ON THE ANALYTICAL DERIVATION OF THE PARETO-OPTIMAL SET WITH APPLICATIONS TO STRUCTURAL DESIGN

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1. Introduction. Often at the earliest stage of an engineering project, a preliminary optimization could be useful, in order to allow the designer to ascertain the envisaged performance of the system under development.

Providing an efficient (analytical) tool to quickly define the Pareto-optimal set could be an extremely valuable chance to make the right design decision at the right time.

The procedure proposed here to obtain the Pareto-optimal set in analytical form refers mostly to design problems described by a limited number of design variables and a limited number of objective functions and constraints.

In the first part of the paper, the analytical derivation of the expression of the Pareto-optimal set for multi-objective optimization problems is dealt with.

According to the knowledge of the authors, in the literature, very few papers exist on this topic and related issues. A survey of current continuous multi-objective optimization concepts and methods is presented in Ref. [19]. Some relevant contributions are given in Ref. [17] and Ref. [20] in which some new formulations of the Fritz John first order conditions are proposed and analyzed. In Ref. [30] first and second order conditions are proposed for a convex multi-objective problem via scalarization and in Ref. [1] some second order conditions are analyzed in detail. In Refs. [17, 20, 30, 1, 32, 26] necessary and/or sufficient conditions are discussed but

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no procedures are introduced nor mentioned to allow the analytical derivation of the Pareto-optimal set. A slightly different formulation of the Fritz-John conditions is used in Ref. [4], where a symbolic algorithm for finding the Pareto front is discussed.

The procedure we propose in the paper is based on the reformulation of the Fritz John conditions for Pareto-optimality (first order conditions). Necessary conditions (a relaxed form of the Fritz John ones) are introduced and used to define the procedure to find analytically the Pareto-optimal set [18].

In the second part of the paper basic engineering examples are used to show the effectiveness of the proposed procedure.

First, the Fonseca and Fleming problem with two design variables and two objective functions has been addressed.

Second, the diameter of two spheres pressed one against the other have been designed to obtain mimimum mass, minumum compliance with the constraint of structural integrity.

Third, an ideal cantilever has been designed in order to minimize both mass and deflection in presence of maximum stress and buckling design constraints. Analytical solutions for optimal beam design are described in Ref. [25] for a single objective problem (min compliance) where the design variable is the area of the beam. In Refs. [13, 21, 6, 10], the multi-objective problem referring to the derivation of an optimal cantilever beam has been introduced and solved by applying numerical optimization methods.

Fourth, the Fonseca and Fleming problem with three design variables, two objective functions and two design constraints has been addressed.

2. The Fritz John necessary conditions. A multi-objective optimization problem can be formulated as (bold characters indicate vectors or matrices), see Ref. [23]

$$\begin{aligned} &\text{minimize} \quad \mathbf{F}(\mathbf{x}) = (\, f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_k(\mathbf{x}) \,) \\ &\text{subject to} \quad \mathbf{x} \in S = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), ..., g_m(\mathbf{x})) \le \mathbf{0}\} \end{aligned} \tag{2.1}$$

The problem (2.1) has n design variables $(x_1, ..., x_n)$, k objective functions $(f_1, ..., f_k)$, and m constraint functions $(g_1, ..., g_m)$.

DEF. *Pareto-optimal solution (global). Given a multi-objective optimization problem (minimization) considering n design variables and k objective functions, the Pareto-optimal solution $\boldsymbol{x}_i$ satisfies the following conditions*

$\nexists \boldsymbol{x}_j$:

$$\begin{cases} f_n(\boldsymbol{x}_j) \le f_n(\boldsymbol{x}_i) \quad n = 1, 2, 3, ..., k \\ \exists l : \ f_l(\boldsymbol{x}_j) < f_l(\boldsymbol{x}_i) \end{cases} \tag{2.2}$$

The Fritz John necessary conditions for Pareto-optimality are reported below (see Ref. [23])

Fritz John conditions. Let the objective and constraint functions of problem (2.1) be continuously differentiable at a decision vector $\mathbf{x}^* \in S$. A necessary condition for $\mathbf{x}^*$ to be Pareto-optimal is that vectors must exists reading $\mathbf{0} \leq \boldsymbol{\lambda} \in \mathbf{R}^k$ and $\mathbf{0} \leq \boldsymbol{\mu} \in \mathbf{R}^m$. If $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq (\mathbf{0}, \mathbf{0})$

$$\left\{ \begin{array}{l} \sum_{i=1}^{k} \lambda_i \nabla f_i(\mathbf{x}^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(\mathbf{x}^*) = \mathbf{0} \\ \\ \mu_j g_j(\mathbf{x}^*) = 0 \quad \forall j = 1, ..., m. \end{array} \right. \tag{2.3}$$

The Fritz John conditions (2.3) become the well known Karush-Kuhn-Tucker (KKT) sufficient conditions for Pareto-optimality if $\boldsymbol{\lambda} \neq \mathbf{0}$ and if objective and constraint functions are convex. Dealing with convex multi-objective optimization problems, every local Pareto-optimal solution is also global Pareto-optimal, see Ref. [23].

Convexity can be easily checked by considering the Hessian H of a function $f(\mathbf{x})$. If the Hessian is positive semi-definite ($H(f(\mathbf{x})) \geq 0$) then $f(\mathbf{x})$ is convex [3].

Actually, the convexity assumption can be relaxed and the KKT sufficient conditions are also valid if the objective functions are pseudo-convex and the constraint functions quasi-convex, see Ref. [17, 20].

A function $f(\mathbf{x})$ is quasi-convex if $f(\beta\mathbf{x}^I + (1-\beta)\mathbf{x}^{II}) \leq \max[f(\mathbf{x}^I),\ f(\mathbf{x}^{II})]$ for all $0 \leq \beta \leq 1$ and for all $\mathbf{x}^I$, $\mathbf{x}^{II}$.

A differentiable function $f(\mathbf{x})$ is pseudo-convex if for all $\mathbf{x}^I, \mathbf{x}^{II}$ such that $\nabla f(\mathbf{x}^I)^T(\mathbf{x}^{II} - \mathbf{x}^I) \geq 0$, we have $f(\mathbf{x}^{II}) \geq f(\mathbf{x}^I)$. Every pseudo-convex function is also quasi-convex [23].

Pseudo-convexity can be checked for a twice differentiable function $f(\mathbf{x})$ by means of the first and second partial derivatives arranged into the bordered determinant

$$|\mathbf{B}| = \begin{vmatrix} 0 & f_1 & f_2 & ... & f_n \\ f_1 & f_{11} & f_{12} & ... & f_{1n} \\ f_2 & f_{21} & f_{22} & ... & f_{2n} \\ ... & ... & ... & ... & ... \\ fn & f_{n1} & f_{n2} & ... & f_{nn} \end{vmatrix} \tag{2.4}$$

along with its leading principal minors

$$|\mathbf{B}_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix} \qquad |\mathbf{B}_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} \quad ... \quad |\mathbf{B}_n| = |\mathbf{B}| \tag{2.5}$$

where $f_i \equiv \partial f / \partial x_i$ and $f_{ij} \equiv \partial^2 f / \partial x_i \partial x_j$.

If $|\mathbf{B}_1| < 0$, $|\mathbf{B}_2| < 0$, ..., $|\mathbf{B}_n| < 0$ for all $\mathbf{x}$ then $f(\mathbf{x})$ is pseudo-convex [28, 29] (sufficient condition).

3. The L matrix. Let $\nabla\mathbf{F}$ and $\nabla\mathbf{G}$ be the matrices of the gradients (Jacobian matrices) of objective and constraint functions respectively

$$
\begin{aligned}
\nabla \mathbf{F} &= [\nabla f_1 \; \nabla f_2 \; ... \; \nabla f_k] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_k}{\partial x_n} \end{bmatrix} \qquad [n \text{ x } k] \\
\nabla \mathbf{G} &= [\nabla g_1 \; \nabla g_2 \; ... \; \nabla g_m] = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} \qquad [n \text{ x } m]
\end{aligned} \tag{3.1}
$$

Let $\mathbf{O}$ be the null matrix $[m \text{ x } k]$ and

$$
\mathbf{G} = \text{diag}(g_1, g_2, ..., g_m) \qquad [m \text{ x } m] \tag{3.2}
$$

a diagonal matrix which has on the main diagonal the constraint functions.
Let $\mathbf{L}$ be the matrix

$$
\mathbf{L} = \begin{bmatrix} \nabla \mathbf{F} & \nabla \mathbf{G} \\ \mathbf{O} & \mathbf{G} \end{bmatrix} \qquad [(n+m) \text{ x } (k+m)] \tag{3.3}
$$

The Fritz John conditions can be simply written as follows.

Fritz John conditions (matrix form). Let the objective and constraint functions of problem (2.1) be continuously differentiable at a decision vector $\mathbf{x}^* \in S$. A necessary condition for $\mathbf{x}^*$ to be Pareto-optimal is the existence of a vector $\boldsymbol{\delta} \in \mathbf{R}^{k+m}$ such that

$$
\begin{gathered}
\mathbf{L} \cdot \boldsymbol{\delta} = \mathbf{0} \\
\text{with} \qquad \mathbf{L} = \mathbf{L}(\mathbf{x}^*) \qquad \text{and} \qquad \boldsymbol{\delta} \geq \mathbf{0} \qquad \textit{and} \qquad \boldsymbol{\delta} \neq \mathbf{0}.
\end{gathered} \tag{3.4}
$$

This definition is completely equivalent to definition (2.3) since $\boldsymbol{\delta}$ is the vector

$$
\boldsymbol{\delta} = \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} \qquad [(k+m) \text{ x } 1] \tag{3.5}
$$

Let us notice that no constraints on the respective values of n, k, m have been set, so $\mathbf{L}$ is in general rectangular.

4. Analytical derivation of the Pareto-optimal set. The matrix form of the Fritz John conditions (3.4) can be employed to derive the analytical expression of the Pareto-optimal set.

All of the following considerations refer to problems in which $n \geq k$, that is the number of design variables is greater or equal than the number of objective functions.

The Fritz John conditions (see Eq. 3.4) can be relaxed by removing the condition $\boldsymbol{\delta} \geq \mathbf{0}$.

The relaxed Fritz John conditions read

$$\mathbf{L} \cdot \boldsymbol{\delta} = \mathbf{0}$$

$$\text{with} \qquad \mathbf{L} = \mathbf{L}(\mathbf{x}^*) \qquad \text{and} \qquad \boldsymbol{\delta} \neq \mathbf{0}. \tag{4.1}$$

This relaxation implies that we are dealing with necessary conditions also in presence of convex objective functions and constraints. So, the analytical expression derived on the basis of the relaxed form contains the actual Pareto-optimal set but also non-Pareto-optimal solutions.

The non-Pareto-optimal solutions have to be eliminated by computing the minimum of each objective function. Such minima obviously define the boundaries of the Pareto-optimal set.

The relaxed Fritz John conditions (Eq. 4.1) are a homogeneous system of linear equations [2]. The trivial solution $\boldsymbol{\delta} = \mathbf{0}$ is obviously of no interest.

If we consider the homogeneous overdetermined system of $(n+m)$ linear equations in $(k+m)$ variables, $\mathbf{L} \cdot \boldsymbol{\delta} = \mathbf{0}$, there will be non-trivial solutions only when the system has enough linearly dependent equations so that the number of independent equations is at most $(k+m)$. But being $(n+m) \geq (k+m)$, the number of independent equations (i.e. $rank(\mathbf{L})$) could be as high as $(k+m)$, in which case the trivial solution $\boldsymbol{\delta} = \mathbf{0}$ is the only one. The matrix $\mathbf{L^T L}$ is positive definite if and only if all the columns of $\mathbf{L}$ are linearly independent [7], i.e. $rank(\mathbf{L}) = k + m$ and a positive definite matrix is always nonsingular. If $\mathbf{L^T L}$ is nonsingular, there is the unique trivial solution $\boldsymbol{\delta} = \mathbf{0}$, but if $\mathbf{L^T L}$ is singular there are infinitely many solutions. This approach gives the exact solution when one does exist.

So, eq. 4.1 admits non-trivial solution if

$$\det(\mathbf{L^T L}) = 0. \tag{4.2}$$

For a square matrix $\mathbf{L}$, this condition is simply obtained by setting to zero its determinant.

If $n < k$ (the number of design variables is smaller than the number of objective functions), $\det(\mathbf{L^T L})$ is always zero and the problem is no longer a minimization problem. The analytical expression of the Pareto-optimal set can be simply derived by directly applying the substitution method [22, 21].

So, we can state the following necessary conditions for Pareto-optimality (relaxed Fritz John conditions)

L-matrix necessary conditions for Pareto-optimality. Let the objective and constraint functions of problem (2.1) be continuously differentiable at a decision vector $\mathbf{x}^* \in S$ and let be $n \geq k$. A necessary condition for $\mathbf{x}^*$ to be Pareto-optimal is that $\det(\mathbf{L}^T\mathbf{L}) = 0$ with $\mathbf{L} = \mathbf{L}(\mathbf{x}^*)$.

By computing the determinant of the $\mathbf{L}$ matrix, the analytical relationship between the design variables that represents the Pareto-optimal set in the design variable space can be obtained.

The flow chart of the proposed procedure to find the Pareto-optimal set in the design variable space is shown in Fig. 4.1.

For a number of special problems, the L-matrix necessary conditions for Pareto-optimality can be further simplified as described is the following subsections.

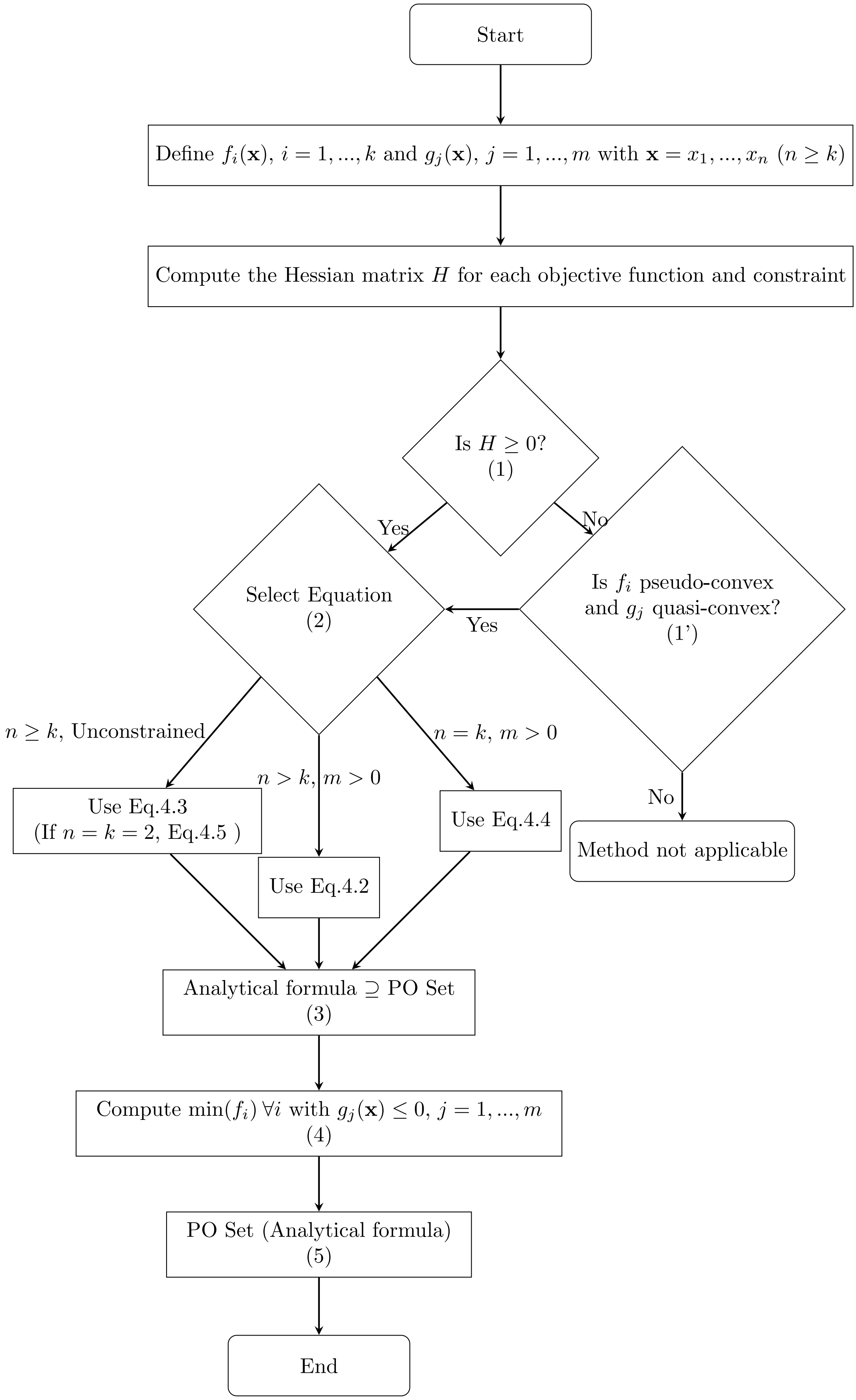


FIG. 4.1. *Flow chart of the proposed procedure.*

4.1. Unconstrained problem. Clearly, if the problem (2.1) is unconstrained, that is there are no constraint functions, the L matrix is simply $\mathbf{L} = \nabla\mathbf{F}$. Hence, the L-matrix condition is

$$\det(\mathbf{L}^T\mathbf{L}) = 0 \qquad \Rightarrow \qquad \det(\nabla\mathbf{F}^T\nabla\mathbf{F}) = 0 \tag{4.3}$$

4.2. Even number of design variables and objective functions. If $n = k$, that is the number of design variables is equal to the number of objective functions, the L matrix is square. So, it is not necessary to multiply it by its transposed matrix and the L-matrix condition is simply, see Eq. (3.2,3.3)

$$\det(\mathbf{L}^T\mathbf{L}) = 0 \quad \Rightarrow \quad \det(\mathbf{L}) = 0 \quad \Rightarrow \quad \left(\prod_{i=1}^{m} g_i\right) \cdot \det(\nabla\mathbf{F}) = 0 \tag{4.4}$$

Hence, the Pareto-optimal set for the constrained problem can be an active constraint and/or the Pareto-optimal set for the unconstrained problem, see Eq. (4.3).

4.2.1. Unconstrained problem with $n = k = 2$. For the simplest multi-objective unconstrained problem with two objective functions and two design variables, the L-matrix condition is

$$\det(\nabla\mathbf{F}^T\nabla\mathbf{F}) = 0 \quad \Rightarrow \quad \det(\nabla\mathbf{F}) = 0 \quad \Rightarrow \quad \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} = \frac{\partial f_1}{\partial x_2} \cdot \frac{\partial f_2}{\partial x_1} \tag{4.5}$$

According to the knowledge of the authors, this formula is original. The formula has been successfully applied by the authors in Ref. [12] to solve the actual and very important engineering problem referring to the trade-off between road-holding and comfort of ground vehicles.

5. Case studies. A number of case studies are presented which address the derivation of the Pareto-optimal set either in the design variable domain or in the objective function domain.

5.1. Case #1. Two design variables, two objective functions, no constraints. The problem proposed by Fonseca and Fleming in [11] has been selected from the ones presented in the literature [27], and used to find analytically the Pareto-optimal set (see Fig. 4.1).

The problem has two design variables and two objective functions and reads

$$\textit{minimize} \quad \begin{pmatrix} f_1(x_1,x_2) = 1 - e^{-[(x_1-1/\sqrt{2})^2+(x_2-1\sqrt{2})^2]} \\ f_2(x_1,x_2) = 1 - e^{-[(x_1+1/\sqrt{2})^2+(x_2+1\sqrt{2})^2]} \end{pmatrix} \tag{5.1}$$

Convexity can be easily verified for the problem given by Eq. 5.1 by computing the Hessian matrix for the two objective functions and checking that $H(f_1(x_1,x_2)) \geq 0$ and $H(f_2(x_1,x_2)) \geq 0$, see Step 1 in Fig. 4.1.

Eq. 4.5 (Step 2 in Fig. 4.1) can be directly applied to obtain the analytical expression of the Pareto front, see Step 3 in Fig. 4.1.

$$(x_1 - 1/\sqrt{2})(x_2 + 1/\sqrt{2}) = (x_1 + 1/\sqrt{2})(x_2 - 1/\sqrt{2}) \quad \Rightarrow \quad x_1 = x_2 \tag{5.2}$$

which has to be limited by the minima of the two objective functions considered separately.

The two minima are computed by directly setting to zero the gradients, being both the Hessian matrixes for f_1 and f_2 positive semi-definite, see Step 4 in Fig. 4.1.

$$
\begin{array}{lllll}
(5.3) & \nabla f_1 = 0 & (H(f_1) \geq 0) & \Rightarrow & x_1 = 1/\sqrt{2}, x_2 = 1/\sqrt{2} \\
 & \nabla f_2 = 0 & (H(f_2) \geq 0) & \Rightarrow & x_1 = -1/\sqrt{2}, x_2 = -1/\sqrt{2}
\end{array}
$$

So the analytical expression of the Pareto-optimal front in the design variables domain (see Step 5 in Fig. 4.1) reads

$$
(5.4) \quad x_1 = x_2 \quad \textit{with} \quad -1/\sqrt{2} \leq x_1 \leq 1/\sqrt{2} \quad \textit{and} \quad -1/\sqrt{2} \leq x_2 \leq 1/\sqrt{2}.
$$

and in the objective functions domain the analytical expression of the Pareto-optimal front (obtained by simple substitution) reads

$$
(5.5) \qquad f_1 = 1 + (f_2 - 1)\, e^{-4+4\sqrt{-\log(1-f_2)}} \quad 0 \leq f_2 \leq 0.982.
$$

Even if the two objective functions are convex, the Pareto front is non-convex, so the Pareto-optimal set cannot be computed by applying a simple weighted sum approach [21]. A state of the art multi-objective genetic algorithm (MOGA)[15] has been used to validate the analytical results. The results are shown in Fig. 5.1. The MOGA requires about ten thousand objective functions evaluation to obtain a reasonable level of accuracy.

5.2. Case #2. Two design variables, two objective functions, one constraint. The problem has two design variables, i.e. the diameters of, respectively, two spheres D_1 and D_2. Two objective functions are to be minimized, namely the total mass M and the relative displacement of the two spheres along the axis of loading y. The design constraint is the maximum stress in one (or both) of the spheres σ_{max}. The problem could refer to the basic design of sintered materials [8]. The elastic stress and deformation produced by the pressure between two spheres has been modelled on the basis of Hertz's theory [31].

The optimization problem reads

$$
(5.6) \qquad
\begin{array}{c}
\textit{minimize} \quad \begin{pmatrix} M(D_1, D_2) = \frac{1}{6}\,\pi\,\rho\,\left({D_1}^3 + {D_2}^3\right) \\ y(D_1, D_2) = k_y\,\sqrt[3]{\frac{P^2\,(D_1+D_2)}{D_1\,D_2\,E^2}} \end{pmatrix} \\
\sigma_{max} = k_\sigma\,\sqrt[3]{\frac{P\,E^2\,(D_1+D_2)^2}{D_1^2\,D_2^2}} \leq \sigma_{adm}
\end{array}
$$

given ρ the material density, E the material modulus of elasticity, σ_{adm} the material admissible stress, $k_\sigma = 0.616$, $k_y = 1.55$ [31].

Pseudo-convexity can be easily verified for the problem in Eq. 5.6 by applying Eq. 2.5. $M(D_1, D_2)$ and $y(D_1, D_2)$ are pseudo-convex, being $|\mathbf{B}_1| < 0$, $|\mathbf{B}_2| < 0$ for all D_1, D_2, see Step 1' in Fig. 4.1.

Eq. 4.4 (Step 2 in Fig. 4.1) can be directly applied to obtain the analytical expression of the Pareto front, see Step 3 in Fig. 4.1.

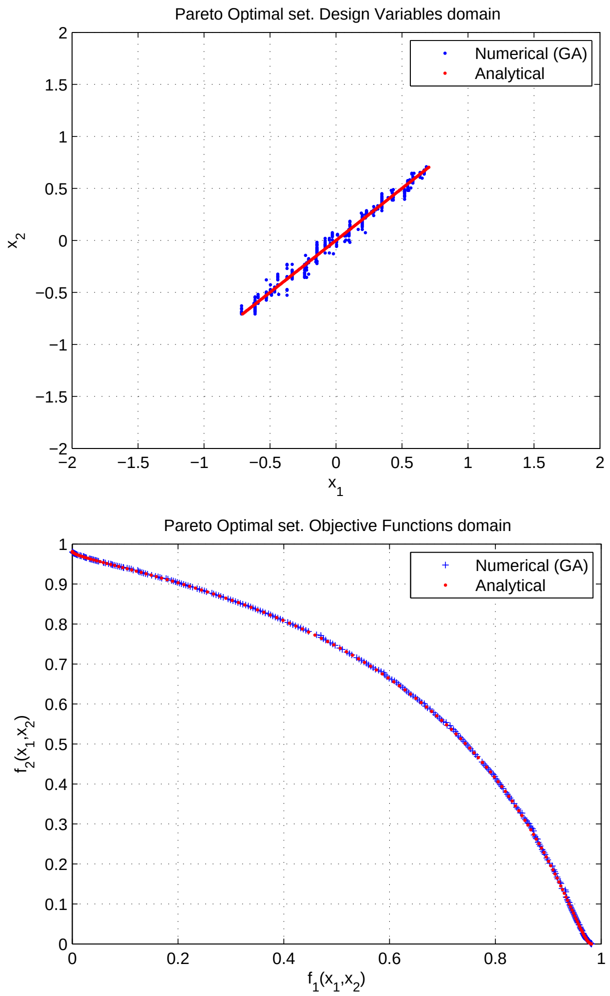


FIG. 5.1. *Analytical vs. numerical solution. Pareto-optimal set into the design variables domain (top) and into the objective functions domain (bottom) for the problem 5.1.*

$$P^2\, k_y\, \pi\, \rho\, \left({D_1}^4 - {D_2}^4\right) \left(\sigma_{adm} - k_\sigma \sqrt[3]{\frac{E^2\, P\, (D_1 + D_2)^2}{{D_1}^2\, {D_2}^2}}\right) = 0 \tag{5.7}$$

The Pareto-optimal front for the constrained problem is coincident with the Pareto-optimal set for the unconstrained problem, see Eq. (4.3), up to the activation of the maximum stress constraint $\sigma_{max} = \sigma_{adm}$ and it reads

$$D_1 = D_2 \quad for \quad D_2 \geq 2\, E \sqrt{\frac{P\, k_\sigma^3}{\sigma_{adm}^3}} \tag{5.8}$$

which has to be limited by the minima of the two objective functions considered separately.

The two minima read, see Step 4 in Fig. 4.1.

$$\begin{array}{lll} min & M(D_1, D_2) & D_1 = D_2 \to 0 \\ min & y(D_1, D_2) & D_1 = D_2 \to \infty \end{array} \tag{5.9}$$

Eqs. 5.9 don't affect the results given in Eq. 5.8, so the analytical expression of the Pareto-optimal set in the design variables domain (see Step 5 in Fig. 4.1) reads

$$D_1 = D_2 \quad for \quad D_2 \geq 2\,E\sqrt{\frac{P\,k_\sigma^3}{\sigma_{adm}^3}} \tag{5.10}$$

and in the objective functions domain the analytical expression of the Pareto-optimal set (obtained by simple substitution) reads

$$y = k_y\,\sqrt[3]{\frac{2\,P^2}{E^2\,(\frac{3\,M}{\pi\,\rho})^{\frac{1}{3}}}} \tag{5.11}$$

A numerical procedure has been used to validate the analytical results. The results are shown in Fig. 5.2.

5.3. Case #3. Two design variables, two objective functions, two (four) constraints. Let us imagine that a cantilever has to be designed. The cantilever, shown in Fig. 5.3, has a rectangular cross section and a force acts at the free end. Let us assume that the optimization problem to be solved is to find the values of the design variables (length b and length h) defining the cantilever cross section[1] in order to minimize both the cantilever mass and the cantilever deflection at its free end. The two constraints refer respectively to the maximum stress at the fixed end (the stess must be less than (or equal to) the admissible stress) and to elastic stability (buckling to be be avoided).

A designer should choose the values defining the cross section of the cantilever in order to get it as light and stiff as possible, avoiding both a failure (due to too high stress) and elastic instability.

In mathematical form, the above optimization problem may be stated as follows.

Given

l	the cantilever length	[m]
b	the beam cross section width	[m]
h	the beam cross section height	[m]
J	$= \frac{1}{12}bh^3$ the flexural moment of inertia of the section (n–n axis)	[m^4]
F	the force applied at the cantilever free end (see Fig.5.3)	[N]
σ_E	the material yield stress	[MPa]
η	safety coefficient ($\geq$1)	[–]
σ_{adm}	$= \sigma_E/h$ the admissible stress at the cantilever fixed end	[MPa]
E	the material modulus of elasticity (Young's modulus)	[MPa]
G	the material modulus of tangential elasticity	[MPa]
ρ	the material density	[kg/m^3]

[1] b and h may vary, respectively, within two well defined ranges

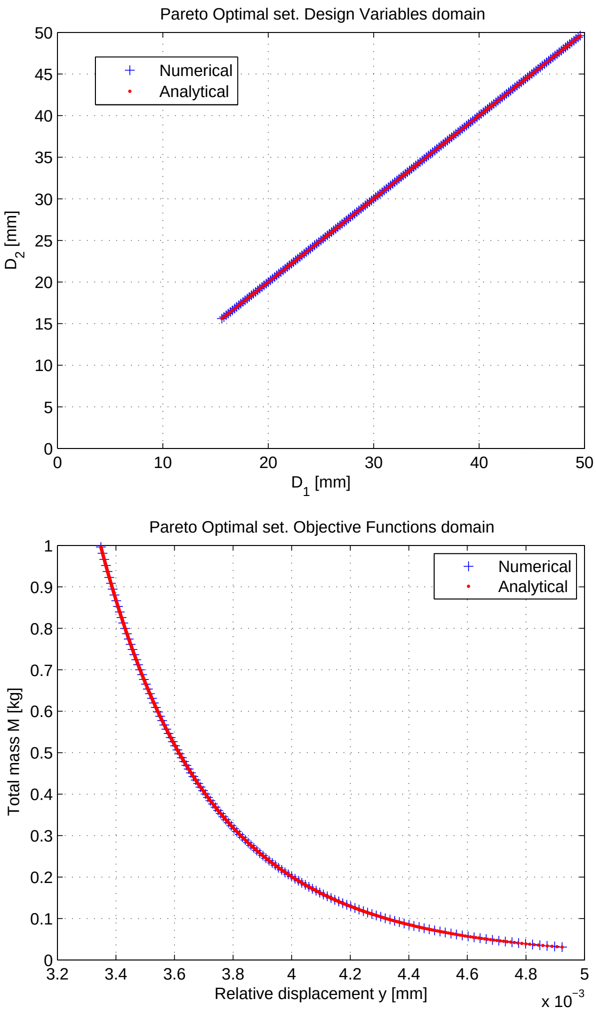


FIG. 5.2. *Analytical vs. numerical solution. Pareto-optimal set into the design variables* (D_1, D_2) *domain (top) and into the objective functions* (y, M) *domain (bottom) for the problem 5.6. Material 100Cr6 Steel,* $\rho = 7800\,kg/m^3$, $E = 210000\,MPa$, $\sigma_{adm} = 2500\,MPa$, $F = 100\,N$, $k_\sigma = 0.616$, $k_y = 1.55$.

and defining

$m = \rho bhl$	the cantilever mass	[kg]
$y = \dfrac{1}{3}\dfrac{Fl^3}{EJ} = 4\dfrac{Fl^3}{Ebh^3}$	the deflection at the free end due to load F	[m]
$\sigma_{max} = 6\dfrac{Fl}{bh^2}$	the maximum stress located at the top of the cross section at the fixed end of the cantilever	[MPa]
$F_{cr} = \dfrac{k_1 b^3 h}{l^2}\sqrt{\left(1 - k_2\dfrac{b}{h}\right)EG}$	the critical load (see [31, 14])	[N]

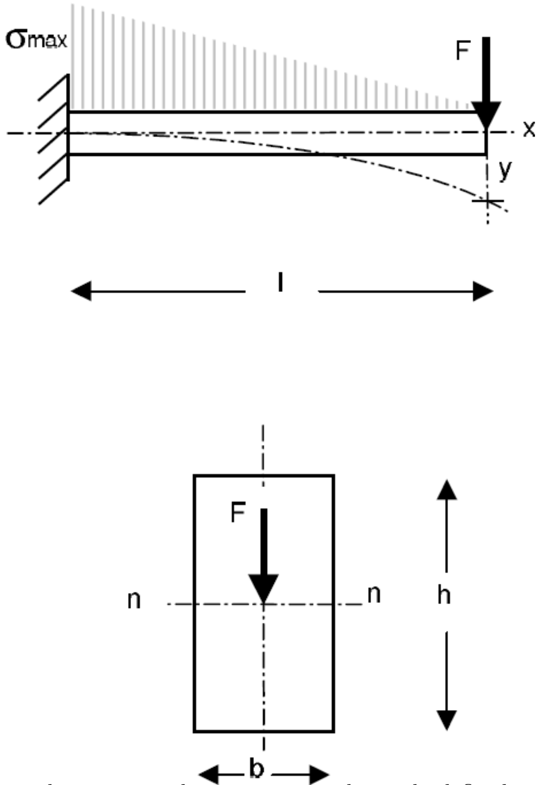


FIG. 5.3. *Cantilever whose rectangular cross section has to be defined in order to minimize both the cantilever mass and the cantilever deflection at the free end.*

find b *and* h *such that*

$$b_{min} \leq b \leq b_{max}$$
$$h_{min} \leq h \leq h_{max}$$

and such that

$$\min \begin{pmatrix} m(b,h) \\ y(b,h) \end{pmatrix} = \min \begin{pmatrix} \rho bhl \\ 4\frac{Fl^3}{Ebh^3} \end{pmatrix} \tag{5.12}$$

subject to

$$\sigma_{max} = 6\frac{Fl}{bh^2} \leq \sigma_{adm} = \sigma_E/\eta \tag{5.13}$$

$$F < F_{cr} = \frac{k_1 b^3 h}{l^2}\sqrt{\left(1 - k_2\frac{b}{h}\right) EG} \tag{5.14}$$

In Fig. 5.4 the results of such a computation are shown. Each point in the rectangle on the plane b–h corresponds a point in the manifold on the plane y–m. The rectangular manifold on the plane b–h is transformed into the manifold on the

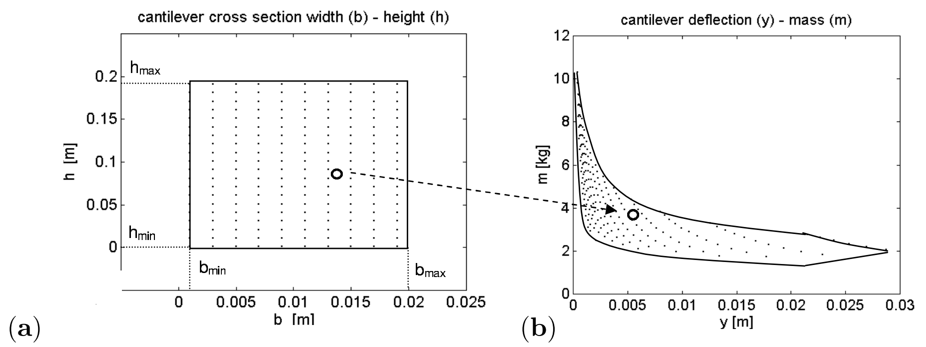


FIG. 5.4. *(**a**) cantilever cross section width b and height h that are considered for the optimization. (**b**) cantilever mass $m(b,h)$ and cantilever deflection at the free end $y(b,h)$. The manifold in (b) accounts for the constraints on the maximum stress (5.13) and on the critical load (5.14). Data: (symbols referring to (5.13) , (5.14) and Fig. 5.3)* $b_{min} = 0.001\,m$, $b_{max} = 0.020\,m$, $h_{min} = 0.001\,m$, $h_{max} = 0.200\,m$, $\rho = 2700\,kg/m^3$ $E = 70000\,MPa$, $G = 27000\,MPa$, $\sigma_E = 160\,MPa$, $\eta = 1$, $F = 1000\,N$, $l = 1\,m$, $k_1 = 0.669$, $k_2 = 0.63$.

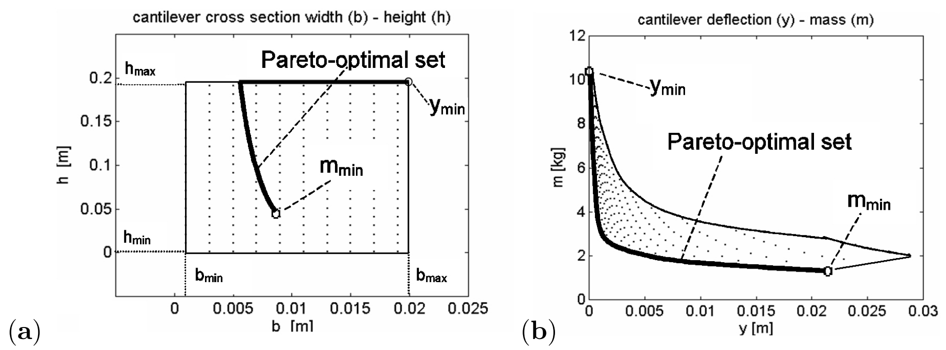


FIG. 5.5. *The two Pareto–optimal sets defined, respectively in the design variable domain (**a**) and in the objective function domain (**b**). Data in Fig.5.4.*

plane y–m. In other words, a transformation is established between the domain of design variables and the domain of objective functions.

As all the possible combinations of design variables b, h have been used to generate $m(b,h)$ $y(b,h)$ (having verified that $m(b,h)$ $y(b,h)$ do satisfy the constraints (5.13) and (5.14)), the question is now how to find the values of b, h which minimize concurrently the mass m and the deflection y at the free end of the cantilever.
All of the solutions corresponding to points in Fig. 5.5 which do not lay on the bold line (defined by the two end points y_{min} and m_{min}) are *wrong solutions* to be discarded by a designer. Conversely, the good or *Pareto-optimal solutions* are those and only those which are represented by points laying on the said bold line, the set of these solutions is the *Pareto–optimal set.*

The task of the designer is that of choosing a solution from the Pareto–optimal set and only from this set (Ref. [15]).

5.3.1. Analytical Solution. The analytical solution of the stated optimization problem can be found by firstly considering the problem as unconstrained, i.e. admissible maximum stress and elastic stability constraints are removed from the problem formulation. The objective functions are continuous and differentiable. The beam deflection y is convex [3]

$$\mathbf{H} = 4Fl^3 \begin{bmatrix} \frac{2}{b^3h^3} & \frac{3}{b^2h^4} \\ \frac{3}{b^2h^4} & \frac{12}{bh^5} \end{bmatrix} \tag{5.15}$$

being the eigenvalues of matrix $\mathbf{H}$ both real and positive.

The beam mass m is not convex, but only pseudo-convex, being m strictly monotonically increasing for any feasible value of b and h ($b > 0$, $h > 0$).

The L-matrix defined by Eq. (3.3) with $\nabla\mathbf{G} = \mathbf{0}$ (no constraints present), reads

$$\mathbf{L} = \begin{bmatrix} h\,l\,\rho & -\frac{4\,F\,l^3}{E\,b^2\,h^3} \\ b\,l\,\rho & -\frac{12\,F\,l^3}{E\,b\,h^4} \end{bmatrix} \tag{5.16}$$

By applying Eq. (4.5), we have

$$-\frac{8\,F\,l^4\,\rho}{E\,b\,h^3} = 0 \tag{5.17}$$

which has solution only if $bh^3 \to \infty$. Such solution not belonging to the set of the positive and finite numbers has no physical meaning. The nonexistence of the Pareto-optimal set can be proved by considering the ϵ-constraint method [16] and by applying the monotonicity principles [24]. The first monotonicity principle is violated for this optimization problem. In fact, if we apply the ϵ-constraint method and we consider the beam mass (m) as objective function and the beam deflection ($y(b,h)$) as (active) design constraint, the objective function ($m(b,h)$) is monotonically increasing by considering both the design variables (b,h), but the design variables are not bounded below by the active constraint on beam deflection ($y(b,h)$).

By including upper and lower bounds for b and h, the Pareto-optimal set for the unconstrained problem is given by the two boundaries on the design variables domain ($b = b_{min}$ and $h = h_{max}$), as shown in Fig. (5.6). The Pareto-optimal set is limited between $\min m = m(b_{min}, h_{min})$ and $\min y = y(b_{max}, h_{max})$.

If a sufficiently large design space is considered, the two design constraints read

$$\sigma = \frac{6\,F\,l}{b\,h^2} \leq \sigma_{max} \tag{5.18}$$

$$F_{cr} = \frac{k_1 b^3 h}{l^2} \sqrt{\left(1 - k_2 \frac{b}{h}\right) EG} \geq F \tag{5.19}$$

that for any given b can be rewritten as

$$h \geq \sqrt{\frac{6\,F\,l}{b\,\sigma_{max}}} \tag{5.20}$$

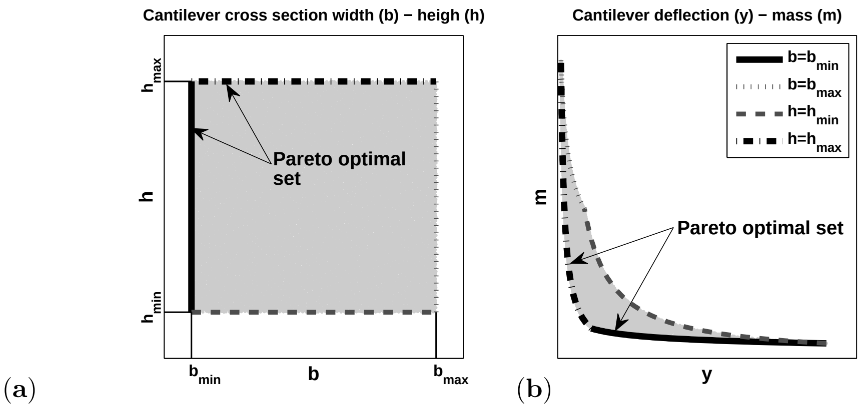


FIG. 5.6. *The two Pareto–optimal sets defined, respectively in the design variable domain (**a**) and in the objective function domain (**b**). Unconstrained problem with $b_{min} \leq b \leq b_{max}$ and $h_{min} \leq h \leq h_{max}$. Data in Fig.5.4.*

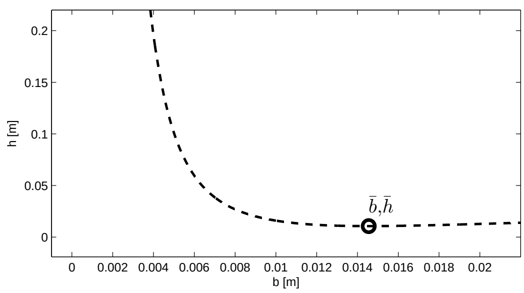


FIG. 5.7. *Equation (5.21) represents the buckling limit. It has a minimum at $\overline{b}$, $\overline{h}$. For $b > \overline{b}$ the buckling constraint equation (5.21) has no physical meaning.*

$$h \geq \frac{b\,\mathrm{k}_2}{2} + \frac{\sqrt{E^2\,G^2\,b^8\,{\mathrm{k}_1}^2\,{\mathrm{k}_2}^2 + 4\,E\,F^2\,G\,l^4}}{2\,E\,G\,b^3\,\mathrm{k}_1} \tag{5.21}$$

Eq. (5.20) represents the limit on the admissible maximum stress, it is a monotonic decreasing function proportional to $1/\sqrt{b}$ and it has limits $h \to \infty$ as $b \to 0$ and $h \to 0$ as $b \to \infty$. Eq. (5.21) represents the limit on buckling. It has limits $h \to \infty$ as $b \to 0$ and $h \to \infty$ as $b \to \infty$. Function (5.21), plotted in Fig. 5.7, is non-monotonic and it has a minimum at

$$\overline{b} = \sqrt[8]{\frac{36\,F^2\,l^4}{7\,E\,G\,k_1^2\,k_2^2}}, \qquad \overline{h} = \sqrt[8]{\frac{7^7\,F^2\,l^4\,k_2^6}{6^6\,E\,G\,k_1^2}} \tag{5.22}$$

It must be noticed that for $b > \overline{b}$ the buckling constraint equation (5.21) has no physical meaning, i.e. the buckling constraint has to be considered only for $b \le \overline{b}$. For $b \to 0$ the equation of the buckling constraint is proportional to $1/b^3$ and, thus, is more binding than the constraint on the maximum stress. For $b = \overline{b}$ and $h = \overline{h}$ the maximum stress on the cantilever beam can be computed by replacing Eq. (5.22) into the expression of the maximum stress (5.18). By considering reasonable values for the system parameters, the computed stress level for $b = \overline{b}$ and $h = \overline{h}$ is unacceptable for any engineering material. Therefore at $b = \overline{b}$ (and above) the constraint on the maximum stress is active.

Since for $b \to 0$ the buckling constraint is active and for $b = \overline{b}$ the constraint on the maximum stress is active and the two constraints functions are continuous and monotonic in the range $0 < b < \overline{b}$, there must be a design solution given by equation

$$\frac{b\,\mathrm{k}_2}{2} + \frac{\sqrt{E^2\,G^2\,b^8\,{\mathrm{k}_1}^2\,{\mathrm{k}_2}^2 + 4\,E\,F^2\,G\,l^4}}{2\,E\,G\,b^3\,\mathrm{k}_1} - \sqrt{\frac{6\,F\,l^2}{b\,\sigma_{max}}} = 0 \tag{5.23}$$

in which the active constraint switches from buckling to maximum stress.

By substituting Eq. (5.20) into the expressions of the mass and of the deflection of the cantilever beam, the following equation representing the maximum stress constraint in the objective functions domain can be obtained

$$m = \frac{9\,\rho\,F\,E}{{\sigma_{\mathrm{max}}}^2}\,y \tag{5.24}$$

Eq. (5.24) is a straight line with positive angular coefficient, so the buckling constraint is the active one up to the maximum acceptable stress level, and the L-matrix of Eq. (3.3) reads

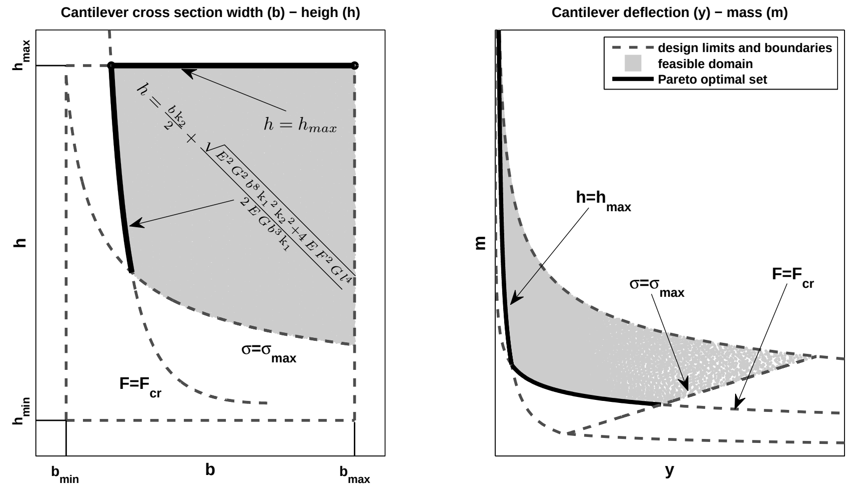
$$\mathbf{L} = \begin{bmatrix} h\,l\,\rho & -\frac{4\,F\,l^3}{E\,b^2\,h^3} & \frac{E\,G\,b^2\,\mathrm{k}_1\,(6\,h-7\,b\,\mathrm{k}_2)}{2\,l^2\,\sqrt{\frac{E\,G\,(h-b\,\mathrm{k}_2)}{h}}} \\ b\,l\,\rho & -\frac{12\,F\,l^3}{E\,b\,h^4} & \frac{E\,G\,b^3\,\mathrm{k}_1\,(2\,h-b\,\mathrm{k}_2)}{2\,h\,l^2\,\sqrt{\frac{E\,G\,(h-b\,\mathrm{k}_2)}{h}}} \\ 0 & 0 & -\frac{F\,l^2-b^3\,h\,\mathrm{k}_1\,\sqrt{\frac{E\,G\,(h-b\,\mathrm{k}_2)}{h}}}{l^2} \end{bmatrix} \tag{5.25}$$

Being $n = k$, we apply Eq. (4.4), so we have

$$h = \frac{b\,\mathrm{k}_2}{2} + \frac{\sqrt{E^2\,G^2\,b^8\,{\mathrm{k}_1}^2\,{\mathrm{k}_2}^2 + 4\,E\,F^2\,G\,l^4}}{2\,E\,G\,b^3\,\mathrm{k}_1} \tag{5.26}$$

Eq. (5.26) is the analytical expression of the Fritz John necessary condition when the buckling constraint is active and it is coincident with the expression of the buckling boundary condition itself (see Eq. (5.21)). The equation of this set in the space of the objective functions can be expressed as

$$y = y\,(m) = y\,(m\,(b,\,h)) \tag{5.27}$$



(**a**) (**b**)

FIG. 5.8. *The two Pareto–optimal sets defined, respectively in the design variables domain (**a**) and in the objective functions domain (**b**) for an optimal cantilever beam with rectangular cross section. The Pareto-optimal set is given by either the buckling condition or the maximum allowed height of the cross section. Data in Fig.5.4.*

In this case the derivation of $y = y(m)$ could not be performed as the analytical derivation is cumbersome.

The Pareto-optimal set is the portion of (5.26) limited by Eq. (5.23).

The first and second derivatives of the function can be found by applying the chain rule

$$\frac{dm}{dy} = \frac{\partial m}{\partial b}\frac{\partial b}{\partial y} + \frac{\partial m}{\partial h}\frac{\partial h}{\partial y} = -\frac{E\, b^2\, h^4\, \rho}{3\, F\, l^2} \tag{5.28}$$

$$\frac{d^2 y}{dm^2} = \frac{\partial\left(\frac{dy}{dm}\right)}{\partial b}\frac{\partial b}{\partial m} + \frac{\partial\left(\frac{dy}{dm}\right)}{\partial h}\frac{\partial h}{\partial m} = \frac{5\, E\, b^3\, h^7\, \rho}{18\, F^2\, l^5} \tag{5.29}$$

Eq. (5.28) and Eq. (5.29) show that this function is decreasing and convex for any value of b and h. Therefore, the points belonging to the buckling boundary condition belong to the Pareto-optimal set [5, 9]. Fig. 5.8 shows the Pareto-optimal set. In the design variable domain, the Pareto-optimal set is composed by the curve representing the buckling boundary condition from the point defined by Eq. (5.23) to the point where $h = h_{max}$ and then on the boundary $h = h_{max}$ from this point to $b = b_{max}$.

A numerical validation is reported in Fig. 5.9.

5.4. Case #4. Three design variables, two objective functions, two constraints. The problem proposed by Fonseca and Fleming in [11] with three design variables has been considered.

The problem has three design variables, two objective functions and two constraints

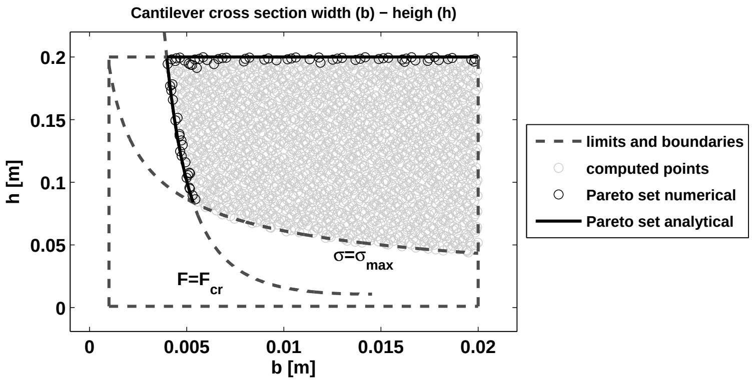


FIG. 5.9. *Pareto-optimal solutions in the design variables domain for an optimal cantilever beam with rectangular section. Numerical validation. Data in Fig.5.4.*

$$\text{(5.30)}\quad minimize \quad \begin{pmatrix} f_1(x_1,x_2,x_3) = 1 - e^{-[(x_1-1/\sqrt{3})^2+(x_2-1\sqrt{3})^2+(x_3-1\sqrt{3})^2]} \\ f_2(x_1,x_2,x_3) = 1 - e^{-[(x_1+1/\sqrt{3})^2+(x_2+1\sqrt{3})^2+(x_3+1\sqrt{3})^2]} \end{pmatrix}$$
$$\tfrac{1}{2} \le x_1 \le \tfrac{1}{2}$$

Convexity can be easily verified for the problem given by Eq. 5.30 by computing the Hessian matrix for the two objective functions and checking that $H(f_1(x_1,x_2,x_3)) \ge 0$ and $H(f_2(x_1,x_2,x_3)) \ge 0$, see Step 1 in Fig. 4.1.

Eq. 4.2 (Step 2 in Fig. 4.1) can be applied to obtain the analytical expression of the Pareto front, see Step 3 in Fig. 4.1.

$$\text{(5.31)}\quad \begin{aligned} &(x_1-x_2)(x_2-x_3)\left(16x_1^4+8x_1^2+5\right)-(x_1-x_3)(x_2-x_3)\left(16x_1^4+8x_1^2+5\right) \\ &\quad -(x_1-x_2)(x_1-x_3)(2x_1-1)^2(2x_1+1)^2 = 0 \end{aligned}$$

which gives

$$\text{(5.32)}\quad \begin{array}{cc} x_1 = x_2 = x_3 & if\, \tfrac{1}{2} \le (x_1 = x_2 = x_3) \le \tfrac{1}{2} \\ x_1 = \tfrac{1}{2},\, x_2 = x_3 & if\, (x_2 = x_3) > \tfrac{1}{2} \\ x_1 = \tfrac{1}{2},\, x_2 = x_3 & if\, (x_2 = x_3) < -\tfrac{1}{2} \end{array}$$

which has to be limited by the minima of the two objective functions taking into account the design constraints.

The two minima have been computed, see Step 4 in Fig. 4.1.

$$\text{(5.33)}\quad \begin{array}{c} \min f_1 \quad \Rightarrow \quad x_1 = \tfrac{1}{2},\, x_2 = 1/\sqrt{3},\, x_3 = 1/\sqrt{3} \\ \min f_2 \quad \Rightarrow \quad x_1 = -\tfrac{1}{2},\, x_2 = -1/\sqrt{3},\, x_3 = -1/\sqrt{3} \end{array}$$

So the analytical expression of the Pareto-optimal front in the design variables domain (see Step 5 in Fig. 4.1) reads

$$\text{(5.34)}\quad \begin{array}{cc} x_1 = x_2 = x_3 & if\, -\tfrac{1}{2} \le (x_1 = x_2 = x_3) \le \tfrac{1}{2} \\ x_1 = \tfrac{1}{2},\, x_2 = x_3 & if\, \tfrac{1}{2} < (x_2 = x_3) \le 1/\sqrt{3} \\ x_1 = -\tfrac{1}{2},\, x_2 = x_3 & if -1/\sqrt{3} \le (x_2 = x_3) < -\tfrac{1}{2} \end{array}$$

A numerical procedure has been used to validate the analytical results. The results are shown in Fig. 5.10.

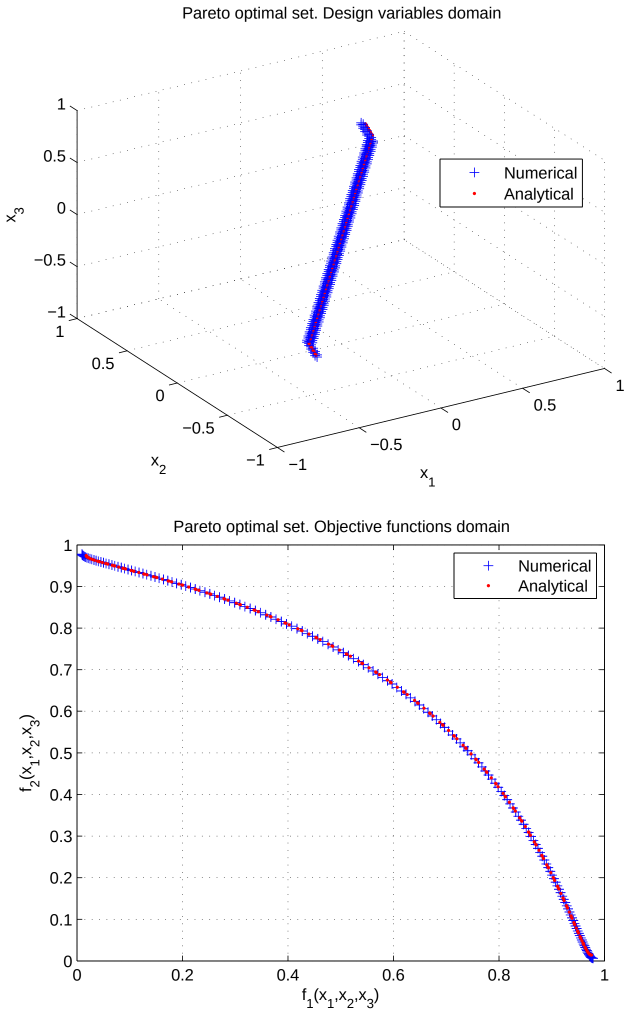


FIG. 5.10. *Analytical vs. numerical solution. Pareto-optimal set into the design variables (x_1, x_2, x_3) domain (top) and into the objective functions (f_1, f_2) domain (bottom) for the problem 5.30.*

6. Conclusion. A procedure has been proposed to find -when possible- the Pareto-optimal set in analytical form in the design variable domain. Both the objective and constraint functions are assumed to be twice differentiable and convex or pseudo-convex. The Fritz John necessary condition for the Pareto-optimality has been re-formulated in matrix form. This formulation has been employed to derive a new necessary condition (the L-matrix necessary condition) that is a relaxed form of the Fritz John one. The L-matrix condition has been applied for the analytical derivation of the Pareto-optimal set if the number of design variables is greater than (or equal to) the number of objective functions.

When two design variables and two objective functions define an optimization problem, the Pareto-optimal set can be computed quite easily by applying the simple formula derived in the paper which seems original and is based on simple partial derivatives. If the number of design variables equals the number of objective functions, the Pareto-optimal set in the design variables domain can be found after the product of the constraint functions times the determinant of the Jacobian of the objective functions.

The proposed procedure for the analytical derivation of Pareto-optimal sets appears to be quite general and can be easily applied to problems with low dimensionality. Obviously a numerical check of the derived analytical Pareto-optimal set is quite useful. Nonetheless the analytical formulation of the Pareto-optimal set provides a strong reference for designers.

An attempt to prove the effectiveness of the proposed procedure has been performed. A number of basic engineering problems have been addressed. First, the the test problem proposed by Fonseca and Fleming with two design variables has been solved analytically. Second, the respective radii of two spheres pressed one against the other have been defined by mimimising both the total mass and the total deflection, with the constraint to preserve their structural integrity. The result is that the diameter of the two spheres must be the same to obtain Pareto-optimal solutions. Third, the (constrained) dimensions of the rectangular cross section of a cantilever beam subject to bending have been defined by minimizing both the mass and the deflection, with the constraint to preserve the structural integrity (referring to both maximum stress and buckling). The result is that the Pareto-optimal set in the design variable domain is defined by either the buckling or the maximum height of the rectangular cross section. Fourth, the test problem proposed by Fonseca and Fleming with three design variables and two inequality constraints has been solved analytically. The procedure proposed in the paper could help designers to make a best preliminary choice at an early stage of a project.

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