NONEXISTENCE RESULTS FOR ELLIPTIC DIFFERENTIAL INEQUALITIES WITH A POTENTIAL IN BOUNDED DOMAINS

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Abstract. In this paper we are concerned with a class of elliptic differential inequalities with a potential in bounded domains both of $\mathbb{R}^m$ and of Riemannian manifolds. In particular, we investigate the effect of the behavior of the potential at the boundary of the domain on nonexistence of nonnegative solutions.

1. Introduction

In this paper we investigate nonexistence of nonnegative solutions to elliptic differential inequalities of the form

$$\frac{1}{a(x)}\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right)+V(x)u^{\sigma}\leq 0 \quad \text{in } \Omega, \tag{1}$$

where Ω is an open relatively compact connected subset of a general $m-$dimensional Riemannian manifold M endowed with a metric tensor g, ∇, div and Δ denote the gradient, the divergence operator and the Laplace-Beltrami operator associated to the metric, respectively. Furthermore, here and in the rest of the paper we assume that $a:\Omega\to\mathbb{R}$ satisfies

$$a>0, \quad a\in \operatorname{Lip}_{\mathrm{loc}}(\Omega), \tag{2}$$

$V>0$ a.e. on Ω, $V\in L^1_{\mathrm{loc}}(\Omega)$, and the constants p and σ satisfy $p>1$, $\sigma>p-1$. Tipically V is *unbounded* at $\partial\Omega$. We explicitly note that some of the results we find are new also for the model equation

$$\Delta u+V(x)u^{\sigma}\leq 0 \quad \text{in } \Omega, \tag{3}$$

in the special case $\Omega\subset\mathbb{R}^m$.

If, differently from what will be the focus of the present paper, we consider the case when $\Omega=\mathbb{R}^m$ or $\Omega=M$, where M is a complete noncompact Riemannian manifold, then there exists an extensive literature concerning nonexistence of nonnegatve solutions of equation (1). We refer to [1], [9], [10], [11] and [12] for a comprehensive description of results related to these (and also more general) problems on $\mathbb{R}^m$. Note that analogous results have also been obtained for degenerate elliptic equations and inequalities (see, e.g., [2], [13]), and for the parabolic companion problems (see, e.g., [12], [14], [15], [16]). The results in the case of a complete Riemannian manifold have a more recent history, in particular we cite the inspiring papers [5] and [6], and the papers [7], [8], [17], [18]. In particular it is showed that equation (1) admits the unique nonnegative solution $u\equiv 0$, assuming certain key assumptions hold, which are concerned with the parameters p, σ and with the behavior of a suitably weighted volume of geodesic balls, with density given by the product of a and of a negative power of the potential V.

In this work we intend to focus our attention on the case where $\Omega\subset M$ is an open relatively compact domain, considering *local weak solutions*, meant in the sense of Definition 2.1 below.

We start with a definition describing the weighted volume growth conditions on special subsets of Ω, contained in a neighborhood of $\partial\Omega$, that will be used in obtaining our nonexistence results for nonnegative

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solutions of (1). We denote the canonical Riemannian measure on M with $d\mu_0$, while we define

$$d\mu = a\, d\mu_0 \tag{4}$$

the weighted measure on M with density a.

Let $d(x) := \operatorname{dist}(x, \partial\Omega)$ for any $x \in \bar{\Omega}$. For every $\delta > 0$ we define

$$\mathcal{S}^\delta := \{x \in \Omega \,:\, d(x) < \delta\}, \qquad \Omega^\delta := \Omega \setminus \overline{\mathcal{S}^\delta}. \tag{5}$$

Recall that $p > 1$, $\sigma > p-1$, $V > 0$ a.e. in Ω and $V \in L^1_{\text{loc}}(\Omega)$ and define

$$\alpha = \frac{p\sigma}{\sigma - p + 1}, \qquad \beta = \frac{p-1}{\sigma - p + 1}. \tag{6}$$

Note that $\alpha > 1$. We introduce the following three weighted volume growth conditions:

i) We say that *condition (HP1)* holds if there exist $C > 0$, $C_0 > 0$, $k \in [0, \beta)$, $\delta_0 \in (0,1)$, $\varepsilon_0 \in \left(0, \frac{\alpha-1}{C_0} \wedge \beta\right)$ such that, for every $\delta \in (0, \delta_0)$ and for every $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta+\varepsilon}\, d\mu \leq C\delta^{\alpha - C_0\varepsilon} |\log \delta|^k. \tag{7}$$

ii) We say that *condition (HP2)* holds if there exist $C > 0$, $C_0 > 0$, $\delta_0 \in (0,1)$, $\varepsilon_0 \in \left(0, \frac{\alpha-1}{C_0} \wedge \beta\right)$ such that, for every $\delta \in (0, \delta_0)$ and for every $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta+\varepsilon}\, d\mu \leq C\delta^{\alpha - C_0\varepsilon} |\log \delta|^\beta \quad \text{and} \quad \int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta-\varepsilon}\, d\mu \leq C\delta^{\alpha - C_0\varepsilon} |\log \delta|^\beta. \tag{8}$$

iii) We say that *condition (HP3)* holds if there exist $C > 0$, $C_0 \geq 0$, $k \geq 0$, $\theta > 0$, $\tau > \max\{\frac{\sigma-p+1}{\sigma}(k+1), 1\}$, $\delta_0 \in (0,1)$, $\varepsilon_0 \in \left(0, \frac{\alpha-1}{C_0} \wedge \beta\right)$ such that, for every $\delta \in (0, \delta_0)$ and for every $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta+\varepsilon}\, d\mu \leq C\delta^{\alpha - C_0\varepsilon} |\log \delta|^k e^{-\varepsilon\theta|\log\delta|^\tau}. \tag{9}$$

Remark 1.1. Observe that, in general, conditions $(HP1)$, $(HP2)$ and $(HP3)$ are mutually independent. In particular, we note that, in general, $(HP3)$ does not imply $(HP1)$; this is essentially due to the fact that constant C in $(HP3)$ must be independent of δ and ε, see Example 4.1. The remaining cases can be easily treated; we leave the details to the interested reader.

Let us discuss some sufficient conditions for (7), (8), (9).

Remark 1.2. i) Suppose that there exist $C > 0, C_0 > 0$, $k \in (0, \beta), \delta_0 \in (0,1)$ such that

$$V(x) \leq C(d(x))^{-C_0} \qquad \text{for all } x \in \mathcal{S}^{\delta_0} \tag{10}$$

and

$$\int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta}\, d\mu \leq C\delta^\alpha |\log \delta|^k \tag{11}$$

for every $\delta \in (0, \delta_0)$, then condition (7) holds.

ii) If

$$a(x) \leq C \quad \text{and} \quad V(x) \geq C(d(x))^{-(\sigma+1)} |\log d(x)|^{-\frac{k}{\beta}} \qquad \text{for all } x \in \mathcal{S}^{\delta_0}, \tag{12}$$

then (7) holds, with $C_0 \geq \sigma + 1$.

iii) Suppose that there exist $C > 0$, $C_0 > 0$, $\delta_0 \in (0,1)$ such that

$$\frac{1}{C}(d(x))^{C_0} \leq V(x) \leq C(d(x))^{-C_0} \qquad \text{for all } x \in \mathcal{S}^{\delta_0} \tag{13}$$

and

$$\int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta}\, d\mu \leq C\delta^\alpha |\log \delta|^\beta \tag{14}$$

for every $\delta \in (0, \delta_0)$, then condition (8) holds. Moreover, condition (14) is satisfied, provided that

$$a(x) \leq C \quad \text{and} \quad V(x) \geq C(d(x))^{-(\sigma+1)} |\log d(x)|^{-1} \qquad \text{for all } x \in \mathcal{S}^{\delta_0}\,. \tag{15}$$

Hence, if (15) and the second inequality of (13) are satisfied, then condition (8) holds,

iv) Suppose there exist $C > 0$, $C_0 \geq 0$, $k \geq 0$, $\theta > 0$, $\tau > \max\{\frac{\sigma-p+1}{\sigma}(k+1), 1\}$, $\delta_0 \in (0,1)$ such that

$$V(x) \leq C(d(x))^{-C_0} e^{-\theta|\log d(x)|^{\tau}} \qquad \text{for all } x \in \mathcal{S}^{\delta_0} \tag{16}$$

and

$$\int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta}\, d\mu \leq C\delta^{\alpha} |\log \delta|^k \tag{17}$$

for every $\delta \in (0, \delta_0)$, then condition (9) holds.

v) If, for some $\epsilon_1 \in (0, \epsilon_0 \wedge \beta)$,

$$a(x) \leq C \quad \text{and} \quad V(x) \geq C d(x)^{-(\sigma+1)} |\log(d(x))|^{-\frac{k}{\beta}} e^{\left(-1+\frac{\beta}{\beta-\epsilon_1}\right)\theta|\log(d(x)|^{\tau}} \qquad \text{for all } x \in \mathcal{S}^{\delta_0}\,,$$

then condition (9) holds.

We can now state our main theorem.

Theorem 1.3. *Let $p > 1$, $\sigma > p-1$, $V \in L^1_{loc}(\Omega)$ with $V > 0$ a.e. in Ω and $a \in \mathrm{Lip}_{loc}(\Omega)$ with $a > 0$ on Ω. Assume that one of the conditions (HP1), (HP2) or (HP3) holds. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{\sigma}_{\mathrm{loc}}(\Omega, V d\mu)$ is a nonnegative weak solution of* (1)*, then $u \equiv 0$ in Ω.*

We remark that, for the case $p = 2$, the weighted volume growth conditions that we assume on geodesic balls are in many cases sharp. In particular, in this direction we construct a counterexample in geodesic balls of Riemannian models (see Section 4.3). In order to construct such counterexample, we will provide some conditions implying that the infimum of the spectrum of the Laplace operator on the space $L^2_V(\Omega) := \{f : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega f^2\, V d\mu < \infty\}$ is 0, a result that can be of independent interest (see Remark 4.4). Such a spectrum is clearly related to the eigenvalue equation

$$\Delta\phi + \lambda V(x)\phi = 0 \quad \text{in } \Omega\,. \tag{18}$$

The rest of the paper is organized as follows. In Section 2 we establish some preliminary technical results, that we put to use in Section 3, where we give the proof of Theorem 1.3. Finally in Section 4 we provide a family of counterexamples.

2. Preliminary results

For any relatively compact domain $D \subset \Omega$ and for any $p > 1$, $W^{1,p}(D)$ is the completion of the space of Lipschitz functions $w : D \to \mathbb{R}$ with respect to the norm

$$\|w\|_{W^{1,p}(D)} = \left(\int_D |\nabla w|^p\, d\mu_0 + \int_D |w|^p\, d\mu_0\right)^{\frac{1}{p}}.$$

For any function $u : \Omega \to \mathbb{R}$ we say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ if for every relatively compact domain $D \subset\subset \Omega$ one has $u_{|D} \in W^{1,p}(D)$.

Definition 2.1. *Let $p > 1$, $\sigma > p-1$, $V > 0$ a.e. in Ω and $V \in L^1_{loc}(\Omega)$. We say that u is a* weak solution *of equation* (1) *if $u \in W^{1,p}_{loc}(\Omega) \cap L^{\sigma}_{loc}(\Omega, V d\mu_0)$ and for every $\varphi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, with $\varphi \geq 0$ a.e. in Ω and compact support, one has*

$$-\int_\Omega a(x)|\nabla u|^{p-2} \left\langle \nabla u, \nabla\left(\frac{\varphi}{a(x)}\right)\right\rangle d\mu_0 + \int_\Omega V(x) u^{\sigma} \varphi\, d\mu_0 \leq 0\,. \tag{19}$$

Remark 2.2. We note that, by (2), $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{\sigma}_{\mathrm{loc}}(\Omega, V d\mu)$ is a weak solution of (1) if and only if it is a weak solution of

$$\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) + a(x)V(x)u^{\sigma} \le 0 \quad \text{in } \Omega,$$

i.e. if and only if for every $\psi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, with $\psi \ge 0$ a.e. in Ω and compact support, one has

$$-\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle \, d\mu + \int_{\Omega} V(x)u^{\sigma}\psi \, d\mu \le 0, \tag{20}$$

where $d\mu$ is the measure on M with density a, as defined in (4).

Indeed, given any nonnegative $\psi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ with compact support, one can choose $\varphi = a\psi$ as a test function in (19) in order to obtain (20). Similarly, given any nonnegative $\varphi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ with compact support, one can insert $\psi = \frac{\varphi}{a}$ in (20) and find (19).

The following two lemmas will be crucial ingredients in the proof the Theorem 1.3 (for their proofs see [7, Lemma 2.3, 2.4]).

Lemma 2.3. *Let $s \ge \frac{p\sigma}{\sigma - p + 1}$ be fixed. Then there exists a constant $C > 0$ such that for every $t \in (0, \min\{1, p-1\})$, every nonnegative weak solution u of equation (1) and every function $\varphi \in \mathrm{Lip}(\Omega)$ with compact support and $0 \le \varphi \le 1$ one has*

$$\frac{t}{p}\int_{\Omega} \varphi^{s} u^{-t-1}|\nabla u|^{p}\chi_D \, d\mu + \frac{1}{p}\int_{\Omega} V u^{\sigma - t}\varphi^{s} \, d\mu \le C t^{-\frac{(p-1)\sigma}{\sigma-p+1}} \int_{\Omega} V^{-\frac{p-t-1}{\sigma-p+1}} |\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}} \, d\mu, \tag{21}$$

where $D = \{x \in \Omega : u(x) > 0\}$, χ_D is the characteristic function of D and $d\mu$ is the measure on M with density a, as defined in (4).

Lemma 2.4. *Let $s \ge \frac{2p\sigma}{\sigma-p+1}$ be fixed. Then there exists a constant $C > 0$ such that for every nonnegative weak solution u of equation (1), every function $\varphi \in \mathrm{Lip}(\Omega)$ with compact support and $0 \le \varphi \le 1$ and every $t \in (0, \min\{1, p-1, \frac{\sigma-p+1}{2(p-1)}\})$ one has*

$$\begin{aligned}\int_{\Omega} \varphi^{s} u^{\sigma} V \, d\mu \le{}& C t^{-\frac{p-1}{p} - \frac{(p-1)^2\sigma}{p(\sigma-p+1)}} \left(\int_{\Omega\setminus K} V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}} |\nabla\varphi|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}} \, d\mu\right)^{\frac{\sigma-(t+1)(p-1)}{p\sigma}} \\ &\left(\int_{\Omega} V^{-\frac{p-t-1}{\sigma-p+1}} |\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}} \, d\mu\right)^{\frac{p-1}{p}} \left(\int_{\Omega\setminus K} \varphi^{s} u^{\sigma} V \, d\mu\right)^{\frac{(t+1)(p-1)}{p\sigma}},\end{aligned} \tag{22}$$

with $K = \{x \in \Omega : \varphi(x) = 1\}$ and $d\mu$ is the measure on M with density a, as defined in (4).

From Lemma 2.4 we immediately deduce

Corollary 2.5. *Under the same assumptions of Lemma 2.4 there exists a constant $C > 0$, independent of u, φ and t, such that*

$$\begin{aligned}&\left(\int_{\Omega} \varphi^{s} u^{\sigma} V \, d\mu\right)^{1 - \frac{(t+1)(p-1)}{p\sigma}} \\ &\le C t^{-\frac{p-1}{p} - \frac{(p-1)^2\sigma}{p(\sigma-p+1)}} \left(\int_{\Omega} V^{-\frac{p-t-1}{\sigma-p+1}} |\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}} \, d\mu\right)^{\frac{p-1}{p}} \left(\int_{\Omega} V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}} |\nabla\varphi|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}} \, d\mu\right)^{\frac{\sigma-(t+1)(p-1)}{p\sigma}}.\end{aligned} \tag{23}$$

3. Proof of Theorem 1.3

We divide the proof of Theorem 1.3 in three cases, depending on which of the conditions (HP1), (HP2) or (HP3) is assumed to hold.

Proof of Theorem 1.3. (*a*) Assume that condition (HP1) holds (see (7)). For any fixed $\delta \in (0, \delta_0)$ let $t := -\frac{1}{\log\delta}$. Fix any $C_1 \geq \max\left\{\frac{4(C_0-p+1)}{p\sigma}, 1\right\}$ with C_0 as in condition (7), define for every $x \in \Omega$

$$\varphi(x) = \begin{cases} 1 & \text{for } d(x) > \delta, \\ \left(\frac{d(x)}{\delta}\right)^{C_1 t} & \text{for } d(x) \leq \delta \end{cases} \tag{24}$$

and for $n \in \mathbb{N}$

$$\eta_n(x) = \begin{cases} 1 & \text{for } d(x) > \frac{\delta}{n}, \\ \frac{2n}{\delta}d(x) - 1 & \text{for } \frac{\delta}{2n} \leq d(x) \leq \frac{\delta}{n}, \\ 0 & \text{for } d(x) < \frac{\delta}{2n}. \end{cases} \tag{25}$$

Let

$$\varphi_n(x) = \eta_n(x)\varphi(x) \qquad \text{for all } x \in \Omega; \tag{26}$$

then $\varphi_n \in \mathrm{Lip}_c(\Omega)$ with $0 \leq \varphi_n \leq 1$. Moreover, we have

$$\nabla\varphi_n = \eta_n\nabla\varphi + \varphi\nabla\eta_n \qquad \text{a.e. in } \Omega\,,$$

and for every $b \geq 1$

$$|\nabla\varphi_n|^b \leq 2^{b-1}\big(|\nabla\varphi|^b + \varphi^b|\nabla\eta_n|^b\big) \qquad \text{a.e. in } \Omega.$$

Now we use φ_n in formula (21) of Lemma 2.3 with any fixed $s \geq \frac{p\sigma}{\sigma-p+1}$ and deduce that, for some positive constant C and for every $n \in \mathbb{N}$ and every small enough $t > 0$, we have

(27)

$$\begin{aligned}
&\int_\Omega V u^{\sigma-t}\varphi_n^s\,d\mu \\
&\leq C t^{-\frac{(p-1)\sigma}{\sigma-p+1}} \int_\Omega V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu \\
&= C t^{-\frac{(p-1)\sigma}{\sigma-p+1}} \int_\Omega V^{-\beta+\frac{t}{\sigma-p+1}}|\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu \\
&\leq C t^{-\frac{(p-1)\sigma}{\sigma-p+1}} 2^{\frac{p(\sigma-t)}{\sigma-p+1}-1}\left[\int_\Omega V^{-\beta+\frac{t}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu + \int_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}} V^{-\beta+\frac{t}{\sigma-p+1}}\varphi^{\frac{p(\sigma-t)}{\sigma-p+1}}|\nabla\eta_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu\right] \\
&\leq C t^{-\frac{(p-1)\sigma}{\sigma-p+1}}[I_1 + I_2],
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \int_{\Omega\setminus\mathcal{S}^\delta} V^{-\beta+\frac{t}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu, \\
I_2 &:= \int_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}} \varphi^{\frac{p(\sigma-t)}{\sigma-p+1}}|\nabla\eta_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}V^{-\beta+\frac{t}{\sigma-p+1}}\,d\mu.
\end{aligned}$$

By (24), (25) and assumption (HP1) with $\varepsilon = \frac{t}{\sigma-p+1}$ (see equation (7)), for every $n \in \mathbb{N}$ and every small enough $t > 0$ we have

$$\begin{aligned}
I_2 &\leq \left(\sup_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}}\varphi\right)^{\frac{p(\sigma-t)}{\sigma-p+1}} \left(\frac{2n}{\delta}\right)^{\frac{p(\sigma-t)}{\sigma-p+1}} \int_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}} V^{-\beta+\frac{t}{\sigma-p+1}}\,d\mu \\
&\leq C\left(\frac{1}{n}\right)^{\frac{p(\sigma-t)}{\sigma-p+1}C_1 t}\left(\frac{2n}{\delta}\right)^{\frac{p(\sigma-t)}{\sigma-p+1}}\left(\frac{\delta}{n}\right)^{\alpha-\frac{C_0 t}{\sigma-p+1}}\left|\log\left(\frac{\delta}{n}\right)\right|^k \\
&\leq C n^{-\alpha+\frac{C_0 t}{\sigma-p+1}+\frac{p(\sigma-t)}{\sigma-p+1}(-C_1 t+1)}\delta^{\alpha-\frac{C_0 t}{\sigma-p+1}-\frac{p(\sigma-t)}{\sigma-p+1}}\left|\log\left(\frac{\delta}{n}\right)\right|^k.
\end{aligned} \tag{28}$$

By our choice of C_1, for every small enough $t > 0$

$$-\alpha + \frac{C_0 t}{\sigma - p + 1} + \frac{p(\sigma - t)}{\sigma - p + 1}(-C_1 t + 1) = \frac{t(C_0 - p\sigma C_1 + pC_1 t - p)}{\sigma - p + 1} \le -\frac{t}{\sigma - p + 1} < 0. \tag{29}$$

Moreover, since $t = -\frac{1}{\log \delta}$, we have

$$\delta^{\alpha - \frac{C_0 t}{\sigma - p + 1} - \frac{p(\sigma - t)}{\sigma - p + 1}} = \delta^{\frac{-C_0 + p}{\sigma - p + 1} t} = e^{\frac{-C_0 + p}{\sigma - p + 1} t \log \delta} = e^{\frac{C_0 - p}{\sigma - p + 1}}$$

In view of (28) and (29) for $\delta > 0$ small enough, and thus $t = -\frac{1}{\log \delta}$ small enough, we obtain

$$I_2 \le C n^{-\frac{t}{\sigma - p + 1}} \left| \log \left(\frac{\delta}{n} \right) \right|^k . \tag{30}$$

In order to estimate I_1, we need the next

Claim: If $f : (0, \infty) \to [0, \infty)$ is a nonnegative decreasing measurable function and (7) holds, then for any $\varepsilon \in (0, \varepsilon_0)$ and for any $\delta \in (0, \delta_0)$ we have

$$\int_{\mathcal{S}^\delta} f(d(x))(V(x))^{-\beta + \varepsilon} \, d\mu \le C \int_0^{\frac{\delta}{2}} f(r) r^{\alpha - C_0 \varepsilon - 1} |\log r|^k \, dr \tag{31}$$

for some positive constant C. In fact, we have that

$$\begin{aligned}
\int_{\mathcal{S}^\delta} f \, V^{-\beta + \varepsilon} d\mu &= \sum_{n=0}^{\infty} \int_{\mathcal{S}^{\frac{\delta}{2^n}} \setminus \mathcal{S}^{\frac{\delta}{2^{n+1}}}} f(d(x)) V^{-\beta + \varepsilon} \, d\mu \\
&\le \sum_{n=0}^{\infty} f\left(\frac{\delta}{2^{n+1}} \right) \int_{\mathcal{S}^{\frac{\delta}{2^n}} \setminus \mathcal{S}^{\frac{\delta}{2^{n+1}}}} V^{-\beta + \varepsilon} d\mu \\
&\le C \sum_{n=0}^{\infty} f\left(\frac{\delta}{2^{n+1}} \right) \left(\frac{\delta}{2^n} \right)^{\alpha - C_0 \varepsilon} \left| \log \left(\frac{\delta}{2^n} \right) \right|^k \\
&\le C \sum_{n=0}^{\infty} f\left(\frac{\delta}{2^{n+1}} \right) \left(\frac{\delta}{2^{n+2}} \right)^{\alpha - C_0 \varepsilon - 1} \frac{\delta}{2^{n+1}} \left| \log \left(\frac{\delta}{2^{n+2}} \right) \right|^k \\
&\le C \sum_{n=0}^{\infty} \int_{\frac{\delta}{2^{n+2}}}^{\frac{\delta}{2^{n+1}}} f(r) r^{\alpha - C_0 \varepsilon - 1} |\log r|^k \, dr = C \int_0^{\frac{\delta}{2}} f(r) r^{\alpha - C_0 \varepsilon - 1} |\log r|^k dr \, .
\end{aligned}$$

Hence the claim has been shown. Moreover, there holds

$$|\nabla \varphi(x)| \le C_1 t \, \delta^{-C_1 t} \, (d(x))^{C_1 t - 1} \quad \text{a.e. in } \Omega \, . \tag{32}$$

Thus, using (31), (32),

$$\begin{aligned}
I_1 &= \int_{\mathcal{S}^\delta} V^{-\beta + \frac{t}{\sigma - p + 1}} (\delta^{-C_1 t} C_1 t (d(x))^{C_1 t - 1})^{\frac{p(\sigma - t)}{\sigma - p + 1}} \, d\mu \\
&\le C \delta^{-\frac{p(\sigma - t)}{\sigma - p + 1} C_1 t} \int_0^{\frac{\delta}{2}} t^{\frac{p(\sigma - t)}{\sigma - p + 1}} r^{\frac{p(\sigma - t)}{\sigma - p + 1}(C_1 t - 1) + \alpha - C_0 \frac{t}{\sigma - p + 1} - 1} |\log r|^k \, dr.
\end{aligned}$$

Now note that

$$\delta^{-\frac{p(\sigma - t)}{\sigma - p + 1} C_1 t} = e^{\frac{p(\sigma - t)}{\sigma - p + 1} C_1} < e^{\frac{p \sigma C_1}{\sigma - p + 1}}$$

and that by our choice of C_1 we have

$$\begin{aligned}
a &:= \frac{p(\sigma - t)}{\sigma - p + 1}(C_1 t - 1) + \alpha - C_0 \frac{t}{\sigma - p + 1} \\
&= \frac{t}{\sigma - p + 1}(-pC_1 t + p\sigma C_1 + p - C_0) \ge \frac{t}{2(\sigma - p + 1)} > 0 \, .
\end{aligned} \tag{33}$$

Then, by the above inequalities and performing the change of variables $\xi := a\log r$, we get

$$\begin{aligned} I_1 &\le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}} \int_0^{\frac{\delta}{2}} r^a(-\log r)^k \, \frac{dr}{r} \\ &\le Ca^{-(k+1)}t^{\frac{p(\sigma-t)}{\sigma-p+1}} \int_{-\infty}^0 e^{\xi}(-\xi)^k \, d\xi \\ &\le C\left(\frac{t}{2(\sigma-p+1)}\right)^{-k-1} t^{\frac{p(\sigma-t)}{\sigma-p+1}} \\ &\le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}-k-1}. \end{aligned} \tag{34}$$

By (27), (30) and (34)

$$\begin{aligned} \int_{\Omega\setminus\mathcal{S}^\delta} Vu^{\sigma-t}\, d\mu &\le \int_\Omega Vu^{\sigma-t}\varphi_n^s \, d\mu \\ &\le Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\left[n^{-\frac{t}{\sigma-p+1}} \left|\log\left(\frac{\delta}{n}\right)\right|^k + t^{\frac{p(\sigma-t)}{\sigma-p+1}-k-1}\right]. \end{aligned} \tag{35}$$

Since $\delta > 0$ is small and fixed, and thus $t = -\frac{1}{\log\delta} < 1$ is also fixed, taking the lim inf as $n \to \infty$ in (35) we obtain

$$\int_{\Omega\setminus\mathcal{S}^\delta} Vu^{\sigma-t} d\mu \le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}-k-1-\frac{(p-1)\sigma}{\sigma-p+1}}. \tag{36}$$

Now observe that, for each small enough $t > 0$,

$$\frac{p(\sigma-t)}{\sigma-p+1} - k - 1 - \frac{(p-1)\sigma}{\sigma-p+1} = \frac{p-1}{\sigma-p+1} - k - \frac{pt}{\sigma-p+1} = \beta - k - \frac{pt}{\sigma-p+1} \ge \delta_* > 0\,.$$

Then, for any fixed sufficiently small $t > 0$, we have

$$\int_\Omega Vu^{\sigma-t}\chi_{\Omega\setminus\mathcal{S}^{e^{-1/t}}}\, d\mu = \int_{\Omega\setminus\mathcal{S}^{e^{-1/t}}} Vu^{\sigma-t}\, d\mu \le Ct^{\delta_*}\,.$$

By Fatou's Lemma, taking the lim inf as $t \to 0^+$ in the previous inequality we obtain

$$\int_\Omega Vu^\sigma \, d\mu \le 0,$$

which implies $u \equiv 0$ in Ω.

(b) Assume that condition (HP2) holds (see (8)). Let the functions φ, η_n and φ_n be defined on M as in formulas (24), (25) and (26), with $\delta \in (0, \delta_0)$, $t = -\frac{1}{\log\delta}$, $C_1 \ge \max\left\{\frac{4(C_0-p+1)}{p\sigma}, 1, \frac{2(p+C_0)}{\sigma-p+1}\right\}$ and C_0 as in condition (8). We now apply formula (23), using the family of functions $\varphi_n \in \mathrm{Lip}_0(M)$ and any fixed $s \ge \frac{2p\sigma}{\sigma-p+1}$. Therefore, we get

$$\begin{aligned} \left(\int_\Omega \varphi_n^s u^\sigma V\, d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}} \le & Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}} \left(\int_\Omega V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu\right)^{\frac{p-1}{p}} \\ & \times \left(\int_\Omega V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}}|\nabla\varphi_n|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}}\, d\mu\right)^{\frac{\sigma-(t+1)(p-1)}{p\sigma}}. \end{aligned}$$

We now need need to estimate

$$\int_\Omega V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu \qquad \text{and} \qquad \int_\Omega V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}}|\nabla\varphi_n|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}}\, d\mu. \tag{37}$$

Arguing as in the first part of the proof of the theorem, under the validity of condition (HP1), with the only difference that the condition $k < \beta$ there is replaced here by $k = \beta$, using (8) we can deduce that

$$\int_\Omega V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu \le C\left[n^{-\frac{t}{\sigma-p+1}}\left|\log\left(\frac{\delta}{n}\right)\right|^\beta + t^{\frac{p(\sigma-t)}{\sigma-p+1}-\beta-1}\right]. \tag{38}$$

In order to estimate the second integral in (37) we start by defining $\Lambda = \frac{(p-1)\sigma t}{(\sigma-p+1)[\sigma-(t+1)(p-1)]}$, and we note that

$$\frac{(p-1)\sigma}{(\sigma-p+1)^2}t < \Lambda < \frac{2(p-1)\sigma}{(\sigma-p+1)^2}t < \varepsilon^* \tag{39}$$

for every small enough $t > 0$, and that

$$\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)} = \beta + \Lambda \qquad \text{and} \qquad \frac{p\sigma}{\sigma-(t+1)(p-1)} = \alpha + \Lambda p\,.$$

By our definition of the functions φ_n, for every $n \in \mathbb{N}$ and every small enough $t > 0$ we have

$$\begin{aligned}\int_\Omega V^{-\beta-\Lambda}|\nabla\varphi_n|^{\alpha+\Lambda p}\,d\mu &\le C\left[\int_\Omega V^{-\beta-\Lambda}{\eta_n}^{\alpha+\Lambda p}|\nabla\varphi|^{\alpha+\Lambda p}\,d\mu + \int_\Omega V^{-\beta-\Lambda}\varphi^{\alpha+\Lambda p}|\nabla\eta_n|^{\alpha+\Lambda p}\,d\mu\right]\\ &\le C\left[\int_{\mathcal{S}^\delta} V^{-\beta-\Lambda}|\nabla\varphi|^{\alpha+\Lambda p}\,d\mu + \int_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}} V^{-\beta-\Lambda}\varphi^{\alpha+\Lambda p}|\nabla\eta_n|^{\alpha+\Lambda p}\,d\mu\right]\\ &:= C(I_1+I_2).\end{aligned} \tag{40}$$

Now we use condition (8) with $\varepsilon = \Lambda$, and we obtain

$$\begin{aligned}I_2 &= \int_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}} V^{-\beta-\Lambda}\varphi^{\alpha+\Lambda p}|\nabla\eta_n|^{\alpha+\Lambda p}\,d\mu\\ &\le \left(\sup_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}}\varphi\right)^{\alpha+\Lambda p}\left(\frac{2n}{\delta}\right)^{\alpha+\Lambda p}\left(\int_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}} V^{-\beta-\Lambda}\,d\mu\right)\\ &\le Cn^{-(\alpha+\Lambda p)C_1 t}\left(\frac{2n}{\delta}\right)^{\alpha+\Lambda p}\left(\frac{\delta}{n}\right)^{\alpha-C_0\Lambda}\left|\log\left(\frac{\delta}{n}\right)\right|^{\beta}\\ &\le Cn^{-(\alpha+\Lambda p)C_1 t+p\Lambda+C_0\Lambda}\delta^{-p\Lambda-C_0\Lambda}\left|\log\left(\frac{\delta}{n}\right)\right|^{\beta}.\end{aligned}$$

By our definition of Λ and (39), choosing $C_1 > 0$ big enough, we easily find

$$\begin{aligned}-C_1(\alpha+\Lambda p)t + \Lambda(p+C_0) &< -C_1\alpha t - \frac{C_1 p(p-1)\sigma}{(\sigma-p+1)^2}t^2 + \frac{2(p+C_0)(p-1)\sigma}{(\sigma-p+1)^2}t\\ &\le -\widehat{C}t < 0,\end{aligned} \tag{41}$$

for some $\widehat{C} > 0$, for any small enough $t > 0$. Moreover by (39), since $t = -\frac{1}{\log\delta}$, we have

$$\delta^{-p\Lambda-C_0\Lambda} \le \delta^{-\frac{2(p-1)\sigma(C_0+p)t}{(\sigma-p-1)^2}} = e^{\frac{2(p-1)\sigma(C_0+p)}{(\sigma-p-1)^2}}.$$

Thus, for any sufficiently small $\delta > 0$,

$$I_2 \le Cn^{-\widehat{C}t}\left|\log\left(\frac{\delta}{n}\right)\right|^{\beta}. \tag{42}$$

In order to estimate I_1 we note that if $f : [0,\infty) \to [0,\infty)$ is a nonnegative decreasing measurable function and (8) holds, then for any small enough $\varepsilon > 0$ and $\delta > 0$ we have

$$\int_{\mathcal{S}^\delta} f(d(x))(V(x))^{-\beta-\varepsilon}\,d\mu \le C\int_0^{\frac{\delta}{2}} f(r)r^{\alpha-C_0\varepsilon-1}|\log r|^\beta\,dr \tag{43}$$

for some positive constant C, see (31). Thus, using (32), for every small enough $t > 0$ we have

$$\begin{aligned}I_1 &\le \int_{\mathcal{S}^\delta} V^{-\beta-\Lambda}\big(C_1 t\delta^{-C_1 t}(d(x))^{C_1 t-1}\big)^{\alpha+\Lambda p}\,d\mu\\ &\le C\int_0^{\frac{\delta}{2}} \delta^{-(\alpha+\Lambda p)C_1 t}t^{\alpha+\Lambda p}r^{(\alpha+\Lambda p)(C_1 t-1)+\alpha-C_0\Lambda-1}|\log r|^\beta\,dr\end{aligned}$$

Now, since $t=-\frac{1}{\log\delta}$, by (39) we have

$$\delta^{-(\alpha+\Lambda p)C_1 t}=e^{(\alpha+\Lambda p)C_1}\le e^{(\alpha+\varepsilon^* p)C_1};$$

moreover, in view of (41), for some $\widehat{C}_1>0$ and for every $t>0$ small enough

$$\widehat{C}t\le b:=(\alpha+\Lambda p)(C_1t-1)+\alpha-C_0\Lambda\le\widehat{C}_1t.$$

With the change of variables $\xi=b\log r$, using the previous inequalities we find

$$I_1\le Ct^{\alpha+\Lambda p}\int_0^{\frac{\delta}{2}}r^b|\log r|^\beta\,\frac{dr}{r}=Ct^{\alpha+\Lambda p}b^{-\beta-1}\left(\int_{-\infty}^0 e^\xi|\xi|^\beta\,d\xi\right)\le Ct^{\alpha+\Lambda p-\beta-1}. \tag{44}$$

From equations (40), (42) and (44) it follows that

$$\int_M V^{-\beta-\Lambda}|\nabla\varphi_n|^{\alpha+\Lambda p}\,d\mu\le C\left[t^{\alpha+\Lambda p-\beta-1}+n^{-\widehat{C}t}\left|\log\left(\frac{\delta}{n}\right)\right|^\beta\right]. \tag{45}$$

From (23), using (38) and (45) we have

$$\begin{aligned}\left(\int_{\Omega\setminus\mathcal{S}^\delta}u^\sigma V\,d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}}&\le\left(\int_\Omega\varphi_n^s u^\sigma V\,d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}}\\&\le Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}}\left(\int_\Omega V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu\right)^{\frac{p-1}{p}}\\&\quad\times\left(\int_\Omega V^{-\beta-\Lambda}|\nabla\varphi_n|^{\alpha+\Lambda p}\,d\mu\right)^{\frac{1}{\alpha+\Lambda p}}\\&\le Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}}\left[n^{-\frac{t}{\sigma-p+1}}\left|\log\left(\frac{\delta}{n}\right)\right|^\beta+t^{\frac{p(\sigma-t)}{\sigma-p+1}-\beta-1}\right]^{\frac{p-1}{p}}\\&\quad\times\left[t^{\alpha+\Lambda p-\beta-1}+n^{-\widehat{C}t}\left|\log\left(\frac{\delta}{n}\right)\right|^\beta\right]^{\frac{1}{\alpha+\Lambda p}}.\end{aligned}$$

By taking the lim inf as $n\to+\infty$ we get

$$\left(\int_{\Omega\setminus\mathcal{S}^\delta}u^\sigma V\,d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}}\le Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}+\frac{(p-1)(\sigma-t)}{\sigma-p+1}-\frac{(\beta+1)(p-1)}{p}+1-\frac{\beta+1}{\alpha+\Lambda p}}$$

for every sufficiently small $t>0$, with $t=-\frac{1}{\log\delta}$. But

$$-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}+\frac{(p-1)(\sigma-t)}{\sigma-p+1}-\frac{(\beta+1)(p-1)}{p}+1-\frac{\beta+1}{\alpha+\Lambda p}=-\frac{(p-1)^2}{p(\sigma-p+1)}t,$$

hence for every small enough $t>0$ we have

$$\left(\int_{\Omega\setminus\mathcal{S}^{e^{-1/t}}}u^\sigma V\,d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}}\le Ct^{-\frac{(p-1)^2}{p(\sigma-p+1)}t}\le C,$$

that is

$$\int_{\Omega\setminus\mathcal{S}^{e^{-1/t}}}u^\sigma V\,d\mu\le C$$

uniformly in t, for $t>0$ sufficiently small. By taking the limit for $t\to0^+$ we deduce

$$\int_\Omega u^\sigma V\,d\mu<+\infty, \tag{46}$$

and thus $u \in L^\sigma(\Omega, V d\mu)$. Now we exploit inequality (22) with the cutoff function φ_n, and using again (38) and (45) we obtain

$$\int_\Omega \varphi_n^s u^\sigma V\,d\mu \le Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}}\left[n^{-\frac{t}{\sigma-p+1}}\left|\log\left(\frac{\delta}{n}\right)\right|^\beta + t^{\frac{p(\sigma-t)}{\sigma-p+1}-\beta-1}\right]^{\frac{p-1}{p}}\left(\int_{\mathcal{S}^\delta}\varphi_n^s u^\sigma V\,d\mu\right)^{\frac{(t+1)(p-1)}{p\sigma}}$$
$$\times\left[t^{\alpha+\Lambda p-\beta-1}+n^{-\widehat{C}t}\left|\log\left(\frac{\delta}{n}\right)\right|^\beta\right]^{\frac{1}{\alpha+\Lambda p}}.$$

Since $\varphi_n \equiv 1$ in $\Omega\setminus\mathcal{S}^\delta$ and $0\le\varphi_n\le 1$ in Ω, for all $n\in\mathbb{N}$

$$\int_{\Omega\setminus\mathcal{S}^\delta} u^\sigma V\,d\mu \le \int_\Omega \varphi_n^s u^\sigma V\,d\mu, \qquad \int_{\mathcal{S}^\delta}\varphi_n^s u^\sigma V\,d\mu \le \int_{\mathcal{S}^\delta} u^\sigma V\,d\mu.$$

Using previous inequalities and taking the lim inf as $n\to+\infty$ we get

$$\int_{\Omega\setminus\mathcal{S}^\delta} u^\sigma V\,d\mu \le Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}+\frac{(p-1)(\sigma-t)}{\sigma-p+1}-\frac{(\beta+1)(p-1)}{p}+1-\frac{\beta+1}{\alpha+\Lambda p}}\left(\int_{\mathcal{S}^\delta} u^\sigma V\,d\mu\right)^{\frac{(t+1)(p-1)}{p\sigma}}$$
$$= Ct^{-\frac{(p-1)^2}{p(\sigma-p+1)}t}\left(\int_{\mathcal{S}^\delta} u^\sigma V\,d\mu\right)^{\frac{(t+1)(p-1)}{p\sigma}} \le C\left(\int_{\mathcal{S}^\delta} u^\sigma V\,d\mu\right)^{\frac{(t+1)(p-1)}{p\sigma}}$$

uniformly for $t>0$ sufficiently small, with $t=-\frac{1}{\log\delta}$. Since $u\in L^\sigma(\Omega,V d\mu)$,

$$\int_{\mathcal{S}^\delta} u^\sigma V\,d\mu \to 0 \quad \text{as } \delta\to 0.$$

Moreover $\frac{(t+1)(p-1)}{p\sigma}\to\frac{p-1}{p\sigma}>0$ as $\delta\to 0$. It follows that

$$\int_\Omega u^\sigma V\,d\mu = \lim_{\delta\to 0}\int_{\Omega\setminus\mathcal{S}^\delta} u^\sigma V\,d\mu = 0,$$

which implies $u\equiv 0$ in Ω.

(c) Assume that condition (HP3) holds (see (9)). Consider the functions φ, η_n and φ_n defined in (24), (25) and (26), with $\delta>0$ small enough, $t=-\frac{1}{\log\delta}$, $C_1>0$ as in b) and C_0 as in condition (9). Arguing as in a), by formula (21) with any fixed $s\ge\frac{p\sigma}{\sigma-p+1}$, we see that

(47)
$$\int_\Omega V u^{\sigma-t}\varphi_n^s\,d\mu \le Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\int_\Omega V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu$$
$$\le Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\left[\int_\Omega V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu + \int_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}} V^{-\frac{p-t-1}{\sigma-p+1}}\varphi^{\frac{p(\sigma-t)}{\sigma-p+1}}|\nabla\eta_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu\right]$$
$$:= Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}[I_1+I_2],$$

for some positive constant C and for every $n\in\mathbb{N}$ and every small enough $t>0$. Now, recalling the definitions of φ and η_n, by condition (9) with $\varepsilon=\frac{t}{\sigma-p+1}$, for every small enough $t>0$ we have

$$I_2 \le \left(\sup_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}}\varphi\right)^{\frac{p(\sigma-t)}{\sigma-p+1}}\left(\frac{2n}{\delta}\right)^{\frac{p(\sigma-t)}{\sigma-p+1}}\int_{\mathcal{S}^{\frac{\delta}{n}}\setminus\mathcal{S}^{\frac{\delta}{2n}}} V^{-\beta+\frac{t}{\sigma-p+1}}\,d\mu$$
$$\le Cn^{-\frac{p(\sigma-t)}{\sigma-p+1}C_1 t}\left(\frac{2n}{\delta}\right)^{\frac{p(\sigma-t)}{\sigma-p+1}}\left(\frac{\delta}{n}\right)^{\alpha-\frac{C_0 t}{\sigma-p+1}}\left|\log\left(\frac{\delta}{n}\right)\right|^k e^{-\frac{\theta t}{\sigma-p+1}\left|\log\left(\frac{\delta}{n}\right)\right|^\tau}$$
$$= Cn^{-\alpha+\frac{C_0 t}{\sigma-p+1}+\frac{p(\sigma-t)}{\sigma-p+1}(-C_1 t+1)}\delta^{\alpha-\frac{C_0 t}{\sigma-p+1}-\frac{p(\sigma-t)}{\sigma-p+1}}\left|\log\left(\frac{\delta}{n}\right)\right|^k e^{-\frac{\theta t}{\sigma-p+1}\left|\log\left(\frac{\delta}{n}\right)\right|^\tau}.$$

Note that since $t = -\frac{1}{\log\delta}$, we have

$$\delta^{\alpha - \frac{C_0 t}{\sigma-p+1} - \frac{p(\sigma-t)}{\sigma-p+1}} = \delta^{\frac{-C_0+p}{\sigma-p+1}t} = e^{\frac{-C_0+p}{\sigma-p+1}t\log\delta} = e^{\frac{C_0-p}{\sigma-p+1}}.$$

Thus, by our choice of C_1, if $t>0$ is sufficiently small, we have

$$I_2 \le Cn^{-\frac{t}{\sigma-p+1}}\left|\log\left(\frac{\delta}{n}\right)\right|^k. \tag{48}$$

In order to estimate I_1 we note that if $f:[0,\infty)\to[0,\infty)$ is a nonnegative measurable decreasing function and (9) holds, then for any small enough $\varepsilon>0$ and any sufficiently large $R>0$ we have

$$\int_{\mathcal{S}^\delta} f(d(x))(V(x))^{-\beta+\varepsilon}\,d\mu \le C\int_0^{\frac{\delta}{2}} f(r)r^{\alpha-C_0\varepsilon-1}|\log r|^k e^{-\varepsilon\theta|\log r|^\tau}\,dr \tag{49}$$

Inequality (49) can be obtained in a similar way as (31).

Now, using (32) and (49) with $\varepsilon = \frac{t}{\sigma-p+1}$, we obtain that for every small enough $t>0$ with $t=-\frac{1}{\log\delta}$

$$\begin{aligned} I_1 &\le \int_{\mathcal{S}^\delta} V^{-\beta+\frac{t}{\sigma-p+1}}\left(C_1 t\delta^{-C_1 t}(d(x))^{C_1 t-1}\right)^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu \\ &\le C\delta^{-\frac{p(\sigma-t)C_1 t}{\sigma-p+1}}\int_0^{\frac{\delta}{2}} t^{\frac{p(\sigma-t)}{\sigma-p+1}} r^{\frac{p(\sigma-t)(C_1t-1)}{\sigma-p+1}+\alpha-\frac{C_0 t}{\sigma-p+1}-1}|\log r|^k e^{-\frac{t\theta}{\sigma-p+1}|\log r|^\tau}\,dr. \end{aligned}$$

Note that

$$\delta^{-\frac{p(\sigma-t)C_1 t}{\sigma-p+1}} = e^{\frac{p(\sigma-t)C_1}{\sigma-p+1}} \le e^{\frac{p\sigma C_1}{\sigma-p+1}}.$$

Thus, with the change of variable $r=e^{-\xi}$, we deduce

$$\begin{aligned} I_1 &\le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}}\int_0^{\frac{\delta}{2}} r^{\frac{p(\sigma-t)(C_1t-1)}{\sigma-p+1}+\alpha-\frac{C_0 t}{\sigma-p+1}}|\log r|^k e^{-\frac{t\theta}{\sigma-p+1}|\log r|^\tau}\,r^{-1}dr. \\ &\le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}}\int_0^{+\infty} e^{-a\xi}\xi^k e^{-\frac{t\theta}{\sigma-p+1}\xi^\tau}\,d\xi, \end{aligned}$$

with a defined in (33). Now recall that by our choice of C_1, for $t>0$ small enough, we have $a>0$. Hence, setting $\rho = \left(\frac{t\theta}{\sigma-p+1}\right)^{\frac{1}{\tau}}\xi$, we have

$$I_1 \le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}}\int_0^{+\infty}\xi^k e^{-\frac{t\theta}{\sigma-p+1}\xi^\tau}\,d\xi = Ct^{\frac{p(\sigma-t)}{\sigma-p+1}}\left(\frac{t\theta}{\sigma-p+1}\right)^{-\frac{k+1}{\tau}}\int_0^{+\infty}\rho^k e^{-\rho^\tau}\,d\rho \le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}-\frac{k+1}{\tau}}. \tag{50}$$

From (47), (48) and (50) we conclude that for every $n\in\mathbb{N}$ and every small enough $t=-\frac{1}{\log\delta}>0$ we have

$$\int_{\Omega\setminus\mathcal{S}^\delta} Vu^{\sigma-t}\,d\mu \le \int_\Omega Vu^{\sigma-t}\varphi_n^s\,d\mu \le Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\left[t^{\frac{p(\sigma-t)}{\sigma-p+1}-\frac{k+1}{\tau}} + n^{-\frac{t}{\sigma-p+1}}\left|\log\left(\frac{\delta}{n}\right)\right|^k\right]$$

for some fixed positive constant C. Passing to the limit as $n\to+\infty$ in the previous inequality yields

$$\int_{\Omega\setminus\mathcal{S}^\delta} Vu^{\sigma-t}\,d\mu \le Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}+\frac{p(\sigma-t)}{\sigma-p+1}-\frac{k+1}{\tau}}. \tag{51}$$

Now note that by our assumptions on τ, k we have

$$-\frac{(p-1)\sigma}{\sigma-p+1}+\frac{p(\sigma-t)}{\sigma-p+1}-\frac{k+1}{\tau} = \frac{\sigma}{\sigma-p+1}-\frac{k+1}{\tau}-\frac{pt}{\sigma-p+1} \ge \frac{1}{2}\left(\frac{\sigma}{\sigma-p+1}-\frac{k+1}{\tau}\right) := \delta_* > 0$$

for every small enough $t=-\frac{1}{\log\delta}>0$. Thus (51) yields

$$\int_{\Omega\setminus\mathcal{S}^{e^{-1/t}}} Vu^{\sigma-t}\,d\mu \le Ct^{\delta_*} \tag{52}$$

for every small enough $t>0$. Passing to the $\liminf$ as t tends to 0^+ in (52), we conclude by an application of Fatou's Lemma that

$$\int_\Omega Vu^\sigma\,d\mu = 0,$$

so that $u \equiv 0$ on Ω. □

4. Counterexamples

To begin, we show that in general hypothesis $(HP3)$ does not imply hypothesis $(HP1)$.

Example 4.1. Let $\sigma > 1, p > 1$, and let $a \in C^1(\Omega), a > 0$ with

$$a(x) := \begin{cases} 1 & \text{if } d(x) > 2\delta^*, \\ d(x)^{\alpha-1} |\log d(x)|^{\beta_0} e^{-\beta\theta|\log d(x)|^\tau} & \text{if } d(x) \le \delta^*, \end{cases}$$

with $\delta^* > 0$, $\beta_0 > \beta$, $\theta > 0$, $\tau > \max\left\{\frac{\sigma-p+1}{\sigma}(\beta_0+1), 1\right\}$. Define

$$V(x) := e^{-\theta|\log d(x)|^\tau} \quad \text{for all } x \in \Omega.$$

Then, in view of (4), for all $\delta > 0$ sufficiently small and every $\varepsilon > 0$

$$\begin{aligned} \int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta+\varepsilon}(x)\, d\mu &\le e^{-\varepsilon\theta|\log\delta|^\tau} \int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} e^{\beta\theta|\log d(x)|^\tau}\, d\mu \\ &\le C e^{-\varepsilon\theta|\log\delta|^\tau} \int_{\delta/2}^{\delta} r^{\alpha-1}|\log r|^{\beta_0}\, dr \;\le\; C e^{-\varepsilon\theta|\log\delta|^\tau} \delta^\alpha |\log\delta|^{\beta_0} \end{aligned}$$

for some positive constant C independent of δ and ε. Thus $(HP3)$ holds.

On the other hand, for every $\delta > 0$ small enough and every $\varepsilon > 0$

$$\begin{aligned} \int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta+\varepsilon}(x)\, d\mu &\ge e^{-\varepsilon\theta 2^\tau|\log\delta|^\tau} \int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} e^{\beta\theta|\log d(x)|^\tau}\, d\mu \\ &\ge C e^{-\varepsilon\theta 2^\tau|\log\delta|^\tau} \int_{\delta/2}^{\delta} r^{\alpha-1}|\log r|^{\beta_0}\, dr \;\ge\; C e^{-\varepsilon\theta 2^\tau|\log\delta|^\tau} \delta^\alpha |\log\delta|^{\beta_0}, \end{aligned} \tag{53}$$

for some positive constant C independent of δ and ε. Passing to the limit in (53) as $\varepsilon \to 0$ we obtain that for every $\delta > 0$ small enough

$$\int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta}(x)\, d\mu \ge C\delta^\alpha |\log\delta|^{\beta_0}. \tag{54}$$

If, by contradiction, $(HP1)$ holds, passing to the limit as $\varepsilon \to 0$ in (7) we obtain that, for every $\delta > 0$ small enough,

$$\int_{\mathcal{S}^\delta \setminus \mathcal{S}^{\delta/2}} V^{-\beta}(x)\, d\mu \le \bar{C}\delta^\alpha |\log\delta|^{\beta}. \tag{55}$$

Since $\beta_0 > \beta$, (54) and (55) are in contrast. So $(HP1)$ cannot hold.

Now we show that if condition $(HP1)$ or $(HP2)$ or $(HP3)$ is not satisfied, then problem (1) can admit a positive nontrivial solution. Before constructing our counterexample, which will only deal with the case $p = 2$, we need some auxiliary results on spectral theory for a *weighted eigenvalue problem* for the Laplace–Beltrami operator. Moreover, we recall the notion of Riemannian model manifolds (see Section 4.2). We should mention that a similar counterexample has been constructed in [6] and in [7], when $\Omega = M$, with M a complete noncompact Riemannian manifold. However, many differences occur in the present situation, due to the fact that Ω is bounded. Furthermore, the study of the first eigenvalue for the weighted eigenvalue problem was not necessary in [6], [7].

4.1. Preliminary results for a weighted eigenvalue problem.

Let $a \equiv 1$, $V \in C(\Omega)$, $V > 0$ in Ω; for any domain $D \subseteq \Omega$ set $L^2_V(D) := \{f : D \to \mathbb{R}$ measurable such that $\int_D f^2\, V d\mu < \infty\}$. For every $\delta > 0$, consider the *weighted* eigenvalue problem

$$\begin{cases} \Delta\phi + \lambda V \phi = 0 & \text{in } \Omega^\delta, \\ \phi = 0 & \text{on } \partial\Omega^\delta, \end{cases} \tag{56}$$

where $\Omega^\delta := \Omega \setminus \overline{\mathcal{S}^\delta}$. It is known that, since $V \in C(\overline{\Omega^\delta})$ and $V > 0$, there exist the first eigenvalue λ_δ and the first eigenfunction $\phi_\delta \in L^2_V(\Omega^\delta) \cap W^{1,2}_0(\Omega^\delta)$ of (56), with

$$\lambda_\delta = \inf_{\phi \in C_c^\infty(\Omega^\delta), \phi \not\equiv 0} \frac{\int_{\Omega^\delta} |\nabla \phi|^2 \, d\mu}{\int_{\Omega^\delta} V \phi^2 \, d\mu}, \tag{57}$$

see e.g. [3]. Moreover, $\lambda_\delta > 0$, $\lambda_{\delta_1} \geq \lambda_{\delta_2}$ if $\delta_1 > \delta_2$, and $\lambda_\delta \to \bar{\lambda}(\Omega)$ as $\delta \to 0^+$, for some $\bar{\lambda}(\Omega) \in [0, \infty)$. Using (57), an easy computation shows that

$$\bar{\lambda}(\Omega) = \inf_{\phi \in C_c^\infty(\Omega), \phi \not\equiv 0} \frac{\int_{\Omega} |\nabla \phi|^2 \, d\mu}{\int_{\Omega} V \phi^2 \, d\mu}. \tag{58}$$

From condition (58), the following lemma immediately follows.

Lemma 4.2. *Let $V \in C(\Omega), V > 0$ in Ω. Suppose that there exists $C > 0$ such that for any $\alpha > 0$ there exists $\phi_\alpha \in W^{1,2}_0(\Omega) \cap L^2_V(\Omega)$, with $\phi_\alpha \not\equiv 0$, for which*

$$\int_\Omega |\nabla \phi_\alpha|^2 d\mu \leq C\alpha \int_\Omega V \phi_\alpha^2 \, d\mu \tag{59}$$

and

$$\int_{\mathcal{S}^{2\delta} \setminus \mathcal{S}^\delta} |\phi_\alpha|^2 d\mu = o(\delta^2) \qquad \text{as } \delta \to 0^+. \tag{60}$$

Then $\bar{\lambda}(\Omega) = 0$.

Proof. For any small $\delta > 0$, consider a Lipschitz cut–off function ψ such that $\psi \equiv 0$ on $\mathcal{S}^\delta$, $\psi \equiv 1$ on $\Omega \setminus \mathcal{S}^{2\delta}$ and $|\nabla \psi| \leq C\delta^{-1}$ for some $C > 1$ independent of δ. Then $\psi \phi_\alpha \in W^{1,2}_0(\Omega)$ with $\operatorname{supp}(\psi\phi_\alpha) \subset \Omega \setminus \mathcal{S}^\delta$ and

$$\int_\Omega |\nabla(\psi\phi_\alpha)|^2 \, d\mu \leq 2 \left(\int_\Omega \psi^2 |\nabla \phi_\alpha|^2 \, d\mu + \int_\Omega \phi_\alpha^2 |\nabla \psi|^2 \, d\mu \right).$$

Now note that, up to choosing $\delta > 0$ small enough, since $\phi_\alpha \in L^2_V(\Omega)$ with $\int_\Omega \phi_\alpha^2 V \, d\mu > 0$,

$$\int_\Omega \psi^2 |\nabla \phi_\alpha|^2 \, d\mu \leq \int_\Omega |\nabla \phi_\alpha|^2 \, d\mu \leq C\alpha \int_\Omega \phi_\alpha^2 V \, d\mu \leq 2C\alpha \int_{\Omega^{2\delta}} \phi_\alpha^2 V \, d\mu \leq 2C\alpha \int_\Omega (\psi\phi_\alpha)^2 V \, d\mu.$$

Moreover,

$$\int_\Omega \phi_\alpha^2 |\nabla \psi|^2 \, d\mu \leq C\delta^{-2} \int_{\mathcal{S}^{2\delta} \setminus \mathcal{S}^\delta} \phi_\alpha^2 \, d\mu = o(1)$$

as $\delta \to 0^+$. In particular, since $\phi_\alpha \in L^2_V(\Omega)$ with $\int_\Omega \phi_\alpha^2 V \, d\mu > 0$, we can choose $\delta > 0$ small enough so that

$$\int_\Omega \phi_\alpha^2 |\nabla \psi|^2 \, d\mu \leq \alpha \int_{\Omega \setminus \mathcal{S}^{2\delta}} \phi_\alpha^2 V \, d\mu \leq \alpha \int_\Omega (\psi\phi_\alpha)^2 V \, d\mu.$$

We conclude that

$$\int_\Omega |\nabla(\psi\phi_\alpha)|^2 \, d\mu \leq C'\alpha \int_\Omega (\psi\phi_\alpha)^2 V \, d\mu$$

Since $\psi\phi_\alpha$ has compact support in Ω, by standard mollification there exists $\varphi_\alpha \in C_c^\infty(\Omega)$ such that

$$\frac{\int_\Omega |\nabla \varphi_\alpha|^2 \, d\mu}{\int_\Omega \varphi_\alpha^2 V \, d\mu} \leq 2 \frac{\int_\Omega |\nabla(\psi\phi_\alpha)|^2 \, d\mu}{\int_\Omega (\psi\phi_\alpha)^2 V \, d\mu} \leq 2C'\alpha.$$

Hence $\bar{\lambda}(\Omega) \leq 2C'\alpha$ for every $\alpha > 0$, and the conclusion follows. □

Using the previous lemma, we show the next result.

Proposition 4.3. *Let $V \in C(\Omega), V > 0$ in Ω. Assume that*

$$C_0[d(x)]^{-\beta_0} \leq V(x) \leq C_1[d(x)]^{-\beta_1} \quad \text{for all } x \in \Omega, \tag{61}$$

for some $C_1 > C_0 > 0$, $\beta_1 \geq \beta_0 > 2$. Then $\bar{\lambda}(\Omega) = 0$.

Proof. For each $\alpha > 0$ let

$$\phi_\alpha(x) := e^{-\sqrt{\alpha}[d(x)]^{-\gamma}}, \quad x \in \Omega,$$

where $\gamma := \frac{\beta_0 - 2}{2}$. Note that $\phi_\alpha \in C(\Omega)$, $\phi_\alpha(x) \to 0$ as $d(x) \to 0^+$ and ϕ_α satisfies (60). Furthermore, using the fact that $x \mapsto d(x)$ is Lipschitz in Ω, we have

$$\nabla\phi_\alpha(x) = \gamma\sqrt{\alpha}[d(x)]^{-\gamma-1}\nabla d(x)\phi_\alpha(x)\,, \quad x \in \Omega\,;$$

therefore, since $|\nabla d(x)| \leq 1$ for a.e. $x \in \Omega$,

$$|\nabla\phi_\alpha(x)|^2 \leq \gamma^2\alpha[d(x)]^{-2\gamma-2}\phi_\alpha^2(x) \qquad \text{a.e. } x \in \Omega\,.$$

Hence, in view of (61),

$$\int_\Omega |\nabla\phi_\alpha(x)|^2\,d\mu \leq \gamma^2\alpha\int_\Omega [d(x)]^{-2\gamma-2}\phi_\alpha^2(x)\,d\mu \leq \frac{\gamma^2\alpha}{C_0}\int_\Omega V(x)\phi_\alpha^2(x)\,d\mu < \infty\,.$$

Consequently, $\phi_\alpha \in W_0^{1,2}(\Omega) \cap L_V^2(\Omega)$, and condition (59) is satisfied with $C = \frac{\gamma^2}{C_0}$. Hence, by Lemma 4.2 the conclusion follows. □

Remark 4.4. Note that $\bar{\lambda}(\Omega)$ is the infimum of the spectrum of the Laplace operator on $L_V^2(\Omega)$, which is nonnegative and may not be achieved. Clearly, such a spectrum is deeply related to the eigenvalue equation (18). Hence Lemma 4.2 and Proposition 4.3 provide sufficient conditions which imply that such infimum is 0.

4.2. **Riemannian models.** Let us fix a point $o \in M$ and denote by $\mathrm{Cut}(o)$ the *cut locus* of o. For any $x \in M \setminus \big[\mathrm{Cut}(o) \cup \{o\}\big]$, one can define the *polar coordinates* with respect to o, see e.g. [4]. Namely, to any point $x \in M \setminus \big[\mathrm{Cut}(o) \cup \{o\}\big]$ there correspond a polar radius $r(x) := dist(x, o)$ and a polar angle $\theta \in \mathbb{S}^{m-1}$ such that the shortest geodesics from o to x starts at o with the direction θ in the tangent space T_oM. Since we can identify T_oM with $\mathbb{R}^m$, θ can be regarded as a point of $\mathbb{S}^{m-1}$.

The Riemannian metric in $M \setminus \big[\mathrm{Cut}(o) \cup \{o\}\big]$ in the polar coordinates reads as

$$ds^2 = dr^2 + A_{ij}(r,\theta)d\theta^i d\theta^j,$$

where $(\theta^1, \ldots, \theta^{m-1})$ are coordinates in $\mathbb{S}^{m-1}$ and (A_{ij}) is a positive definite matrix. It is not difficult to see that the Laplace-Beltrami operator in polar coordinates has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \mathcal{F}(r,\theta)\frac{\partial}{\partial r} + \Delta_{S_r}, \tag{62}$$

where $\mathcal{F}(r,\theta) := \frac{\partial}{\partial r}\big(\log\sqrt{A(r,\theta)}\big)$, $A(r,\theta) := \det(A_{ij}(r,\theta))$, Δ_{S_r} is the Laplace-Beltrami operator on the submanifold $S_r := \partial B_r(o) \setminus \mathrm{Cut}(o)$ with $B_r(o) \equiv B_r := \{x \in M \,:\, \rho(x) < r\}$.

M is a *manifold with a pole*, if it has a point $o \in M$ with $\mathrm{Cut}(o) = \emptyset$. The point o is called *pole* and the polar coordinates (r,θ) are defined in $M \setminus \{o\}$.

A manifold with a pole is a *spherically symmetric manifold* or a *model*, if the Riemannian metric is given by

$$ds^2 = dr^2 + \psi^2(r)d\theta^2, \tag{63}$$

where $d\theta^2$ is the standard metric in $\mathbb{S}^{m-1}$, and

$$\psi \in \mathcal{A} := \Big\{f \in C^\infty((0,\infty)) \cap C^1([0,\infty)) : f'(0) = 1,\, f(0) = 0,\, f > 0 \text{ in } (0,\infty)\Big\}. \tag{64}$$

In this case, we write $M \equiv M_\psi$; furthermore, we have $\sqrt{A(r,\theta)} = \psi^{m-1}(r)$, so the boundary area of the geodesic sphere ∂S_R is computed by

$$S(R) = \omega_m\psi^{m-1}(R),$$

ω_m being the area of the unit sphere in $\mathbb{R}^m$. Also, the volume of the ball $B_R(o)$ is given by

$$\mu(B_R(o)) = \int_0^R S(\xi)d\xi\,.$$

Moreover we have

$$\Delta = \frac{\partial^2}{\partial r^2} + (m-1)\frac{\psi'}{\psi}\frac{\partial}{\partial r} + \frac{1}{\psi^2}\Delta_{\mathbb{S}^{m-1}},$$

or equivalently

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'}{S}\frac{\partial}{\partial r} + \frac{1}{\psi^2}\Delta_{\mathbb{S}^{m-1}},$$

where $\Delta_{\mathbb{S}^{m-1}}$ is the Laplace-Beltrami operator in $\mathbb{S}^{m-1}$.

Observe that for $\psi(r) = r$, $M = \mathbb{R}^m$, while for $\psi(r) = \sinh r$, M is the $m-$dimensional hyperbolic space $\mathbb{H}^m$.

4.3. **Construction of a positive nontrivial solution.** Let $M \equiv M_\psi$ be a model, $\Omega = B_1(o) \subset M_\psi$, $a \equiv 1$, $p = 2$. Clearly, the choice $M = \mathbb{R}^m$ is possibile. Let $0 < \epsilon < \frac{1}{\sigma-1}, C > 0$ and define

$$V(x) \equiv V(r) := C(1-r)^{-(\sigma+1)}|\log(1-r)|^{-1-\epsilon(\sigma-1)},$$

with $r \equiv \rho(x)$. It is direct to see that, for some $C_1 > 0, C_2 > 0$,

$$C_1 \delta^{\frac{2\sigma}{\sigma-1}} |\log\delta|^{\frac{1}{\sigma-1}+\epsilon} \leq \int_{B_{1-\delta}\setminus B_{1-2\delta}} V^{-\frac{1}{\sigma-1}}\, d\mu \leq C_2 \delta^{\frac{2\sigma}{\sigma-1}} |\log\delta|^{\frac{1}{\sigma-1}+\epsilon} \tag{65}$$

for any $\delta \in \left(0, \frac{1}{2}\right)$. Hence condition $(HP1)$ or $(HP2)$ or $(HP3)$ cannot be satisfied. On the other hand, V satisfies condition (61), therefore

$$\bar{\lambda}(\Omega) = 0\,. \tag{66}$$

We claim that there exists a positive solution of (1). In order to prove the claim, we argue in three steps.

Step 1. Define

$$\zeta(x) \equiv \zeta(r) := (1-r)|\log(1-r)|^{\lambda} \quad \text{for all } \ x \in B_1\,. \tag{67}$$

We have that

$$\zeta'(r) = -|\log(1-r)|^{\lambda} + \lambda|\log(1-r)|^{\lambda-1}\,,$$

$$\zeta''(r) = \frac{\lambda}{1-r}|\log(1-r)|^{\lambda-2}[(\lambda-1) - |\log(1-r)|]\,.$$

So,

$$\begin{aligned}
\Delta\zeta + V(r)\zeta^\sigma =& \zeta''(r) + (m-1)\frac{\psi'}{\psi}\zeta'(r) + V(r)\zeta^\sigma(r) \\
=& \frac{|\log(1-r)|^{\lambda-2}}{1-r}\left\{\lambda(\lambda-1) - \lambda|\log(1-r)| + \lambda(m-1)\frac{\psi'(r)}{\psi(r)}(1-r)|\log(1-r)|\right. \\
& \left. -(m-1)\frac{\psi'(r)}{\psi(r)}(1-r)|\log(1-r)|^2 + C|\log(1-r)|^{(\sigma-1)(\lambda-\epsilon)+1}\right\}\,.
\end{aligned} \tag{68}$$

In view of (68), if we take $0 < \lambda < \epsilon$ and $\delta > 0$ small enough, we get

$$\Delta\zeta + V\zeta^\sigma \leq 0 \quad \text{in } \ B_1 \setminus B_{1-\delta}\,. \tag{69}$$

Moreover, observe that

$$\zeta' < 0 \quad \text{in } \ [r_0, 1)\,, \tag{70}$$

for $r_0 := 1 - \delta$, if $\delta > 0$ is small enough.

Step 2. For any $\rho \in (0,1)$ let λ_ρ and w_ρ be the first eigenvalue and, respectively, the first eigenfunction of problem

$$\begin{cases} \Delta w_\rho + \lambda_\rho V w_\rho \,=\, 0 & \text{in } \ B_\rho, \\ w_\rho \,=\, 0 & \text{on } \partial B_\rho\,, \end{cases} \tag{71}$$

that is problem (56) with $\Omega = B_1$, $\delta = 1-\rho$. It is known that $\lambda_\rho > 0$ and that the corresponding eigenfunction w_ρ is radial, i.e. $w_\rho = w_\rho(r)$, and does not change sign in B_ρ. We can suppose that

$w_\rho(0)=1$; so, $w_\rho>0$ in B_ρ. Hence we have that

$$w_\rho''+\frac{S'(r)}{S(r)}w_\rho'+\lambda_\rho V w_\rho \,=\, 0, \quad 0<r<\rho\,, \tag{72}$$

with $w_\rho(\rho)=0$, $w_\rho(0)=1$, $w_\rho'(0)=0$, $w_\rho>0$ in $(0,\rho)$. From (72) if follows that

$$(S(r)w_\rho')'+\lambda_\rho S(r)Vw_\rho=0\,.$$

Consequently, $(Sw_\rho')'\leq 0$; so, the function $r\mapsto S(r)w_\rho'(r)$ is decreasing in $[0,\rho]$. Since it vanishes at $r=0$, we have that $S(r)w_\rho'(r)\leq 0$; therefore, $w_\rho'(r)\leq 0$ for all $r\in(0,\rho)$. Hence the function $r\mapsto w_\rho(r)$ is decreasing in $(0,\rho)$. Thus,

$$0<w_\rho\leq 1 \quad \text{in } \ (0,\rho)\,. \tag{73}$$

Hence, w_ρ is a positive solution of

$$\Delta w_\rho+\lambda_\rho V w_\rho^\sigma\leq 0 \quad \text{in } \ B_\rho\,. \tag{74}$$

We claim that there exists a sequence $\{\rho_k\}\subset(0,1)$ such that $\rho_k\to 1$ and $w_{\rho_k}\to 1$ as $k\to\infty$ in $C^1_{\text{loc}}((0,1))$. In fact, set $\rho_n:=1-\frac{1}{n}$. Thanks to (71) and (73) with $\rho=\rho_n$, by standard elliptic regularity theory, there exists a subsequence $\{\rho_{n_k}\}\equiv\{\rho_k\}\subset\{\rho_n\}$ such that $\{w_{\rho_k}\}$ converges in $C^\infty_{\text{loc}}(B_1)$ to a function w. Moreover, using (66), we can infer that w solves

$$\Delta w=0 \quad \text{in } \ B_1\,;$$

therefore,

$$\begin{cases} w''+\frac{S'(r)}{S(r)}w'=0 & \text{in}\,(0,1),\\ w(0)=1\,. \end{cases} \tag{75}$$

Observe that all the solutions of the O.D.E.

$$w''+\frac{S'(r)}{S(r)}w'=0$$

are given by

$$w(r)=C_1\int_r^1\frac{d\xi}{S(\xi)}+C_2 \quad (r\in(0,1])$$

for $C_1,C_2\in\mathbb{R}$. However, $w(r)$ diverges as $r\to 0^+$, if $C_1\neq 0$. Thus, the only bounded solution of (75) is $w\equiv 1$, which corresponds to the choice $C_1=0$, $C_2=1$. Therefore, we can infer that

$$w_{\rho_k}\to 1 \quad \text{in } \ C^\infty_{\text{loc}}(B_1) \ \text{ as } \ k\to\infty\,. \tag{76}$$

Step 3. Fix $\delta>0$ so that (69) and (70) hold, choose $\rho\in(0,1)$ such that $\rho>r_0$ and

$$\frac{w_\rho'(r_0)}{w_\rho(r_0)}>\frac{\zeta'(r_0)}{\zeta(r_0)}\,. \tag{77}$$

This is possible, since, thanks to (76),

$$\frac{w_{\rho_k}'(r_0)}{w_{\rho_k}(r_0)}\to 0 \quad \text{as } \ k\to\infty,$$

whereas, by (70),

$$\frac{\zeta'(r_0)}{\zeta(r_0)}<0\,.$$

Set

$$\theta:=\inf_{[r_0,\rho)}\frac{\zeta}{w_\rho}\,.$$

Since $\lim_{r\to\rho^-}\frac{\zeta(r)}{w_\rho(r)}\to\infty$, we deduce that $\theta=\frac{\zeta(\xi)}{w_\rho(\xi)}$ for some $\xi\in[r_0,\rho)$. Actually, $\xi>r_0$. In fact, thanks to (77),

$$\left(\frac{\zeta}{w_\rho}\right)'(r_0)=\frac{\zeta'(r_0)w_\rho(r_0)-\zeta(r_0)w_\rho'(r_0)}{w_\rho^2(r_0)}<0\,.$$

So, $\frac{\zeta}{w_\rho}$ is strictly decreasing in a neighborhood of r_0 and cannot have a minimum point at r_0. Hence, $\xi\in(r_0,\rho)$ and $\left(\frac{\zeta}{w_\rho}\right)'(\xi)=0$. This implies that

$$\zeta(\xi)=\theta w_\rho(\xi)\,,\quad \zeta'(\xi)=\theta w_\rho'(\xi)\,. \tag{78}$$

Define

$$\widetilde{u}(x)\equiv\widetilde{u}(r):=\begin{cases}\theta w_\rho(r) & \text{for all } r\in(0,\xi),\\ \zeta(r) & \text{for all } r\in[\xi,1).\end{cases}$$

In view of (78) we have that $\widetilde{u}\in C^1(B_1)$; hence, in particular, $\widetilde{u}\in W^{1,2}_{\text{loc}}(\Omega)$. By (74),

$$\Delta\widetilde{u}+\frac{\lambda_\rho}{\theta^{\sigma-1}}V\widetilde{u}^\sigma\le 0\quad\text{in }\ B_\xi\,. \tag{79}$$

From (69) and (79) we obtain

$$\Delta\widetilde{u}+\gamma V\widetilde{u}^\sigma\le 0\quad\text{in }\ B_1\,, \tag{80}$$

where $\gamma:=\min\left\{\frac{\lambda_\rho}{\theta^{\sigma-1}},1\right\}$. Let $u:=\gamma^{\frac{1}{\sigma-1}}\widetilde{u}$. Thanks to (80) we get

$$\Delta u+Vu^\sigma\le 0\quad\text{in }\ B_1\,.$$

So, we have exhibited a positive solution.

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