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Research Article

Global Conservative and Multipeakon Conservative Solutions for the Modified Camassa-Holm System with Coupling Effects

Huabin Wen,[1] Yujuan Wang,[2] Yongduan Song,[2] and Hamid Reza Karimi[3]

[1] *School of Electrical Engineering, Beijing Jiaotong University, Beijing 100044, China*
[2] *School of Automation, Chongqing University, Chongqing 400044, China*
[3] *Department of Engineering, Faculty of Engineering and Science, University of Agder, 4898 Grimstad, Norway*

Correspondence should be addressed to Yongduan Song; ydsong@cqu.edu.cn

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This paper investigates the continuation of solutions to the modified coupled two-component Camassa-Holm system after wave breaking. The underlying problem is rather challenging due to the mutual coupling effect between two components in the system. By introducing a novel transformation that makes use of a skillfully defined characteristic and a set of newly defined variables, the original system is converted into a Lagrangian equivalent system, from which the global conservative solution is obtained, which further allows for the establishment of the multipeakon conservative solution of the system. The results obtained herein are deemed useful for understanding the inevitable phenomenon near wave breaking.

1. Introduction

We consider here the following modified coupled two-component Camassa-Holm system with peakons [1]:

$$\begin{aligned} &m_t + 2mu_x + m_x u + (mv)_x + nv_x = 0, \quad t > 0,\ x \in R,\\ &n_t + 2nv_x + n_x v + (nu)_x + mu_x = 0, \quad t > 0,\ x \in R,\\ &m = u - u_{xx}, \quad t > 0,\ x \in R,\\ &n = v - v_{xx}, \quad t > 0,\ x \in R. \end{aligned} \tag{1}$$

System (1) is a modified version of the new coupled two-component Camassa-Holm system in the following equation; namely,

$$\begin{aligned} &m_t = 2mu_x + m_x u + (mv)_x + nv_x, \quad t > 0,\ x \in R,\\ &n_t = 2nv_x + n_x v + (nu)_x + mu_x, \quad t > 0,\ x \in R,\\ &m = u - u_{xx}, \quad t > 0,\ x \in R,\\ &n = v - v_{xx}, \quad t > 0,\ x \in R, \end{aligned} \tag{2}$$

which, as an extension of the Camassa-Holm (CH) equation, has been established by Fu and Qu to allow for peakon solitons in the form of a superposition of multipeakons. By parameterizing $\tilde{t} = -t$ for system (2), it then takes the form of (1), which can be rewritten as a Hamiltonian system

$$\frac{\partial}{\partial t}\begin{pmatrix} m \\ n \end{pmatrix} = -\begin{pmatrix} \partial m + m\partial & \partial m + n\partial \\ \partial n + m\partial & \partial n + n\partial \end{pmatrix}\begin{pmatrix} \dfrac{\delta H}{\delta m} = u \\ \dfrac{\delta H}{\delta n} = v \end{pmatrix} \tag{3}$$

with the Hamiltonian $H = (1/2)\int (mG*m + nG*n)dx$, where $G * m = u$, $G * n = v$, and $G = (1/2)e^{-|x|}$. Particularly, when $u = 0$ (or $v = 0$), the degenerated (1) has the same peakon solitons as the CH equation. We are interested in such system because it exhibits the following conserved quantities, as can be easily verified:

$$\begin{aligned} &E_1(u) = \int_R u dx, \qquad E_2(v) = \int_R v dx,\\ &E_3(u) = \int_R m dx, \qquad E_4(u) = \int_R n dx,\\ &E_5(u, v) = \int_R \left(u^2 + u_x^2 + v^2 + v_x^2\right) dx. \end{aligned} \tag{4}$$

Note that, when $u = v$, system (1) is reduced to the scalar Camassa-Holm equation as follows:

$$m_t + 4mu_x + 2m_x u = 0. \tag{5}$$

The CH equation, which models the unidirectional propagation of shallow water waves over a flat bottom, has a bi-Hamiltonian structure [3] and is completely integrable [4–6]. The CH equation has attracted considerable attention because it has peaked solitons [4, 7] and experiences wave breaking [4, 8]. The presence of breaking waves means that the solution remains bounded while its slope becomes unbounded in finite time [8, 9]. After wave breaking, the solutions of the CH equation can be continued uniquely as either global conservative [10–13] or global dissipative solutions [14].

As one of the integrable multicomponent generalizations of the CH equation, system (1) has been shown to be locally well posed with global strong solutions which blow up in finite time [1, 2]. Moreover, the existence issue for a class of local weak solutions for the modified coupled CH2 system was also addressed in [1]. It has been known that the continuation of solutions for the system beyond wave breaking has been a challenging problem. In our recent work [15], the problem of continuation beyond wave breaking for the modified coupled CH2 system was studied by applying an approach that reformulates the system (1) into a semilinear system of O.D.E. taking values in a Banach space. Such treatment makes it possible to investigate the continuity of the solutions beyond collision time, leading to the uniquely global solutions of this system. Also the global dissipative and multipeakon dissipative solutions of this system have been established in [16, 17], while, as far as the authors' concern, there is no effort made in the literature on the study of the global conservative as well as multipeakon conservative solutions of such system, another important feature associated with the system. Motivated by our recent work [15–17], in this paper we develop a new approach to establish a global and stable solution for the modified coupled CH2 system, which is conservative and further allows for the construction of the multipeakon conservative solution. The approach utilized in this paper makes use of a novel system transformation, which is different from [15] and is based on a skillfully defined characteristic and a set of newly introduced variables, where the associated energy is introduced as an additional variable so as to obtain a well-posed initial-value problem, facilitating the study on the behavior of wave breaking. It should be stressed that both global stable solution and multipeakon solution are important aspects related to the solutions near wave breaking, while there is no effort made in the literature on the study of multipeakon property of system (1), which is another motivation of this work. Our inspiration of investing the underlying issue mainly also stems from the early work [10, 11] in the study of the global conservative solution of the CH equation and [13] where the multipeakon solution is obtained for the CH equation. In this work a coupled system is dealt with where the mutual effect between two components makes the analysis more complicated than a single one as considered in [10, 11, 13]. By utilizing the novel transformation method, the inherent difficulty is circumvented and then the global conservative solutions of (1) are obtained, which then allows for the establishment of the multipeakon conservative solution of system (1).

The remainder of this paper is organized as follows. Section 2 presents the basic equations. In Section 3, by introducing a set of Lagrangian variables, we transform the original system into an equivalent semilinear system and derive the global solutions of the equivalent system. We obtain a global continuous semigroup of weak conservative solutions for the original system in Section 4 and the multipeakon conservative solution in Section 5.

2. The Original System

We first introduce an operator $\Lambda = (1 - \partial_x^2)^{-1}$, which can be expressed by its associated Green's function $G = (1/2)e^{-|x|}$ such as $\Lambda f(x) = G * f(x) = (1/2)\int_R e^{-|x-x'|} f(x')dx'$, for all $f \in L^2(R)$, where $*$ denotes the spatial convolution. Thus we can rewrite (1) as a form of a quasilinear evolution equation:

$$\begin{aligned}
&u_t + (u+v)\,u_x + G * (uv_x) + \partial_x G \\
&\quad * \left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right) = 0, \quad t > 0,\ x \in R, \\
&v_t + (u+v)\,v_x + G * (u_x v) + \partial_x G \\
&\quad * \left(v^2 + \frac{1}{2}v_x^2 + u_x v_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2\right) = 0, \quad t > 0,\ x \in R.
\end{aligned} \tag{6}$$

Let us define P_1, P_2, P_3, and P_4 as

$$\begin{aligned}
P_1(t,x) &= G * (uv_x) = \frac{1}{2}\int_R e^{-|x-x'|}(uv_x)(t,x')\,dx', \\
P_2(t,x) &= G * \left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right) \\
&= \frac{1}{2}\int_R e^{-|x-x'|}\left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right) \\
&\qquad \times (t,x')\,dx', \\
P_3(t,x) &= G * (vu_x) = \frac{1}{2}\int_R e^{-|x-x'|}(vu_x)(t,x')\,dx', \\
P_4(t,x) &= G * \left(v^2 + \frac{1}{2}v_x^2 + u_x v_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2\right) \\
&= \frac{1}{2}\int_R e^{-|x-x'|}\left(v^2 + \frac{1}{2}v_x^2 + u_x v_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2\right) \\
&\qquad \times (t,x')\,dx'.
\end{aligned} \tag{7}$$

Then (1) can be rewritten as

$$\begin{aligned}
&u_t + (u+v)\,u_x + P_1 + P_{2,x} = 0, \quad t > 0,\ x \in R, \\
&v_t + (u+v)\,v_x + P_3 + P_{4,x} = 0, \quad t > 0,\ x \in R.
\end{aligned} \tag{8}$$

For regular solutions, we get that the total energy

$$E(t) = \int_R u^2(t,x) + u_x^2(t,x) + v^2(t,x) + v_x^2(t,x)\,dx \quad (9)$$

is constant in time. Thus (8) possesses the H^1-norm conservation law defined as

$$\|z\|_{H^1} = \|u\|_{H^1} + \|v\|_{H^1} = \left(\int_R \left[u^2 + u_x^2 + v^2 + v_x^2\right] dx\right)^{1/2}, \quad (10)$$

where $z(t,x) = (u,v)(t,x)$ denotes the solution of system (8). Note that $z = (u,v) \in H^1 \times H^1$, and so Young's inequality ensures that $P_1, P_2, P_3, P_4 \in H^1$.

3. Global Solutions of the Lagrangian Equivalent System

We reformulate system (8) as follows. For a given initial data $y(0,\xi)$, we define the corresponding characteristic $y(t,\xi)$ as the solution of

$$y_t(t,\xi) = (u+v)\left(t, y(t,\xi)\right), \quad (11)$$

and we define the Lagrangian cumulative energy distribution H as

$$H(t,\xi) = \int_{-\infty}^{y(t,\xi)} \left(u^2 + u_x^2 + v^2 + v_x^2\right)(t,x)\,dx. \quad (12)$$

It is not hard to check that

$$\begin{aligned}&\left(u^2 + u_x^2 + v^2 + v_x^2\right)_t + \left((u+v)\left(u^2 + u_x^2 + v^2 + v_x^2\right)\right)_x \\ &\quad = \left(u^3 - 2uP_2 + v^3 - 2vP_4\right)_x.\end{aligned} \quad (13)$$

Then it follows from (11) and (13) that

$$\frac{dH}{dt} = \left[\left(u^3 - 2uP_2 + v^3 - 2vP_4\right)\left(t, y(t,\xi)\right)\right]_{-\infty}^{\xi}. \quad (14)$$

Throughout the following, we use the notation

$$\begin{aligned}U(t,\xi) &= u\left(t, y(t,\xi)\right), & V(t,\xi) &= v\left(t, y(t,\xi)\right),\\ M(t,\xi) &= u_x\left(t, y(t,\xi)\right), & N(t,\xi) &= v_x\left(t, y(t,\xi)\right).\end{aligned} \quad (15)$$

In the following, we drop the variable t for simplification. Here, we take y as an increasing function for any fixed time t for granted (later on we will prove this). Then after the change of variables $x = y(t,\xi)$ and $x' = y(t,\xi')$, we obtain the following expressions for P_i and $P_{i,x}$ $(i = 1,2,3,4)$; namely,

$$\begin{aligned}P_1(t,\xi) &= P_1\left(t, y(t,\xi)\right)\\ &= \frac{1}{2}\int_R e^{-|y(\xi)-y(\xi')|}\left[(UN)\,y_\xi\right](\xi')\,d\xi',\end{aligned} \quad (16)$$

$$\begin{aligned}P_{1,x}(t,\xi) &= P_{1,x}\left(t, y(t,\xi)\right)\\ &= -\frac{1}{2}\int_R \operatorname{sgn}(\xi-\xi')\,e^{-|y(\xi)-y(\xi')|}\left[(UN)\,y_\xi\right]\\ &\qquad \times(\xi')\,d\xi',\end{aligned} \quad (17)$$

$$\begin{aligned}P_2(t,\xi) &= P_2\left(t, y(t,\xi)\right)\\ &= \frac{1}{2}\int_R e^{-|y(\xi)-y(\xi')|}\\ &\qquad \times\left[\left(U^2 + \frac{1}{2}M^2 + MN + \frac{1}{2}V^2 - \frac{1}{2}N^2\right)y_\xi\right]\\ &\qquad \times(\xi')\,d\xi',\\ P_{2,x}(t,\xi) &= P_{2,x}\left(t, y(t,\xi)\right)\\ &= -\frac{1}{2}\int_R \operatorname{sgn}(\xi-\xi')\,e^{-|y(\xi)-y(\xi')|}\\ &\qquad \cdot\left[\left(U^2 + \frac{1}{2}M^2 + MN + \frac{1}{2}V^2 - \frac{1}{2}N^2\right)y_\xi\right]\\ &\qquad \times(\xi')\,d\xi',\end{aligned}$$

$$\begin{aligned}P_3(t,\xi) &= P_3\left(t, y(t,\xi)\right)\\ &= \frac{1}{2}\int_R e^{-|y(\xi)-y(\xi')|}\left[(VM)\,y_\xi\right](\xi')\,d\xi',\\ P_{3,x}(t,\xi) &= P_{3,x}\left(t, y(t,\xi)\right)\\ &= -\frac{1}{2}\int_R \operatorname{sgn}(\xi-\xi')\,e^{-|y(\xi)-y(\xi')|}\left[(VM)\,y_\xi\right]\\ &\qquad \times(\xi')\,d\xi',\end{aligned}$$

$$\begin{aligned}P_4(t,\xi) &= P_4\left(t, y(t,\xi)\right)\\ &= \frac{1}{2}\int_R e^{-|y(\xi)-y(\xi')|}\\ &\qquad \times\left[\left(V^2 + \frac{1}{2}N^2 + MN + \frac{1}{2}U^2 - \frac{1}{2}M^2\right)y_\xi\right]\\ &\qquad \times(\xi')\,d\xi',\\ P_{4,x}(t,\xi) &= P_{4,x}\left(t, y(t,\xi)\right)\\ &= -\frac{1}{2}\int_R \operatorname{sgn}(\xi-\xi')\,e^{-|y(\xi)-y(\xi')|}\\ &\qquad \cdot\left[\left(V^2 + \frac{1}{2}N^2 + MN + \frac{1}{2}U^2 - \frac{1}{2}M^2\right)y_\xi\right]\\ &\qquad \times(\xi')\,d\xi'.\end{aligned} \quad (18)$$

Since $H_\xi = (u^2 + u_x^2 + v^2 + v_x^2) \circ y y_\xi$, then P_2, $P_{2,x}$, P_4, and $P_{4,x}$ can be rewritten as

$$\begin{aligned}
P_2(t,\xi) &= P_2(t, y(\xi)) \\
&= \frac{1}{4}\int_R e^{-|y(\xi)-y(\xi')|}\left[H_\xi + \left(U^2 + 2MN - N^2\right) y_\xi\right] \\
&\qquad \times \left(\xi'\right) d\xi', \\
P_{2,x}(t,\xi) &= P_{2,x}(t, y(\xi)) \\
&= -\frac{1}{4}\int_R \operatorname{sgn}\left(\xi - \xi'\right) e^{-|y(\xi)-y(\xi')|} \\
&\qquad \times \left[H_\xi + \left(U^2 + 2MN - N^2\right) y_\xi\right] \\
&\qquad \times \left(\xi'\right) d\xi', \\
P_4(t,\xi) &= P_4(t, y(\xi)) \\
&= \frac{1}{4}\int_R e^{-|y(\xi)-y(\xi')|}\left[H_\xi + \left(V^2 + 2MN - M^2\right) y_\xi\right] \\
&\qquad \times \left(\xi'\right) d\xi', \\
P_{4,x}(t,\xi) &= P_{4,x}(t, y(\xi)) \\
&= -\frac{1}{4}\int_R \operatorname{sgn}\left(\xi - \xi'\right) e^{-|y(\xi)-y(\xi')|} \\
&\qquad \times \left[H_\xi + \left(V^2 + 2MN - M^2\right) y_\xi\right] \times \left(\xi'\right) d\xi'.
\end{aligned} \tag{19}$$

From the definition of the characteristic, it is not hard to check that

$$\begin{aligned}
U_t(t,\xi) &= u_t(t,y) + u_x(t,y)\, y_t(t,\xi) \\
&= \left(-P_1 - P_{2,x}\right) \circ y(t,\xi), \\
V_t(t,\xi) &= v_t(t,y) + v_x(t,y)\, y_t(t,\xi) \\
&= \left(-P_3 - P_{4,x}\right) \circ y(t,\xi), \\
M_t(t,\xi) &= u_{xt}(t,y) + u_{xx}(t,y)\, y_t(t,\xi) \\
&= \left(-\frac{1}{2}M^2 - \frac{1}{2}N^2 + U^2 + \frac{1}{2}V^2 - P_{1,x} - P_2\right) \\
&\qquad \circ y(t,\xi), \\
N_t(t,\xi) &= v_{xt}(t,y) + v_{xx}(t,y)\, y_t(t,\xi) \\
&= \left(-\frac{1}{2}N^2 - \frac{1}{2}M^2 + V^2 + \frac{1}{2}U^2 - P_{3,x} - P_4\right) \\
&\qquad \circ y(t,\xi).
\end{aligned} \tag{20}$$

We introduce another variable $\varsigma(t,\xi)$ with $\varsigma(t,\xi) = y(t,\xi) - \xi$. It will turn out that $\varsigma \in L^\infty(R)$. With these new variables, we now derive an equivalent system of (8) as follows:

$$\begin{aligned}
\varsigma_t &= U + V, \\
U_t &= -P_1 - P_{2,x}, \\
V_t &= -P_3 - P_{4,x}, \\
M_t &= \left(-\frac{1}{2}M^2 - \frac{1}{2}N^2 + U^2 + \frac{1}{2}V^2 - P_{1,x} - P_2\right), \\
N_t &= \left(-\frac{1}{2}N^2 - \frac{1}{2}M^2 + V^2 + \frac{1}{2}U^2 - P_{3,x} - P_4\right), \\
H_t &= U^3 - 2UP_2 + V^3 - 2VP_4,
\end{aligned} \tag{21}$$

where P_1 and P_3 are given in (18), while P_2, $P_{2,x}$, P_4, and $P_{4,x}$ are given in (19). We regard system (21) as a system of ordinary differential equations in the Banach space

$$E = W \times H^1 \times H^1 \times L^2 \times L^2 \times W \tag{22}$$

endowed with the norm

$$\|X\|_E = \|\varsigma\|_W + \|U\|_{H^1} + \|V\|_{H^1} + \|M\|_{L^2} + \|N\|_{L^2} + \|H\|_W, \tag{23}$$

for any $X = (\varsigma, U, V, M, N, H) \in E$. Here $W = \{f \in C(R) \cap L^\infty(R) \mid f_\xi \in L^2(R)\}$ is a Banach space with the norm given by $\|f\|_W = \|f\|_{L^\infty(R)} + \|f_\xi\|_{L^2(R)}$. Note that $H^1(R) \subset W$.

Differentiating (21) with respect to the variable ξ yields

$$\begin{aligned}
\varsigma_{\xi t} &= U_\xi + V_\xi, \\
U_{\xi t} &= \frac{1}{2}H_\xi + \left(\frac{1}{2}U^2 + MN - N^2 - P_2 - P_{1,x}\right) y_\xi, \\
V_{\xi t} &= \frac{1}{2}H_\xi + \left(\frac{1}{2}V^2 + MN - M^2 - P_4 - P_{3,x}\right) y_\xi, \\
H_{\xi t} &= \left(3U^2 - 2P_2\right) U_\xi - 2UP_{2,x} y_\xi \\
&\qquad + \left(3V^2 - 2P_4\right) V_\xi - 2VP_{4,x} y_\xi,
\end{aligned} \tag{24}$$

which are semilinear with respect to the variables y_ξ, U_ξ, V_ξ, and H_ξ.

To obtain the uniqueness of solutions, one proceeds as follows. By proving that all functions on the right-hand side of (21) are locally Lipschitz continuous, the local existence of solutions will follow from the standard theory of ordinary differential equations in Banach spaces. In a second step, we will then prove that this local solution can be extended globally in time. Note that global solutions of (21) may not exist for all initial data in E. However they exist when the initial data $\overline{X} = (\overline{\varsigma}, \overline{U}, \overline{V}, \overline{M}, \overline{N}, \overline{H})$ belongs to the set Γ which is defined as follows.

Definition 1. The set Γ is composed of all $X = (\varsigma, U, V, M, N, H) \in E$ such that

(i)

$$(\varsigma, U, V, H) \in \left[W^{1,\infty}(R)\right]^4, \tag{25}$$

(ii)

$$\begin{aligned} &y_\xi \geq 0, H_\xi \geq 0, \quad y_\xi + H_\xi > 0 \quad \text{a.e.},\\ &\lim_{\xi \to -\infty} H(\xi) = 0, \end{aligned} \tag{26}$$

(iii)

$$y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2 \quad \text{a.e.}, \tag{27}$$

where $W^{1,\infty}(R) = \{f \in C(R) \cap L^\infty(R) \mid f_\xi \in L^\infty(R)\}$ and $y(\xi) = \varsigma(\xi) + \xi$.

Lemma 2. *Let $\mathfrak{R}_1 : E \to W$ and let $\mathfrak{R}_2 : E \to H^1$, or let $\mathfrak{R}_2 : E \to W$ be two locally Lipschitz maps. Then, the product $X \to \mathfrak{R}_1(X)\mathfrak{R}_2(X)$ is also a locally Lipschitz map from E to H^1 or from E to W.*

Theorem 3. *Given initial data $\overline{X} = (\overline{\varsigma}, \overline{U}, \overline{V}, \overline{M}, \overline{N}, \overline{H}) \in E$, there exists a time T depending only on $\|\overline{X}\|_E$ such that the system* (21) *admits a unique solution in $C^1([0,T], E)$.*

Proof. To obtain the local existence of solutions, it suffices to show that $F(X)$, given by

$$\begin{aligned} F(X) = \Big(&U + V, -P_1 - P_{2,x}, -P_3 - P_{4,x}, -\frac{1}{2}M^2 - \frac{1}{2}N^2 + U^2\\ &+ \frac{1}{2}V^2 - P_{1,x} - P_2, -\frac{1}{2}N^2 - \frac{1}{2}M^2 + V^2 + \frac{1}{2}U^2\\ &-P_{3,x} - P_4, U^3 - 2UP_2 + V^3 - 2VP_4\Big) \end{aligned} \tag{28}$$

with $X = (\varsigma, U, V, M, N, H)$, is a Lipchitz function on any bounded set of E which is a Banach space.

Our main task is to prove the Lipschitz continuity of P_i and $P_{i,x}$ $(i = 1, 2, 3, 4)$ given by (18) and (19) from E to $H^1(R)$. We first prove that $P_{2,x}$ given in (19) is locally Lipschitz continuous from E to $H^1(R)$ and the others follow in the same way. Let us write

$$\begin{aligned} &P_{2,x}(\xi)\\ &= -\frac{e^{-\varsigma(\xi)}}{4}\int_R \chi_{\{\xi'<\xi\}} e^{-|\xi-\xi'|} e^{\varsigma(\xi')}\\ &\qquad \times \left[H_\xi + \left(U^2 + 2MN - N^2\right)\left(1 + \varsigma_\xi\right)\right](\xi')\, d\xi'\\ &\quad + \frac{e^{\varsigma(\xi)}}{4}\int_R \chi_{\{\xi'>\xi\}} e^{-|\xi-\xi'|} e^{-\varsigma(\xi')}\\ &\qquad \times \left[H_\xi + \left(U^2 + 2MN - N^2\right)\left(1 + \varsigma_\xi\right)\right](\xi')\, d\xi', \end{aligned} \tag{29}$$

where χ_Ω denotes the indicator function of a given set Ω. Let

$$\begin{aligned} &P_{2,x}^1(\xi)\\ &= -\frac{e^{-\varsigma(\xi)}}{4}\int_R \chi_{\{\xi'<\xi\}} e^{-|\xi-\xi'|} e^{\varsigma(\xi')}\\ &\qquad \times \left[H_\xi + \left(U^2 + 2MN - N^2\right)\left(1 + \varsigma_\xi\right)\right](\xi')\, d\xi',\\ &P_{2,x}^2(\xi)\\ &= \frac{e^{\varsigma(\xi)}}{4}\int_R \chi_{\{\xi'>\xi\}} e^{-|\xi-\xi'|} e^{-\varsigma(\xi')}\\ &\qquad \times \left[H_\xi + \left(U^2 + 2MN - N^2\right)\left(1 + \varsigma_\xi\right)\right](\xi')\, d\xi'. \end{aligned} \tag{30}$$

We rewrite $P_{2,x}^1(\xi)$ as

$$P_{2,x}^1(\xi) = -\frac{e^{-\varsigma(\xi)}}{2}\Lambda \circ R(X)(\xi), \tag{31}$$

where R is the operator from E to $L^2(R)$ given as

$$R(X)(\xi) = \chi_{\{\xi'<\xi\}} e^{\varsigma}\left[H_\xi + \left(U^2 + 2MN - N^2\right)\left(1 + \varsigma_\xi\right)\right]. \tag{32}$$

Since the operator Λ (given in Section 2) is linear and continuous from $H^{-1}(R)$ to $H^1(R)$ and $L^2(R)$ is continuously embedded in $H^{-1}(R)$, we have $\Lambda \circ R(X) \in H^1$. It is not hard to check that R is locally Lipschitz continuous from E to $L^2(R)$ and therefore from E to $H^{-1}(R)$. Thus $\Lambda \circ R$ is locally Lipschitz continuous from E to $H^1(R)$. Since the mapping $X \to e^{-\varsigma}$ is locally Lipschitz continuous from E to W, by Lemma 2, we deduce that $P_{2,x}^1(\xi)$ is locally Lipschitz continuous from E to $H^1(R)$. Similarly, $P_{2,x}^2(\xi)$ is also locally Lipschitz continuous and therefore $P_{2,x}(\xi)$ is locally Lipschitz continuous. One proceeds in the same way and proves that P_1, $P_{1,x}$, P_3, and $P_{3,x}$ defined by (18) and P_2, P_4, and $P_{4,x}$ defined by (19) are locally Lipschitz continuous from E to $H^1(R)$. We rewrite the solutions of (21) as

$$X(t) = \overline{X} + \int_0^t F(X(\tau))\, d\tau. \tag{33}$$

Thus the theorem follows from the standard contraction argument of ordinary differential equations. □

It remains to prove the existence of global solutions of (21). Theorem 3 gives us the existence of local solutions of (21) for initial data in E. In the following, we will only consider initial data that belongs to $\widetilde{E}$ given by $\widetilde{E} = E \cap [(W^{1,\infty}(R))^3 \cap (L^2)^2 \cap W^{1,\infty}(R)]$. To obtain that the solution of (21) belongs to $\widetilde{E}$, we have to specify the initial condition for (24). Let

$$\begin{aligned} \Omega = \Big\{&\xi \in R \mid \left|\overline{\varsigma}_\xi(\xi)\right| \leq \left\|\overline{\varsigma}_\xi\right\|_{L^\infty}, \left|\overline{U}_\xi(\xi)\right| \leq \left\|\overline{U}_\xi\right\|_{L^\infty},\\ &\left|\overline{V}_\xi(\xi)\right| \leq \left\|\overline{V}_\xi\right\|_{L^\infty}, \left|\overline{H}_\xi(\xi)\right| \leq \left\|\overline{H}_\xi\right\|_{L^\infty}\Big\}. \end{aligned} \tag{34}$$

We have $\text{meas}(\Omega^c) = 0$. For $\xi \in \Omega^c$, $(\varsigma_\xi, U_\xi, V_\xi, H_\xi)(0,\xi)$ is taken as $(0,0,0,0)$, and $(\varsigma_\xi, U_\xi, V_\xi, H_\xi)(0,\xi)$ is given as $(\overline{\varsigma}_\xi, \overline{U}_\xi, \overline{V}_\xi, \overline{H}_\xi)(\xi)$, for $\xi \in \Omega$.

The global existence of the solution for initial data in Γ relies essentially on the fact that the set Γ is preserved by the flow as the next lemma shows.

Lemma 4. *Given initial data $\overline{X} = (\overline{\varsigma}, \overline{U}, \overline{V}, \overline{M}, \overline{N}, \overline{H}) \in \Gamma$, one considers the local solution $X(t) = (\varsigma, U, V, M, N, H)(t) \in C([0,T], E)$ of (21) with initial data $\overline{X}$ for some $T > 0$. One then gets that $X(t) \in \Gamma$ for all $t \in [0,T]$. Moreover, for a.e. $t \in [0,T]$, $y_\xi(t,\xi) > 0$, for a.e. $\xi \in R$, and $\lim_{\xi \to \pm\infty} H(t,\xi)$ exists and is independent of time for all $t \in [0,T]$.*

Proof. We first show that $X(t) \in \Gamma$ for all $t \in [0,T]$. For any given initial data $\overline{X} \in \widetilde{E}$, we get that the local solution $(\varsigma, U, V, H)(t)$ of (21) belongs to $[W^{1,\infty}(R)]^4$, which satisfies (25) for all $t \in [0,T]$. We now show that (27) holds for any $\xi \in \Omega$ and therefore a,e.. Consider a fixed $\xi \in \Omega$ and drop it in the notation if there is no ambiguity. On the one hand, it follows from (24) that

$$\begin{aligned}
&\left(y_\xi H_\xi\right)_t \\
&\quad = y_{\xi t} H_\xi + y_\xi H_{\xi t} \\
&\quad = \left(U_\xi + V_\xi\right) H_\xi + y_\xi \left[\left(3U^2 - 2P_2\right) U_\xi + \left(3V^2 - 2P_4\right) V_\xi \right. \\
&\qquad\qquad \left. -2UP_{2,x} y_\xi - 2VP_{4,x} y_\xi\right] \\
&\quad = U_\xi H_\xi + V_\xi H_\xi + 3U^2 U_\xi y_\xi - 2P_2 U_\xi y_\xi + 3V^2 V_\xi y_\xi \\
&\qquad - 2P_4 V_\xi y_\xi - 2UP_{2,x} y_\xi^2 - 2VP_{4,x} y_\xi^2,
\end{aligned} \tag{35}$$

and, on the other hand,

$$\begin{aligned}
&\left(y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2\right)_t \\
&\quad = 2y_\xi y_{\xi t} U^2 + 2y_\xi^2 U U_t + 2U_\xi U_{\xi t} + 2y_\xi y_{\xi t} V^2 \\
&\qquad + 2y_\xi^2 V V_t + 2V_\xi V_{\xi t} \\
&\quad = U_\xi H_\xi + V_\xi H_\xi + 3U^2 U_\xi y_\xi + 3V^2 V_\xi y_\xi - 2P_2 U_\xi y_\xi \\
&\qquad - 2P_4 V_\xi y_\xi - 2UP_{2,x} y_\xi^2 - 2VP_{4,x} y_\xi^2.
\end{aligned} \tag{36}$$

Hence, $\left(y_\xi H_\xi\right)_t = \left(y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2\right)_t$. Notice that $y_\xi H_\xi(0) = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2)(0)$; then $y_\xi H_\xi(t) = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2)(t)$ for all $t \in [0,T]$ and (27) has been proved. We now prove the inequalities in (26). Set $t^* = \sup\{t \in [0,T] | y_\xi(t') \geq 0 \text{ for all } t' \in [0,t]\}$. Assume that $t^* < T$. Since $y_\xi(t)$ is continuous with respect to t, we have $y_\xi(t^*) = 0$. It follows from (27) that $U_\xi(t^*) = V_\xi(t^*) = 0$. Furthermore, (24) implies that $y_{\xi t}(t^*) = U_\xi(t^*) + V_\xi(t^*) = 0$ and $y_{\xi tt}(t^*) = (U_{\xi t} + V_{\xi t})(t^*) = H_\xi(t^*)$. If $H_\xi(t^*) = 0$, then $(y_\xi, U_\xi, V_\xi, H_\xi)(t^*) = (0,0,0,0)$ which implies that $(y_\xi, U_\xi, V_\xi, H_\xi)(t) = 0$ for all $t \in [0,T]$ by the uniqueness of the solution of system (24). This contradicts the fact that $y_\xi(0) + H_\xi(0) > 0$ for all $\xi \in \Omega$. If $H_\xi(t^*) < 0$, then $y_{\xi tt}(t^*) < 0$. Since $y_\xi(t^*) = y_{\xi t}(t^*) = 0$, there exists a neighborhood ϖ of t^* such that $y_\xi(t) < 0$ for all $t \in \varpi/\{t^*\}$. This contradicts the definition of t^*. Hence, $H_\xi(t^*) > 0$. We now have $y_{\xi tt}(t^*) > 0$, which conversely implies that $y_\xi(t) > 0$ for all $t \in \varpi/\{t^*\}$, which contradicts the fact that $t^* < T$. Thus we have proved $y_\xi(t) \geq 0$ for all $t \in [0,T]$. We now prove that $H_\xi \geq 0$ for all $t \in [0,T]$. This follows from (27) when $y_\xi(t) > 0$. If $y_\xi(t) = 0$, then $U_\xi(t) = V_\xi(t) = 0$ from (27). As we have seen, $H_\xi < 0$ would imply that $y_\xi(t') < 0$ for some t' in a punctured neighborhood of t, which is impossible. Hence, $H_\xi \geq 0$ for all $t \in [0,T]$. Now we get that $y_\xi(t) + H_\xi(t) \geq 0$ for all $t \in [0,T]$. If $y_\xi(t') + H_\xi(t') = 0$ for some t', it then follows that $(y_\xi, U_\xi, V_\xi, H_\xi)(t') = 0$ which implies that $(y_\xi, U_\xi, V_\xi, H_\xi)(t) = 0$ for all $t \in [0,T]$, which contradicts the fact that $y_\xi(0) + H_\xi(0) > 0$ for all $\xi \in \Omega$. Hence, $y_\xi(t) + H_\xi(t) > 0$. This completes the proof that $X(t) \in \Gamma$ for all $t \in [0,T]$.

We now prove that $y_\xi(t,\xi) > 0$ for almost all t. Define the set $\Theta = \{(t,\xi) \in [0,T] \times R \mid y_\xi(t,\xi) = 0\}$. It follows from Fubini's theorem that

$$\text{meas}(\Theta) = \int_R \text{meas}\left(\Theta_\xi\right) d\xi = \int_{[0,T]} \text{meas}\left(\Theta_t\right) dt, \tag{37}$$

where $\Theta_\xi = \{t \in [0,T] \mid y_\xi(t,\xi) = 0\}$ and $\Theta_t = \{\xi \in R \mid y_\xi(t,\xi) = 0\}$. From the above proof, we know that, for all $\xi \in \Omega$, Θ_ξ consists of isolated points that are countable. This means that $\text{meas}(\Theta_\xi) = 0$. Since $\text{meas}(\Omega^c) = 0$, it thus follows from (37) that $\text{meas}(\Theta_t) = 0$ for almost every $t \in [0,T]$. This implies that $y_\xi(t,\xi) > 0$ for almost all t and therefore $y(t,\xi)$ is strictly increasing and invertible with respect to ξ.

For any given $t \in [0,T]$, since $H_\xi(t) \geq 0$ and $H(t,\xi) \in L^\infty(R)$, we know that $H(t,\pm\infty)$ exist. We have the following:

$$H(t,\xi) = H(0,\xi) + \int_0^t \left(U^3 - 2P_2 U + V^3 - 2P_4 V\right)(\tau,\xi)\, d\tau. \tag{38}$$

Let $\xi \to \pm\infty$. Since U, V, P_2, P_4 are bounded in $L^\infty([0,T] \times R)$ and $\lim_{\xi \to \pm\infty} U(t,\xi) = \lim_{\xi \to \pm\infty} V(t,\xi) = 0$ as $U(t,\cdot)$, $V(t,\cdot) \in H^1(R)$ for all $t \in [0,T]$, (38) implies that $H(t,\pm\infty) = H(0,\pm\infty)$ for all $t \in [0,T]$. Since $\overline{X} \in \Gamma$, it follows that $H(0,\pm\infty) = 0$ for all $t \in [0,T]$. □

Theorem 5. *For any initial data $\overline{X} = (\overline{y}, \overline{U}, \overline{V}, \overline{M}, \overline{N}, \overline{H}) \in \Gamma$, there exists a unique global solution $X(t) = (y, U, V, M, N, H)(t) \in C^1(R^+, E)$ for the system (21). Moreover, for all $t \in R^+$, we have $X(t) \in \Gamma$, which constructs a continuous semigroup.*

Proof. To ensure that the local solution $X = (\varsigma, U, V, M, N, H) \in C([0,T], E)$ of system (21) can be extended to a global solution, it suffices to show that

$$\sup_{t \in [0,T)} \|\varsigma(t,\cdot), U(t,\cdot), V(t,\cdot), M(t,\cdot), N(t,\cdot), H(t,\cdot)\|_E < \infty. \tag{39}$$

Since $H(t,\xi)$ is an increasing function with respect to ξ for all t and $\lim_{\xi\to\infty}H(t,\xi) = \lim_{\xi\to\infty}H(0,\xi)$, we have $\sup_{t\in[0,T)}\|H(t,\cdot)\|_{L^\infty(R)} = \|\overline{H}\|_{L^\infty(R)} < \infty$. We now consider a fixed $t \in [0,T)$ and drop it for simplification. Since $U_\xi(\xi) = V_\xi(\xi) = 0$ when $y_\xi(\xi) = 0$ and $y_\xi(\xi) > 0$, for a. e. ξ, it follows from (27) that

$$\begin{aligned} U^2(\xi) &= 2\int_{-\infty}^{\xi} U(\xi')U_\xi(\xi')\,d\xi' \\ &= 2\int_{\{\xi'<\xi\mid y_\xi(\xi')>0\}} U(\xi')U_\xi(\xi')\,d\xi' \\ &\le \int_{\{\xi'<\xi\mid y_\xi(\xi')>0\}} \left(y_\xi U^2 + \frac{U_\xi^2}{y_\xi}\right)(\xi')\,d\xi' \\ &\le \int_R H_\xi(\xi')\,d\xi' = H(\xi), \end{aligned} \tag{40}$$

which implies that

$$\sup_{t\in[0,T)}\left\|U^2(t,\cdot)\right\|_{L^\infty} \le \sup_{t\in[0,T)}\|H(t,\cdot)\|_{L^\infty(R)} = \left\|\overline{H}\right\|_{L^\infty(R)} < \infty, \tag{41}$$

and therefore

$$\sup_{t\in[0,T)}\|U(t,\cdot)\|_{L^\infty} < \infty. \tag{42}$$

Similarly,

$$\sup_{t\in[0,T)}\|V(t,\cdot)\|_{L^\infty} < \infty. \tag{43}$$

We can obtain from the governing equation (21) that

$$\left|\varsigma(t,\xi)\right| \le \left|\varsigma(0,\xi)\right| + \sup_{t\in[0,T)}\left(\|U(t,\cdot)\|_{L^\infty} + \|V(t,\cdot)\|_{L^\infty}\right)T, \tag{44}$$

and then $\sup_{t\in[0,T)}\|\varsigma(t,\cdot)\|_{L^\infty} < \infty$. We can also get from the governing equation (21) that

$$\sup_{t\in[0,T)}\|M(t,\cdot)\|_{L^\infty} < \infty, \qquad \sup_{t\in[0,T)}\|N(t,\cdot)\|_{L^\infty} < \infty. \tag{45}$$

From the identity $H_\xi = (U^2+M^2+V^2+N^2)y_\xi$, we can deduce that

$$\left|\left(U^2+2MN-N^2\right)y_\xi\right| \le \left(U^2+M^2+N^2+N^2\right)y_\xi \le 2H_\xi, \tag{46}$$

which implies that

$$\begin{aligned} &\left|P_{2,x}\right| \\ &\le \frac{1}{2}\left|\int_R e^{-|y(\xi)-y(\xi')|}\left[H_\xi + \left(U^2+2MN-N^2\right)y_\xi\right](\xi')\,d\xi'\right| \\ &\le \frac{1}{2}\left|\int_R e^{-|y(\xi)-y(\xi')|}3H_\xi(\xi')\,d\xi'\right| \\ &\le C\sup_{t\in[0,T)}\|H(t,\cdot)\|_{L^\infty(R)} < \infty. \end{aligned} \tag{47}$$

Therefore, $\|P_{2,x}\|_{L^\infty} < \infty$. It is not hard to know that $\|P_{2,x}\|_{L^2} \le C\|e^{-y(\xi)}\|_{L^2}\cdot\sup_{t\in[0,T)}\|H(t,\cdot)\|_{L^\infty(R)} < \infty$. Similarly, one can obtain that the bounds hold for P_1, $P_{1,x}$, P_2, P_3, $P_{3,x}$, P_4, and $P_{4,x}$. Let

$$\begin{aligned} Z(t) = {} & \|U(t,\cdot)\|_{L^2} + \left\|U_\xi(t,\cdot)\right\|_{L^2} + \|V(t,\cdot)\|_{L^2} \\ & + \left\|V_\xi(t,\cdot)\right\|_{L^2} + \left\|\varsigma_\xi(t,\cdot)\right\|_{L^2} + \left\|H_\xi(t,\cdot)\right\|_{L^2}. \end{aligned} \tag{48}$$

Using the integrated version of (21) and (24), after taking the L^2-norms on both sides, we obtain

$$Z(t) \le Z(0) + C\int_0^t Z(\tau)\,d\tau. \tag{49}$$

It follows from Gronwall's inequality that $\sup_{t\in[0,T)}Z(t) < \infty$. Hence, we infer that the map $S_t : \Gamma \to \Gamma\times R^+$ defined as

$$S_t\left(\overline{X}\right) = X(t) \tag{50}$$

generates a continuous semigroup from the standard theory of ordinary differential equations. □

4. Global Conservative Solutions of the Original System

We show that the global solution of the equivalent system (21) yields a global conservative solution of the original system (8), which constructs a continuous semigroup in this section.

To obtain the global conservative solution of the original system, we have to establish the correspondence between the Lagrangian equivalent system and the original system.

Let us first introduce the subsets F and F_α of Γ given by

$$\begin{aligned} F &= \{X = (y,U,V,M,N,H) \in \Gamma \mid y + H \in G\}, \\ F_\alpha &= \{X = (y,U,V,M,N,H) \in \Gamma \mid y + H \in G_\alpha\}, \end{aligned} \tag{51}$$

where G is defined as

$$\begin{aligned} G = \{&f \text{ is invertible} \mid f - Id, \\ &f^{-1} - Id \text{ both belong to } W^{1,\infty}(R)\}. \end{aligned} \tag{52}$$

And, for any $\alpha > 1$, the subsets G_α of G are given by

$$G_\alpha = \left\{f \in G \mid \|f - Id\|_{W^{1,\infty}(R)} + \left\|f^{-1} - Id\right\|_{W^{1,\infty}(R)} \le \alpha\right\}, \tag{53}$$

with a useful characterization. If $f \in G_\alpha$ ($\alpha \ge 0$), then $1/(1+\alpha) \le f_\xi \le 1+\alpha$ a.e. Conversely, if f is absolutely continuous, $f - Id \in L^\infty(R)$ and there exists $c \ge 1$ such that $1/c \le f_\xi \le c$ a.e., and then $f \in G_\alpha$ for some α depending only on c and $\|f - Id\|_{L^\infty(R)}$. With this useful characterization of G_α, it is not hard to prove that the space F is preserved by the governing equation (21). Notice that the map $(f,X) \to X\circ f$ defines a group action of G on F; we consider the quotient space F/G of F with respect to the group action. The equivalence relation on F is defined as follows: for any $X, X' \in F$, if there exists

$f \in G$ such that $X' = X \circ f$, we claim that X and X' are equivalent.

We denote the projection $\Pi : F \to F/G$ by $\Pi(X) = [X]$. For any $X = (y, U, V, M, N, H) \in F$, we introduce the mapping $\Upsilon : F \to F_0$ given by $\Upsilon(X) = X \circ (y + H)^{-1}$. It is not hard to prove that $\Upsilon(X) = X$ when $X \in F_0$ and $\Upsilon(X \circ f) = \Upsilon(X)$ for any $X \in F$ and $f \in G$. Hence, we can define the map $\widetilde{\Upsilon} : F/G \to F_0$ as $\widetilde{\Upsilon}([X]) = \Upsilon(X)$, for any representative $[X] \in F/G$ of $X \in F$. For any $X \in F_0$, we have $\widetilde{\Upsilon} \circ \Pi(X) = \Upsilon(X) = X$. Hence, $\widetilde{\Upsilon} \circ \Pi_{|F_0} = Id_{|F_0}$ and any topology defined on F_0 is naturally transported into F/G by this isomorphism. That is, if we equip F_0 with the metric induced by the E-norm; that is, $d_{F_0}(X, X') = \|X - X'\|_E$, for all $X, X' \in F_0$, which is complete, then the topology on F/G is defined by a complete metric given by $d_{F/G}([X], [X']) = \|\Upsilon(X) - \Upsilon(X')\|_E$ for any $[X], [X'] \in F/G$. Let us denote by $S : F \times R^+ \to F$ the continuous semigroup which to any initial data $\overline{X} \in F$ associates the solution $X(t)$ of (21). The system (8) is invariant with respect to relabeling. That is, for any $t > 0$, $S_t(X \circ f) = S_t(X) \circ f$, for an $X \in F$ and $f \in G$. Thus the map $\widetilde{S}_t : F/G \to F/G$ given by $\widetilde{S}_t([X]) = [S_t X]$ is well-defined, which generates a continuous semigroup.

To obtain a semigroup of solution for (8), we have to consider the space D, which characterizes the solutions in the original system:

$$D = \left\{(z, \mu) \mid z \in H^1(R) \times H^1(R), \; \mu_{\mathrm{ac}} = \left(u^2 + u_x^2 + v^2 + v_x^2\right) dx\right\}, \tag{54}$$

where $z = (u, v)$ and μ is a positive finite Radon measure with μ_{ac} as its absolute continuous part.

We now establish a bijection between F/G and D to transport the continuous semigroup obtained in the Lagrangian equivalent system (functions in F/G) into the original system (functions in D).

We first introduce the mapping $L : D \to F/G$, which transforms the original system into the Lagrangian equivalent system defined as follows.

Definition 6. For any $(z, \mu) \in D$, let

$$y(\xi) = \sup\{y \mid \mu(-\infty, y) + y < \xi\}, \tag{55}$$

$$\begin{aligned} U(\xi) &= u \circ y(\xi), \qquad V(\xi) = v \circ y(\xi), \\ M(\xi) &= u_x \circ y(\xi), \qquad N(\xi) = v_x \circ y(\xi), \end{aligned} \tag{56}$$

$$H(\xi) = \xi - y(\xi), \tag{57}$$

with $z = (u, v)$. We define $L(z, \mu) \in F/G$ as the equivalence class of (y, U, V, M, N, H).

Remark 7. From the definition of y, U, V, M, N, H in (55)–(57), we can check that $X = (y, U, V, M, N, H) \in E$, which also satisfies (25). Moreover, we have $y + H = Id$ from (57), which implies that $X = (y, U, V, M, N, H) \in F_0$. Furthermore, if μ is absolutely continuous, then $\mu = (u^2 + u_x^2 + v^2 + v_x^2)dx$ and

$$\int_{-\infty}^{y(\xi)} \left(u^2 + u_x^2 + v^2 + v_x^2\right) dx + y(\xi) = \xi, \tag{58}$$

for all $\xi \in R$.

We are led to the mapping M, which corresponds to the transformation from the Lagrangian equivalent system into the original system. In the other direction, we obtain the energy density μ in the original system, by pushing forward by y the energy density $H_\xi d\xi$ in the Lagrangian equivalent system, where the push-forward $f_\# \nu$ of a measure ν by a measurable function f is defined as

$$f_\# \nu(B) = \nu\left(f^{-1}(B)\right), \tag{59}$$

for all the Borel set B. Give any element $[X] \in F/G$, and let (z, μ) be defined as

$$z(x) = Z(\xi) \quad \text{for any } \xi \text{ such that } x = y(\xi), \tag{60}$$

$$\mu = y_\# \left(H_\xi d\xi\right), \tag{61}$$

where $z(x) = (u, v)(x)$ and $Z(\xi) = (U, V)(\xi)$. We get that $(z, \mu) \in D$, which does not depend on the representative $X = (y, U, V, M, N, H) \in F$ of $[X]$ that we choose. We denote by $M : F/G \to D$ the map to any $[X] \in F/G$ and $(z, \mu) \in D$ given by (60)-(61), which conversely transforms the Lagrangian equivalent system into the original system.

We claim that the transformation from the original system into the Lagrangian equivalent system is a bijection.

Theorem 8. *The maps M and L are well-defined and $L^{-1} = M$. That is,*

$$L \circ M = Id_{F/G}, \qquad M \circ L = Id_D. \tag{62}$$

Proof. Let $[X]$ in F/G be given. We consider $X = (y, U, V, M, N, H) = \widetilde{\Upsilon}([X])$ as a representative of $[X]$ and (z, μ) given by (60)-(61) for this particular X. From the definition of $\widetilde{\Upsilon}$, we have $X \in F_0$. Let $\overline{X} = (\overline{y}, \overline{U}, \overline{V}, \overline{M}, \overline{N}, \overline{H})$ be the representative of $L(z, \mu)$ in F_0 given by Definition 6. We have to prove that $\overline{X} = X$ and therefore $L \circ M = Id_{F/G}$. Let

$$g(x) = \sup\{\xi \in R \mid y(\xi) < x\}. \tag{63}$$

Using the fact that y is increasing and continuous, it follows that

$$y(g(x)) = x \tag{64}$$

and $y^{-1}((-\infty, x)) = (-\infty, g(x))$. From (61) and since $H(-\infty) = 0$, for any $x \in R$, we get the following:

$$\mu((-\infty, x)) = \int_{y^{-1}((-\infty, x))} H_\xi d\xi = \int_{-\infty}^{g(x)} H_\xi d\xi = H(g(x)). \tag{65}$$

Since $X \in F_0$ and $y + H = Id$, we have

$$\mu((-\infty, x)) + x = g(x). \tag{66}$$

From the definition of $\overline{y}$, it follows that

$$\overline{y}(\xi) = \sup\{x \in R \mid g(x) < \xi\}. \tag{67}$$

For any given $\xi \in R$, using the fact that y is increasing and (64), it follows that $\overline{y}(\xi) \le y(\xi)$. If $\overline{y}(\xi) < y(\xi)$, there then exists x such that $\overline{y}(\xi) < x < y(\xi)$ and (67) implies that $g(x) \ge \xi$. Conversely, since y is increasing, then $x = y(g(x)) < y(\xi)$ implies that $g(x) < \xi$, which gives us a contradiction. Hence, we have $\overline{y} = y$. Since $y + H = Id$, it follows directly from the definitions that $\overline{H} = H, \overline{U} = U, \overline{V} = V$, $\overline{M} = M$, and $\overline{N} = N$. We thus proved that $L \circ M = Id_{F/G}$.

Given (z, μ) in D, we denote by (y, U, V, M, N, H) the representative of $L(z, \mu)$ in F_0 given by Definition 6. Let $(\overline{z}, \overline{\mu}) = M \circ L(z, \mu)$ and g be defined as before by (63). The same computation that leads to (66) now gives

$$\overline{\mu}((-\infty, x)) + x = g(x). \tag{68}$$

Given $\xi \in R$, we consider an increasing sequence x_i converging to $y(\xi)$ which is guaranteed by (55) such that $\mu((-\infty, x_i)) + x_i < \xi$. Let i tend to infinity. Since $F(x) = \mu((-\infty, x))$ is lower semi-continuous, we have $\mu((-\infty, y(\xi))) + y(\xi) \le \xi$. Take $\xi = g(x)$ and then we get

$$\mu((-\infty, x)) + x \le g(x). \tag{69}$$

By the definition of g, there exists an increasing sequence ξ_i converging to $g(x)$ such that $y(\xi_i) < x$. It follows from the definition of y in (55) that $\mu((-\infty, x)) + x \ge \xi_i$. Passing to the limit, we obtain $\mu((-\infty, x)) + x \ge g(x)$ which, together with (69), yields

$$\mu((-\infty, x)) + x = g(x). \tag{70}$$

We obtain that $\overline{\mu} = \mu$ by comparing (70) and (68). It is clear from the definitions that $\overline{z} = z$. Hence, $(\overline{z}, \overline{\mu}) = (z, \mu)$ and $M \circ L = Id_D$. □

The topology defined in F/G can be transported into D, which is guaranteed by the fact that we have established a bijection between the two equivalent systems. We define the metric d_D on D as

$$d_D((z, \mu), (\overline{z}, \overline{\mu})) = d_{F/G}(L(z, \mu), L(\overline{z}, \overline{\mu})), \tag{71}$$

which makes the bijection L between D and F/G into an isometry. Since F/G equipped with $d_{F/G}$ is a complete metric space, D equipped with the metric d_D is also a complete metric space. For each $t \in R$, we define the mapping $T_t : D \to D$ as

$$T_t = M\widetilde{S}_t L. \tag{72}$$

Then a continuous semigroup of conservative weak solutions for the original system is obtained as the following theorem shows.

Theorem 9. *Let $(\overline{z}, \overline{\mu}) \in D$ be given. If one denotes by $t \to (z(t), \mu(t)) = T_t(\overline{z}, \overline{\mu})$ the corresponding trajectory, then $z = (u, v)$ is a weak solution of the modified coupled two-component Camassa-Holm equation* (8)*, which constructs a continuous semigroup. Moreover, μ is a weak solution of the following transport equation for the energy density:*

$$\mu_t + [(u + v)\mu]_x = \left(u^3 - 2P_2 u + v^3 - 2P_4 v\right)_x. \tag{73}$$

Furthermore, for all t, it holds that

$$\mu(t)(R) = \mu(0)(R) \tag{74}$$

and, for almost all t,

$$\begin{aligned} \mu(t)(R) &= \mu_{ac}(t)(R) = \|z(t)\|_{H^1}^2 \\ &= \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 = \mu(0)(R). \end{aligned} \tag{75}$$

Thus the unique solution described here is a conservative weak solution of the system (8).

Proof. To prove that $z = (u, v)$ is a weak solution of the original system (8), it suffices to show that, for all $\phi \in C^\infty(R^+ \times R)$ with compact support,

$$\begin{aligned} &\int_{R^+\times R} \left[-u\phi_t + (u + v)u_x\phi\right](t, x)\,dx\,dt \\ &\quad = -\int_{R^+\times R} \left[\left(P_1 + P_{2,x}\right)\phi\right](t, x)\,dx\,dt, \\ &\int_{R^+\times R} \left[-v\phi_t + (u + v)v_x\phi\right](t, x)\,dx\,dt \\ &\quad = -\int_{R^+\times R} \left[\left(P_3 + P_{4,x}\right)\phi\right](t, x)\,dx\,dt, \end{aligned} \tag{76}$$

where P_1, $P_{2,x}$, P_3, and $P_{4,x}$ are given by (8). Let the solution $(y, U, V, M, N, H)(t)$ of (21) be a representative of $L(z(t), \mu(t))$. On the one hand, since $y(t, \xi)$ is Lipschitz continuous and invertible with respect to ξ, for almost all t, we then can use the change of variables $x = y(t, \xi)$ and obtain

$$\begin{aligned} &\int_{R^+\times R} \left[-u\phi_t + (u + v)u_x\phi\right](t, x)\,dx\,dt \\ &\quad = \int_{R^+\times R} \Big[-\left(Uy_\xi\right)(t, \xi)\,\phi_t(t, y(t, \xi)) \\ &\qquad + \left((U + V)U_\xi\right)(t, \xi)\,\phi(t, y(t, \xi))\Big]\,d\xi\,dt. \end{aligned} \tag{77}$$

By using the identities $y_t = U + V$ and $y_{\xi t} = U_\xi + V_\xi$, it then follows from (21) that

$$\begin{aligned} &\int_{R^+\times R} \left[-Uy_\xi\phi_t(t, y) + (U + V)U_\xi\phi(t, y)\right]d\xi\,dt \\ &\quad = \frac{1}{2}\int_{R^+\times R^2} \Big\{-UNy_\xi + \frac{1}{2}\operatorname{sgn}\left(\xi - \xi'\right) \\ &\qquad \times \left[H_\xi + \left(U^2 + 2MN - N^2\right)y_\xi\right]\Big\}\left(\xi'\right) \\ &\qquad \cdot e^{-|y(\xi) - y(\xi')|}\phi(t, y(\xi))\,y_\xi(\xi)\,d\xi'\,d\xi\,dt. \end{aligned} \tag{78}$$

On the other hand, using the change of variables $x = y(t,\xi)$ and $x' = y(t,\xi')$ and since y is increasing with respect to ξ, we have the following:

$$\begin{aligned}
&-\int_{R^+\times R}\left[\left(P_1+P_{2,x}\right)\phi\right](t,x)\,dx\,dt\\
&=\frac{1}{2}\int_{R^+\times R^2}\left[-uv_x+\operatorname{sgn}\left(\xi-\xi'\right)\right.\\
&\qquad\left.\times\left(u^2+\frac{1}{2}u_x^2+u_xv_x+\frac{1}{2}v^2-\frac{1}{2}v_x^2\right)\right]\\
&\quad\times\left(t,y\left(\xi'\right)\right)\cdot e^{-|y(\xi)-y(\xi')|}\\
&\quad\times\phi\left(t,y\left(\xi\right)\right)y_\xi\left(\xi'\right)y_\xi\left(\xi\right)d\xi'd\xi\,dt.
\end{aligned}\tag{79}$$

We obtain from (27) that

$$\begin{aligned}
&-\int_{R^+\times R}\left[\left(P_1+P_{2,x}\right)\phi\right](t,x)\,dx\,dt\\
&=\frac{1}{2}\int_{R^+\times R^2}\left\{-UNy_\xi+\frac{1}{2}\operatorname{sgn}\left(\xi-\xi'\right)\right.\\
&\qquad\left.\times\left[H_\xi+\left(U^2+2MN-N^2\right)y_\xi\right]\right\}\left(\xi'\right)\\
&\quad\cdot e^{-|y(\xi)-y(\xi')|}\phi\left(t,y\left(\xi\right)\right)y_\xi\left(\xi\right)d\xi'd\xi\,dt.
\end{aligned}\tag{80}$$

By comparing (78) and (80), we know that

$$\begin{aligned}
&\int_{R^+\times R}\left[-Uy_\xi\phi_t\left(t,y\right)+\left(U+V\right)U_\xi\phi\left(t,y\right)\right]d\xi\,dt\\
&\qquad=-\int_{R^+\times R}\left[\left(P_1+P_{2,x}\right)\phi\right](t,x)\,dx\,dt.
\end{aligned}\tag{81}$$

Hence, the first identity in (76) holds. The second identity in (76) follows in the same way. One can easily check that $\mu(t)$ is solution of (73). From the definition μ in (61), we get

$$\mu\left(t\right)\left(R\right)=\int_R H_\xi d\xi=H\left(t,\infty\right),\tag{82}$$

which is constant in time from Lemma 2. Thus, we have proved (74).

Since $y_\xi(t,\xi) > 0$ a.e., for almost every $\xi \in R$, it then follows from (27) that

$$\begin{aligned}
\mu\left(t\right)\left(B\right)&=\int_{y^{-1}(B)}H_\xi d\xi\\
&=\int_{y^{-1}(B)}\left(U^2+\frac{U_\xi^2}{y_\xi^2}+V^2+\frac{V_\xi^2}{y_\xi^2}\right)y_\xi\,d\xi,
\end{aligned}\tag{83}$$

for any Borel set B. Since y is one-to-one and $u_x\circ yy_\xi = U_\xi$, $v_x\circ yy_\xi = V_\xi$ a.e. and then (83) implies that

$$\mu\left(t\right)\left(B\right)=\int_B\left(u^2+u_x^2+v^2+v_x^2\right)(t,x)\,dx.\tag{84}$$

Hence, (75) is proved (and the solution is conservative), which completes the proof. □

5. Multipeakon Conservative Solutions of the Original System

In this section, we will derive a new system of ordinary differential equations for the multipeakon solutions which is well posed even when collisions occur, and the variables (y,U,V,M,N,H) will be used to characterize multipeakons in a way that avoids the problems related to blowing up.

Solutions of the modified coupled two-component Camassa-Holm system may experience wave breaking in the sense that the solution develops singularities in finite time, while keeping the H^1-norm finite. Continuation of the solution beyond wave breaking imposes significant challenge as can be illustrated in the case of multipeakons, which are special solutions of the modified coupled two-component Camassa-Holm system of the following form:

$$\begin{aligned}
u\left(t,x\right)&=\sum_{i=1}^n p_i\left(t\right)e^{-|x-q_i(t)|},\\
v\left(t,x\right)&=\sum_{i=1}^n r_i\left(t\right)e^{-|x-q_i(t)|},
\end{aligned}\tag{85}$$

where $(p_i(t), r_i(t), q_i(t))$ satisfy the explicit system of ordinary differential equations:

$$\begin{aligned}
\dot p_i&=\sum_{j=1,i\neq j}^n\left(p_ip_j+r_ir_j\right)\operatorname{sgn}\left(q_j-q_i\right)e^{-|q_i-q_j|},\\
\dot r_i&=\sum_{j=1,i\neq j}^n\left(p_ip_j+r_ir_j\right)\operatorname{sgn}\left(q_j-q_i\right)e^{-|q_i-q_j|},\\
\dot q_i&=-\sum_{j=1}^n\left(p_j+r_j\right)e^{-|q_i-q_j|}.
\end{aligned}\tag{86}$$

Peakons interact in a way similar to that of solitons of the CH equation, and wave breaking may appear when at least two of the q_i coincide. Clearly, if the q_i remain distinct, the system (86) allows for a global smooth solution. It is not hard to see that $z = (u, v)$ is a global weak solution of system (8) by inserting that solution into (85). In the case where $p_i(0)$ and $r_i(0)$ have the same sign for all $i = 1, 2, \ldots, n$, (86) admits a unique global solution, where the $q_i(t)$ remain distinct and the peakons are traveling in the same direction. However, when two peakons have opposite signs, collisions may occur, and, if so, the system (86) blows up.

Let us consider initial data $\overline{z} = (\overline{u}, \overline{v})$ given by

$$\begin{aligned}
\overline{u}\left(x\right)&=\sum_{i=1}^n p_ie^{-|x-\xi_i|},\\
\overline{v}\left(x\right)&=\sum_{i=1}^n r_ie^{-|x-\xi_i|}.
\end{aligned}\tag{87}$$

Without loss of generality, we assume that the p_i and r_i are all nonzero and that the ξ_i are all distinct. The aim is to characterize the unique and global weak solution from Theorem 9 with initial data (87) explicitly. Since the variables

p_i and r_i blow up at collisions, they are not appropriate to define a multipeakon in the form of (85). We consider the following characterization of multipeakons given as continuous solutions $z = (u, v)$, which are defined on intervals $[y_i, y_{i+1}]$ as the solutions of the Dirichlet problem

$$z - z_{xx} = 0, \tag{88}$$

with boundary conditions $z(t, y_i(t)) = z_i(t)$ and $z(t, y_{i+1}(t)) = z_{i+1}(t)$. The variables y_i denote the position of the peaks, and the variables z_i denote the values of z at the peaks. In the following we will show that this property persists for conservative solutions.

Let us set $A = R \setminus \{\xi_1, \ldots, \xi_n\}$. The next lemma gives us the functions $\overline{U}, \overline{V}$, and $\overline{H}$ which belong to $C^2(A)$ (they even belong to $C^\infty(A)$).

Lemma 10. *Let $\overline{X} = (\overline{y}, \overline{U}, \overline{V}, \overline{M}, \overline{N}, \overline{H}) \in F$ such that $(\overline{y}, \overline{U}, \overline{V}, \overline{H}) \in [C^2(A)]^4$ is given, and then the solution (y, U, V, H) of (21) with initial data $\overline{X}$ belongs to $C^1(R^+, [C^2(A)]^4)$.*

Proof. To prove this Lemma, one proceeds as in Theorem 3 by using the contraction argument and replacing E by

$$\overline{E} = E \cap \left[\left(C^2(A) \right)^3 \cap \left(L^2 \right)^2 \cap C^2(A) \right], \tag{89}$$

endowed with the norm

$$\begin{aligned} \|X\|_{\overline{E}} = {} & \|X\|_E + \left\| y - Id \right\|_{W^{2,\infty}(A)} + \|U\|_{W^{2,\infty}(A)} \\ & + \|V\|_{W^{2,\infty}(A)} + \|H\|_{W^{2,\infty}(A)}. \end{aligned} \tag{90}$$

Our main task is to prove the Lipschitz continuity of P_i and $P_{i,x}$ $(i = 1, 2, 3, 4)$ from $\overline{E}$ to $H^1(R) \cap C^2(A)$. We first show that $P_{2,x}$ is Lipschitz continuous from $\overline{E}$ to $H^1(R) \cap C^2(A)$ and the others follow in the same way. Given a bounded set $B = \{X \in \overline{E} \mid \|X\|_{\overline{E}} \le C_B\}$ where C_B is a positive constant, from Theorem 3, we get that

$$\left\| P_{2,x}(X) - P_{2,x}\left(\overline{X}\right) \right\|_{L^\infty(R)} \le C \left\| X - \overline{X} \right\|_E \le C \left\| X - \overline{X} \right\|_{\overline{E}} \tag{91}$$

for a constant C depending only on C_B. We can compute the derivative of $P_{2,x}$ given as

$$P_{2,x\xi} = -\frac{1}{2} H_\xi + \left(P_2 - \frac{1}{2} U^2 - MN + N^2 \right) \left(1 + \varsigma_\xi \right). \tag{92}$$

From Lemma 2, $P_{2,x\xi}$ is Lipschitz continuous from $\overline{E}$ to $C(A)$ and therefore $P_{2,x}$ is Lipschitz continuous from $\overline{E}$ to $C^1(A)$. Similarly, we obtain the same results for P_1, P_2, P_3, and P_4 and $P_{1,x}, P_{3,x}, P_{4,x}$. We can also compute the derivative of $P_{2,x\xi}$ on A as follows:

$$\begin{aligned} P_{2,x\xi\xi} = {} & -\frac{1}{2} H_{\xi\xi} + \left(P_{2,x} y_\xi - U U_\xi - M_\xi N - M N_\xi + 2 N N_\xi \right) y_\xi \\ & + \left(P_2 - \frac{1}{2} U^2 - MN + N^2 \right) y_{\xi\xi}. \end{aligned} \tag{93}$$

Since $P_{2,x\xi\xi}$ is locally Lipschitz maps from $\overline{E}$ to $C(A)$, we then get that $P_{2,x}$ is locally Lipschitz continuous from $\overline{E}$ to $C^2(A)$. The same results can be obtained for the other P_i and $P_{i,x}$ $(i = 1, 2, 3, 4)$ by the same way. From the standard contraction argument, the local existence of solutions of (21) can be proved in $\overline{E}$. As far as global existence is concerned, $\|X\|_{W^{1,\infty}(R)}$ does not blow up for initial data in $W^{1,\infty}(R)$. For the second derivative, for any $\xi \in A$, we get that

$$\begin{aligned} y_{\xi\xi t} = {} & U_{\xi\xi} + V_{\xi\xi}, \\ U_{\xi\xi t} = {} & \frac{1}{2} H_{\xi\xi} + \left(\frac{1}{2} U^2 + MN - N^2 - P_2 - P_{1,x} \right) y_{\xi\xi} \\ & + \left(U U_\xi + M_\xi N + M N_\xi - 2 N N_\xi \right) y_\xi \\ & - \left(P_{2,x} + P_1 - UN \right) y_\xi^2, \\ V_{\xi\xi t} = {} & \frac{1}{2} H_{\xi\xi} + \left(\frac{1}{2} V^2 + MN - M^2 - P_4 - P_{3,x} \right) y_{\xi\xi} \\ & + \left(V V_\xi + N_\xi M + N M_\xi - 2 M N_\xi \right) y_\xi \\ & - \left(P_{4,x} + P_3 - VM \right) y_\xi^2, \\ H_{\xi\xi t} = {} & -\left(2 U P_{2,x} + 2 V P_{4,x} \right) y_{\xi\xi} + \left(3U^2 - 2P_2 \right) U_{\xi\xi} \\ & + \left(3V^2 - 2P_4 \right) V_{\xi\xi} + 6 U U_\xi^2 - 4 P_{2,x} U_\xi y_\xi \\ & + 6 V V_\xi^2 - 4 P_{4,x} V_\xi y_\xi - 2 U P_2 y_\xi^2 + U^3 y_\xi^2 \\ & + 2 UMN y_\xi^2 - 2 U N^2 y_\xi^2 + U H_\xi y_\xi - 2 V P_4 y_\xi^2 \\ & + V^3 y_\xi^2 + 2 VMN y_\xi^2 - 2 V M^2 y_\xi^2 + V H_\xi y_\xi. \end{aligned} \tag{94}$$

The system (94) is affine with respect to $y_{\xi\xi}, U_{\xi\xi}, V_{\xi\xi}, H_{\xi\xi}$. Thus, we get that

$$\begin{aligned} & \left\| X_{\xi\xi}(t, \cdot) \right\|_{L^\infty(A)} \\ & \quad \le \left\| X_{\xi\xi}(0, \cdot) \right\|_{L^\infty(A)} + C + C \int_0^t \left\| X_{\xi\xi}(\tau, \cdot) \right\|_{L^\infty(A)} d\tau, \end{aligned} \tag{95}$$

where C is a constant depending only on $\sup_{t \in [0,T)} \|X\|_{W^{1,\infty}(R)}$ which is bounded on any time interval $[0, T)$. It follows from Gronwall's lemma that $\|X\|_{W^{2,\infty}(A)}$ does not blow up and therefore the solution is globally defined in $\overline{E}$. □

We now prove that $\overline{X} = (\overline{y}, \overline{U}, \overline{V}, \overline{M}, \overline{N}, \overline{H})$ is a representative of $z = (u, v)$ in the Lagrangian system; that is, $[\overline{X}] = L(\overline{z}, \overline{\mu})$, where $\overline{X} = (\overline{y}, \overline{U}, \overline{V}, \overline{M}, \overline{N}, \overline{H})$ is given by

$$\overline{y}(\xi) = \xi, \tag{96a}$$

$$\begin{aligned} & \overline{U}(\xi) = \overline{u}(\xi), \qquad \overline{V}(\xi) = \overline{v}(\xi), \\ & \overline{M}(\xi) = \overline{u}_x(\xi), \qquad \overline{N}(\xi) = \overline{v}_x(\xi), \end{aligned} \tag{96b}$$

$$\overline{H}(\xi) = \int_{-\infty}^{\xi} \left(\overline{u}^2 + \overline{u}_x^2 + \overline{v}^2 + \overline{v}_x^2 \right) dx. \tag{96c}$$

We first check that $\overline{X} \in F$. Since $\overline{z} = (\overline{u}, \overline{v})$ is a multipeakon, we get that $\overline{z} = (\overline{u}, \overline{v}) \in W^{1,\infty}(R) \cap H^1(R)$ from (87). Hence, $\overline{U}, \overline{V}$, and $\overline{H}$ all belong to $W^{1,\infty}(R)$ while $\overline{y} - Id$ is identically zero. Due to the exponential decay of $(\overline{u}, \overline{v})$ and $(\overline{u}_x, \overline{v}_x)$ and $\overline{H}_\xi \in L^\infty(R)$, we get that $\overline{H}_\xi \in L^2(R)$. The properties (25)–(27) are straightforward to check. It is not hard to check that $M([\overline{X}]) = (\overline{z}, (\overline{u}^2 + \overline{u}_x^2 + \overline{v}^2 + \overline{v}_x^2)dx)$ and, therefore, since $L \circ M = Id$, we get that $[\overline{X}] = L(\overline{z}, (\overline{u}^2 + \overline{u}_x^2 + \overline{v}^2 + \overline{v}_x^2)dx)$.

Theorem 11. *Let initial data be given in* (87). *The solution given by Theorem 9 satisfies* $z - z_{xx} = 0$ *between the peaks.*

Proof. Let us first prove that $u - u_{xx} = 0$. Assuming that $y_\xi(t, \xi) \neq 0$, we get that

$$u_x \circ y = M, \qquad u_{xx} \circ y = \frac{M_\xi}{y_\xi} = \frac{\left(U_{\xi\xi} y_\xi - y_{\xi\xi} U_\xi\right)}{y_\xi^3}, \tag{97}$$

and therefore

$$\left(u - u_{xx}\right) \circ y = \frac{\left(U y_\xi^3 - U_{\xi\xi} y_\xi + y_{\xi\xi} U_\xi\right)}{y_\xi^3}. \tag{98}$$

We set

$$R = U y_\xi^3 - U_{\xi\xi} y_\xi + y_{\xi\xi} U_\xi. \tag{99}$$

For a given $\xi \in A$, differentiating (99) with respect to t and after using (21), (24), and (94), we obtain

$$\begin{aligned}
\frac{dR}{dt} &= 3U y_\xi^2 y_{\xi t} + U_t y_\xi^3 - U_{\xi\xi t} y_\xi - U_{\xi\xi} y_{\xi t} + y_{\xi\xi t} U_\xi + y_{\xi\xi} U_{\xi t} \\
&= 2UU_\xi y_\xi^2 + 2UV_\xi y_\xi^2 - \frac{H_{\xi\xi} y_\xi}{2} \\
&\quad - \left(M_\xi N + N_\xi M - 2NN_\xi\right) y_\xi^2 - U_{\xi\xi} V_\xi \\
&\quad + V_{\xi\xi} U_\xi + \frac{H_\xi y_{\xi\xi}}{2} \\
&= 2U\left(U_\xi + V_\xi\right) y_\xi^2 - 2N\left(M_\xi - N_\xi\right) y_\xi^2 \\
&\quad - \frac{H_{\xi\xi} y_\xi}{2} + \frac{H_\xi y_{\xi\xi}}{2}.
\end{aligned} \tag{100}$$

Differentiating (27) with respect to ξ, we get

$$\begin{aligned}
y_{\xi\xi} H_\xi + y_\xi H_{\xi\xi} = {} & 2y_\xi y_{\xi\xi} U^2 + 2y_\xi^2 U U_\xi + 2U_\xi U_{\xi\xi} \\
& + 2y_\xi y_{\xi\xi} V^2 + 2y_\xi^2 V V_\xi + 2V_\xi V_{\xi\xi}.
\end{aligned} \tag{101}$$

After inserting the value of $y_\xi H_{\xi\xi}$ given by (101) into (100) and multiplying the equation by y_ξ, we obtain that

$$\begin{aligned}
y_\xi \cdot \frac{dR}{dt} &= U U_\xi y_\xi^3 - U_\xi U_{\xi\xi} y_\xi + \left(H_\xi y_\xi - y_\xi^2 U^2 - y_\xi^2 V^2 - V_\xi^2\right) y_{\xi\xi} \\
&\quad + U V_\xi y_\xi^3 - U_{\xi\xi} V_\xi y_\xi + U_\xi V_\xi y_{\xi\xi}.
\end{aligned} \tag{102}$$

It follows from (27), and since $y_{\xi t} = (U_\xi + V_\xi)$, that

$$y_\xi \cdot \frac{dR}{dt} = y_{\xi t} \cdot R. \tag{103}$$

We claim that, for any time t such that $y_\xi(t) \neq 0$,

$$\frac{d}{dt}\left(\frac{R}{y_\xi}\right) = \frac{R_t y_\xi - y_{\xi t} R}{y_\xi^2} = 0. \tag{104}$$

We have to prove that R/y_ξ is C^1 in time. Since

$$\begin{aligned}
\frac{R}{y_\xi} &= U y_\xi^2 - U_{\xi\xi} + \frac{y_{\xi\xi} U_\xi}{y_\xi} \\
&= U y_\xi^2 - U_{\xi\xi} + \frac{y_{\xi\xi} U_\xi}{y_\xi + H_\xi} + \frac{y_{\xi\xi} M H_\xi}{y_\xi + H_\xi} = \frac{J\left(X, X_\xi, X_{\xi\xi}\right)}{y_\xi + H_\xi},
\end{aligned} \tag{105}$$

for some polynomial J and $X \in C^1(R, \overline{E})$, we get that X, X_ξ, and $X_{\xi\xi}$ are C^1 in time. Since $X(t)$ remains in Γ, for all t, from (26), we have $y_\xi + H_\xi > 0$ and therefore $1/(y_\xi + H_\xi)$ is C^1 in time, which implies that R/y_ξ is C^1 in time. Hence, it holds that

$$R(t, \xi) = K(\xi)\, y_\xi(t, \xi), \tag{106}$$

for some constant $K(\xi)$ which is independent of time, which leads to

$$y_\xi^2 \left(u - u_{xx}\right) \circ y = K(\xi). \tag{107}$$

For the multipeakons at time $t = 0$, we have $y(0, \xi) = \xi$ and $(u - u_{xx})(0, \xi) = 0$ for all $\xi \in A$. Hence,

$$\frac{R}{y_\xi}(t, \xi) = 0, \tag{108}$$

for all time t and all $\xi \in A$. Thus, $(u - u_{xx})(t, \xi) = 0$. Similarly, $(v - v_{xx})(t, \xi) = 0$. □

For solutions with multipeakon initial data, we have the following result. If $y_\xi(t, \xi)$ vanishes at some point $\overline{\xi}$ in the interval (ξ_i, ξ_{i+1}), then $y_\xi(t, \xi)$ vanishes everywhere in (ξ_i, ξ_{i+1}). Moreover, for given initial multipeakon solution $\overline{z}(x) = (\overline{u}, \overline{v})(x) = (\sum_{i=1}^n p_i e^{-|x-\xi_i|}, \sum_{i=1}^n r_i e^{-|x-\xi_i|})$, let (y, U, V, M, N, H) be the solution of system (21) with initial data $(\overline{y}, \overline{U}, \overline{V}, \overline{M}, \overline{N}, \overline{H})$ given by (96a), (96b) and (96c), and then, between adjacent peaks, if $x_i = y(t, \xi_i) \neq x_{i+1} = y(t, \xi_{i+1})$, the solution $z(t, x) = (u, v)(t, x)$ is twice differentiable with respect to the space variable and we have $(z - z_{xx}) = 0$, for $x \in (x_i, x_{i+1})$.

We now start the derivation of a system of ordinary differential equations for multipeakons.

From (21), we get that, for each $i = 1, 2 \ldots, n$,

$$\begin{aligned}
\frac{dy_i}{dt} &= u_i + v_i,\\
\frac{du_i}{dt} &= -P_{1,i} - P_{2,xi},\\
\frac{dv_i}{dt} &= -P_{3,i} - P_{4,xi},\\
\frac{dH_i}{dt} &= u_i^3 - 2u_iP_{2,i} + v_i^3 - 2v_iP_{4,i},
\end{aligned} \tag{109}$$

where $(y_i, u_i, v_i, H_i) = (y, U, V, H)(t, \xi_i)$, $P_{k,i} = P_k(t, \xi_i)$, and $P_{k,xi} = P_{k,x}(t, \xi_i)$, $(k = 1, 2, 3, 4)$, respectively. Since the function $y(t, \cdot)$ is invertible, for almost every t, we can use the change of variables $x = y(t, \xi)$ such that $P_{k,i}$ and $P_{k,xi}(k = 1, 2, 3, 4)$ can be rewritten as

$$\begin{aligned}
P_{1,i} &= \frac{1}{2}\int_R e^{-|y_i - x|}\left(uv_x\right)dx,\\
P_{1,xi} &= -\frac{1}{2}\int_R \operatorname{sgn}\left(y_i - x\right)e^{-|y_i - x|}\left(uv_x\right)dx,\\
P_{2,i} &= \frac{1}{2}\int_R e^{-|y_i - x|}\left(u^2 + \frac{1}{2}u_x^2 + u_xv_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right)dx,\\
P_{2,xi} &= -\frac{1}{2}\int_R \operatorname{sgn}\left(y_i - x\right)e^{-|y_i - x|}\\
&\quad\times\left(u^2 + \frac{1}{2}u_x^2 + u_xv_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right)dx,\\
P_{3,i} &= \frac{1}{2}\int_R e^{-|y_i - x|}\left(vu_x\right)dx,\\
P_{3,xi} &= -\frac{1}{2}\int_R \operatorname{sgn}\left(y_i - x\right)e^{-|y_i - x|}\left(vu_x\right)dx,\\
P_{4,i} &= \frac{1}{2}\int_R e^{-|y_i - x|}\left(v^2 + \frac{1}{2}v_x^2 + u_xv_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2\right)dx,\\
P_{4,xi} &= -\frac{1}{2}\int_R \operatorname{sgn}\left(y_i - x\right)e^{-|y_i - x|}\\
&\quad\times\left(v^2 + \frac{1}{2}v_x^2 + u_xv_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2\right)dx.
\end{aligned} \tag{110}$$

Between two adjacent peaks located at y_i and y_{i+1}, we know that $z = (u, v)$ satisfies $(z - z_{xx}) = 0$ and therefore $z = (u, v)$ can be written as

$$z(x) = \begin{pmatrix} u(x)\\ v(x)\end{pmatrix} = \begin{pmatrix} A_ie^x + B_ie^{-x}\\ C_ie^x + D_ie^{-x}\end{pmatrix}, \tag{111}$$

for $x \in [y_i, y_{i+1}]$, $i = 1, 2, \ldots, n-1$, where the constants A_i, B_i, C_i, and D_i depend on u_i, u_{i+1}, v_i, v_{i+1}, y_i, and y_{i+1} and read

$$\begin{aligned}
A_i &= \frac{e^{-\overline{y}_i}}{2}\left[\frac{\overline{u}_i}{\cosh\left(\delta y_i\right)} + \frac{\delta u_i}{\sinh\left(\delta y_i\right)}\right],\\
B_i &= \frac{e^{\overline{y}_i}}{2}\left[\frac{\overline{u}_i}{\cosh\left(\delta y_i\right)} - \frac{\delta u_i}{\sinh\left(\delta y_i\right)}\right],\\
C_i &= \frac{e^{-\overline{y}_i}}{2}\left[\frac{\overline{v}_i}{\cosh\left(\delta y_i\right)} + \frac{\delta v_i}{\sinh\left(\delta y_i\right)}\right],\\
D_i &= \frac{e^{\overline{y}_i}}{2}\left[\frac{\overline{v}_i}{\cosh\left(\delta y_i\right)} - \frac{\delta v_i}{\sinh\left(\delta y_i\right)}\right],
\end{aligned} \tag{112}$$

where

$$\begin{aligned}
\overline{y}_i &= \frac{1}{2}\left(y_i + y_{i+1}\right), \qquad \delta y_i = \frac{1}{2}\left(y_i - y_{i+1}\right),\\
\overline{u}_i &= \frac{1}{2}\left(u_i + u_{i+1}\right), \qquad \delta u_i = \frac{1}{2}\left(u_i - u_{i+1}\right),\\
\overline{v}_i &= \frac{1}{2}\left(v_i + v_{i+1}\right), \qquad \delta v_i = \frac{1}{2}\left(v_i - v_{i+1}\right).
\end{aligned} \tag{113}$$

Thus, the constants A_i, B_i, C_i, and D_i uniquely determine $z = (u, v)$ on the interval $[y_i, y_{i+1}]$, and we compute

$$\begin{aligned}
\delta H_i &= H_{i+1} - H_i\\
&= \int_{y_i}^{y_{i+1}}\left(u^2 + u_x^2 + v^2 + v_x^2\right)dx\\
&= 2\overline{u}_i^2\tanh\left(\delta y_i\right) + 2\delta u_i^2\coth\left(\delta y_i\right)\\
&\quad + 2\overline{v}_i^2\tanh\left(\delta y_i\right) + 2\delta v_i^2\coth\left(\delta y_i\right)\\
&= \delta H_{1i} + \delta H_{2i},
\end{aligned} \tag{114}$$

where $\delta H_{1i} = 2\overline{u}_i^2\tanh(\delta y_i) + 2\delta u_i^2\cosh(\delta y_i)$ and $\delta H_{2i} = 2\overline{v}_i^2\tanh(\delta y_i) + 2\delta v_i^2\cosh(\delta y_i)$. At this point, we can get some more understanding of what is happening at a time of collision. Let t^* be a time when the two peaks located at y_i and y_{i+1} collide, that is, such that $\lim_{t\to t^*}\delta y_i(t) = 0$. Since the solution $z = (u, v)$ remains in H^1 for all time, the function $z = (u, v)$ remains continuous so that we have $\lim_{t\to t^*}\delta u_i(t) = \lim_{t\to t^*}\delta v_i(t) = 0$, and, when t tends to t^*, A_i, B_i, C_i, and D_i may have a finite limit. However, we know that the first derivative blows up, which implies that $\lim_{t\to t^*}B_i = -\lim_{t\to t^*}A_i = \infty$ and $\lim_{t\to t^*}D_i = -\lim_{t\to t^*}C_i = \infty$. Thus, δu_i and δv_i tend to zero, respectively, but are slower than δy_i. Indeed, let t tend to t^* in (114), and then, to first order in δy_i, we obtain that

$$\sqrt{\delta u_i^2 + \delta v_i^2} = \sqrt{\frac{\delta H_i}{2}}\cdot\sqrt{\delta y_i} + \circ\left(\delta y_i\right), \tag{115}$$

which implies that δu_i and δv_i tend to zero at the same rate as $\sqrt{\delta y_i}$. We now turn to the computation of $P_{k,i}$ $(k = 1, 2, 3, 4)$ given by (110). Let us write $z = (u, v)$ as

$$\begin{aligned} z(t,x) &= (u(t,x), v(t,x)) \\ &= \left(\sum_{j=0}^{n} \left(A_j e^{x} + B_j e^{-x} \right) \chi_{(y_j, y_{j+1})}(x), \right. \\ &\qquad \left. \sum_{j=0}^{n} \left(C_j e^{x} + D_j e^{-x} \right) \chi_{(y_j, y_{j+1})}(x) \right). \end{aligned} \tag{116}$$

We have sets $y_0 = -\infty$, $y_{n+1} = \infty$, $u_0 = u_{n+1} = 0$, $v_0 = v_{n+1} = 0$, $A_0 = u_1 e^{-y_1}$, $B_0 = 0$, $A_n = 0$, $B_n = u_n e^{y_n}$, and $C_0 = v_1 e^{-y_1}$, $D_0 = 0$, $C_n = 0$, $D_n = v_n e^{y_n}$. We have

$$\begin{aligned} uv_x &= \sum_{j=0}^{n} \left(A_j C_j e^{2x} - A_j D_j + B_j C_j - B_j D_j e^{-2x} \right) \chi_{(y_j, y_{j+1})}, \\ u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2 &= \sum_{j=0}^{n} \left(\left(\frac{3}{2}A_j^2 + A_j C_j \right) e^{2x} \right. \\ &\quad + \left(A_j B_j - A_j D_j - B_j C_j + 2C_j D_j \right) \\ &\quad \left. + \left(\frac{3}{2}B_j^2 + B_j D_j \right) e^{-2x} \right) \chi_{(y_j, y_{j+1})}, \\ vu_x &= \sum_{j=0}^{n} \left(A_j C_j e^{2x} - C_j B_j + D_j A_j - B_j D_j e^{-2x} \right) \chi_{(y_j, y_{j+1})}, \\ v^2 + \frac{1}{2}v_x^2 + u_x v_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2 &= \sum_{j=0}^{n} \left(\left(\frac{3}{2}C_j^2 + A_j C_j \right) e^{2x} \right. \\ &\quad + \left(C_j D_j - C_j B_j - D_j A_j + 2A_j B_j \right) \\ &\quad \left. + \left(\frac{3}{2}D_j^2 + D_j B_j \right) e^{-2x} \right) \chi_{(y_j, y_{j+1})}. \end{aligned} \tag{117}$$

We set

$$k_{ij} = \begin{cases} -1 & \text{if } i \le j, \\ 1 & \text{otherwise.} \end{cases} \tag{118}$$

By inserting (117) into (110), we get

$$\begin{aligned} P_{1,i} &= \frac{1}{2}\sum_{j=0}^{n} \int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} \\ &\qquad \times \left(A_j C_j e^{2x} - A_j D_j + B_j C_j - B_j D_j e^{-2x} \right) dx, \\ P_{2,i} &= \frac{1}{2}\sum_{j=0}^{n} \int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} \left(\left(\frac{3}{2}A_j^2 + A_j C_j \right) e^{2x} \right. \\ &\qquad + \left(A_j B_j - A_j D_j \right. \\ &\qquad \left. - B_j C_j + 2C_j D_j \right) \\ &\qquad \left. + \left(\frac{3}{2}B_j^2 + B_j D_j \right) e^{-2x} \right) dx, \\ P_{3,i} &= \frac{1}{2}\sum_{j=0}^{n} \int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} \\ &\qquad \times \left(A_j C_j e^{2x} - C_j B_j + D_j A_j - B_j D_j e^{-2x} \right) dx, \\ P_{4,i} &= \frac{1}{2}\sum_{j=0}^{n} \int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i - x)} \\ &\qquad \times \left(\left(\frac{3}{2}C_j^2 + A_j C_j \right) e^{2x} \right. \\ &\qquad + \left(C_j D_j - C_j B_j - D_j A_j + 2A_j B_j \right) \\ &\qquad \left. + \left(\frac{3}{2}D_j^2 + D_j B_j \right) e^{-2x} \right). \end{aligned} \tag{119}$$

It follows from (112) and (114) that

$$\begin{aligned} A_j^2 &= \left[\frac{e^{-\overline{y}_j}}{2} \left(\frac{\overline{u}_j}{\cosh(\delta y_j)} + \frac{\delta u_j}{\sinh(\delta y_j)} \right) \right]^2 \\ &= \frac{e^{-2\overline{y}_j}}{\sinh^2(2\delta y_j)} \left[\overline{u}_j^2 \sinh^2(\delta y_j) \right. \\ &\qquad + 2\overline{u}_j \delta u_j \sinh(\delta y_j) \cosh(\delta y_j) \\ &\qquad \left. + \delta u_j^2 \cosh^2(\delta y_j) \right] \\ &= \frac{e^{-2\overline{y}_j}}{4\sinh(2\delta y_j)} \left[\delta H_{1j} + 4\overline{u}_j \delta u_j \right], \end{aligned} \tag{120}$$

$$\begin{aligned} A_j B_j &= \frac{e^{-\overline{y}_j}}{2} \left(\frac{\overline{u}_j}{\cosh(\delta y_j)} + \frac{\delta u_j}{\sinh(\delta y_j)} \right) \\ &\quad \cdot \frac{e^{\overline{y}_j}}{2} \left(\frac{\overline{u}_j}{\cosh(\delta y_j)} - \frac{\delta u_j}{\sinh(\delta y_j)} \right) \end{aligned}$$

$$= \frac{1}{4\sinh\left(2\delta y_j\right)}\left[2\overline{u}_j^2\tanh\left(\delta y_j\right) - 2\delta u_j^2\coth\left(\delta y_j\right)\right]$$
$$= \frac{1}{4\sinh\left(2\delta y_j\right)}\left[4\overline{u}_j^2\tanh\left(\delta y_j\right) - \delta H_{1j}\right], \tag{121}$$

$$A_jC_j = \frac{e^{-\overline{y}_j}}{2}\left(\frac{\overline{u}_j}{\cosh\left(\delta y_j\right)} + \frac{\delta u_j}{\sinh\left(\delta y_j\right)}\right) \cdot \frac{e^{-\overline{y}_j}}{2}\left(\frac{\overline{v}_j}{\cosh\left(\delta y_j\right)} + \frac{\delta v_j}{\sinh\left(\delta y_j\right)}\right)$$
$$= \frac{e^{-2\overline{y}_j}}{2\sinh\left(2\delta y_j\right)}\left[\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) + \delta u_j\overline{v}_j + \delta v_j\overline{u}_j + \delta u_j\delta v_j\coth\left(\delta y_j\right)\right], \tag{122}$$

$$A_jD_j = \frac{e^{-\overline{y}_j}}{2}\left(\frac{\overline{u}_j}{\cosh\left(\delta y_j\right)} + \frac{\delta u_j}{\sinh\left(\delta y_j\right)}\right) \cdot \frac{e^{\overline{y}_j}}{2}\left(\frac{\overline{v}_j}{\cosh\left(\delta y_j\right)} - \frac{\delta v_j}{\sinh\left(\delta y_j\right)}\right)$$
$$= \frac{1}{2\sinh\left(2\delta y_j\right)}\left[\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) + \delta u_j\overline{v}_j - \delta v_j\overline{u}_j - \delta u_j\delta v_j\coth\left(\delta y_j\right)\right], \tag{123}$$

$$B_jC_j = \frac{e^{\overline{y}_j}}{2}\left(\frac{\overline{u}_j}{\cosh\left(\delta y_j\right)} - \frac{\delta u_j}{\sinh\left(\delta y_j\right)}\right) \cdot \frac{e^{-\overline{y}_j}}{2}\left(\frac{\overline{v}_j}{\cosh\left(\delta y_j\right)} + \frac{\delta v_j}{\sinh\left(\delta y_j\right)}\right)$$
$$= \frac{1}{2\sinh\left(2\delta y_j\right)}\left[\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) + \delta v_j\overline{u}_j - \delta u_j\overline{v}_j - \delta u_j\delta v_j\coth\left(\delta y_j\right)\right], \tag{124}$$

$$B_jD_j = \frac{e^{\overline{y}_j}}{2}\left(\frac{\overline{u}_j}{\cosh\left(\delta y_j\right)} - \frac{\delta u_j}{\sinh\left(\delta y_j\right)}\right) \cdot \frac{e^{\overline{y}_j}}{2}\left(\frac{\overline{v}_j}{\cosh\left(\delta y_j\right)} - \frac{\delta v_j}{\sinh\left(\delta y_j\right)}\right)$$
$$= \frac{e^{2\overline{y}_j}}{2\sinh\left(2\delta y_j\right)}\left[\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) - \delta u_j\overline{v}_j - \delta v_j\overline{u}_j + \delta u_j\delta v_j\coth\left(\delta y_j\right)\right], \tag{125}$$

$$B_j^2 = \left[\frac{e^{\overline{y}_j}}{2}\left(\frac{\overline{u}_j}{\cosh\left(\delta y_j\right)} - \frac{\delta u_j}{\sinh\left(\delta y_j\right)}\right)\right]^2$$
$$= \frac{e^{2\overline{y}_j}}{4\sinh\left(2\delta y_j\right)}\left[\delta H_{1j} - 4\overline{u}_j\delta u_j\right]. \tag{126}$$

Thus, from (120)–(126), we can obtain that

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)}A_j^2e^{2x}dx = \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{2\left(2+k_{ij}\right)\sinh\left(2\delta y_j\right)}\sinh\left(\left(2+k_{ij}\right)\delta y_j\right) \times\left[\delta H_{1j} + 4\overline{u}_j\delta u_j\right], \tag{127}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)}A_jC_je^{2x}dx = \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{\left(2+k_{ij}\right)\sinh\left(2\delta y_j\right)}\sinh\left(\left(2+k_{ij}\right)\delta y_j\right) \cdot\left[\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) + \delta u_j\overline{v}_j + \delta v_j\overline{u}_j + \delta u_j\delta v_j\coth\left(\delta y_j\right)\right], \tag{128}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)}A_jB_jdx = \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{2\sinh\left(2\delta y_j\right)}\sinh\left(\delta y_j\right) \times\left[4\overline{u}_j^2\tanh\left(\delta y_j\right) - \delta H_{1j}\right], \tag{129}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)}A_jD_jdx = \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{\sinh\left(2\delta y_j\right)}\sinh\left(\delta y_j\right) \cdot\left[\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) + \delta u_j\overline{v}_j - \delta v_j\overline{u}_j - \delta u_j\delta v_j\coth\left(\delta y_j\right)\right], \tag{130}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)}B_jC_jdx = \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{\sinh\left(2\delta y_j\right)}\sinh\left(\delta y_j\right) \cdot\left[\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) + \delta v_j\overline{u}_j - \delta u_j\overline{v}_j - \delta u_j\delta v_j\coth\left(\delta y_j\right)\right], \tag{131}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)} C_j D_j dx = \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{2\sinh\left(2\delta y_j\right)} \sinh\left(\delta y_j\right)\left[4\overline{v}_j^2\tanh\left(\delta y_j\right) - \delta H_{2j}\right], \tag{132}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)} B_j^2 e^{-2x} dx = \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{2\left(k_{ij}-2\right)\sinh\left(2\delta y_j\right)} \sinh\left(\left(k_{ij}-2\right)\delta y_j\right) \times\left[\delta H_{1j} - 4\overline{u}_j\delta u_j\right], \tag{133}$$

$$\int_{y_j}^{y_{j+1}} e^{-k_{ij}(y_i-x)} B_j D_j e^{-2x} dx = \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{\left(k_{ij}-2\right)\sinh\left(2\delta y_j\right)} \sinh\left(\left(k_{ij}-2\right)\delta y_j\right) \cdot\left[\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) - \delta u_j\overline{v}_j - \delta v_j\overline{u}_j + \delta u_j\delta v_j\coth\left(\delta y_j\right)\right]. \tag{134}$$

It thus follows from (127)–(134) that

$$\begin{aligned} P_{1,i} = \sum_{j=0}^{n} \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{6\cosh\left(\delta y_j\right)} \Big[&2k_{ij}\overline{u}_j\overline{v}_j\sinh^2\left(\delta y_j\right)\tanh\left(\delta y_j\right) \\ &+ 2k_{ij}\delta u_j\delta v_j\sinh\left(\delta y_j\right)\cosh\left(\delta y_j\right) \\ &+ 2\delta u_j\overline{v}_j\cosh^2\left(\delta y_j\right) \\ &+ 2\delta v_j\overline{u}_j\cosh^2\left(\delta y_j\right) \\ &+ \delta u_j\overline{v}_j + \delta v_j\overline{u}_j - 3\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) \\ &+ 3\delta u_j\delta v_j\coth\left(\delta y_j\right)\Big], \end{aligned}$$

$$\begin{aligned} P_{2,i} = \sum_{j=0}^{n} \frac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{4\cosh\left(\delta y_j\right)} \Big[&\delta H_{1j}\cosh^2\left(\delta y_j\right) \\ &+ 4k_{ij}\overline{u}_j\delta u_j\sinh^2\left(\delta y_j\right) \\ &+ 2\overline{u}_j^2\tanh\left(\delta y_j\right) + 4\overline{v}_j^2\tanh\left(\delta y_j\right) \\ &- \delta H_{2j} - \frac{4}{3}\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) \\ &+ \frac{8}{3}\delta u_j\delta v_j\coth\left(\delta y_j\right) \\ &+ \frac{4}{3}\overline{u}_j\overline{v}_j\cosh\left(\delta y_j\right)\sinh\left(\delta y_j\right) \\ &+ \frac{4}{3}\delta u_j\delta v_j\coth\left(\delta y_j\right)\cosh^2\left(\delta y_j\right) \\ &+ \frac{4}{3}k_{ij}\delta u_j\overline{v}_j\sinh^2\left(\delta y_j\right) \\ &+ \frac{4}{3}k_{ij}\delta v_j\overline{u}_j\sinh^2\left(\delta y_j\right)\Big]. \end{aligned} \tag{135}$$

We can also write $P_{1,i}$ and $P_{2,i}$ as

$$P_{1,i} = \sum_{j=0}^{n} P_{1,ij}, \qquad P_{2,i} = \sum_{j=0}^{n} P_{2,ij}, \tag{136}$$

where

$$P_{1,ij} = \begin{cases} \dfrac{1}{6}u_1v_1e^{y_1-y_i}, & \text{for } j=0, \\ \dfrac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{6\cosh\left(\delta y_j\right)} \\ \times\Big[2k_{ij}\overline{u}_j\overline{v}_j\sinh^2\left(\delta y_j\right)\tanh\left(\delta y_j\right) \\ \quad +2k_{ij}\delta u_j\delta v_j\sinh\left(\delta y_j\right)\cosh\left(\delta y_j\right) \\ \quad +2\delta u_j\overline{v}_j\cosh^2\left(\delta y_j\right) \\ \quad +2\delta v_j\overline{u}_j\cosh^2\left(\delta y_j\right) + \delta u_j\overline{v}_j \\ \quad +\delta v_j\overline{u}_j - 3\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) \\ \quad +3\delta u_j\delta v_j\coth\left(\delta y_j\right)\Big], \\ \qquad \text{for } j=1,\ldots,n-1, \\ -\dfrac{1}{6}u_nv_ne^{-y_n+y_i} \quad \text{for } j=n, \end{cases}$$

$$P_{2,ij} = \begin{cases} \dfrac{1}{4}u_1^2e^{y_1-y_i} + \dfrac{1}{6}u_1v_1e^{y_1-y_i}, \quad \text{for } j=0, \\ \dfrac{e^{-k_{ij}y_i}\cdot e^{k_{ij}\overline{y}_j}}{4\cosh\left(\delta y_j\right)} \\ \times\Big[\delta H_{1j}\cosh^2\left(\delta y_j\right) \\ \quad +4k_{ij}\overline{u}_j\delta u_j\sinh^2\left(\delta y_j\right) + 2\overline{u}_j^2\tanh\left(\delta y_j\right) \\ \quad +4\overline{v}_j^2\tanh\left(\delta y_j\right) - \delta H_{2j} - \dfrac{4}{3}\overline{u}_j\overline{v}_j\tanh\left(\delta y_j\right) \\ \quad +\dfrac{8}{3}\delta u_j\delta v_j\coth\left(\delta y_j\right) \\ \quad +\dfrac{4}{3}\overline{u}_j\overline{v}_j\cosh\left(\delta y_j\right)\sinh\left(\delta y_j\right) \\ \quad +\dfrac{4}{3}\delta u_j\delta v_j\cdot\coth\left(\delta y_j\right)\cosh^2\left(\delta y_j\right) \\ \quad +\dfrac{4}{3}k_{ij}\delta u_j\overline{v}_j\sinh^2\left(\delta y_j\right) \\ \quad +\dfrac{4}{3}k_{ij}\delta v_j\overline{u}_j\sinh^2\left(\delta y_j\right)\Big], \\ \qquad \text{for } j=1,\ldots,n-1, \\ \dfrac{1}{4}u_n^2e^{-y_n+y_i} + \dfrac{1}{6}u_nv_ne^{-y_n+y_i}, \\ \qquad \text{for } j=n. \end{cases} \tag{137}$$

The terms $P_{3,i}$, $P_{4,i}$, and $P_{k,ix}$ $(k = 1, 2, 3, 4)$ can be computed in the same way and we have

$$P_{1,ix} = -\sum_{j=0}^{n} k_{ij} P_{1,ij}, \qquad P_{2,ix} = -\sum_{j=0}^{n} k_{ij} P_{2,ij},$$
$$P_{3,ix} = -\sum_{j=0}^{n} k_{ij} P_{3,ij}, \qquad P_{4,ix} = -\sum_{j=0}^{n} k_{ij} P_{4,ij}. \tag{138}$$

The result can be summarized in the following theorem.

Theorem 12. *Assume that* $\overline{y}_i = \xi_i$, $\overline{z}_i = (\overline{u}_i, \overline{v}_i) = (\overline{u}(\xi_i), \overline{v}(\xi_i))$ *and* $\overline{H}_i = \int_{-\infty}^{\xi_i} (\overline{u}^2 + \overline{u}_x^2 + \overline{v}^2 + \overline{v}_x^2) dx$ *for* $i = 1, \ldots, n$ *with a multipeakon initial data* $\overline{z} = (\overline{u}, \overline{v})$ *as given by* (87). *Then, there exists a global solution* (y_i, u_i, v_i, H_i) *of* (109), (136), *and* (138) *with initial data* $(\overline{y}_i, \overline{u}_i, \overline{v}_i, \overline{H}_i)$. *On each interval* $[y_i(t), y_{i+1}(t)]$, *one defines* $z(t, x) = (u, v)(t, x)$ *as the solution of the Dirichlet problem* $z - z_{xx} = 0$ *with boundary conditions* $z(t, y_i(t)) = z_i(t)$ *and* $z(t, y_{i+1}(t)) = z_{i+1}(t)$ *for each time* t. *Thus* $z = (u, v)$ *is a conservative solution of the modified coupled two-component Camassa-Holm system, which is the multipeakon conservative solution.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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