Hilbert spaces. In particular, we have been inspired by [28], where an optimal recovery result for CS, with non-uniform random subsampling based on the local coherence, is proved for the Haar and Fourier discrete bases in dimension one and two. In this theoretical analysis, we will assume the test functions to be selected *independently*. This is required in order to apply Chernoff's bounds for the sum of random matrices in Theorem 3.8 (see also Remark 2.4).

The main results of the paper can be thus summarized. First, we prove sufficient conditions for the RISP, depending on suitable hypotheses on the local a-coherence. Then, recovery error estimates for the CORSING algorithm are provided. In particular, in Theorem 3.8 we show that a sufficient condition for the RISP to hold with high probability in a given s-sparse set is that m and s be linearly dependent, up to logarithmic factors. On the contrary, at the moment we are only able to prove (Theorem 3.10) a *uniform* RISP (i.e., a RISP holding in *all* possible s-sparse sets) assuming a quadratic dependence between m and s, although we conjecture that, as in CS, the dependence on s should be linear. Exploiting these theorems, we prove a recovery result in expectation (Theorem 3.15) and two in probability (Theorem 3.16 and Theorem 3.18). In particular, we check the hypotheses on the local a-coherence in the case of a one-dimensional advection-diffusion-reaction equation employing the hierarchical multiscale basis in [51, 19] and the Fourier sine basis.

Outline of the paper. In Section 2, we formally introduce the CORSING, defining all the input/output variables involved in the algorithm. The theoretical analysis based on the RISP is presented in Section 3, and an application of the theory to a one-dimensional advection-diffusion-reaction equation is discussed in Section 4. In Section 5, we provide some numerical results, and we draw some conclusions in Section 6.

2. CORSING

In this section, after setting up the notation, we describe the COmpRessed Solving procedure, in short, CORSING, first introduced in [11].

2.1. **Notation.** Let $\mathbb{N} := \{1, 2, 3, \dots\}$ be the set of positive natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Consider two separable Hilbert spaces over the field $\mathbb{R}$,

$$U := \overline{\text{span}\{\psi_j\}}_{j\in\mathbb{N}} \quad \text{and} \quad V := \overline{\text{span}\{\varphi_q\}}_{q\in\mathbb{N}},$$

generated by the bases $\{\psi_j\}_{j\in\mathbb{N}}$ and $\{\varphi_q\}_{q\in\mathbb{N}}$, respectively, and equipped with the inner products $(\cdot,\cdot)_U$ and $(\cdot,\cdot)_V$, where the closures are made with respect to the norms $\|\cdot\|_U$ and $\|\cdot\|_V$ induced by the corresponding inner products. Given two positive integers N and M, we define the finite dimensional truncations of U and V, which represent the *trial* and *test* space, respectively, as

$$U^N := \text{span}\{\psi_j\}_{j\in[N]} \quad \text{and} \quad V^M := \text{span}\{\varphi_q\}_{q\in[M]},$$

where $[k] := \{1, \dots, k\}$ for every $k \in \mathbb{N}$. In particular, $[\infty] = \mathbb{N}$. We denote the span of the basis functions relative to a given subset of indices $\mathcal{S} \subseteq [N]$ as

$$U^N_{\mathcal{S}} := \text{span}\{\psi_j\}_{j\in\mathcal{S}}.$$

Given a positive integer $s \leq N$, we also define the set U^N_s of s-sparse functions of U^N with respect to the basis $\{\psi_j\}_{j\in[N]}$ as the set of all functions that are linear

combinations of at most s basis functions, namely

$$U_s^N := \bigcup_{\mathcal{S}\subseteq[N];\,|\mathcal{S}|=s} U_{\mathcal{S}}^N.$$

We stress that U_s^N is not a vector space. Indeed, the sum of two s-sparse elements is in general $2s$-sparse. The sets $V_{\mathcal{T}}^M$ and V_m^M are defined analogously, for every $\mathcal{T} \subseteq [M]$ and $m \leq M$.

We denote by U^* and V^* the dual spaces of U and V, respectively.

The bases $\{\psi_j\}_{j\in\mathbb{N}}$ and $\{\varphi_q\}_{q\in\mathbb{N}}$ are assumed to be *Riesz* bases (see, e.g., [16]). In particular, we require the existence of two constants $0 < c_\psi \leq C_\psi$ such that

$$c_\psi \|\mathbf{u}\|_2^2 \leq \left\| \sum_{j\in\mathbb{N}} u_j \psi_j \right\|_U^2 \leq C_\psi \|\mathbf{u}\|_2^2, \quad \forall \mathbf{u} \in \ell^2. \tag{1}$$

A relation analogous to (1) is assumed to hold for $\{\varphi_q\}_{q\in\mathbb{N}}$, with constants $0 < c_\varphi \leq C_\varphi$.

This assumption allows us to introduce the *reconstruction* and *decomposition* operators associated with a basis, which allow us to switch between functions and their corresponding coefficients in the basis expansion.

Definition 2.1. The *reconstruction operator* $\Psi : \ell^2 \to U$ related to the basis $\{\psi_j\}_{j\in\mathbb{N}}$ of U associates with a sequence $\mathbf{u} = (u_j)_{j\in\mathbb{N}} \in \ell^2$ the linear combination

$$u = \Psi\mathbf{u} := \sum_{j=1}^{\infty} u_j \psi_j.$$

The *decomposition operator* $\Psi^* : U \to \ell^2$ applied to a given function $u \in U$ is defined component-wise as

$$(\Psi^* u)_k := (u, \psi_k^*)_U, \quad \forall k \in \mathbb{N},$$

where $\{\psi_k^*\}_{k\in\mathbb{N}}$ is the basis biorthogonal to $\{\psi_j\}_{j\in\mathbb{N}}$, namely, $(\psi_j, \psi_k^*)_U = \delta_{jk}$, $\forall j, k \in \mathbb{N}$.

The reconstruction operator Φ and the decomposition operator Φ^* associated with the basis $\{\varphi_q\}_{q\in\mathbb{N}}$ of V are defined analogously.

Remark 2.2. We observe that $\Psi\Psi^* = Id_U$ and $\Psi^*\Psi = Id_{\ell^2}$.

2.2. **The general reference problem.** Consider the following problem

$$\text{find } u \in U : a(u, v) = \mathcal{F}(v), \quad \forall v \in V, \tag{2}$$

where $a : U \times V \to \mathbb{R}$ is a bilinear form and $\mathcal{F} \in V^*$. We assume $a(\cdot,\cdot)$ to fulfill the following three conditions

$$\exists \alpha > 0 : \quad \inf_{u\in U} \sup_{v\in V} \frac{a(u,v)}{\|u\|_U \|v\|_V} \geq \alpha, \tag{3}$$

$$\exists \beta > 0 : \quad \sup_{u\in U} \sup_{v\in V} \frac{|a(u,v)|}{\|u\|_U \|v\|_V} \leq \beta, \tag{4}$$

$$\sup_{u\in U} a(u,v) > 0, \quad \forall v \in V \setminus \{0\}.$$

These assumptions imply the existence and uniqueness of the solution to (2), thanks to a generalization of the Lax-Milgram lemma due to Nečas [32], [37, Theorem 5.1.2].

To simplify the notation, when an infimum or a supremum of a fraction $f(x)/g(x)$ over a given set X is considered, the zeros of $g(x)$ are understood to be removed from X.

Our goal is to approximate the solution to (2), by merging the classical Petrov-Galerkin formulation (sometimes also called non-standard Galerkin method) [6, 37, 25] with CS techniques [23, 12].

2.3. **Main hypotheses.** We will use three assumptions throughout the article.

Hypothesis 1 (Riesz bases)**.** *The trial basis* $\{\psi_j\}_{j\in\mathbb{N}}$ *and the test basis* $\{\varphi_q\}_{q\in\mathbb{N}}$ *are Riesz bases, i.e., they fulfill relation* (1) *with constants* $0 < c_\psi \leq C_\psi$ *and* $0 < c_\varphi \leq C_\varphi$*, respectively.*

Hypothesis 1 is needed for the operators Ψ, Ψ^*, Φ, and Φ^* to be well defined (see Definition 2.1), but this requirement can be relaxed in the case of the trial basis. Indeed, throughout the article, we just consider the restriction of Ψ from ℓ^2 to $\mathbb{R}^N$, namely the operator $\Psi^N := \Psi|_{\mathbb{R}^N} : \mathbb{R}^N \to U^N$, and the restriction of Ψ^* from U to U^N, namely $\Psi^{N,*} := \Psi^*|_{U^N} : U^N \to \mathbb{R}^N$. A sufficient condition for the existence of operators Ψ^N and $\Psi^{N,*}$ is that there exist two constants $0 < c_\psi^N \leq C_\psi^N$ such that relation (1) holds for every $\mathbf{u} \in \mathbb{R}^N$ (in place of $\mathbf{u} \in \ell^2$). Within this more general setting, the constants are allowed to degenerate, i.e., we may have that $c_\psi^N \to 0$ and $C_\psi^N \to +\infty$ as $N \to +\infty$. However, for easiness of presentation, we assume Hypothesis 1 to hold also for the trial basis.

We generalize the notion of local coherence (see, e.g., [28]) to bilinear forms defined over Hilbert spaces.

Definition 2.3 (Local a-coherence $\boldsymbol{\mu}^N$)**.** Given $N \in \mathbb{N} \cup \{\infty\}$, the real-valued sequence $\boldsymbol{\mu}^N$ defined as

$$\mu_q^N := \sup_{j\in[N]} |a(\psi_j, \varphi_q)|^2, \quad \forall q \in \mathbb{N},$$

is called *local a-coherence of* $\{\psi_j\}_{j\in[N]}$ *with respect to* $\{\varphi_q\}_{q\in\mathbb{N}}$.

Notice that $\boldsymbol{\mu}^N$ is a bounded sequence, with a uniform bound with respect to N. Indeed, exploiting the continuity (4) of the bilinear form $a(\cdot,\cdot)$ and Hypothesis 1, we have

$$\mu_q^N = \sup_{j\in[N]} |a(\psi_j, \varphi_q)|^2 \leq \beta^2 \sup_{j\in[N]} \|\psi_j\|_U^2 \cdot \|\varphi_q\|_V^2 \leq \beta^2 C_\psi C_\varphi, \quad \forall q, N \in \mathbb{N}.$$

The second hypothesis concerns the local a-coherence.

Hypothesis 2 (Summability of $\boldsymbol{\mu}^N$)**.** *The local a-coherence of* $\{\psi_j\}_{j\in[N]}$ *with respect to* $\{\varphi_q\}_{q\in\mathbb{N}}$ *fulfills the summability condition*

$$\|\boldsymbol{\mu}^N\|_1 < +\infty,$$

or, equivalently, $\boldsymbol{\mu}^N \in \ell^1$.

Notice that Hypothesis 2 does not hinge on the ordering considered for the elements of the truncated trial basis $\{\psi_j\}_{j\in[N]}$.

The last hypothesis concerns an explicit upper bound to the local a-coherence.

Algorithm 2.1

PROCEDURE $\widehat{u} = \mathsf{CORSING}\ (N,\ s,\ \boldsymbol{\nu}^N,\ \widehat{\gamma},\ \widehat{C},\ \overline{\gamma},\ \overline{C})$

1. **Definition of M and m**
> $M \leftarrow \widehat{C} s^{\widehat{\gamma}} N$;
> $m \leftarrow \overline{C} s^{\overline{\gamma}} \|\boldsymbol{\nu}^{N,M}\|_1 \log(N/s)$;
2. **Test selection**
> $\mathbf{p} \leftarrow \boldsymbol{\nu}^{N,M}/\|\boldsymbol{\nu}^{N,M}\|_1$;
> Draw $\tau_1, \ldots, \tau_m$ independently at random from $[M]$ according to the probability $\mathbf{p}$;
3. **Assembly**
> Build $\mathbf{A}$, $\mathbf{f}$ and $\mathbf{D}$, defined in (6) and (7), respectively;
4. **Recovery**
> Find an approximate solution $\widehat{\mathbf{u}}$ to $\arg\min_{\mathbf{v}\in\mathbb{R}^N} \|\mathbf{D}(\mathbf{A}\mathbf{v} - \mathbf{f})\|_2^2$, s.t. $\|\mathbf{v}\|_0 \le s$;
> $\widehat{u} \leftarrow \Psi\widehat{\mathbf{u}}$.

Hypothesis 3 (Upper bound $\boldsymbol{\nu}^N$)**.** *For every $N \in \mathbb{N}$, we assume to have a computable componentwise upper bound $\boldsymbol{\nu}^N$ to the local a-coherence $\boldsymbol{\mu}^N$, i.e., a real-valued sequence such that*

$$\mu_q^N \le \nu_q^N, \quad \forall q \in \mathbb{N}.$$

For every $M \in \mathbb{N}$, we define the vector $\boldsymbol{\nu}^{N,M} \in \mathbb{R}^M$ as the restriction of $\boldsymbol{\nu}^N$ to the first M components. Moreover, we require that

- *the vector $\boldsymbol{\nu}^{N,M}/\|\boldsymbol{\nu}^{N,M}\|_1$ is efficiently computable for every $N, M \in \mathbb{N}$;*
- *there exists a real bivariate polynomial P such that*

$$\|\boldsymbol{\nu}^{N,M}\|_1 \lesssim P(\log N, \log M).$$

The upper bound $\boldsymbol{\nu}^N$ need not be sharp.

As usual, with notation $x \sim y$, $x \lesssim y$ or $x \gtrsim y$, it is understood that there exists a constant $C > 0$ not depending on x and y, such that $x = Cy$, $x \le Cy$ or $x \ge Cy$, respectively.

2.4. **The CORSING procedure.** The CORSING procedure is summarized in Algorithm 2.1. Let us now describe in more detail the input/output variables and the main steps of the method.

INPUT.

- N: dimension of the trial space;
- $s \ll N$: number of trial coefficients to recover;
- upper bound $\boldsymbol{\nu}^N$ in Hypothesis 3 and four positive constants $\widehat{\gamma}$, $\widehat{C}$, $\overline{\gamma}$, and $\overline{C}$, used to select the dimension M of the test space and the m tests to perform.

OUTPUT.

- $\widehat{u} \in U_s^N$: approximate s-sparse solution to (2).

1. Definition of M and m. The test space dimension M and the number m of tests to perform are chosen as functions of N and s as

$$M = \widehat{C} s^{\widehat{\gamma}} N, \quad m = \overline{C} s^{\overline{\gamma}} \|\boldsymbol{\nu}^{N,M}\|_1 \log(N/s).$$

In Section 3, we prove the existence of suitable values for the constants $\widehat{\gamma}$, $\widehat{C}$, $\overline{\gamma}$, and $\overline{C}$ that ensure the CORSING algorithm to recover the best s-term approximation to u in expectation and in probability. In Section 4, we perform a sensitivity analysis on the constants $\widehat{C}$ and $\overline{C}$ for some specific differential problems and with $\overline{\gamma} = 1, 2$. Numerical evidence shows that $\overline{\gamma} = 1$ is a valid choice, but proving this from a theoretical viewpoint still remains an open problem. On the contrary, the value of $\widehat{\gamma}$ seems to depend on the trial and test bases considered (see Section 4).

2. Test selection. In order to formalize the test selection procedure, we introduce a probability space $(\Omega, \mathcal{E}, \mathbb{P})$ and consider $\tau_1, \dots, \tau_m$ as i.i.d. discrete random variables taking values in $[M]$, namely

$$\tau_i : \Omega \to [M], \quad \forall i \in [m].$$

Moreover, given a vector $\mathbf{p} = (p_q)_{q \in [M]} \in [0,1]^M$ such that $\|\mathbf{p}\|_1 = 1$, the probability law is defined as

$$\mathbb{P}\{\tau_i = q\} = p_q, \quad \forall q \in [M].$$

Throughout the paper, the vector $\mathbf{p}$ will be assumed to be of the form

$$\mathbf{p} := \frac{\boldsymbol{\nu}^{N,M}}{\|\boldsymbol{\nu}^{N,M}\|_1}, \tag{5}$$

where the values for $\boldsymbol{\nu}^{N,M}$ are known from Hypothesis 3.

3. Assembly. In this phase, we build the *stiffness matrix* $\mathbf{A} \in \mathbb{R}^{m \times N}$ and the *load vector* $\mathbf{f} \in \mathbb{R}^m$ associated with the Petrov-Galerkin discretization of (2), defined as

$$A_{ij} := a(\psi_j, \varphi_{\tau_i}), \quad f_i := \mathcal{F}(\varphi_{\tau_i}), \quad \forall j \in [N], \forall i \in [m]. \tag{6}$$

Moreover, the matrix $\mathbf{D} \in \mathbb{R}^{m \times m}$ is a diagonal preconditioner, depending on the vector $\mathbf{p}$ as

$$D_{ik} := \frac{\delta_{ik}}{\sqrt{m p_{\tau_i}}}, \quad \forall i \in [m]. \tag{7}$$

4. Recovery. The vector of trial coefficients $\widehat{\mathbf{u}}$ is an approximate solution to

$$\operatorname*{arg\,min}_{\mathbf{v} \in \mathbb{R}^N} \|\mathbf{D}(\mathbf{A}\mathbf{v} - \mathbf{f})\|_2^2, \quad \text{s.t.} \quad \|\mathbf{v}\|_0 \le s, \tag{8}$$

where $\|\mathbf{u}\|_0 = |\{j : u_j \neq 0\}|$ is the so called ℓ^0-*norm*. Consequently, the CORSING solution is defined as $\widehat{u} := \Psi \widehat{\mathbf{u}}$. An equivalent functional formulation of (8) is

$$\operatorname*{arg\,min}_{v \in U_s^N} \sum_{i=1}^{m} \frac{1}{m p_{\tau_i}} (a(v, \varphi_{\tau_i}) - \mathcal{F}(\varphi_{\tau_i}))^2. \tag{9}$$

The procedure defined by (8) (or, equivalently, (9)) has been proved to be generally NP-hard, [30], but fortunately, there are several ways to efficiently and accurately approximate its solutions under particular circumstances, e.g., when the RIP holds. These strategies can be divided in two main families: convex relaxation techniques, such as the well known ℓ^1-minimization, also known as Basis Pursuit (BP) [13], and greedy algorithms [47, 31]. In this paper, we focus on greedy techniques

and, in particular, we employ the Orthogonal Matching Pursuit (OMP) algorithm [29, 34]. For recent results concerning its accuracy, we refer to [52, 17].

Using OMP, we can easily control the parameter s, i.e., the sparsity of the compressed solution $\widehat{u}$, by employing a stopping criterion based on sparsity. The time complexity of the OMP algorithm is easily estimated, namely $\mathcal{O}(smN)$ for basic implementations. All the numerical experiments made in this work are performed using the OMP-BOX MATLAB® package, version 10 - see [44, 43]. For a comparison between OMP and ℓ^1-minimization for CORSING, we refer to [11].

A valuable alternative to OMP (not addressed here) is the Hard Thresholding Pursuit (HTP), introduced in [26]. HTP is a greedy strategy very similar to OMP, which easily allows for a control on the sparsity level, s, and has been shown to be competitive with OMP [8].

Remark 2.4. The CORSING procedure described in Algorithm 2.1 is almost identical to the R-CORSING method discussed in [11]. There is only one difference. Here, we assume the independence of the random variables $\tau_1, \dots, \tau_m$ (hence allowing repetitions). In [11] the same random variables are selected through successive drawings from an urn without replacement (see [11, Algorithm 4.1]), in order to avoid repetitions. The hypothesis of independence is needed to simplify the theoretical analysis of the next section and to apply Chernoff's bounds.

3. Theoretical analysis

3.1. **Preliminary results.** The main statistical tools employed in this paper are Chernoff's bounds for matrices. They were introduced by H. Chernoff during the early 50's in the scalar form [14], and generalized to the matrix setting by R. Ahlswede and A. Winter in 2003 [5]. These bounds have been recently refined in 2012 by J. Tropp in [48].

First, we present the main result employed in our analysis. The proof of the following theorem can be found in [48, Corollary 5.2].

Theorem 3.1 (Matrix Chernoff's bounds)**.** *Consider a finite sequence of i.i.d. random, symmetric $s \times s$ real matrices $\mathbf{X}^1, \dots, \mathbf{X}^m$ such that*

$$0 \leq \lambda_{\min}(\mathbf{X}^i) \text{ and } \lambda_{\max}(\mathbf{X}^i) \leq R \quad \text{almost surely}, \quad \forall i \in [m].$$

Define $\overline{\mathbf{X}} := \frac{1}{m} \sum_{i=1}^{m} \mathbf{X}^i$, $E_{\min} := \lambda_{\min}(\mathbb{E}[\mathbf{X}^i])$ *and* $E_{\max} := \lambda_{\max}(\mathbb{E}[\mathbf{X}^i])$. *Then,*

$$\mathbb{P}\{\lambda_{\min}(\overline{\mathbf{X}}) \leq (1-\delta) E_{\min}\} \leq s \exp\left(-\frac{m \rho_\delta E_{\min}}{R}\right), \quad \forall \delta \in [0,1], \tag{10}$$

$$\mathbb{P}\{\lambda_{\max}(\overline{\mathbf{X}}) \geq (1+\delta) E_{\max}\} \leq s \exp\left(-\frac{m \widetilde{\rho}_\delta E_{\max}}{R}\right), \quad \forall \delta \geq 0,$$

with

$$\rho_\delta := (1-\delta)\log(1-\delta) + \delta, \quad \widetilde{\rho}_\delta := (1+\delta)\log(1+\delta) - \delta. \tag{11}$$

□

Notice that both constants $\rho_\delta, \widetilde{\rho}_\delta \sim \delta^2$ when $\delta \to 0$.

We conclude this section by recalling a result that will be repeatedly used in the next proofs.

Lemma 3.2. *If $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ are symmetric and $\mathbf{B}$ is also positive definite, it holds*

$$\lambda_{\min}(\mathbf{B}^{-\frac{1}{2}}\mathbf{A}\mathbf{B}^{-\frac{1}{2}}) = \inf_{\mathbf{u}\in\mathbb{R}^d} \frac{\mathbf{u}^\intercal\mathbf{A}\mathbf{u}}{\mathbf{u}^\intercal\mathbf{B}\mathbf{u}}, \tag{12}$$

$$\lambda_{\max}(\mathbf{B}^{-\frac{1}{2}}\mathbf{A}\mathbf{B}^{-\frac{1}{2}}) = \sup_{\mathbf{u}\in\mathbb{R}^d} \frac{\mathbf{u}^\intercal\mathbf{A}\mathbf{u}}{\mathbf{u}^\intercal\mathbf{B}\mathbf{u}}. \tag{13}$$

3.2. **Non-uniform restricted inf-sup property.** In this section, we deal with the core of our paper, namely an analysis of the CORSING algorithm.

We denote the space of vectors of $\mathbb{R}^N$ supported in $\mathcal{S} \subseteq [N]$ as $\mathbb{R}^N_{\mathcal{S}}$, namely

$$\mathbb{R}^N_{\mathcal{S}} := \{\mathbf{u} \in \mathbb{R}^N : u_j = 0,\ \forall j \notin \mathcal{S}\}.$$

Moreover, we introduce some further notation.

Definition 3.3 (Matrices $\mathbf{K}$, $\mathbf{K}_{\mathcal{S}}$ and $\mathbf{A}_{\mathcal{S}}$)**.** We define the matrix $\mathbf{K} \in \mathbb{R}^{N\times N}$ as

$$K_{jk} := (\psi_j, \psi_k)_U.$$

and its restriction $\mathbf{K}_{\mathcal{S}} \in \mathbb{R}^{s\times s}$ to $\mathcal{S} := \{\sigma_1, \dots, \sigma_s\} \subseteq [N]$ as

$$(K_{\mathcal{S}})_{jk} := (\psi_{\sigma_j}, \psi_{\sigma_k})_U.$$

Moreover, we denote by $\mathbf{A}_{\mathcal{S}} \in \mathbb{R}^{m\times s}$ the submatrix of $\mathbf{A}$ consisting only of the columns with indices in $\mathcal{S}$.

We observe that $\mathbf{K}$ is symmetric and positive definite (s.p.d.) and fulfills

$$\mathbf{u}^\intercal\mathbf{K}\mathbf{u} = \|\Psi\mathbf{u}\|_U^2, \quad \forall \mathbf{u} \in \mathbb{R}^N, \tag{14}$$

where the reconstruction operator in (14) is implicitly restricted from ℓ^2 to $\mathbb{R}^N$ (equivalently, the vector $\mathbf{u}$ is extended to ℓ^2 by adding zeros for $j > N$). Morever, when basis $\{\psi_j\}_{j\in\mathbb{N}}$ is orthonormal, the matrix $\mathbf{K}$ is the identity. The matrix $\mathbf{K}_{\mathcal{S}}$ is also s.p.d. and it satisfies the relation

$$\mathbf{u}_{\mathcal{S}}^\intercal\mathbf{K}_{\mathcal{S}}\mathbf{u}_{\mathcal{S}} = \mathbf{u}^\intercal\mathbf{K}\mathbf{u}, \quad \forall \mathbf{u} \in \mathbb{R}^N_{\mathcal{S}},$$

where $\mathbf{u}_{\mathcal{S}} \in \mathbb{R}^s$ is the restriction of $\mathbf{u}$ to $\mathcal{S}$, namely $(u_{\mathcal{S}})_j = u_{\sigma_j}$, for every $j \in [s]$. In this section, we fix a subset $\mathcal{S} := \{\sigma_1, \dots, \sigma_s\} \subseteq [N]$ of cardinality s.

We introduce the Gram matrix $\mathbf{G}^\infty$ relative to the restriction of $a(\cdot,\cdot)$ to $U^N_{\mathcal{S}} \times V$.

Definition 3.4 (Matrix $\mathbf{G}^\infty$)**.** Define the matrix $\mathbf{G}^\infty \in \mathbb{R}^{s\times s}$ such that

$$G^\infty_{jk} := \sum_{q=1}^{\infty} a(\psi_{\sigma_j}, \varphi_q)a(\psi_{\sigma_k}, \varphi_q), \quad \forall j,k \in [s],$$

where the series are well defined thanks to Hypothesis 2 and $|G^\infty_{jk}| \le \|\boldsymbol{\mu}^N\|_1$, for every $j,k \in [s]$.

The first lemma provides a relation between the inf-sup constant α associated with the bilinear form $a(\cdot,\cdot)$ and the Gram matrix $\mathbf{G}^\infty$.

Lemma 3.5. *Suppose that the bilinear form $a(\cdot,\cdot)$ fulfills the inf-sup property* (3)*. Then, it holds*

$$\lambda_{\min}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{G}^\infty\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}) \ge c_\varphi\alpha^2.$$

Proof. Recalling property (3), relation (14), and the Riesz basis property for the test functions, we obtain

$$\begin{aligned}\alpha &\leq \inf_{u\in U}\sup_{v\in V}\frac{a(u,v)}{\|u\|_U\|v\|_V} \leq \inf_{u\in U_{\mathcal{S}}^N}\sup_{v\in V}\frac{a(u,v)}{\|u\|_U\|v\|_V}\\ &\leq \frac{1}{\sqrt{c_\varphi}}\inf_{\mathbf{u}\in\mathbb{R}_{\mathcal{S}}^N}\sup_{\mathbf{v}\in\ell^2}\frac{1}{\|\mathbf{K}^{\frac{1}{2}}\mathbf{u}\|_2\|\mathbf{v}\|_2}\sum_{q=1}^{\infty}a(\Psi\mathbf{u},\varphi_q)v_q\\ &= \frac{1}{\sqrt{c_\varphi}}\inf_{\mathbf{u}\in\mathbb{R}_{\mathcal{S}}^N}\frac{1}{\|\mathbf{K}^{\frac{1}{2}}\mathbf{u}\|_2}\left[\sum_{q=1}^{\infty}a(\Psi\mathbf{u},\varphi_q)^2\right]^{\frac{1}{2}}.\end{aligned}$$

The last equality can be deduced by applying the definition of operator norm

$$\sup_{\mathbf{v}\in\ell^2}\frac{1}{\|\mathbf{v}\|_2}\sum_{q=1}^{\infty}a(\Psi\mathbf{u},\varphi_q)v_q = \|(a(\Psi\mathbf{u},\varphi_q))_{q\in\mathbb{N}}\|_{(\ell^2)^*}$$

and by identifying $(\ell^2)^*$ with ℓ^2. Now, since all the quantities involved in the chain of inequalities are positive, we can square the terms, obtaining

$$\begin{aligned}c_\varphi\alpha^2 &\leq \inf_{\mathbf{u}\in\mathbb{R}_{\mathcal{S}}^N}\frac{1}{\mathbf{u}^\intercal\mathbf{K}\mathbf{u}}\sum_{q=1}^{\infty}a(\Psi\mathbf{u},\varphi_q)^2 = \inf_{\mathbf{u}\in\mathbb{R}^s}\frac{1}{\mathbf{u}^\intercal\mathbf{K}_{\mathcal{S}}\mathbf{u}}\sum_{q=1}^{\infty}\left[\sum_{j=1}^{s}u_j a(\psi_{\sigma_j},\varphi_q)\right]^2\\ &= \inf_{\mathbf{u}\in\mathbb{R}^s}\frac{1}{\mathbf{u}^\intercal\mathbf{K}_{\mathcal{S}}\mathbf{u}}\sum_{q=1}^{\infty}\sum_{j=1}^{s}\sum_{k=1}^{s}u_j u_k a(\psi_{\sigma_j},\varphi_q)a(\psi_{\sigma_k},\varphi_q)\\ &= \inf_{\mathbf{u}\in\mathbb{R}^s}\frac{1}{\mathbf{u}^\intercal\mathbf{K}_{\mathcal{S}}\mathbf{u}}\sum_{j=1}^{s}\sum_{k=1}^{s}u_j u_k\sum_{q=1}^{\infty}a(\psi_{\sigma_j},\varphi_q)a(\psi_{\sigma_k},\varphi_q)\\ &= \inf_{\mathbf{u}\in\mathbb{R}^s}\frac{\mathbf{u}^\intercal\mathbf{G}^\infty\mathbf{u}}{\mathbf{u}^\intercal\mathbf{K}_{\mathcal{S}}\mathbf{u}} = \lambda_{\min}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{G}^\infty\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}),\end{aligned}$$

where we employed Hypothesis 2 and relation (12). □

The second lemma provides a recipe on how to choose the truncation level M on the tests, after selecting N and s.

Lemma 3.6. *Under the same hypotheses as in Lemma 3.5, we fix a real number $\widehat{\delta}\in(0,1)$. Then, if $M\in\mathbb{N}$ satisfies the truncation condition*

$$s\sum_{q>M}\mu_q^N \leq c_\varphi\alpha^2\lambda_{\min}(\mathbf{K}_{\mathcal{S}})\widehat{\delta}, \tag{15}$$

the following inequality holds

$$\lambda_{\min}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{G}^M\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}) \geq (1-\widehat{\delta})c_\varphi\alpha^2,$$

where $\mathbf{G}^M\in\mathbb{R}^{s\times s}$ is the truncated version of $\mathbf{G}^\infty$, namely

$$G_{jk}^M := \sum_{q=1}^{M}a(\psi_{\sigma_j},\varphi_q)a(\psi_{\sigma_k},\varphi_q).$$

Proof. First, consider the splitting $\mathbf{G}^\infty = \mathbf{G}^M + \mathbf{T}^M$, where $\mathbf{T}^M$ corresponds to the tail of the series identifying $\mathbf{G}^\infty$,

$$T_{jk}^M = \sum_{q>M}a(\psi_{\sigma_j},\varphi_q)a(\psi_{\sigma_k},\varphi_q).$$

Now, notice that

$$\begin{aligned}\lambda_{\min}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{G}^{M}\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}) &= \lambda_{\min}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}(\mathbf{G}^{\infty}-\mathbf{T}^{M})\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}) \\ &\geq \lambda_{\min}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{G}^{\infty}\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}) - \lambda_{\max}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{T}^{M}\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}})\end{aligned}$$

The inequality can be proved using Lemma 3.2. Applying Lemma 3.5, we obtain

$$\lambda_{\min}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{G}^{M}\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}) \geq c_{\varphi}\alpha^{2}(1-\lambda_{\max}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{T}^{M}\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}})/(c_{\varphi}\alpha^{2})).$$

Thus, the thesis is proved if we bound the maximum eigenvalue of the tail as follows

$$\lambda_{\max}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{T}^{M}\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}) \leq \widehat{\delta}c_{\varphi}\alpha^{2}.$$

For this purpose, employing relations (13) and (12), Hypothesis 2, and condition (15), we compute

$$\begin{aligned}\lambda_{\max}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{T}^{M}\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}) &= \sup_{\mathbf{u}\in\mathbb{R}^{s}}\frac{\mathbf{u}^{\intercal}\mathbf{T}^{M}\mathbf{u}}{\mathbf{u}^{\intercal}\mathbf{K}_{\mathcal{S}}\mathbf{u}} \\ &= \sup_{\mathbf{u}\in\mathbb{R}^{s}}\frac{1}{\mathbf{u}^{\intercal}\mathbf{K}_{\mathcal{S}}\mathbf{u}}\sum_{j=1}^{s}\sum_{k=1}^{s}u_{j}u_{k}\sum_{q>M}a(\psi_{\sigma_{j}},\varphi_{q})a(\psi_{\sigma_{k}},\varphi_{q}) \\ &= \sup_{\mathbf{u}\in\mathbb{R}^{s}}\frac{1}{\mathbf{u}^{\intercal}\mathbf{K}_{\mathcal{S}}\mathbf{u}}\sum_{q>M}\left[\sum_{j=1}^{s}u_{j}a(\psi_{\sigma_{j}},\varphi_{q})\right]^{2} \\ &\leq \sup_{\mathbf{u}\in\mathbb{R}^{s}}\frac{\mathbf{u}^{\intercal}\mathbf{u}}{\mathbf{u}^{\intercal}\mathbf{K}_{\mathcal{S}}\mathbf{u}}s\sum_{q>M}\mu_{q}^{N} = \frac{1}{\lambda_{\min}(\mathbf{K}_{\mathcal{S}})}s\sum_{q>M}\mu_{q}^{N}.\end{aligned}$$

□

Remark 3.7. Truncation condition (15) depends on the ordering of the test functions $\{\varphi_q\}_{q\in\mathbb{N}}$. This dependence can be removed by considering a set $\mathcal{T}$ of test indices defined as

$$\mathcal{T} := \underset{\widetilde{\mathcal{T}}\subseteq\mathbb{N},\,|\widetilde{\mathcal{T}}|=M}{\arg\max}\sum_{q\in\widetilde{\mathcal{T}}}\mu_{q}^{N}. \tag{16}$$

Then, condition (15) is replaced by

$$s\sum_{q\in\mathcal{T}^{c}}\mu_{q}^{N} \leq c_{\varphi}\alpha^{2}\lambda_{\min}(\mathbf{K}_{\mathcal{S}})\widehat{\delta}.$$

Optimization problem (16) could be very difficult (or even impossible) to solve in practice, but it can drive the choice of the ordering in the multi-dimensional case.

This lemma provides a sufficient condition on the truncation parameter M that ensures an arbitrarily small decrease of the inf-sup constant α by a factor $[c_{\varphi}(1-\widehat{\delta})]^{\frac{1}{2}}$. Moreover, a value M that fulfills (15) always exists thanks to Hypothesis 2. We notice that condition (15) becomes more and more restrictive as $\widehat{\delta}\to 0$. In particular, the choice $\widehat{\delta}=0$ is not allowed, since condition (15) would hold true only in the very special case when $\boldsymbol{\mu}^{N}$ has a finite number of nonzero elements. Moreover, condition (15) becomes more and more suboptimal as $\widehat{\delta}\to 1$, since the minimum eigenvalue of $\mathbf{K}_{\mathcal{S}}^{-1/2}\mathbf{G}^{M}\mathbf{K}_{\mathcal{S}}^{-1/2}$ is always nonnegative. The choice $\widehat{\delta}=1$ has been excluded because, in the following developments, the minimum eigenvalue is not allowed to vanish (see Theorems 3.8 and 3.10).

Relation (15) can be interpreted as a sufficient condition for the space V^M to be δ-proximal for $U_{\mathcal{S}}^N$, with constant $\delta = \widehat{\delta}^{\,1/2}$ (see [20]). Furthermore, condition (15) is related to the *balancing property* presented in [4], providing a criterion to control the distortion with respect to the ℓ^∞-norm associated with the truncation of an infinite-dimensional isometry of ℓ^2 to a finite-dimensional map from $\mathbb{R}^N$ to $\mathbb{R}^M$.

Now, we prove the main result of this section.

Theorem 3.8 (Non-uniform RISP)**.** *Let the truncation condition in Lemma 3.6 hold. Then, for every $0 < \varepsilon < 1$ and $\overline{\delta} \in (0,1)$, provided that*

$$m \geq \widetilde{C}_{\mathcal{S}}\, s \|\boldsymbol{\nu}^{N,M}\|_1 \log(s/\varepsilon),$$

where $\widetilde{C}_{\mathcal{S}} := [\rho_{\overline{\delta}}(1-\widehat{\delta})c_\varphi \alpha^2 \lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1}$ and $\rho_{\overline{\delta}}$ is defined according to (11), the following non-uniform RISP holds with probability greater than or equal to $1-\varepsilon$

$$\inf_{\mathbf{u}\in\mathbb{R}^s} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}\mathbf{A}_{\mathcal{S}}\mathbf{u}}{\|\mathbf{K}_{\mathcal{S}}^{\frac{1}{2}}\mathbf{u}\|_2 \|\mathbf{v}\|_2} > \widetilde{\alpha} > 0, \tag{17}$$

where $\widetilde{\alpha} := [(1-\widehat{\delta})(1-\overline{\delta})c_\varphi]^{\frac{1}{2}}\alpha$ and $\mathbf{D}$ is defined in (7).

Proof. The proof is organized as follows. First, we show that the inf-sup in (17) can be interpreted as the square root of the minimum eigenvalue of the sample mean of a sequence of certain i.i.d. random matrices $\mathbf{X}^{\tau_1}, \dots, \mathbf{X}^{\tau_m}$. Then, we compute the expectation of $\mathbf{X}^{\tau_i}$ and show that the maximum eigenvalue of $\mathbf{X}^{\tau_i}$ is uniformly bounded. Finally, we apply matrix Chernoff's bound (10).

Let us discuss each step of the proof in detail. First, we compute

$$\begin{aligned}\inf_{\mathbf{u}\in\mathbb{R}^s} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}\mathbf{A}_{\mathcal{S}}\mathbf{u}}{\|\mathbf{K}_{\mathcal{S}}^{\frac{1}{2}}\mathbf{u}\|_2 \|\mathbf{v}\|_2} &= \inf_{\mathbf{u}\in\mathbb{R}^s} \frac{1}{\|\mathbf{K}_{\mathcal{S}}^{\frac{1}{2}}\mathbf{u}\|_2} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}\mathbf{A}_{\mathcal{S}}\mathbf{u}}{\|\mathbf{v}\|_2} \\ &= \inf_{\mathbf{u}\in\mathbb{R}^s} \frac{\|\mathbf{D}\mathbf{A}_{\mathcal{S}}\mathbf{u}\|_2}{\|\mathbf{K}_{\mathcal{S}}^{\frac{1}{2}}\mathbf{u}\|_2} = [\lambda_{\min}(\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}\mathbf{A}_{\mathcal{S}}^\intercal \mathbf{D}^2 \mathbf{A}_{\mathcal{S}} \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}})]^{\frac{1}{2}}.\end{aligned}$$

The second equality hinges on the definition of operator norm combined with the identification of $(\mathbb{R}^m)^*$ with $\mathbb{R}^m$ while the third one exploits (12).

Relying on the following relation,

$$(\mathbf{A}_{\mathcal{S}}^\intercal \mathbf{D}^2 \mathbf{A}_{\mathcal{S}})_{jk} = \frac{1}{m}\sum_{i=1}^m \frac{1}{p_{\tau_i}} a(\psi_{\sigma_j}, \varphi_{\tau_i}) a(\psi_{\sigma_k}, \varphi_{\tau_i})$$

we define the matrices $\mathbf{H}^{\tau_i} \in \mathbb{R}^{s\times s}$ with $H_{jk}^{\tau_i} := \frac{1}{p_{\tau_i}} a(\psi_{\sigma_j}, \varphi_{\tau_i}) a(\psi_{\sigma_k}, \varphi_{\tau_i})$ and

$$\mathbf{X}^{\tau_i} := \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}} \mathbf{H}^{\tau_i} \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}},$$

so that

$$\overline{\mathbf{X}} := \frac{1}{m}\sum_{i=1}^m \mathbf{X}^{\tau_i} = \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}} \mathbf{A}_{\mathcal{S}}^\intercal \mathbf{D}^2 \mathbf{A}_{\mathcal{S}} \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}.$$

Thus, it holds

$$\inf_{\mathbf{u}\in\mathbb{R}^s} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}\mathbf{A}_{\mathcal{S}}\mathbf{u}}{\|\mathbf{K}_{\mathcal{S}}^{\frac{1}{2}}\mathbf{u}\|_2 \|\mathbf{v}\|_2} = [\lambda_{\min}(\overline{\mathbf{X}})]^{\frac{1}{2}}. \tag{18}$$

With a view to Chernoff's bounds, we estimate $\mathbb{E}[\mathbf{X}^{\tau_i}]$ and the corresponding minimum eigenvalue. A direct computation yields

$$\mathbb{E}[H_{jk}^{\tau_i}] = \sum_{q=1}^{M} \mathbb{P}\{\tau_i = q\} H_{jk}^{q} = \sum_{q=1}^{M} p_q \frac{1}{p_q} a(\psi_{\sigma_j}, \varphi_q) a(\psi_{\sigma_k}, \varphi_q) = G_{jk}^{M}.$$

As a consequence, we have

$$\mathbb{E}[\mathbf{X}^{\tau_i}] = \mathbb{E}[\mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}} \mathbf{H}^{\tau_i} \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}}] = \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}} \mathbb{E}[\mathbf{H}^{\tau_i}] \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}} = \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}} \mathbf{G}^{M} \mathbf{K}_{\mathcal{S}}^{-\frac{1}{2}},$$

and, from Lemma 3.6

$$\lambda_{\min}(\mathbb{E}[\mathbf{X}^{\tau_i}]) \geq (1 - \widehat{\delta}) c_\varphi \alpha^2. \tag{19}$$

Our aim is now to bound $\lambda_{\max}(\mathbf{X}^{\tau_i})$ from above. We have

$$\begin{aligned}
\lambda_{\max}(\mathbf{X}^{\tau_i}) &= \sup_{\mathbf{u}\in\mathbb{R}^s} \frac{\mathbf{u}^\intercal \mathbf{H}^{\tau_i} \mathbf{u}}{\mathbf{u}^\intercal \mathbf{K}_{\mathcal{S}} \mathbf{u}} \leq \sup_{\mathbf{u}\in\mathbb{R}^s} \frac{\mathbf{u}^\intercal \mathbf{u}}{\mathbf{u}^\intercal \mathbf{K}_{\mathcal{S}} \mathbf{u}} \sup_{\mathbf{u}\in\mathbb{R}^s} \frac{\mathbf{u}^\intercal \mathbf{H}^{\tau_i} \mathbf{u}}{\mathbf{u}^\intercal \mathbf{u}} \\
&= [\lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1} \sup_{\mathbf{u}\in\mathbb{R}^s} \frac{1}{\mathbf{u}^\intercal \mathbf{u}} \sum_{j=1}^{s} \sum_{k=1}^{s} u_j u_k \frac{1}{p_{\tau_i}} a(\psi_{\sigma_j}, \varphi_{\tau_i}) a(\psi_{\sigma_k}, \varphi_{\tau_i}) \\
&= [\lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1} \frac{1}{p_{\tau_i}} \sup_{\mathbf{u}\in\mathbb{R}^s} \frac{1}{\mathbf{u}^\intercal \mathbf{u}} \left[\sum_{j=1}^{s} u_j a(\psi_{\sigma_j}, \varphi_{\tau_i}) \right]^2 \\
&\leq [\lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1} \frac{\|\boldsymbol{\nu}^{N,M}\|_1}{\nu_{\tau_i}^{N}} \sum_{j=1}^{s} a(\psi_{\sigma_j}, \varphi_{\tau_i})^2 \leq [\lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1} \, s \, \|\boldsymbol{\nu}^{N,M}\|_1.
\end{aligned} \tag{20}$$

The first line follows from (13). The last line exploits Cauchy-Schwarz inequality combined with definition (5) of $\mathbf{p}$, and Hypothesis 3.

Now, we compute the probability of failure of satisfying (17), i.e.,

$$\begin{aligned}
\mathbb{P}\left\{ \inf_{\mathbf{u}\in\mathbb{R}^s} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D} \mathbf{A}_{\mathcal{S}} \mathbf{u}}{\|\mathbf{K}_{\mathcal{S}}^{\frac{1}{2}} \mathbf{u}\|_2 \|\mathbf{v}\|_2} \leq \widetilde{\alpha} \right\} &= \mathbb{P}\left\{ \lambda_{\min}(\overline{\mathbf{X}}) \leq (1 - \overline{\delta})(1 - \widehat{\delta}) c_\varphi \alpha^2 \right\} \\
&\leq \mathbb{P}\{\lambda_{\min}(\overline{\mathbf{X}}) \leq (1 - \overline{\delta}) \lambda_{\min}(\mathbb{E}[\mathbf{X}^{\tau_i}])\} \\
&\leq s \exp\left(- \frac{m \rho_{\overline{\delta}} \lambda_{\min}(\mathbb{E}[\mathbf{X}^{\tau_i}])}{s \|\boldsymbol{\nu}^{N,M}\|_1 [\lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1}} \right) \\
&\leq s \exp\left(- \frac{m \rho_{\overline{\delta}} (1 - \widehat{\delta}) c_\varphi \alpha^2}{s \|\boldsymbol{\nu}^{N,M}\|_1 [\lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1}} \right).
\end{aligned} \tag{21}$$

The first equality relies on (18) and on the definition of $\widetilde{\alpha}$. The first inequality in the second line hinges on (19), while the second inequality is the first matrix Chernoff bound (10), where the uniform estimate (20) has been employed. The final inequality follows from (19).

The thesis is finally proved on estimating that

$$s \exp\left(- \frac{m \rho_{\overline{\delta}} (1 - \widehat{\delta}) c_\varphi \alpha^2}{s \|\boldsymbol{\nu}^{N,M}\|_1 [\lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1}} \right) \leq \varepsilon \iff m \geq \widetilde{C}_{\mathcal{S}} \, s \|\boldsymbol{\nu}^{N,M}\|_1 \log(s/\varepsilon),$$

with $\widetilde{C}_{\mathcal{S}} := [\rho_{\overline{\delta}} (1 - \widehat{\delta}) c_\varphi \alpha^2 \lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1}$.

□

The parameter $\overline{\delta}$ controls the reduction of inf-sup constant due to the randomized selection of the reduced test space (spanned by the functions $\varphi_{\tau_1}, \dots, \varphi_{\tau_m}$). In particular, Theorem 3.8 provides a sufficient condition on m, such that the RISP constant looses at most a portion $(1-\overline{\delta})^{1/2}$ of the minimum eigenvalue of the matrix $\mathbf{K}_{\mathcal{S}}^{-1/2}\mathbf{G}^M\mathbf{K}_{\mathcal{S}}^{-1/2}$ (recall Lemma 3.6). When $\overline{\delta} \to 0$, the lower bound on m degenerates, since the constant $\widetilde{C}_{\mathcal{S}}$ grows with order $\overline{\delta}^{-2}$. When $\overline{\delta} \to 1$, the RISP constant $\widetilde{\alpha}$ tends to vanish. In the limit case $\overline{\delta} = 1$, Theorem 3.8 gives a sufficient condition for the RISP constant to be nonzero with high probability.

Remark 3.9. Due to the independence of $\tau_1, \dots, \tau_m$, repetitions of the outcomes can occur during the drawing selection. This does not seem to be very efficient, since the matrix $\mathbf{A}$ has some repeated rows, which introduce redundant information. We show how to get rid of the repeated indices, further compressing the CORSING discretization. Among the indices $\tau_1, \dots, \tau_m$, consider only the nonrepeated ones $\widetilde{\tau}_1, \dots, \widetilde{\tau}_{\widetilde{m}}$. Of course, $\widetilde{m} \le m$. Then, we define a modified version of the stiffness matrix $\widetilde{\mathbf{A}}$ and of the load vector $\widetilde{\mathbf{f}}$ as

$$\widetilde{A}_{ij} := a(\varphi_{\widetilde{\tau}_i}, \psi_j), \quad \widetilde{f}_i := \mathcal{F}(\varphi_{\widetilde{\tau}_i}), \quad \forall i \in [\widetilde{m}],\ \forall j \in [N].$$

Moreover, we slightly modify the preconditioner $\widetilde{\mathbf{D}}$ as

$$\widetilde{D}_{ik} := \delta_{ik}\sqrt{\frac{r_i}{m p_{\widetilde{\tau}_i}}}, \quad \forall i,k \in [\widetilde{m}],$$

where $r_i := |\{k \in [m] : \tau_k = \widetilde{\tau}_i\}|$ is the number of outcomes of $\widetilde{\tau}_i$. It is not difficult to verify that $\mathbf{A}^\intercal\mathbf{D}^2\mathbf{A} = \widetilde{\mathbf{A}}^\intercal\widetilde{\mathbf{D}}^2\widetilde{\mathbf{A}}$, and thus the proof of Theorem 3.8 can be replicated to prove an analogous RISP result for $\widetilde{\mathbf{D}}\widetilde{\mathbf{A}}$. This leads to a further computational reduction considering the system $\widetilde{\mathbf{D}}\widetilde{\mathbf{A}} = \widetilde{\mathbf{D}}\widetilde{\mathbf{f}}$, of dimension $\widetilde{m} \times N$.

3.3. **Uniform restricted inf-sup property.** We extend the results in the previous Section to the uniform case, i.e., we aim at proving the RISP over U_s^N, instead of $U_{\mathcal{S}}^N$, for a fixed subset $\mathcal{S} \subseteq [N]$ with $|\mathcal{S}| = s$. For this purpose, we use the non-uniform Theorem 3.8 and a union bound.

First, we introduce the set Σ_s^N of s-sparse vectors of $\mathbb{R}^N$, namely

$$\Sigma_s^N := \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\|_0 \le s\} \equiv \bigcup_{\mathcal{S}\subseteq[N];\,|\mathcal{S}|=s} \mathbb{R}_{\mathcal{S}}^N.$$

The following theorem provides a sufficient condition for the uniform RISP to hold.

Theorem 3.10 (Uniform RISP)**.** *Given $\widehat{\delta} \in (0,1)$, choose $M \in \mathbb{N}$ such that the following truncation condition is fulfilled*

$$s\sum_{q>M} \mu_q^N \le c_\varphi \alpha^2 \kappa_s \widehat{\delta},$$

where

$$\kappa_s := \min_{\mathcal{S}\subseteq[N];\,|\mathcal{S}|=s} \lambda_{\min}(\mathbf{K}_{\mathcal{S}}). \tag{22}$$

Then, for every $0 < \varepsilon < 1$ and $\overline{\delta} \in (0,1)$, provided

$$m \ge \widetilde{C}_s\, s\|\boldsymbol{\nu}^{N,M}\|_1[s\log(eN/s) + \log(s/\varepsilon)], \tag{23}$$

with

$$\widetilde{C}_s := [\rho_{\overline{\delta}}(1-\widehat{\delta})c_\varphi \alpha^2 \kappa_s]^{-1} \tag{24}$$

and $\rho_{\overline{\delta}}$ *as in* (11)*, the following uniform* s*-sparse RISP holds with probability greater than or equal to* $1-\varepsilon$

$$\inf_{\mathbf{u}\in\Sigma_s^N} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}\mathbf{A}\mathbf{u}}{\|\mathbf{K}^{\frac{1}{2}}\mathbf{u}\|_2 \|\mathbf{v}\|_2} > \widetilde{\alpha} > 0,$$

where $\widetilde{\alpha} := [(1-\widehat{\delta})(1-\overline{\delta})c_\varphi]^{\frac{1}{2}}\alpha$.

Proof. First, we define the event where the RISP holds non-uniformly over a single subset $\mathcal{S} \subseteq [N]$ with $|\mathcal{S}| = s$:

$$\Omega_{\mathcal{S}} := \left\{ \omega \in \Omega : \inf_{\mathbf{u}\in\mathbb{R}^s} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}(\omega)\mathbf{A}_{\mathcal{S}}(\omega)\mathbf{u}}{\|\mathbf{K}_{\mathcal{S}}^{\frac{1}{2}}\mathbf{u}\|_2 \|\mathbf{v}\|_2} > \widetilde{\alpha} \right\},$$

where the dependence of $\mathbf{A}_{\mathcal{S}}$ and $\mathbf{D}$ on ω has been highlighted. Analogously, we define the event where the RISP holds uniformly

$$\Omega_s := \left\{ \omega \in \Omega : \inf_{\mathbf{u}\in\Sigma_s^N} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}(\omega)\mathbf{A}(\omega)\mathbf{u}}{\|\mathbf{K}^{\frac{1}{2}}\mathbf{u}\|_2 \|\mathbf{v}\|_2} > \widetilde{\alpha} \right\}. \tag{25}$$

In particular, the following relation holds

$$\Omega_s = \bigcap_{\mathcal{S}\subseteq[N];\, |\mathcal{S}|=s} \Omega_{\mathcal{S}},$$

and, thanks to the subadditivity of $\mathbb{P}$ and De Morgan's laws, we have

$$\mathbb{P}(\Omega_s^c) = \mathbb{P}\left(\left(\bigcap \Omega_{\mathcal{S}}\right)^c\right) = \mathbb{P}\left(\bigcup \Omega_{\mathcal{S}}^c\right) \leq \sum_{\mathcal{S}\subseteq[N];\, |\mathcal{S}|=s} \mathbb{P}(\Omega_{\mathcal{S}}^c). \tag{26}$$

Now, the non-uniform inequality (21) and the definition (22) of κ_s, yield the following uniform upper bound

$$\mathbb{P}(\Omega_{\mathcal{S}}^c) \leq s \exp\left(- \frac{m\rho_{\overline{\delta}}(1-\widehat{\delta})c_\varphi \alpha^2}{s\|\boldsymbol{\nu}^{N,M}\|_1 [\lambda_{\min}(\mathbf{K}_{\mathcal{S}})]^{-1}} \right) \leq s \exp\left(- \frac{m\rho_{\overline{\delta}}(1-\widehat{\delta})c_\varphi \alpha^2}{s\|\boldsymbol{\nu}^{N,M}\|_1 \kappa_s^{-1}} \right). \tag{27}$$

Moreover, Stirling's formula furnishes the following upper bound

$$|\{\mathcal{S} \subseteq [N] : |\mathcal{S}| = s\}| = \binom{N}{s} = \frac{N!}{s!(N-s)!} \leq \frac{N^s}{s!} \leq \left(\frac{eN}{s}\right)^s. \tag{28}$$

Combining (26), (27) and (28), we finally obtain the uniform estimate

$$\mathbb{P}(\Omega_s^c) \leq \left(\frac{eN}{s}\right)^s s \exp\left(- \frac{m\rho_{\overline{\delta}}(1-\widehat{\delta})c_\varphi \alpha^2}{s\|\boldsymbol{\nu}^{N,M}\|_1 \kappa_s^{-1}} \right). \tag{29}$$

Simple algebraic manipulations show that the right hand-side of (29) is less than or equal to ε if and only if relation (23) holds. □

We note that the sufficient condition (23) is, in general, too pessimistic. Indeed, in the classical literature on CS, e.g., [23, 12], the optimal asymptotically dependence of m on s is linear. Likely, this lack of optimality is due to the union bound, that is a very rough estimate. We expect that it is possible to achieve the optimal behavior by using more advanced techniques, such as those described in

[27, Chapter 12] and [38] in the case of Bounded Orthonormal Systems. This will be investigated in the future.

Remark 3.11. The quantity κ_s can be bounded from below by the lower Riesz constant c_ψ. Indeed, for every $\mathcal{S} \subseteq [N]$, it holds

$$\lambda_{\min}(\mathbf{K}_{\mathcal{S}}) = \min_{\mathbf{u} \in \mathbb{R}^N_{\mathcal{S}}} \frac{\mathbf{u}^\intercal \mathbf{K} \mathbf{u}}{\mathbf{u}^\intercal \mathbf{u}} \geq \min_{\mathbf{u} \in \mathbb{R}^N} \frac{\mathbf{u}^\intercal \mathbf{K} \mathbf{u}}{\mathbf{u}^\intercal \mathbf{u}} = \min_{\mathbf{u} \in \mathbb{R}^N} \frac{\|\Psi \mathbf{u}\|_U^2}{\|\mathbf{u}\|_2^2} \geq c_\psi.$$

Moreover, recalling (24), we have

$$\widetilde{C}_s \leq [\rho_{\widehat{\delta}}(1-\widehat{\delta})\alpha^2 c_\varphi c_\psi]^{-1}.$$

Therefore, $\widetilde{C}_s$ is uniformly bounded from above with respect to s. We also notice that $\kappa_s = 1$ when $\{\psi_j\}_{j \in \mathbb{N}}$ is an orthonormal basis.

3.4. **Recovery error analysis.** In this section, we deal with the analysis of the recovery error associated with the CORSING procedure, computed with respect to the trial norm $\|\cdot\|_U$, i.e., the quantity $\|\widehat{u} - u\|_U$. Notice that this error is a random variable, depending on the extracted indices $\tau_1, \dots, \tau_m$. Our aim is to compare the recovery error with the best s-term approximation error of the exact solution u in U^N, i.e., the quantity $\|u^s - u\|_U$, where

$$u^s := \operatorname*{arg\,min}_{w \in U^N_s} \|w - u\|_U. \tag{30}$$

Due to the s-sparsity constraint in the recovery procedure (8), u^s is the best result that CORSING can ideally provide.[1]

For this purpose, we show that the uniform $2s$-sparse RISP implies a recovery result, depending on a random preconditioned residual (Lemma 3.12), whose second moment is controlled by the square of the best s-term approximation error (Lemma 3.13). Afterwards, in Theorem 3.15, we prove that the best s-term approximation error dominates the first moment of the error associated with a truncated version of the CORSING solution and, finally, we provide two recovery error estimates that holds with high probability in Theorems 3.16 and 3.18.

In the analysis carried out here, the CORSING solution $\widehat{u}$ is supposed to solve the minimization problem (9) exactly. In particular, we are not taking into account the additional approximation error introduced by the OMP algorithm (or by other solvers that could be employed to approximate (9), such as ℓ^1-minimization).

In the following, a key quantity is the preconditioned random residual

$$\mathcal{R}(v) := \left[\frac{1}{m} \sum_{i=1}^m \frac{1}{p_{\tau_i}} [a(v, \varphi_{\tau_i}) - \mathcal{F}(\varphi_{\tau_i})]^2 \right]^{\frac{1}{2}}, \quad \forall v \in U. \tag{31}$$

Now, we prove the two lemmas.

Lemma 3.12. *If the uniform $2s$-sparse RISP*

$$\inf_{\mathbf{u} \in \Sigma^N_{2s}} \sup_{\mathbf{v} \in \mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D} \mathbf{A} \mathbf{u}}{\|\mathbf{K}^{\frac{1}{2}} \mathbf{u}\|_2 \|\mathbf{v}\|_2} > \widetilde{\alpha} > 0, \tag{32}$$

holds, then the CORSING procedure computes a solution $\widehat{u}$ such that

$$\|\widehat{u} - u^s\|_U < \frac{2}{\widetilde{\alpha}} \mathcal{R}(u^s).$$

[1]The quantity in (30) is actually a minimum and not an infimum, since the function $w \mapsto \|w - u\|_U$ is convex and U^N_s is a finite union of linear subspaces.

Proof. Define $\widehat{\mathbf{u}} := \Psi^* \widehat{u}$ and $\mathbf{u}^s := \Psi^* u^s$. Then, casting (25) in Ω_{2s}, since $\widehat{\mathbf{u}} - \mathbf{u}^s$ is at most $2s$-sparse, and thanks to the RISP property (32), we have

$$\|\widehat{u} - u^s\|_U = \|\mathbf{K}^{\frac{1}{2}}(\widehat{\mathbf{u}} - \mathbf{u}^s)\|_2 < \frac{1}{\widetilde{\alpha}} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}\mathbf{A}(\widehat{\mathbf{u}} - \mathbf{u}^s)}{\|\mathbf{v}\|_2} = \frac{1}{\widetilde{\alpha}} \|\mathbf{D}\mathbf{A}(\widehat{\mathbf{u}} - \mathbf{u}^s)\|_2.$$

Moreover, the last norm can be bounded as

$$\begin{aligned}
\|\mathbf{D}\mathbf{A}(\widehat{\mathbf{u}} - \mathbf{u}^s)\|_2^2 &= \frac{1}{m}\sum_{i=1}^m \frac{1}{p_{\tau_i}} a(\widehat{u} - u^s, \varphi_{\tau_i})^2 \\
&= \frac{1}{m}\sum_{i=1}^m \frac{1}{p_{\tau_i}} [a(\widehat{u}, \varphi_{\tau_i}) - \mathcal{F}(\varphi_{\tau_i}) - a(u^s, \varphi_{\tau_i}) + \mathcal{F}(\varphi_{\tau_i})]^2 \\
&\leq \frac{2}{m}\sum_{i=1}^m \frac{1}{p_{\tau_i}} \{[a(\widehat{u}, \varphi_{\tau_i}) - \mathcal{F}(\varphi_{\tau_i})]^2 + [a(u^s, \varphi_{\tau_i}) - \mathcal{F}(\varphi_{\tau_i})]^2\} \\
&\leq \frac{4}{m}\sum_{i=1}^m \frac{1}{p_{\tau_i}} [a(u^s, \varphi_{\tau_i}) - \mathcal{F}(\varphi_{\tau_i})]^2 = 4\mathcal{R}(u^s)^2,
\end{aligned}$$

where the last inequality exploits the optimality of $\widehat{u}$. □

Lemma 3.13. *The following upper bound holds*

$$\mathbb{E}[\mathcal{R}(u^s)^2] \leq C_\varphi \beta^2 \|u^s - u\|_U^2, \tag{33}$$

where β is the continuity constant of $a(\cdot,\cdot)$ defined in (4).

Proof. Thanks to (2), the residual (31) becomes

$$\mathcal{R}(u^s)^2 = \frac{1}{m}\sum_{i=1}^m p_{\tau_i}^{-1} a(u^s - u, \varphi_{\tau_i})^2,$$

Thus, in expectation, we obtain

$$\mathbb{E}[\mathcal{R}(u^s)^2] = \frac{1}{m}\sum_{i=1}^m \mathbb{E}[p_{\tau_i}^{-1} a(u^s - u, \varphi_{\tau_i})^2]. \tag{34}$$

Each term in the last summation can be bounded as

$$\mathbb{E}[p_{\tau_i}^{-1} a(u^s - u, \varphi_{\tau_i})^2] = \sum_{q=1}^M p_q^{-1} a(u^s - u, \varphi_q)^2 p_q \leq \sum_{q=1}^\infty a(u^s - u, \varphi_q)^2. \tag{35}$$

Now, notice that

$$\begin{aligned}
\|a(u^s - u, \cdot)\|_{V^*} = \sup_{v\in V} \frac{|a(u^s - u, v)|}{\|v\|_V} &\geq \frac{1}{\sqrt{C_\varphi}} \sup_{\mathbf{v}\in\ell^2} \frac{|\sum_{q=1}^\infty v_q a(u^s - u, \varphi_q)|}{\|\mathbf{v}\|_2} \\
&= \left[\frac{1}{C_\varphi}\sum_{q=1}^\infty a(u^s - u, \varphi_q)^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Plugging this equality and (35) in (34), and thanks to (4), we have

$$\mathbb{E}[\mathcal{R}(u^s)^2] \leq \|a(u^s - u, \cdot)\|_{V^*}^2 \leq C_\varphi \beta^2 \|u^s - u\|_U^2.$$

□

If an upper bound of the form $\|u\|_U \leq \mathcal{K}$ is known, a near-optimal recovery result holds in expectation for a truncation of the CORSING solution. This truncation is obtained through the operator $\mathcal{T}_\mathcal{K} : U \to U$ defined as

$$\mathcal{T}_\mathcal{K} w := \begin{cases} w & \text{if } \|w\|_U \leq \mathcal{K}, \\ \mathcal{K} w / \|w\|_U & \text{if } \|w\|_U > \mathcal{K}, \end{cases} \qquad \forall w \in U. \tag{36}$$

Using (2) and (3), a possible choice of $\mathcal{K}$ is $\|\mathcal{F}\|_{V^*}/\alpha$.

Then, we have the following lemma whose proof is straightforward.

Lemma 3.14. *$\mathcal{T}_\mathcal{K}$ is 1-Lipschitz, with respect to $\|\cdot\|_U$, for every $\mathcal{K} > 0$.*

Employing an argument similar to that used in [18, 15], we show an upper bound to the error associated with the truncated CORSING solution.

Theorem 3.15 (Error estimate in expectation)**.** *Let $\mathcal{K} > 0$ be such that $\|u\|_U \leq \mathcal{K}$. Given $\widehat{\delta} \in (0,1)$, choose $M \in \mathbb{N}$ such that the truncation condition*

$$2s \sum_{q>M} \mu_q^N \leq c_\varphi \alpha^2 \kappa_{2s} \widehat{\delta}, \tag{37}$$

is fulfilled, and fix $\overline{\delta} \in (0,1)$.

Then, for every $0 < \varepsilon < 1$, provided

$$m \geq 2\,\widetilde{C}_{2s}\; s \|\boldsymbol{\nu}^{N,M}\|_1 [2s \log(eN/(2s)) + \log(2s/\varepsilon)], \tag{38}$$

with $\widetilde{C}_{2s}$ defined analogously to (24) and $\widetilde{\alpha} = [(1-\widehat{\delta})(1-\overline{\delta})c_\varphi]^{\frac{1}{2}}\alpha$, the truncated CORSING solution $\mathcal{T}_\mathcal{K}\widehat{u}$ fulfills

$$\mathbb{E}[\|\mathcal{T}_\mathcal{K}\widehat{u} - u\|_U] < \left(1 + \frac{2\beta C_\varphi^{1/2}}{\widetilde{\alpha}}\right) \|u^s - u\|_U + 2\mathcal{K}\varepsilon,$$

where β is the continuity constant of $a(\cdot,\cdot)$ defined in (4).

Proof. First, recalling the definition (25) of the event Ω_s, and considering the partitioning $\Omega = \Omega_{2s} \cup \Omega_{2s}^c$, we have the splitting

$$\mathbb{E}[\|\mathcal{T}_\mathcal{K}\widehat{u} - u\|_U] = \int_{\Omega_{2s}} \|\mathcal{T}_\mathcal{K}(\widehat{u} - u)\|_U \,\mathrm{d}\mathbb{P} + \int_{\Omega_{2s}^c} \|\mathcal{T}_\mathcal{K}\widehat{u} - u\|_U \,\mathrm{d}\mathbb{P}.$$

Then, the second term is easily bounded as

$$\int_{\Omega_{2s}^c} \|\mathcal{T}_\mathcal{K}\widehat{u} - u\|_U \,\mathrm{d}\mathbb{P} \leq 2\mathcal{K}\varepsilon.$$

Indeed, thanks to the adopted choice of m, Theorem 3.10 guarantees $\mathbb{P}(\Omega_{2s}^c) \leq \varepsilon$. Moreover, $\|\mathcal{T}_\mathcal{K}\widehat{u} - u\|_U \leq 2\mathcal{K}$, since both $\|\mathcal{T}_\mathcal{K}\widehat{u}\|_U$ and $\|u\|_U$ are less than or equal to $\mathcal{K}$.

Now, employing Lemma 3.14 and the triangle inequality, we have

$$\int_{\Omega_{2s}} \|\mathcal{T}_\mathcal{K}(\widehat{u} - u)\|_U \,\mathrm{d}\mathbb{P} \leq \int_{\Omega_{2s}} \|\widehat{u} - u\|_U \,\mathrm{d}\mathbb{P} \leq \int_{\Omega_{2s}} \|\widehat{u} - u^s\|_U \,\mathrm{d}\mathbb{P} + \int_{\Omega_{2s}} \|u^s - u\|_U \,\mathrm{d}\mathbb{P}.$$

The second integral on the right hand side is less than or equal to the best s-term approximation error $\|u^s - u\|_U$. In order to bound the first integral, we apply Lemmas 3.12 and 3.13, obtaining

$$\int_{\Omega_{2s}} \|\widehat{u} - u^s\|_U \,\mathrm{d}\mathbb{P} < \frac{2}{\widetilde{\alpha}} \int_{\Omega_{2s}} \mathcal{R}(u^s) \,\mathrm{d}\mathbb{P} \leq \frac{2}{\widetilde{\alpha}} \mathbb{E}[\mathcal{R}(u^s)] \leq \frac{2\beta C_\varphi^{1/2}}{\widetilde{\alpha}} \|u^s - u\|_U,$$

where the last relation follows on applying Jensen's inequality to (33). Notice that Lemma 3.12 can be employed since the $2s$-sparse RISP holds on the restricted domain Ω_{2s}. Combining all the inequalities yields the thesis. □

We provide a first recovery estimate in probability. This is asymptotically optimal, but the constant grows like the inverse of the square root of the probability of failure, which is rather pessimistic. This dependence will be removed in Theorem 3.18, at the price of a higher regularity of the solution u and of the trial functions.

Theorem 3.16 (Error estimate in probability)**.** *Given $\widehat{\delta} \in (0,1)$, choose $M \in \mathbb{N}$ such that the truncation condition* (37) *is fulfilled. Then, for every $0 < \varepsilon \leq 1/2$ and $\overline{\delta} \in (0,1)$, provided*

$$m \geq 2\widetilde{C}_{2s}\, s\|\boldsymbol{\nu}^{N,M}\|_1[2s\log(eN/(2s)) + \log(2s/\varepsilon)],$$

with $\widetilde{C}_{2s}$ defined analogously to (24)*, with probability greater than or equal to $1-2\varepsilon$, the CORSING procedure computes a solution $\widehat{u}$ such that*

$$\|\widehat{u} - u\|_U < \left(1 + \frac{2\beta C_\varphi^{1/2}}{\widetilde{\alpha}\sqrt{\varepsilon}}\right)\|u^s - u\|_U$$

where $\widetilde{\alpha} := [(1-\widehat{\delta})(1-\overline{\delta})c_\varphi]^{\frac{1}{2}}\alpha$ and β is the continuity constant of $a(\cdot,\cdot)$ defined in (4)*.*

Proof. Define $e_s := \|u^s - u\|_U$ and the random variables $Z := \|\widehat{u} - u\|_U$ and $Z_s := \|\widehat{u} - u^s\|_U$. Moreover, consider the quantity

$$b_s := \left(1 + \frac{2\beta C_\varphi^{1/2}}{\widetilde{\alpha}\sqrt{\varepsilon}}\right)e_s. \tag{39}$$

The goal is to show that $\mathbb{P}\{Z \geq b_s\} \leq 2\varepsilon$. The triangle inequality implies $Z \leq Z_s + e_s$. Thus,

$$\mathbb{P}\{Z \geq b_s\} \leq \mathbb{P}\{Z_s \geq b_s - e_s\}.$$

Moreover, defining the event Ω_{2s} according to (25) and denoting by I_A the indicator function of a generic set A, we have

$$\begin{aligned}\mathbb{P}\{Z_s \geq b_s - e_s\} = \mathbb{E}[I_{\{Z_s \geq b_s - e_s\}}] &= \int_{\Omega_{2s}} I_{\{Z_s \geq b_s - e_s\}}\,\mathrm{d}\mathbb{P} + \int_{\Omega_{2s}^c} I_{\{Z_s \geq b_s - e_s\}}\,\mathrm{d}\mathbb{P} \\ &\leq \int_{\Omega_{2s}} I_{\{Z_s \geq b_s - e_s\}}\,\mathrm{d}\mathbb{P} + \mathbb{P}\{\Omega_{2s}^c\}.\end{aligned}$$

Theorem 3.10 implies $\mathbb{P}\{\Omega_{2s}^c\} \leq \varepsilon$. Moreover, employing Lemmas 3.12 and 3.13, we can bound the first integral as

$$\begin{aligned}\int_{\Omega_{2s}} I_{\{Z_s \geq b_s - e_s\}}\,\mathrm{d}\mathbb{P} &\leq \int_{\Omega_{2s}} I_{\{(2/\widetilde{\alpha})\mathcal{R}(u^s) > b_s - e_s\}}\,\mathrm{d}\mathbb{P} \\ &< \mathbb{E}\left[\frac{4\mathcal{R}(u^s)^2}{\widetilde{\alpha}^2(b_s - e_s)^2}\right] \leq \frac{4C_\varphi\beta^2 e_s^2}{\widetilde{\alpha}^2(b_s - e_s)^2} = \varepsilon,\end{aligned}$$

where the last equality follows from (39).

□

It is possible to remove the dependence on ε in the error estimate of Theorem 3.16 by employing Hoeffding's bound for the sum of random variables and assuming a higher regularity of the solution u and of the trial functions. The idea is to employ Hoeffding's bound to improve the tail estimate for the centered random variable $\mathcal{R}(u^s)^2 - \mathbb{E}[\mathcal{R}(u^s)^2]$. With this aim, we introduce a "stronger" norm on the trial space and a "weaker" norm on the test space.

We recall Hoeffding's bound for the sum of random variables (see [27, Theorem 7.20]).

Theorem 3.17 (Hoeffding's bound)**.** *Let $X_1, \dots, X_m$ be independent random variables, with $\mathbb{E}[X_i] = 0$ and $|X_i| \leq B_i$, almost surely for every $i \in [m]$. Then, for every $t > 0$, it holds*

$$\mathbb{P}\left\{ \sum_{i=1}^{m} X_i \geq t \right\} \leq \exp\left(- \frac{t^2}{2\sum_{i=1}^{m} B_i^2} \right).$$

□

Now, consider two spaces $\widetilde{U} \subseteq U$ and $\widetilde{V} \supseteq V$, endowed with norm $\|\cdot\|_{\widetilde{U}}$ and $\|\cdot\|_{\widetilde{V}}$, respectively, such that $a(\cdot,\cdot)$ is continuous, namely,

$$\exists \widetilde{\beta} > 0 : \quad |a(u,v)| \leq \widetilde{\beta} \|u\|_{\widetilde{U}} \|v\|_{\widetilde{V}}, \quad \forall u \in \widetilde{U},\ \forall v \in \widetilde{V}. \tag{40}$$

The following recovery estimate in probability holds.

Theorem 3.18 (Error estimate in probability)**.** *Let $\widetilde{U} \subseteq U$, and $\widetilde{V} \supseteq V$ be such that (40) is fulfilled, and assume that $u \in \widetilde{U}$, $\{\psi_j\}_{j\in\mathbb{N}} \subseteq \widetilde{U}$, and that there exists a constant $A > 0$ such that*

$$\|\varphi_q\|_{\widetilde{V}}^2 \leq A\, \nu_q^N, \quad \forall q \in \mathbb{N}. \tag{41}$$

Given $\widehat{\delta} \in (0,1)$, choose $M \in \mathbb{N}$ such that the truncation condition (37) is fulfilled. Then, for every $0 < \varepsilon \leq 1/2$ and $\overline{\delta} \in (0,1)$, provided

$$m \geq 2\widetilde{C}_{2s}\, s \max\{\|\boldsymbol{\nu}^{N,M}\|_1, 1\}\, [2s \log(eN/(2s)) + \log(2s/\varepsilon)], \tag{42}$$

with $\widetilde{C}_{2s}$ defined analogously to (24), with probability greater than or equal to $1-2\varepsilon$, the CORSING procedure computes a solution $\widehat{u} \in \widetilde{U}$ such that

$$\|\widehat{u} - u\|_U \leq \frac{4\beta C_\varphi^{1/2}}{\widetilde{\alpha}} \|u - u^s\|_U + \frac{2\widetilde{\beta} A^{1/2} \|\boldsymbol{\nu}^{N,M}\|_1^{1/4}}{\widetilde{\alpha}} \|u - u^s\|_{\widetilde{U}},$$

where $\widetilde{\alpha} := [(1-\widehat{\delta})(1-\overline{\delta})c_\varphi]^{\frac{1}{2}} \alpha$, and u^s is defined as in (30).

Proof. First, since

$$\mathcal{R}(u^s)^2 - \mathbb{E}[\mathcal{R}(u^s)^2] = m^{-1} \sum_{i=1}^{m} (p_{\tau_i}^{-1} |a(u-u^s, \varphi_{\tau_i})|^2 - \mathbb{E}[\mathcal{R}(u^s)^2]),$$

we define

$$X_i := m^{-1}(p_{\tau_i}^{-1} |a(u-u^s, \varphi_{\tau_i})|^2 - \mathbb{E}[\mathcal{R}(u^s)^2]).$$

Let $e_s := \|u - u^s\|_U$ and $\widetilde{e}_s := \|u - u^s\|_{\widetilde{U}}$. Employing Lemma 3.13, relation (40), and noticing that $u - u^s \in \widetilde{U}$ (recall that the trial functions belong to $\widetilde{U}$), we have

$$\begin{aligned} |X_i| &= m^{-1}|p_{\tau_i}^{-1}|a(u-u^s,\varphi_{\tau_i})|^2 - \mathbb{E}[\mathcal{R}(u^s)^2]| \\ &\leq m^{-1}(p_{\tau_i}^{-1}|a(u-u^s,\varphi_{\tau_i})|^2 + C_\varphi \beta^2 e_s^2) \\ &\leq m^{-1}(\widetilde{\beta}^2 \widetilde{e}_s^2 \frac{\|\varphi_{\tau_i}\|_{\widetilde{V}}^2}{\nu_{\tau_i}^N} \|\boldsymbol{\nu}^{N,M}\|_1 + C_\varphi \beta^2 e_s^2). \end{aligned}$$

Thanks to hypothesis (41), we obtain a uniform bound $|X_i| \leq B$, with

$$B = m^{-1}(\widetilde{\beta}^2 A \|\boldsymbol{\nu}^{N,M}\|_1 \widetilde{e}_s^2 + C_\varphi \beta^2 e_s^2).$$

As a consequence, recalling Lemma 3.12 and the splitting $\Omega = \Omega_{2s} \cup \Omega_{2s}^c$ as in Theorem 3.15, we obtain, for any generic constant $E > 0$,

$$\begin{aligned} \mathbb{P}\{\|\widehat{u} - u\|_U \geq E\} &\leq \mathbb{P}\{\|\widehat{u} - u^s\|_U \geq E - e_s\} \\ &\leq \underbrace{\mathbb{P}\left\{ \frac{2}{\widetilde{\alpha}} \mathcal{R}(u^s) \geq E - e_s \right\}}_{=:P} + \varepsilon \end{aligned}$$

A further application of Lemma 3.13 yields

$$\begin{aligned} P &= \mathbb{P}\left\{ \mathcal{R}(u^s)^2 - \mathbb{E}[\mathcal{R}(u^s)^2] \geq \frac{\widetilde{\alpha}^2}{4}(E - e_s)^2 - \mathbb{E}[\mathcal{R}(u^s)^2] \right\} \\ &\leq \mathbb{P}\left\{ \mathcal{R}(u^s)^2 - \mathbb{E}[\mathcal{R}(u^s)^2] \geq \underbrace{\frac{\widetilde{\alpha}^2}{4}(E - e_s)^2 - C_\varphi \beta^2 e_s^2}_{=:t} \right\}. \end{aligned}$$

Resorting to Hoeffding's bound (Theorem 3.17), we have

$$P \leq \exp\left(-\frac{t^2}{2mB^2} \right).$$

Finally, standard estimates show that the right hand-side is equal to ε if and only if

$$E = e_s + \left\{ \frac{4\beta^2 C_\varphi}{\widetilde{\alpha}^2} \left[1 + \left(\frac{2\log \varepsilon^{-1}}{m} \right)^{\frac{1}{2}} \right] e_s^2 + \frac{4\widetilde{\beta}^2 A \|\boldsymbol{\nu}^{N,M}\|_1}{\widetilde{\alpha}^2} \left(\frac{2\log \varepsilon^{-1}}{m} \right)^{\frac{1}{2}} \widetilde{e}_s^2 \right\}^{\frac{1}{2}}.$$

Recalling hypothesis (42), we notice that

$$\frac{2\log \varepsilon^{-1} \max\{\|\boldsymbol{\nu}^{N,M}\|_1, 1\}}{m} \leq 1,$$

and we simplify the estimate as follows

$$E \leq \left(1 + \frac{2\sqrt{2}\beta C_\varphi^{1/2}}{\widetilde{\alpha}} \right) e_s + \frac{2\widetilde{\beta} A^{1/2} \|\boldsymbol{\nu}^{N,M}\|_1^{1/4}}{\widetilde{\alpha}} \widetilde{e}_s$$

which, in turn, implies the thesis.

□

The error estimate of Theorem 3.18 does not depend on ε, but we have a factor $\|\boldsymbol{\nu}^{N,M}\|_1^{1/4}$, which diverges for $N, M \to +\infty$. However, in view of Hypothesis 3, its growth is at most polylogarithmic.

We conclude this section with a useful corollary, dealing with a particular truncation condition. In practice, this corollary provides sufficient conditions for Theorem 3.15 to hold. We will apply this result to some examples in Section 4.

Corollary 3.19. *Suppose that there exist two positive constants $C_{\boldsymbol{\mu}}$ and $\widehat{\gamma}$ such that*

$$\sum_{q>M} \mu_q^N \leq C_{\boldsymbol{\mu}} \left(\frac{N}{M}\right)^{1/\widehat{\gamma}}, \quad \forall M \in \mathbb{N}. \tag{43}$$

Then, for every $\varepsilon \in (0, 2^{-1/3}]$ and for $s \leq 2N/e$ there exist two positive constants $\widehat{C}$ and $\overline{C}$ such that, for

$$M \geq \widehat{C} s^{\widehat{\gamma}} N \quad \text{and} \quad m \geq \overline{C} s \|\boldsymbol{\nu}^{N,M}\|_1 [s \log(N/s) + \log(s/\varepsilon)], \tag{44}$$

the CORSING solution $\widehat{u}$ fulfills

$$\mathbb{E}[\|\mathcal{T}_{\mathcal{K}} \widehat{u} - u\|_U] < \left(1 + \frac{4\beta}{\alpha}\left[\frac{C_\varphi}{c_\varphi}\right]^{\frac{1}{2}}\right) \|u^s - u\|_U + 2\mathcal{K}\varepsilon,$$

for every $\mathcal{K} > 0$ such that $\|u\|_U \leq \mathcal{K}$, with $\mathcal{T}_{\mathcal{K}}$ defined as in (36) and where α and β are defined by (3) and (4), respectively. In particular, two possible upper bounds for the constants $\widehat{C}$ and $\overline{C}$ are

$$\widehat{C} \leq \left(\frac{4C_{\boldsymbol{\mu}}}{\kappa_{2s} c_\varphi \alpha^2}\right)^{\widehat{\gamma}} \quad \text{and} \quad \overline{C} \leq \frac{105}{\kappa_{2s} c_\varphi \alpha^2},$$

respectively, with κ_s defined in (22).

Proof. The idea is to choose $\overline{\delta} = \widehat{\delta} = 1/2$ and, as anticipated, to apply Theorem 3.15. First, notice that assumption (43) is consistent with Hypothesis 2, on passing to the limit for $M \to +\infty$. In view of Theorem 3.15, we show that the second inequality in (44) implies (38) with a suitable choice of $\overline{C}$. Moreover, the truncation condition (37), on which Theorem 3.15 relies on, is implied by

$$2sC_{\boldsymbol{\mu}} \left(\frac{N}{M}\right)^{1/\widehat{\gamma}} \leq \frac{c_\varphi \alpha^2 \kappa_{2s}}{2},$$

that, in turn, is equivalent to

$$M \geq \left(\frac{4C_{\boldsymbol{\mu}}}{\kappa_{2s} c_\varphi \alpha^2}\right)^{\widehat{\gamma}} s^{\widehat{\gamma}} N.$$

Moreover, thanks to the assumptions on ε and s, we have

$$\begin{aligned} &\varepsilon \leq 2^{-1/3} \Longrightarrow \log(2s/\varepsilon) \leq 4 \log(s/\varepsilon), \\ &s \leq 2N/e \Longrightarrow \log(eN/(2s)) \leq 2 \log(N/s). \end{aligned}$$

Thus, recalling the right-hand side of (38), we have

$$\begin{aligned} 2\,\widetilde{C}_{2s}\; s\|\boldsymbol{\nu}^{N,M}\|_1 [2s \log(eN/(2s)) + \log(2s/\varepsilon)] \\ \leq 8\,\widetilde{C}_{2s}\; s\|\boldsymbol{\nu}^{N,M}\|_1 [s \log(N/s) + \log(s/\varepsilon)], \end{aligned}$$

where $\widetilde{C}_{2s}$ is defined analogously to (24). In particular, if $\overline{C}$ in (44) is chosen such that

$$\overline{C} \leq 8\,\widetilde{C}_{2s} = \frac{32}{(1 - \log 2)\kappa_{2s} c_\varphi \alpha^2} \leq \frac{105}{\kappa_{2s} c_\varphi \alpha^2},$$

then (38) holds. Moreover, relation $\widetilde{\alpha} = [(1-\widehat{\delta})(1-\overline{\delta})c_\varphi]^{\frac{1}{2}}\alpha$ yields $\widetilde{\alpha} = \frac{1}{2}c_\varphi^{1/2}\alpha$, so that the quantity $2C_\varphi^{1/2}\beta/\widetilde{\alpha}$ in Theorem 3.15 can be replaced by $4C_\varphi^{1/2}\beta/(c_\varphi^{1/2}\alpha)$. □

Remark 3.20. The assumptions $\varepsilon \leq 2^{-1/3} \approx 0.79$ and $s \leq 2N/e \approx 0.74N$ made in Corollary 3.19 are quite weak and they are chosen in such a way that the upper bounds to $\widehat{C}$ and $\overline{C}$ are easy to derive. Of course, more restrictive hypotheses on ε and s would give sharper upper bounds for the asymptotic constants. Moreover, the parameters $\widehat{\delta}$ and $\overline{\delta}$ could be chosen differently from $\overline{\delta} = \widehat{\delta} = 1/2$ and this would lead to different values for the constant in the recovery error estimate.

Remark 3.21. If $\varepsilon \geq s^{s+1}/N^s$, then $s\log(N/s) + \log(s/\varepsilon) \leq 2s\log(N/s)$ and the term $\log(s/\varepsilon)$ disappears from the inequality on m by doubling the constant $\overline{C}$, giving the trend

$$m \geq \overline{C}\|\boldsymbol{\nu}^{N,M}\|_1 s^2 \log(N/s),$$

claimed in Algorithm 2.1. This assumption on ε is not restrictive, since $s \ll N$ guarantees $s^{s+1}/N^s \ll 1$.

Remark 3.22. A result analogous to Corollary 3.19 holds in probability by resorting to Theorem 3.16 (or Theorem 3.18) instead of Theorem 3.15 in the proof.

4. Application to a 1D advection-diffusion-reaction equation

In this section, we apply the general theory presented in Section 3 to elliptic one-dimensional problems, such as the Poisson equation and an advection-diffusion-reaction (ADR) equation.

We adopt Corollary 3.19 as the main tool. In particular, we provide estimates for α, β, κ_s, $C_{\boldsymbol{\mu}}$, $\widehat{\gamma}$, $\boldsymbol{\nu}^N$ and $\|\boldsymbol{\nu}^{N,M}\|_1$, and then deduce suitable hypotheses on m and M such that the CORSING method recovers the best s-term approximation u^s to u. All the recovery results of the section are given in expectation, but they can be easily converted in probability (see Remark 3.22).

Let us first fix the notation. Consider $\mathcal{D} = (0,1)$, $U = V = H_0^1(\mathcal{D})$ and

$$(u,v)_U = (u,v)_V = \int_{\mathcal{D}} u'(x)v'(x)\,\mathrm{d}x,$$

resulting in $\|\cdot\|_U = \|\cdot\|_V = |\cdot|_{H^1(\mathcal{D})}$, the $H^1(\mathcal{D})$-seminorm. Moreover, we introduce two Hilbert bases of $H_0^1(\mathcal{D})$. The first one is the hierarchical multiscale basis [51, 19], defined as

$$\mathcal{H}_{\ell,k}(x) := 2^{-\frac{\ell}{2}}\mathcal{H}(2^\ell x - k), \quad \forall x \in [0,1],$$

for every $\ell \in \mathbb{N}_0$, $k = 0, \dots, 2^\ell - 1$ and with $\mathcal{H}(x) := \max(0, \frac{1}{2} - |x - \frac{1}{2}|)$, for any $x \in [0,1]$, ordered according to the lexicographic mapping

$$j \mapsto (\ell(j), k(j)) := (\lfloor \log_2(j) \rfloor, j - 2^{\lfloor \log_2(j) \rfloor}).$$

The second one is the rescaled sine function basis

$$\mathcal{S}_r(x) := \frac{\sqrt{2}}{r\pi}\sin(r\pi x), \quad \forall x \in [0,1],\ \forall r \in \mathbb{N}.$$

For further details concerning these bases, see [11, Section 5]. It is easy to check that both bases are orthonormal with respect to $(\cdot,\cdot)_U$.

With reference to [11], when the following combination of trial and test functions is employed

$$\psi_j = \mathcal{H}_{\ell(j),k(j)}, \quad \varphi_q = \mathcal{S}_q,$$

we denote the approach by CORSING $\mathcal{HS}$. On the contrary, when the roles of the trial and test functions are switched, we denote it by CORSING $\mathcal{SH}$. In both cases, $\mathcal{HS}$ and $\mathcal{SH}$, we observe that $c_\varphi = C_\varphi = 1$ and that $\mathbf{K} = \mathbf{I}$. Thus, in particular, from (22), $\kappa_s = 1$.

As the reference problem, we consider the one-dimensional ADR equation over $\mathcal{D}$, with Dirichlet boundary conditions

$$\begin{cases} -u'' + bu' + \eta u = f & \text{in } \mathcal{D} \\ u(0) = u(1) = 0, \end{cases} \tag{45}$$

with $b, \eta \in \mathbb{R}$ and $f : (0,1) \to \mathbb{R}$, corresponding to the weak problem

$$\text{find } u \in H^1_0(\mathcal{D}): \quad (u', v') + b(u', v) + \eta(u, v) = (f, v), \quad \forall v \in H^1_0(\mathcal{D}), \tag{46}$$

where $(\cdot, \cdot)$ denotes the standard inner product in $L^2(\mathcal{D})$.

4.1. **The Poisson equation ($\mathcal{HS}$).** First, we deal with the Poisson equation, corresponding to (45) with $b = \eta = 0$, whose weak formulation is

$$\text{find } u \in H^1_0(\mathcal{D}): \quad a_\Delta(u, v) = (f, v), \quad \forall v \in H^1_0(\mathcal{D}). \tag{47}$$

where $a_\Delta(u, v) := (u', v')$. In such a case, we denote the local a-coherence by $\boldsymbol{\mu}^N_\Delta$. The inf-sup and continuity constants of $a_\Delta(\cdot, \cdot)$ are $\alpha = \beta = 1$.

We can prove the following result for the CORSING $\mathcal{HS}$ procedure applied to (47).

Proposition 4.1. *Fix a maximum hierarchical level $L \in \mathbb{N}$, corresponding to $N = 2^{L+1} - 1$. Then, for every $\varepsilon \in (0, 2^{-1/3}]$ and $s \leq 2N/e$, and provided that*

$$M \geq \widehat{C} s N, \quad m \geq \overline{C} s \log M [s \log(N/s) + \log(s/\varepsilon)],$$

for suitable constants $\overline{C}$ and $\widehat{C}$, a valid upper bound $\boldsymbol{\nu}^N$ is

$$\nu^N_q := \frac{8}{\pi q}, \quad \forall q \in \mathbb{N},$$

and the CORSING $\mathcal{HS}$ solution to (47) fulfills

$$\mathbb{E}[|\mathcal{T}_\mathcal{K} \widehat{u} - u|_{H^1(\mathcal{D})}] < 5 |u^s - u|_{H^1(\mathcal{D})} + 2\mathcal{K}\varepsilon,$$

for every $\mathcal{K} > 0$ such that $|u|_{H^1(\mathcal{D})} \leq \mathcal{K}$, with $\mathcal{T}_\mathcal{K}$ defined as in (36). In particular, two possible upper bounds for $\widehat{C}$ and $\overline{C}$ are

$$\widehat{C} \leq \frac{320}{3\pi^2} \approx 10.8 \quad \text{and} \quad \overline{C} \leq \frac{840}{\pi} \left(1 + \frac{1}{\log 3}\right) \approx 511.$$

Proof. An explicit computation yields the exact stiffness matrix entries (the dependence of ℓ and k on j is omitted)

$$a_\Delta(\mathcal{H}_{\ell,k}, \mathcal{S}_q) = \frac{4\sqrt{2}}{\pi} \frac{2^{\frac{\ell}{2}}}{q} \sin\left(\frac{\pi q}{2^\ell}\left(k + \frac{1}{2}\right)\right) \sin^2\left(\frac{\pi}{4}\frac{q}{2^\ell}\right). \tag{48}$$

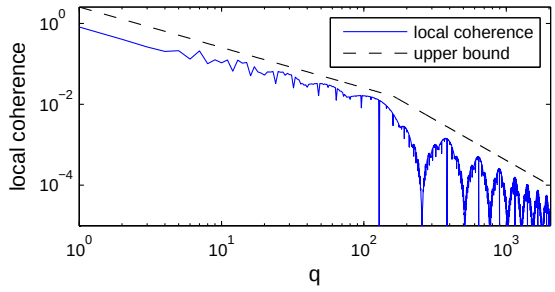

FIGURE 1. Sharpness of the upper bound (49) with $N = 127$ and $M = 2047$.

Using Definition 2.3, employing the inequalities $\sin^2(x) \leq 1$ on the first sine and $\sin^4(x) \leq \min\{1, |x|\}$ on the second sine, for every $x \in \mathbb{R}$, we have

$$|a_\Delta(\mathcal{H}_{\ell,k}, \mathcal{S}_q)|^2 \leq \frac{32}{\pi^2}\frac{2^\ell}{q^2}\sin^4\left(\frac{\pi}{4}\frac{q}{2^\ell}\right) \leq \min\left\{\frac{32}{\pi^2}\frac{2^\ell}{q^2}, \frac{8}{\pi q}\right\},$$

and, thus, we obtain the upper bound

$$\mu^N_{\Delta,q} \leq \min\left\{\frac{32}{\pi^2}\frac{2^L}{q^2}, \frac{8}{\pi q}\right\}. \tag{49}$$

Figure 1 shows that this bound is sharp. Considering the first argument of the minimum in (49), on noticing that $2^L = (N+1)/2$, we obtain

$$\begin{aligned}\sum_{q>M}\mu^N_{\Delta,q} &\leq \frac{32}{\pi^2}\frac{N+1}{2}\sum_{q>M}\frac{1}{q^2} \leq \frac{16}{\pi^2}(N+1)\left[\frac{1}{(M+1)^2} + \int_{M+1}^{\infty}\frac{1}{q^2}\,\mathrm{d}q\right] \\ &= \frac{16}{\pi^2}\frac{N+1}{M+1}\left[\frac{1}{M+1}+1\right] \leq \frac{20}{\pi^2}\frac{N+1}{M+1} \leq \frac{80}{3\pi^2}\frac{N}{M}.\end{aligned}$$

The fourth and fifth relations hinges on the assumption $L \geq 1$, that implies $N \geq 3$. Consequently, assuming $M \geq N$ we have also $M \geq 3$. This implies $1/(M+1) \leq 1/4$ (fourth relation) and $(N+1)/(M+1) \leq 4N/(3M)$ (fifth relation). Thus, in view of Corollary 3.19, we can pick

$$C_{\boldsymbol{\mu}} = \frac{80}{3\pi^2} \quad \text{and} \quad \widehat{\gamma} = 1.$$

Now, to bound $\|\boldsymbol{\nu}^{N,M}\|_1$, which is required by Corollary 3.19, we deal with the second argument of the minimum in (49) and set

$$\nu^N_q := \frac{8}{\pi q}.$$

This choice leads to the estimate

$$\begin{aligned}\|\boldsymbol{\nu}^{N,M}\|_1 &= \frac{8}{\pi}\sum_{q=1}^{M}\frac{1}{q} \leq \frac{8}{\pi}\left[1 + \int_1^M \frac{1}{q}\,\mathrm{d}q\right] = \frac{8}{\pi}(1+\log M) \\ &\leq \frac{8}{\pi}\left(1+\frac{1}{\log 3}\right)\log M, \end{aligned} \tag{50}$$

since $M \geq 3$. Thus, combining the lower bound for m and M in Corollary 3.19 with (50), we conclude the proof. □

Remark 4.2. The upper bound $\sin^4(x) \leq \min\{1, |x|\}$ can be improved as $\sin^4(x) \leq \min\{1, 0.68\,|x|\}$, which can be easily checked numerically. This change leads to rescale the value of $\overline{C}$ by a factor 0.68, i.e., $\overline{C} \approx 347$.

Remark 4.3. The choice $\nu_q^N = 8/(\pi q)$ is suboptimal. If we choose the sharper upper bound

$$\nu_q^N = \min\left\{\frac{32}{\pi^2}\frac{2^L}{q^2}, \frac{8}{\pi q}\right\},$$

the term $\log M$ in the lower bound to m can be replaced by $\log N$. Indeed, in this case

$$\|\boldsymbol{\nu}^{N,M}\|_1 \lesssim \sum_{q=1}^{N}\frac{1}{q} + N\sum_{q=N+1}^{M}\frac{1}{q^2} \lesssim \log N + N\left(\frac{1}{N} - \frac{1}{M}\right) \lesssim \log N + 1 - \frac{1}{s} \lesssim \log N.$$

Moreover, thanks to Hypothesis 3, we notice that the choice $\nu_q^N \sim 1/q$ is the slowest algebraic decay allowed, since for $\nu_q^N \sim 1/q^r$, with $r < 1$, the quantity $\|\boldsymbol{\nu}^{N,M}\|_1$ diverges algebraically in M.

4.2. **ADR equation ($\mathcal{HS}$).** We consider problem (45) and state the following result.

Proposition 4.4. *Fix a maximum hierarchical level $L \in \mathbb{N}$, corresponding to $N = 2^{L+1} - 1$. Then, for every $\varepsilon \in (0, 2^{-1/3}]$ and $s \leq 2N/e$, and provided that*

$$M \gtrsim sN, \quad \frac{|b|}{M} \lesssim 1, \quad \frac{|\eta|}{M^2} \lesssim 1,$$

$$m \gtrsim s(\log M + |b|^2 + |\eta|^2)[s\log(N/s) + \log(s/\varepsilon)],$$

a valid upper bound $\boldsymbol{\nu}^N$ is

$$\nu_q^N \sim \frac{1}{q} + \frac{|b|^2}{q^3} + \frac{|\eta|^2}{q^5}, \quad \forall q \in \mathbb{N},$$

and the CORSING $\mathcal{HS}$ solution to (46)*, with $\eta > -2$, fulfills*

$$\mathbb{E}[|\mathcal{T}_{\mathcal{K}}\widehat{u} - u|_{H^1(\mathcal{D})}] < \left(1 + \frac{4 + 2\sqrt{2}|b| + 2|\eta|}{1 + \min(0, \eta/2)}\right)|u^s - u|_{H^1(\mathcal{D})} + 2\mathcal{K}\varepsilon,$$

for every $\mathcal{K} > 0$ such that $|u|_{H^1(\mathcal{D})} \leq \mathcal{K}$, with $\mathcal{T}_{\mathcal{K}}$ defined as in (36)*.*

Proof. The argument is the same as in Proposition 4.1, thus we will just highlight the different parts. The precise values of the asymptotic constants will not be tracked during the proof.

First, a straightforward computation gives

$$a(\mathcal{H}_{\ell,k}, \mathcal{S}_q) = \frac{4\sqrt{2}}{\pi}\frac{2^{\frac{\ell}{2}}}{q}\sin^2\left(\frac{\pi}{4}\frac{q}{2^\ell}\right)\left[\left(1 + \frac{\eta}{(\pi q)^2}\right)\sin\left(\frac{\pi q}{2^\ell}(k + \tfrac{1}{2})\right) - \frac{b}{\pi q}\cos\left(\frac{\pi q}{2^\ell}(k + \tfrac{1}{2})\right)\right].$$

Hence, using the same upper bounds as in Proposistion 4.1, we obtain

$$|a(\mathcal{H}_{\ell,k}, \mathcal{S}_q)|^2 \lesssim \min\left\{\frac{2^\ell}{q^2}, \frac{1}{q}\right\}\left(1 + \frac{|b|^2}{q^2} + \frac{|\eta|^2}{q^4}\right),$$

and, consequently,

$$\mu_q^N \lesssim \min\left\{\frac{N}{q^2}, \frac{1}{q}\right\}\left(1 + \frac{|b|^2}{q^2} + \frac{|\eta|^2}{q^4}\right). \tag{51}$$

Considering the first argument of the minimum in (51), yields

$$\begin{aligned}\sum_{q>M} \mu_q^N &\lesssim N\left[\sum_{q>M}\frac{1}{q^2} + |b|^2\sum_{q>M}\frac{1}{q^4} + |\eta|^2\sum_{q>M}\frac{1}{q^6}\right]\\ &\lesssim N\left[\frac{1}{M} + \frac{|b|^2}{M^3} + \frac{|\eta|^2}{M^5}\right] \lesssim \frac{N}{M}.\end{aligned}$$

The second inequality hinges on estimates of the sums by suitable integrals, whereas the third one is implied by the hypotheses $|b|/M \lesssim 1$ and $|\eta|/M^2 \lesssim 1$.

Now, considering the second argument of the minimum in (51), we have the upper bound

$$\nu_q^N \sim \frac{1}{q} + \frac{|b|^2}{q^3} + \frac{|\eta|^2}{q^5}, \quad \forall q \in \mathbb{N},$$

and, consequently, the ℓ^1-norm of its truncation fulfills

$$\|\boldsymbol{\nu}^{N,M}\|_1 \sim \sum_{q=1}^{M}\frac{1}{q} + \sum_{q=1}^{M}\frac{|b|^2}{q^3} + \sum_{q=1}^{M}\frac{|\eta|^2}{q^5} \lesssim \log M + |b|^2 + |\eta|^2.$$

Finally, we notice that (3) and (4) hold with

$$\alpha = 1 + \min\left(0, \frac{\eta}{2}\right), \quad \beta = 1 + \frac{|b|}{\sqrt{2}} + \frac{|\eta|}{2},$$

thanks to the Poincaré inequality

$$\sqrt{2}\|v\|_{L^2(\mathcal{D})} \le |v|_{H^1(\mathcal{D})}, \quad \forall v \in H_0^1(\mathcal{D}).$$

The thesis is now a direct consequence of Corollary 3.19. □

4.3. **The Poisson equation ($\mathcal{SH}$).** We prove a recovery result for the CORSING $\mathcal{SH}$ method applied to the Poisson problem (47).

Proposition 4.5. *For every $\varepsilon \in (0, 2^{-1/3}]$ and $s \le 2N/e$, there exist two positive constants $\overline{C}$ and $\widehat{C}$ such that, and provided*

$$M \ge \widehat{C}\sqrt{s}N, \quad m \ge \overline{C}s\log(M)[s\log(N/s) + \log(s/\varepsilon)],$$

with M of the form $M = 2^{L+1} - 1$ for some $L \in \mathbb{N}$, a valid upper bound $\boldsymbol{\nu}^N$ is

$$\nu_q^N = \frac{1}{2^{\ell(q)-1}}, \quad \forall q \in \mathbb{N},$$

and the CORSING $\mathcal{SH}$ solution to (47) fulfills

$$\mathbb{E}[|\mathcal{T}_{\mathcal{K}}\widehat{u} - u|_{H^1(\mathcal{D})}] \le 5|u^s - u|_{H^1(\mathcal{D})} + 2\mathcal{K}\varepsilon,$$

for every $\mathcal{K} > 0$ such that $|u|_{H^1(\mathcal{D})} \le \mathcal{K}$, with $\mathcal{T}_{\mathcal{K}}$ defined as in (36) and where α and β are defined by (3) and (4), respectively. In particular, two possible upper bounds for $\widehat{C}$ and $\overline{C}$ are

$$\widehat{C} \le \sqrt{\frac{2}{3}}\pi \approx 2.57 \quad \text{and} \quad \overline{C} \le \frac{210\log_2(e)\log(4)}{\log(3)} \approx 382.$$

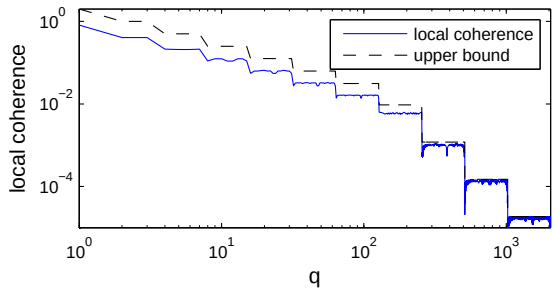

FIGURE 2. Sharpness of the upper bound (52) with $N = 127$ and $M = 2047$.

Proof. The proof is analogous to that of Proposition 4.1. We highlight only the main differences. First, notice that

$$a_\Delta(\mathcal{S}_j, \mathcal{H}_{\ell(q),k(q)}) = a_\Delta(\mathcal{H}_{\ell(q),k(q)}, \mathcal{S}_j).$$

Moving from (48) and employing the inequality $\sin^4(x) \leq \min\{|x|^4, |x|^2\}$, for every $x \in \mathbb{R}$, we obtain

$$\mu^N_{\Delta,q} \leq \min\left\{\frac{\pi^2}{8}\frac{N^2}{2^{3\ell(q)}}, \frac{1}{2^{\ell(q)-1}}\right\}. \tag{52}$$

Figure 2 shows the sharpness of this bound.

Considering the first argument of the minimum in (52), and since $M = 2^{L+1} - 1$, we have that

$$\sum_{q>M} \mu^N_{\Delta,q} \leq \frac{\pi^2}{8} N^2 \sum_{\ell>L} \sum_{k=0}^{2^\ell-1} \frac{1}{2^{3\ell}} = \frac{\pi^2}{8} N^2 \sum_{\ell>L} \frac{1}{2^{2\ell}} = \frac{\pi^2}{8} \frac{N^2}{2^{2(L+1)}} \sum_{\ell\geq 0} \frac{1}{2^{2\ell}} \leq \frac{\pi^2}{6}\left(\frac{N}{M}\right)^2$$

where the change of variable $q \mapsto (\ell, k)$ has been used. Thus, if follows that

$$C_{\boldsymbol{\mu}} = \frac{\pi^2}{6} \quad \text{and} \quad \widehat{\gamma} = \frac{1}{2}.$$

Now, by considering the second argument of the minimum in (52), we select

$$\nu^N_q := \frac{1}{2^{\ell-1}}$$

and conclude the proof by computing

$$\begin{aligned}\|\boldsymbol{\nu}^{N,M}\|_1 &= \sum_{\ell=0}^{L} \sum_{k=0}^{2^\ell-1} \frac{1}{2^{\ell-1}} = 2(L+1) = 2\log_2(e)\log(M+1) \\ &\leq 2\log_2(e)\frac{\log(M+1)}{\log(M)}\log(M) \leq \frac{2\log_2(e)\log(4)}{\log(3)}\log(M),\end{aligned}$$

since $M \geq 3$, thanks to $L \geq 1$. □

Remark 4.6. The choice of $\mathbf{p}$ prompted by Proposition 4.5 (i.e., $p_q \sim 2^{-\ell(q)}$) coincides with that in [11], in the R-CORSING $\mathcal{SH}$ case, for the corresponding parameter $\mathbf{w}$, tuned via a trial-and-error procedure.

We also provide a recovery result in high probability for the CORSING $\mathcal{SH}$ approach, as an application of Theorem 3.18.

Proposition 4.7. *Let $f \in L^2(\mathcal{D})$. Given $s \leq N$ and $0 < \varepsilon \leq 1/2$, choose $M, m \in \mathbb{N}$ such that*

$$M \gtrsim \sqrt{s}N, \quad m \gtrsim s\log(M)[s\log(N/s) + \log(s/\varepsilon)],$$

and $\nu_q^N \sim 1/2^{\ell(q)}$.

Then, with probability greater than or equal to $1-2\varepsilon$, the CORSING $\mathcal{SH}$ solution $\widehat{u}$ to the Poisson problem (47) fulfills

$$|\widehat{u} - u|_{H^1(\mathcal{D})} \lesssim |u - u^s|_{H^1(\mathcal{D})} + \log(M)^{\frac{1}{4}}|u - u^s|_{H^2(\mathcal{D})},$$

where $|w|_{H^2(\mathcal{D})} := \|w''\|_{L^2(\mathcal{D})}$ is the $H^2(\mathcal{D})$-seminorm.

Proof. We apply Theorem 3.18. Employing the same argument as in Corollary 3.19, we have that $M \gtrsim \sqrt{s}N$ guarantees the truncation condition (37).

Now, we set $\widetilde{U} := H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ and $\widetilde{V} := L^2(\mathcal{D})$, equipped with the norm $\|\cdot\|_{\widetilde{U}} := |\cdot|_{H^2(\mathcal{D})}$ and $\|\cdot\|_{\widetilde{V}} := \|\cdot\|_{L^2(\mathcal{D})}$, respectively. Of course, we have the continuity of the bilinear form (u', v') with respect to the adopted (semi)norms, $f \in L^2(\mathcal{D})$ implies $u \in H^2(\mathcal{D})$ (see, e.g., [37, Section 6.1.3]), and the trial functions belong to $H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. Moreover, a direct computation shows that

$$\|\varphi_q\|_{L^2(\Omega)}^2 \sim 2^{-2\ell(q)} \lesssim 2^{-\ell(q)} \sim \nu_q^N.$$

The thesis follows by recalling that $\|\boldsymbol{\nu}^{N,M}\|_1 \sim \log(M)$ (see the proof of Proposition 4.5). □

The recovery estimate of Proposition 4.7 degenerates as $M \to +\infty$, due to the factor $\log(M)^{1/4}$. Nonetheless, this factor grows at an extremely slow rate. For example, $\log(10^{35})^{1/4} \leq 3$. We also notice that the same result does not hold in the $\mathcal{HS}$ setting, due to the low regularity of the trial functions (they only belong to $H^s(\mathcal{D})$, with $s < 3/2$). A possible remedy could be to replace the hierarchical multiscale basis with a family of more regular wavelets.

4.4. **ADR equation ($\mathcal{SH}$).** Considerations analogous to those made in the $\mathcal{HS}$ case hold in the advective/reactive case. It suffices to notice that

$$(u', v') + b(u', v) + \eta(u, v) = (v', u') - b(v', u) + \eta(v, u), \quad \forall u, v \in H_0^1(\mathcal{D}).$$

4.5. **CORSING $\mathcal{HS}$ or $\mathcal{SH}$? A comparison.** The theoretical analysis carried out in the previous sections shows that both CORSING $\mathcal{HS}$ and $\mathcal{SH}$ are able to recover the best s-term approximation to the solution u to an ADR problem with constant coefficients, provided that $m \gtrsim s^2\log(M)\log(N)$. Comparing the $\mathcal{HS}$ and the $\mathcal{SH}$ setting, we have two different truncation conditions. In the $\mathcal{HS}$ case, it is required that $M \gtrsim sN$, whereas in the $\mathcal{SH}$ case we have $M \gtrsim \sqrt{s}N$. However, this difference has only a slight impact on the number m of test functions.

From the accuracy viewpoint, the best choice is the one that minimizes the best s-term approximation error of the solution with respect to the trial basis. In principle, the $\mathcal{HS}$ setting is more suited when the solution exhibits localized phenomena, such as boundary layers or small features. The $\mathcal{SH}$ one is more advantageous when the solution is periodic and smooth, due to the consequent fast decay of the Fourier coefficients, or when the solution exhibits global periodic multiscale phenomena. In

Section 5, we will focus on the $\mathcal{HS}$ case, due to our interest in assessing the ability of CORSING to capture localized features in a non-periodic scenario. For further numerical experiments about the $\mathcal{SH}$ setting, we refer the reader to [11].

Remark 4.8 (Application to more general cases). The main difficulty of the analysis of CORSING is the derivation of the upper bound $\boldsymbol{\nu}^N$ to the local a-coherence. For instance, in dealing with the ADR equation with non-constant coefficients, a highly oscillatory diffusion coefficient can considerably deteriorate $\boldsymbol{\nu}^N$. One possibility to tackle this issue is to expand the non-constant coefficient with respect to a suitable basis and then to exploit Propositions 4.1 and 4.5.

Considering the extension to higher-dimensional problems, first results are provided in [11, Section 6] where CORSING is applied to the two-dimensional ADR equation with constant coefficients, with hierarchical pyramids and tensor product of sine functions. Some numerical results concerning the application of CORSING to the three-dimensional Poisson equation and to the two-dimensional stationary Stokes problem, along with a numerical assessment on the local a-coherence can be found in [10].

5. Numerical experiments

We validate the above theoretical results by both a qualitative and a quantitative analysis. For a more complete numerical assessment of CORSING, we refer to [11].

All the computations have been performed using MATLAB® R2013a 64-bit (version 8.1.0.604) on a MacBook Pro equipped with a 3GHz Intel Core i7 processor and 8GB of RAM.

We recall that the OMP-BOX MATLAB® package (version 10) has been employed in the recovery phase. The stopping criterion employed is based on the sparsity level s, which corresponds to the number of OMP iterations.

5.1. Sensitivity analysis of the RISP constant.

We investigate the sensitivity of $\widetilde{\alpha}$ to the constant $\overline{C}$ on the Poisson problem (47), in the setting $\mathcal{HS}$. We fix the hierarchical level to $L = 14$, corresponding to $N = 32767$. We consider the values $s = 1, 2, 3, 4, 5$ and choose $M = sN$, while selecting m according to one of the following rules

$$\begin{aligned} &\text{Rule 1:} \quad m = \lceil \overline{C} s^2 \log M \log(N/s) \rceil, \\ &\text{Rule 2:} \quad m = \lceil \overline{C} s \log M \log(N/s) \rceil, \\ &\text{Rule 3:} \quad m = \lceil \overline{C} s \log(N/s) \rceil. \end{aligned} \tag{53}$$

Rule 1 is the one derived in this paper, corresponding to $\overline{\gamma} = 2$. Rule 2 is associated with $\overline{\gamma} = 1$, and Rule 3 is the asymptotically optimal lower bound that a general sparse recovery procedure requires to be stable (see [27, Proposition 10.7]). For each choice of M and m, we repeat the following experiment 50 times: first, extract $\tau_1, \dots, \tau_m \in [M]$ i.i.d. with probability $p_q \sim 1/q$ and build the corresponding matrices $\mathbf{D}$ and $\mathbf{A}$; then, generate 1000 random subsets $\mathcal{S}_1, \dots, \mathcal{S}_{1000} \subseteq [N]$ of cardinality s and compute the non-uniform RISP constant $\widetilde{\alpha}_{\mathcal{S}_k}$ for every $k \in [1000]$, corresponding to the minimum singular value of $\mathbf{DA}$, using the `svd` command; finally, approximate the uniform RISP constant as

$$\widetilde{\alpha} \approx \min_{k \in [1000]} \widetilde{\alpha}_{\mathcal{S}_k}.$$

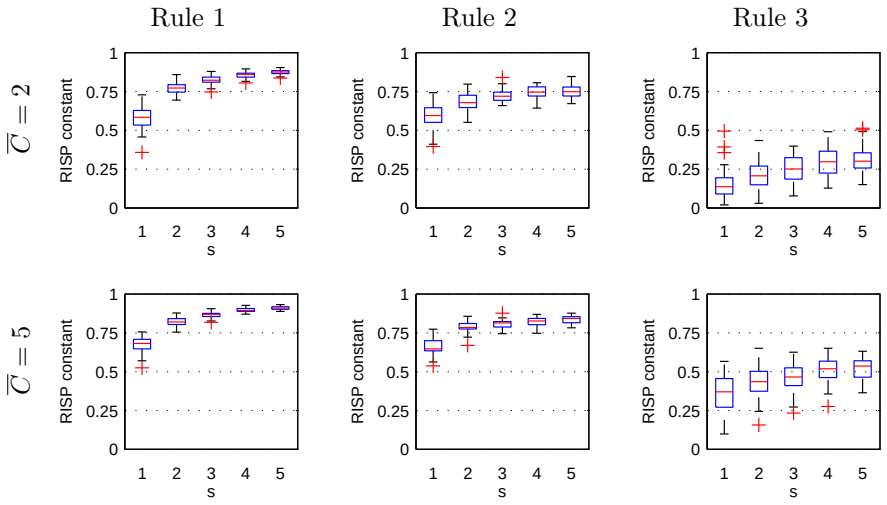

FIGURE 3. Sensitivity analysis of the RISP constant, with $M = sN$ and m defined according to (53).

We consider the three trends in (53) and $\overline{C} = 2$ or 5. The corresponding six boxplots relative to the 50 different values of $\widetilde{\alpha}$, computed for each s, are shown in Figure 3, where the crosses represent the outliers.

For Rule 1 and 2, $\widetilde{\alpha}$ shows a similar behavior since both trends are approaching the value of the inf-sup constant, $\alpha = 1$, when s grows. We notice that the values computed for Rule 1 are more concentrated around the mean, implying that $\overline{\gamma} = 2$ is a too conservative choice. For Rule 3, $\widetilde{\alpha}$ exhibits the lowest values, though the corresponding boxplots are quite aligned and have similar size, especially for $\overline{C} = 5$, where $\widetilde{\alpha}$ seems to stabilize around the value $\alpha/2$. For $\overline{C} = 2$, $\widetilde{\alpha}$ approaches the value $\alpha/4$, even though the presence of too many outliers suggests that the RISP is not being satisfied for a reasonable value of ε. However, since Rule 3 is quite satisfactory, especially for $\overline{C} = 5$, the quantity $\log M$ does not seem to be really necessary in Rule 2. Moreover, Rule 1 is penalized by both the $\log M$ term and the extra s factor.[2]

5.2. **CORSING validation .** We test CORSING $\mathcal{HS}$ on the one-dimensional Poisson equation (47), choosing the forcing term so that the exact solution be

$$u(x) := \widetilde{u}_{0.2,0.7,1000}(x) + 0.3 \cdot \widetilde{u}_{0.4,0.4005,2000}(x), \quad \forall x \in [0,1] \tag{54}$$

with

$$\begin{aligned}
\widetilde{u}_{x_1,x_2,t}(x) &:= \overline{u}_{x_1,x_2,t}(x) - e_{x_1,x_2,t}(x),\\
e_{x_1,x_2,t}(x) &:= x\,\overline{u}_{x_1,x_2,t}(1) + (1-x)\,\overline{u}_{x_1,x_2,t}(0),\\
\overline{u}_{x_1,x_2,t}(x) &:= \arctan(t(x-x_1)) - \arctan(t(x-x_2)),
\end{aligned}$$

[2]In CS, there are cases where the upper bound to the RIP constant is much better for randomly chosen supports $\mathcal{S}_k$ than for the worst case (see, e.g., [27, Chapter 14]). Similar phenomena could also occur for the RISP constant. Thus, the results of these numerical experiments could be too optimistic.

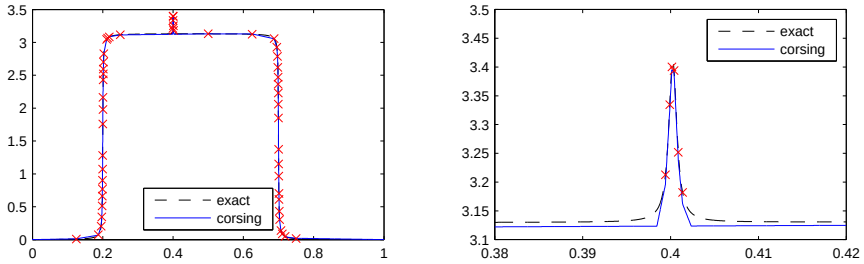

FIGURE 4. Left: comparison between u defined in (54) (dashed line) and $\widehat{u}$ (solid line). Right: a zoom in on the spike-shaped detail of u. Crosses correspond to the selected trial functions.

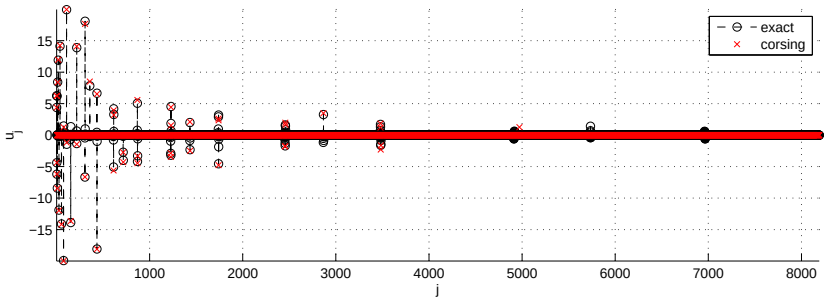

FIGURE 5. Comparison between $\mathbf{u}$ (circles) and $\widehat{\mathbf{u}}$ (crosses).

for every $x \in [0,1]$, $0 \leq x_1 < x_2 \leq 1$ and $t \in \mathbb{R}$. This particular solution is designed so as to exhibit two boundary layers at $x = 0.2$ and $x = 0.7$, and a small spike-shaped detail at $x = 0.4$ (see Figure 4). The hierarchical multiscale basis is particularly suited to capture these sharp features. We fix $L = 12$, corresponding to $N = 8191$, $s = 50$, $M = sN$, and $m = 1200$. According to Proposition 4.1, we choose the upper bound $\nu_q^N = 1/q$.

In Figure 4, we compare u (dashed line) and $\widehat{u}$ (solid line). The exact solution is well recovered. Both boundary layers are correctly captured and also the spike-shaped feature is successfully detected. More quantitatively, the best 50-term relative error is $|u - u_{50}|_{H^1}/|u|_{H^1} \approx 0.092$ and the relative error of the CORSING solution is $|u - \widehat{u}|_{H^1}/|u|_{H^1} \approx 0.111$. Thus, via CORSING, we loose only 21% of the best possible accuracy.

Figures 5 and 6, highlight that CORSING is able to find the most important coefficients of $\mathbf{u}$. In particular, in Figure 5, the coefficients of $\mathbf{u}$ and $\widehat{\mathbf{u}}$ are plotted according to the lexicographic ordering, whereas in Figure 6 they are shown in two dimensions: level ℓ is the vertical axis, and each level is divided horizontally into 2^ℓ parts, corresponding to $k = 0, \ldots, 2^\ell - 1$, (left to right). The color plots refer to $|u_{\ell,k}|$ (left) and $|\widehat{u}_{\ell,k}|$ (right), in logarithmic scale. It is remarkable the capability of CORSING in detecting the localized features of the solution (see the isolated vertical line in Figure 6 (right)).

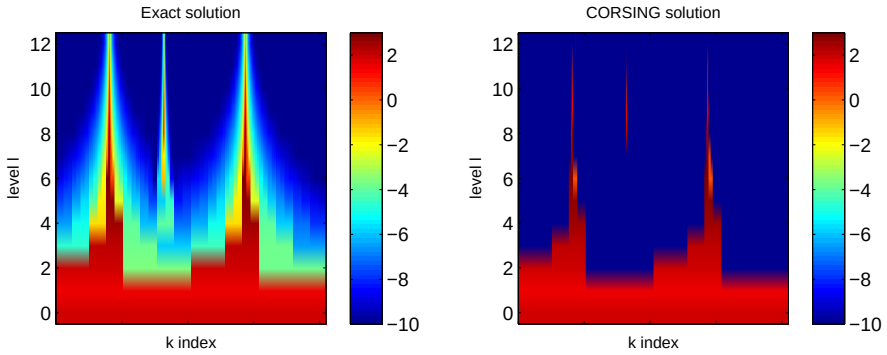

FIGURE 6. 2D color plot of $|u_{\ell,k}|$ and $|\widehat{u}_{\ell,k}|$ in logarithmic scale.

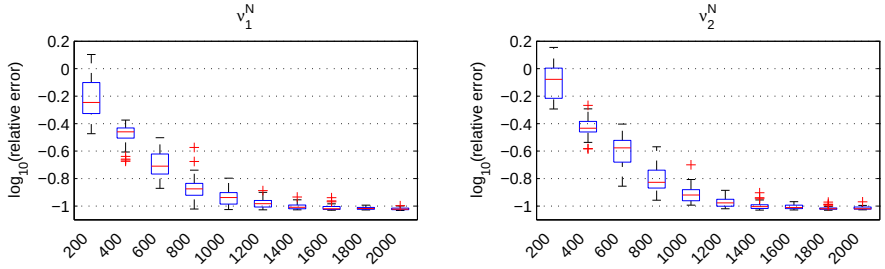

FIGURE 7. CORSING performance for the two choices of the upper bound $\boldsymbol{\nu}_1^N$ and $\boldsymbol{\nu}_2^N$ defined in (55). On the abscissa different values of m, on the ordinate the relative error with respect to the $H^1(\mathcal{D})$-seminorm computed over 50 runs of CORSING.

5.3. **Sharpness of the upper bound $\boldsymbol{\nu}^N$.** In this experiment, we assess to which extent the sharpness of the upper bound $\boldsymbol{\nu}^N$ affects the CORSING performance. We consider the same problem as in Section 5.2, keeping the choices $L = 12$ (corresponding to $N = 8191$), $s = 50$, and $M = sN$. Then, we let m assume the values $200, 400, \ldots, 2000$. For each value of m, we apply the CORSING procedure employing two different upper bounds, namely

$$\boldsymbol{\nu}_1^N = \left(\min\left\{ \frac{32}{\pi^2} \frac{2^L}{q^2}, \frac{8}{\pi q} \right\} \right)_{q\in\mathbb{N}} \quad \text{and} \quad \boldsymbol{\nu}_2^N = \left(\frac{8}{\pi q} \right)_{q\in\mathbb{N}}. \tag{55}$$

The first one corresponds to (49) and is very sharp (Figure 1), whereas the second one, employed in Proposition 4.1, is slightly looser. Comparing the two expressions, we see that $\boldsymbol{\nu}_2^N$ looses its sharpness for $q \geq 4/\pi \cdot 2^L \approx 5215$ (i.e., when $\boldsymbol{\nu}_1^N \leq \boldsymbol{\nu}_2^N$). In practice, this has the effect of favoring the highest frequencies during the test selection.

The results are summarized in Figure 7 and Table 1. In Figure 7, we have the boxplot corresponding to the relative errors computed with respect to the $H^1(\mathcal{D})$-seminorm, against m. Each value of m corresponds to 50 runs of CORSING. In Table 1, we show the mean and the unbiased standard deviation of the computed relative errors, for each value of m.

	$\boldsymbol{\nu}_1^N$		$\boldsymbol{\nu}_2^N$	
m	Mean	Std. dev.	Mean	Std. dev.
200	6.37e-01	2.32e-01	8.51e-01	2.57e-01
400	3.37e-01	5.35e-02	3.79e-01	5.81e-02
600	2.04e-01	4.67e-02	2.60e-01	6.01e-02
800	1.39e-01	2.86e-02	1.61e-01	3.69e-02
1000	1.17e-01	1.59e-02	1.22e-01	1.72e-02
1200	1.05e-01	8.44e-03	1.07e-01	8.25e-03
1400	9.96e-02	5.08e-03	1.00e-01	6.03e-03
1600	9.76e-02	4.52e-03	9.85e-02	3.70e-03
1800	9.63e-02	2.00e-03	9.63e-02	2.56e-03
2000	9.53e-02	1.81e-03	9.67e-02	2.65e-03

TABLE 1. Mean value and (unbiased) standard deviation of the relative errors corresponding to same data of Figure 7.

We notice that the results associated with the sharper upper bound $\boldsymbol{\nu}_1^N$ are better than those provided by $\boldsymbol{\nu}_2^N$. Nevertheless, the difference does not seem to be significative from a qualitative viewpoint. In particular, the two error decays in Figure 7 exhibit the same trend. From Table 1, we notice that the CORSING performance is better in the first case with respect to the mean values and to the standard deviation, but the difference is not impressive.

This experiment seems to justify the use of upper bounds that are not completely sharp.

5.4. **Convergence analysis.** We now perform a convergence analysis of CORSING $\mathcal{HS}$ applied to (47), showing that the mean error shares the same trend as the best s-term approximation error, as predicted by the theoretical results. In particular, the forcing term f is chosen such that the exact solution be

$$u(x) := C_u(1-x)(\exp(100x) - 1),$$

where C_u is chosen such that $|u|_{H^1} = 1$. We take $L = 11$, corresponding to $N = 4095$. For $s = 4, 8, 16, 32$, we define $M = sN$ and $m = \lceil \overline{C} s^{\overline{\gamma}} \log M \log(N/s) \rceil$ for $\overline{\gamma} = 1, 2$, and for different values of $\overline{C}$. For every combination of $\overline{\gamma}$ and $\overline{C}$, we run 100 CORSING experiments and show the mean error obtained $\pm$ the standard deviation, computed using the unbiased estimator. In the case $\overline{\gamma} = 1$, we select $\overline{C} = 0.25, 0.5, 0.75$, whereas for $\overline{\gamma} = 2$, we consider $\overline{C} = 0.01, 0.03, 0.05$. The values of $\overline{C}$ are smaller for $\overline{\gamma} = 2$, in order to ensure that $m < N$ for every s.

The results are shown in Figure 8. The mean error reaches the best s-term approximation rate, that is proportional to $1/s$.

6. CONCLUSIONS

We presented a rigorous formalization and provided a theoretical analysis of the CORSING (COmpRessed SolvING) method [11]. Our analysis essentially relies on the concepts of local a-coherence and restricted inf-sup property (RISP). In particular, we showed how suitable hypotheses on the local a-coherence are sufficient to guarantee the RISP. As a consequence, we provided estimates of the CORSING solution with respect to the best s-term approximation error in expectation (Theorem 3.15)

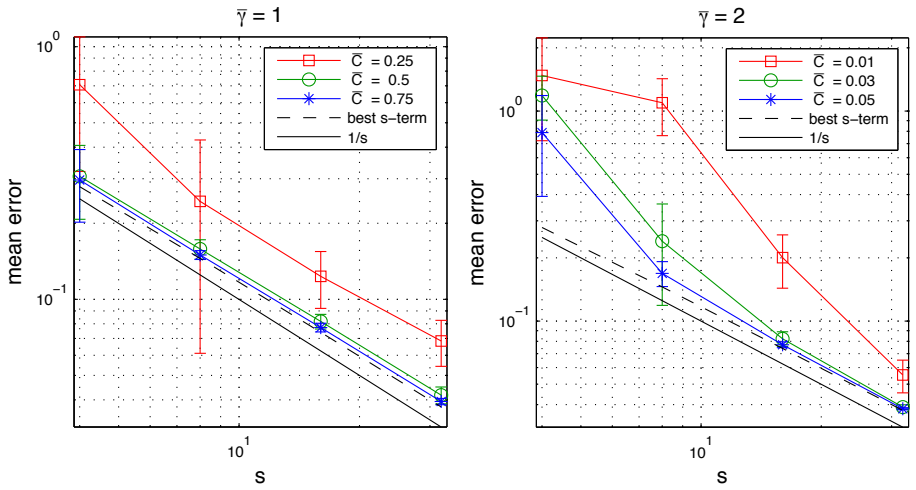

FIGURE 8. Convergence analysis: mean error ± standard deviation and best s-term approximation error. Case $\overline{\gamma} = 1$ (left) and $\overline{\gamma} = 2$ (right).

and in probability (Theorems 3.16 and 3.18). This general theory has been applied to the case of the one-dimensional ADR equation with constant coefficients, and numerical experiments confirm the theoretical results.

Important issues are still open. For instance, the application of our theoretical results to more general cases, such as one-dimensional ADR equation with non-constant coefficients and the two- or three-dimensional case, is not a trivial extension of the results presented here (see Remark 4.8). Another open issue is to assess whether the definition (5) for the probability distribution on the set of test basis functions is the best possible. For example, in the CORSING $\mathcal{HS}$ case, a valuable alternative could be to fully sample at low frequencies and subsample only at higher frequencies, in the spirit of the multi-level sampling schemes studied in [4, 42].

We think that the CORSING method can unveil its full potential when dealing with PDE problems where the stiffness matrix is hard to sparsify, such as nonlocal problems [36, 21]. In this case, it could be competitive with state-of-the-art solvers like *multigrid* [49] or *wavelet methods* [19], whose objective is to sparsify the stiffness matrix as much as possible.

However, this first theoretical analysis of the method highlights the importance of the local a-coherence and the RISP as powerful picklocks, capable to cast the compressed sensing philosophy into the PDEs setting, and these have to be analyzed for each specific problem.

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