Creep analysis of compact cross-sections cast in consecutive stages – Part 2: Algebraic methods

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Available online 17 May 2014

1. Introduction

Over the past few decades, construction practitioners have been facing the challenges of high quality demand and high labour cost. In order to overcome these challenges practitioners often develop structures that combine precast elements with cast in place concrete components. The use of this technique gives a monolithic quality to the structure (see for instance [1]).

Similarly, many countries aspire to maintain the traditional identity of built-up areas. As a result, local administrators promote structural retrofitting of existing reinforced concrete structures to enhance their earthquake resistance, to improve their strength to meet new structural demands or new code requirements, and to retrofit damaged structural elements. Moreover, structural retrofitting is adopted to overcome insufficient strength of the materials in new concrete structures resulting from oversight errors and lack of proper quality control. A common technique adopted to improve the bearing capacity of structural elements is to increase the reinforced concrete cross-section. Concrete jacketing of beams and columns is a specific method used to increase the cross-sectional area (see for instance [2–5]).

In all these cases a reliable evaluation of the stress redistribution that occurs in the cross section because of creep and shrinkage of concrete is important to guarantee an accurate forecast of the behaviour of the structure under service loads and at the ultimate load (see [6]).

A general approach utilised to evaluate the stress and strain time evolution of concrete compact cross-sections cast or prestressed in consecutive stages under long term loading was presented in a previous paper [7]. The overall cross-section was made of reinforced concrete, prestressed concrete or steel parts added at distinct stages of the construction process. Moreover, the cross-section could be prestressed several times during construction and after gaining the final shape. This approach led to a system of Volterra integral equations (see for instance [8,9]), whose convolution integral (that is the closed form solution) cannot be determined because of the complexity of the creep function usually adopted to describe concrete behaviour ([10,11]). The system of Volterra integral equations was therefore solved by means of a refined step-by-step time integration method (based on the techniques suggested by classic numerical analysis [12]). The method gives rise to an error whose value can be minimised through a suitable choice of the time discretization procedure. This approach is complicated and cumbersome, hardly implementable in a computer program and too complex for a common engineer. Therefore, this paper illustrates simplified versions of the algebraic

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Notation

x_1 and y_1 principal axes of piece of concrete 1 (the first to be cast)
A_{c1} and J_{c1} area of piece of concrete 1 and its second moment of area with respect to the x_1 axis. The cross-section change because of grouting of tendon 1 (when post-tensioned) is not taken into account
A_{p1} cross-sectional area of tendon 1
x_2 and y_2 principal axes of piece of concrete 2 (the second to be cast)
A_{c2} and J_{c2} area of piece of concrete 2 and its second moment of area with respect to the x_2 axis. The cross-section change because of grouting of tendon 2 is not taken into account
A_{p2} cross-sectional area of tendon 2
E_{c1} and E_{c2} reference elastic moduli (for instance the elastic moduli at the age of 28 days) of concrete 1 and 2 respectively
$E_{c1}(t)$ and $E_{c2}(T)$ elastic moduli of concrete 1 at age t and of concrete 2 at age T respectively
E_{p1} and E_{p2} elastic moduli of the tendons
G_1 centroid of piece of concrete 1
G_2 E-weighted centroid of the final cross section
r ratio between the relaxation loss and the initial prestressing of the prestressing steel
r age of piece of concrete 1
T age of piece of concrete 2
y_{p1} position of tendon 1 on y_1 axis
y_{p2} position of tendon 2 on y_2 axis
y_{c1} position of ΔX_{II} on y_1 axis (therefore y_{c1} is negative in Fig. 2)
y_{c2} position of ΔX_{II} on y_2 axis
ΔX_I^{ld} stress resultant variation in tendon 1, positive when acting according to Fig. 2, caused by the "ld" load
ΔX_{IV}^{ld} stress resultant variation in tendon 2, positive when acting according to Fig. 2, caused by the "ld" load
ΔN_i and ΔM_i^* internal axial force and bending moment variation due to an external long term load. These axial force and bending moment usually follow from a linear elastic structural analysis and therefore act in the centroid of the cross section, i.e. point G_1 or G_2 depending on the current stage of construction. These vectors are positive when acting according to Fig. 2
ΔN_i and ΔM_i internal axial force and bending moment variation due to any external long term load acting at point G_1 (i.e. $\Delta M_i = \Delta M_i^* - \Delta N_i \cdot y_{\Delta N_i}$ when ΔN_i acts at point G_2, $\Delta M_i = \Delta M_i^*$ otherwise. See Fig. 2). These vectors are positive when acting according to Fig. 2
ΔX_{II}^{ld} and ΔX_{III}^{ld} stress resultants in piece of concrete 2 (and in tendon 2, if any) measured on the contact surface between the two pieces of concrete, caused by the "ld" load
δ_{jk}^{sec} term of the flexibility matrix: the axial strain (or the curvature) present in the homogeneous piece "sec" of the cross section, in the point where ΔX_j acts (positive when concordant to ΔX_j), due to $\Delta X_k = 1$
δ_j^{ld} non-compatible strain (or non-compatible curvature) on the contact surface where ΔX_j acts, caused by the "ld" load (positive when acting according to ΔX_j)
$\chi_{c1}(t, t_0^*)$ and $\chi_{c2}(T, T_0^*)$ aging coefficients of concrete 1 and 2 respectively
$\varphi_{c1}(t, t_0^*)$ and $\varphi_{c2}(T, T_0^*)$ creep coefficients of concrete 1 and 2 respectively

methods discussed in [13–15] that allow to overcome the inability to solve the complex numerical integration.

In a following paper the output of the computer program, written according to the more refined solution suggested in the previous paper [7], will be compared with the outcomes of this approach to verify the accuracy of the latter.

2. The approach to problem-solving

The assumptions adopted in the following are:

1. The cross-section is made of two individual homogeneous pieces of concrete (indexes $c1$ and $c2$) or another generic linear viscoelastic material (or an elastic material when setting its creep coefficient to zero) whose constitutive law is a Volterra integral equation approximated by the following algebraic expression (see Fig. 2):

$$\begin{aligned}
\varepsilon_{c1}(x_1, y_1, t) &= \frac{\sigma_{c1}(x_1, y_1, t_0^*)}{E_{c1}(t_0^*)}\varphi_{c1}(t, t_0^*)[1 - \chi_{c1}(t, t_0^*)] \\
&\quad + \frac{\sigma_{c1}(x_1, y_1, t)}{E_{c1}(t_0^*)}[1 + \chi_{c1}(t, t_0^*) \cdot \varphi_{c1}(t, t_0^*)] \\
\varepsilon_{c2}(x_2, y_2, T) &= \frac{\sigma_{c2}(x_2, y_2, T_0^*)}{E_{c2}(T_0^*)}\varphi_{c2}(T, T_0^*)[1 - \chi_{c2}(T, T_0^*)] \\
&\quad + \frac{\sigma_{c2}(x_2, y_2, T)}{E_{c2}(T_0^*)}[1 + \chi_{c2}(T, T_0^*) \cdot \varphi_{c2}(T, T_0^*)] \qquad (1)
\end{aligned}$$

where time t is the age of piece of concrete 1 (the oldest) and time T is the age of piece of concrete 2, related one another by means of the construction history (see Fig. 1). $E_{c1}(t_0^*)$ and $E_{c2}(T_0^*)$ are the elastic moduli measured at the onset of loading, $\varphi_{c1}(t, t_0^*)$ and $\varphi_{c2}(T, T_0^*)$ are the creep coefficients and $\chi_{c1}(t, t_0^*)$ and $\chi_{c2}(T, T_0^*)$ are the aging coefficients.

Both concrete pieces hold a tendon (subscripts $p1$ and $p2$) whose constitutive law is linear elastic (at least under long term service loads):

$$\varepsilon_{p1}(t) = \frac{\sigma_{p1}(t)}{E_{p1}}; \quad \varepsilon_{p2}(t) = \frac{\sigma_{p2}(t)}{E_{p2}} \qquad (2)$$

2. No bond slip can occur among the parts which make up the cross-section (external and unbonded internal prestressing and composite steel-concrete beams with flexible connections are therefore not considered).
3. The Bernoulli–Navier hypothesis (an initially plane beam section which is perpendicular to the beam reference axis remains plane and perpendicular to the beam's axis in the deformed configuration) applies to each individual homogeneous part of the cross-section. This assumption is commonly adopted (and accepted) when dealing with compact cross sections in the service stage (as it is the case of the application presented in the following).
4. The internal axial force and bending moment act on a plane of symmetry of the cross section (out-of-plane bending is not taken into account, not to complicate too much the solving system).

The application of the presented approximate solution is therefore restricted to cross-sections cast in two stages. That is, precast prestressed concrete beams (or steel or timber beams) with a cast-in-situ slab (prestressed or not) or jacketed beams and columns (i.e. the cases most frequently found in practical applications).

The stress and strain of the cross-section will be evaluated in the following time intervals (see Fig. 1 that refers to a precast prestressed beam with a cast-in-situ prestressed slab):

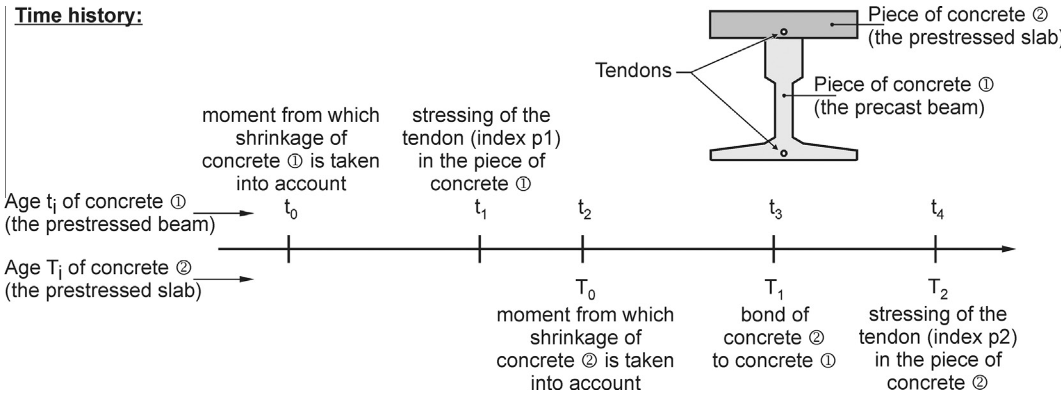

Fig. 1. Correlation between age scales of the first (age t) and the second (age T) piece of concrete for execution stages and service.

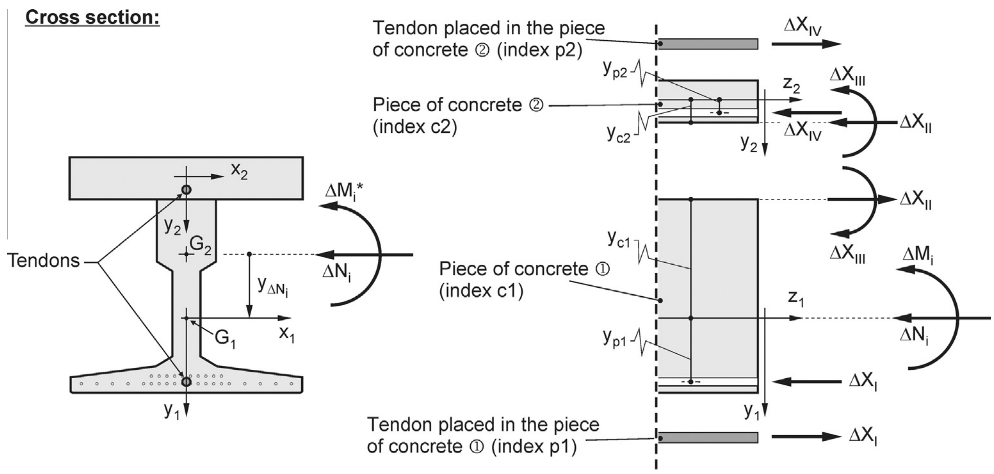

Fig. 2. Division of the cross section in four homogeneous pieces.

1. moment from which shrinkage of piece of concrete 1 (the precast beam) is taken into account [time t_0], i.e. end of concrete curing
2. i-th load that starts acting at time $t_i^g \leqslant t_1^-$, before the bonding of tendon 1 to the surrounding concrete. Usually in post-tensioning the dead load starts acting at tensioning of the tendon, i.e. before grouting, that is when the tendon is still unbonded (time $t_i^g = t_1^-$)
3. tendon 1 release from the prestressing bed (in the case of prestressed elements), or post-tensioning and grouting of tendon 1 [time t_1]
4. i-th load that starts acting at time $t_i^g : t_1 \leqslant t_i^g \leqslant t_2$, after bond of tendon 1 to the surrounding concrete has started acting. This is for instance the case of the dead load of a pre-tensioned beam $(t_i^g = t_1^+)$
5. moment from which shrinkage of piece of concrete 2 (the slab) is taken into account [time t_2], i.e. end of concrete curing
6. bond of piece of concrete 2 to piece of concrete 1 [time t_3]
7. i-th load that starts acting before bonding of tendon 2 to the surrounding concrete, but after bonding of piece of concrete 2 to piece of concrete 1 (at time $t_i^g : t_3 \leqslant t_i^g \leqslant t_4$)
8. post-tensioning and grouting of tendon 2 [time t_4]
9. i-th load that starts acting after bonding of tendon 2 to the surrounding concrete (at $t_i^g > t_4$)

The problem is solved by means of the force method: the cross section is divided into four homogeneous pieces (i.e. the two pieces of concrete and the two tendons). The stress resultants ΔX_j (three axial forces and one bending moment) mutually transferred between them are the redundant unknowns (see Fig. 2). These unknowns can be computed by enforcing compatibility at the contact surfaces between the different homogeneous pieces.

In the following, δ_{jk}^{sec} represents the axial strain (or the curvature) of the homogeneous piece, or "sec", of the cross section at the point where ΔX_j acts (positive when concordant to ΔX_j), due to $\Delta X_k = 1$. The corresponding terms of the flexibility matrix are:

$$
\begin{aligned}
&\delta_{11}^{p1} = \frac{1}{E_{p1}A_{p1}}; \quad \delta_{11}^{c1} = \frac{1}{E_{c1}A_{c1}} + \frac{y_{p1}^2}{E_{c1}J_{c1}} \\
&\delta_{22}^{c1} = \frac{1}{E_{c1}A_{c1}} + \frac{y_{c1}^2}{E_{c1}J_{c1}}; \quad \delta_{22}^{c2} = \frac{1}{E_{c2}A_{c2}} + \frac{y_{c2}^2}{E_{c2}J_{c2}} \\
&\delta_{33}^{c1} = \frac{1}{E_{c1}J_{c1}}; \quad \delta_{33}^{c2} = \frac{1}{E_{c2}J_{c2}} \\
&\delta_{44}^{p2} = \frac{1}{E_{p2}A_{p2}}; \quad \delta_{44}^{c2} = \frac{1}{E_{c2}A_{c2}} + \frac{y_{p2}^2}{E_{c2}J_{c2}} \\
&\delta_{13}^{c1} = \delta_{31}^{c1} = \frac{y_{p1}}{E_{c1}J_{c1}}; \quad \delta_{12}^{c1} = \delta_{21}^{c1} = -\frac{1}{E_{c1}A_{c1}} - \frac{y_{c1}y_{p1}}{E_{c1}J_{c1}} \\
&\delta_{23}^{c1} = \delta_{32}^{c1} = -\frac{y_{c1}}{E_{c1}J_{c1}}; \quad \delta_{23}^{c2} = \delta_{32}^{c2} = -\frac{y_{c2}}{E_{c2}J_{c2}} \\
&\delta_{34}^{c2} = \delta_{43}^{c2} = -\frac{y_{p2}}{E_{c2}J_{c2}}; \quad \delta_{24}^{c2} = \delta_{42}^{c2} = \frac{1}{E_{c2}A_{c2}} + \frac{y_{c2}y_{p2}}{E_{c2}J_{c2}} \\
&\delta_{14}^{c1} = \delta_{41}^{c1} = \delta_{14}^{c2} = \delta_{41}^{c2} = \delta_{12}^{c2} = \delta_{21}^{c2} = \delta_{13}^{c2} = \delta_{31}^{c2} = \delta_{24}^{c1} = \delta_{42}^{c1} = \delta_{34}^{c1} = \delta_{43}^{c1} = 0
\end{aligned} \tag{3}
$$

where E_{c1} and E_{c2} are the reference elastic moduli (for instance the elastic moduli at the age of 28 days) of concrete 1 and 2 respectively.

McHenry's superposition principle [16] and the theorems of linear viscoelasticity [11] apply to each piece of concrete. It is therefore possible to subdivide the redundant unknowns in parts (each one related to a single load or a single non-compatible strain), calculate the values, and then superimpose the outputs to get the overall cross section behaviour.

δ_j^{ld} is the non-compatible strain (or non-compatible curvature) on the contact surface where ΔX_j acts, caused by the "ld" load (positive when acting according to ΔX_j). When "ld" is the shrinkage of piece of concrete 1 we get:

$$\begin{aligned}
&\delta_{10}^{sh1}(t) = \varepsilon_{sh}(t) - \varepsilon_{sh}(t_1) \\
&\delta_{20}^{sh1}(t) = -\varepsilon_{sh}(t) + \varepsilon_{sh}(t_3) \\
&\delta_{30}^{sh1}(t) = 0 \text{ (uniform shrinkage does not give rise to a curvature inside piece of concrete 1)} \\
&\delta_{40}^{sh1}(t) = 0 \text{ (tendon 2 is not connected to piece of concrete 1)}
\end{aligned} \tag{4}$$

whereas when dealing with shrinkage of piece of concrete 2 we get:

$$\begin{aligned}
&\delta_{10}^{sh2}(T) = 0 \text{ (tendon 1 is not connected to piece of concrete 2)} \\
&\delta_{20}^{sh2}(T) = \varepsilon_{sh}(T) - \varepsilon_{sh}(T_1) \\
&\delta_{30}^{sh2}(T) = 0 \text{ (uniform shrinkage does not give rise to a curvature inside piece of concrete 2)} \\
&\delta_{40}^{sh2}(T) = \varepsilon_{sh}(T) - \varepsilon_{sh}(T_2)
\end{aligned} \tag{5}$$

G_1 is the centroid of piece of concrete 1, G_2 is the E-weighted centroid of the final cross section. The variations of internal axial force, ΔN_i, and bending moment, ΔM_i^*, due to an external long term load usually follow from a linear elastic structural analysis. Therefore, the loads act at the centroid of the cross section, i.e. point G_1 or G_2 (according to Fig. 2) depending on the current stage of construction. For the sake of simplicity, the internal axial force and bending moment due to any external long term load will be assumed to act at G_1 and will be named ΔN_i and ΔM_i. (i.e. $\Delta M_i = \Delta M_i^* - \Delta N_i \cdot y_{\Delta N_i}$ when ΔN_i acts at point G_2, $\Delta M_i = \Delta M_i^*$ otherwise. See Fig. 2). Therefore, when dealing with any long term load that gives rise to ΔN_i and ΔM_i, terms δ_j^{ld} are:

$$\begin{aligned}
&\delta_{10,i}^{g} = \frac{\Delta N_i}{E_{c1}(t_i^g) \cdot A_{c1}} - \frac{\Delta M_i}{E_{c1}(t_i^g) \cdot J_{c1}} y_{p1} \\
&\delta_{20,i}^{g} = -\frac{\Delta N_i}{E_{c1}(t_i^g) \cdot A_{c1}} + \frac{\Delta M_i}{E_{c1}(t_i^g) \cdot J_{c1}} y_{c1} \\
&\delta_{30,i}^{g} = -\frac{\Delta M_i}{E_{c1}(t_i^g) \cdot J_{c1}} \\
&\delta_{40,i}^{g} = 0 \text{ (tendon 2 is not connected to the piece of concrete 1)}
\end{aligned} \tag{6}$$

Terms δ_j^{ld} due to tensioning of tendon 1 are:

$$\begin{aligned}
&\delta_{10}^{p1} = \bar{\varepsilon}_{p1} \\
&\delta_{20}^{p1} = 0 \\
&\delta_{30}^{p1} = 0 \\
&\delta_{40}^{p1} = 0
\end{aligned} \tag{7}$$

where $\bar{\varepsilon}_{p1}$ is the non-compatible strain due to tensioning, that is:

$$\bar{\varepsilon}_{p1} = \frac{X_{p1}(1-r)}{E_{p1}A_{p1}} = X_{p1} \cdot (1-r) \cdot \delta_{11}^{p1} \tag{8}$$

when dealing with pre-tensioning ($\bar{\varepsilon}_{p1}$ is the elongation of the tendon), or:

$$\begin{aligned}
\bar{\varepsilon}_{p1} &= \frac{X_{p1}|_{z=z*} \cdot (1-r)}{E_{p1}A_{p1}} + \left[\frac{X_{p1}|_{z=z*} \cdot (1-r)}{E_{c1}(t_1) \cdot A_{c1}} + \frac{X_{p1}|_{z=z*} \cdot (1-r) \cdot y_{p1}^2}{E_{c1}(t_1) \cdot J_{c1}}\right] \\
&= X_{p1}|_{z=z*} \cdot (1-r) \cdot \left(\delta_{11}^{p1} + \delta_{11}^{c1} \cdot \frac{E_{c1}}{E_{c1}(t_1)}\right)
\end{aligned} \tag{9}$$

when dealing with post-tensioning (in this case $\bar{\varepsilon}_{p1}$ is the elongation of the tendon plus the shortening of the surrounding concrete).

Term r is the ratio between the relaxation loss and the initial stress of the prestressing steel (it is approximately equal to 0.03 when dealing with low relaxation tendons). In Eq. (8) X_{p1} is the prestressing force applied by the jack in the prestressing bed at stressing of the tendon. In Eq. (9) $X_{p1}|_{z=z*}$ is the prestressing force applied by the jack at the anchorage minus the friction loss up to cross section z^* (see the notation and Fig. 2 for the other terms).

Post-tensioning of tendon 2 causes the following non-compatible strains:

$$\begin{aligned}
&\delta_{10}^{p2} = 0 \\
&\delta_{20}^{p2} = 0 \\
&\delta_{30}^{p2} = 0 \\
&\delta_{40}^{p2} = \bar{\varepsilon}_{p2}
\end{aligned} \tag{10}$$

where:

$$\bar{\varepsilon}_{p2} = \frac{X_{p2}|_{z=z*} \cdot (1-r)}{E_{p2}A_{p2}} + \frac{1}{E_{c2}(T_2)}\left[\frac{X_{p2}|_{z=z*} \cdot (1-r)}{A_{\sec}} + \frac{X_{p2}|_{z=z*} \cdot (1-r) \cdot y_{p\sec}^2}{J_{\sec}}\right] \tag{11}$$

and $X_{p2}|_{z=z*}$ is the prestressing force applied by the jack at the anchorage minus the friction loss up to cross section z^*, $y_{p\sec}$ is the distance of tendon 2 from G_2 (i.e. $y_{p\sec} = y_{c1} - y_{\Delta N_i} + y_{c2} - y_{p2}$), $A_{\sec}$ and $J_{\sec}$ are the E-weighted area and the E-weighted second moment of area (with respect to its principal axis parallel to x_1, passing through point G_2) of the final cross section.

3. The compatibility equations

The events related to changes in the shape of the cross section modify the compatibility equations and require the recalculation of the unknowns (for the same single load) in every time interval between two consecutive shape changes.

3.1. Prestressing of tendon 1

The unknowns that refer to prestressing of tendon 1 will be indicated with superscript p1.

3.1.1. $t_1 \leqslant t < t_3$

Compatibility (starting at tendon release from the prestressing bed in pre-tensioning or at grouting of tendon 1 in post-tensioning) between tendon 1 and the adjacent concrete fibres yields the following term:

$$\varepsilon_{p1}(t) - \bar{\varepsilon}_{p1} + \varepsilon_{c1}(t, y_{p1}) = 0 \tag{12}$$

On the basis of Eqs. (1)–(3), Eq. (12) becomes:

$$\begin{aligned}
&\Delta X_I^{p1}(t) \cdot \delta_{11}^{p1} - \bar{\varepsilon}_{p1} + \Delta X_I^{p1}(t_1) \cdot \delta_{11}^{c1} \cdot \frac{E_{c1}}{E_{c1}(t_1)} \cdot \varphi_{c1}(t, t_1)[1 - \chi_{c1}(t, t_1)] \\
&+ \Delta X_I^{p1}(t) \cdot \delta_{11}^{c1} \cdot \frac{E_{c1}}{E_{c1}(t_1)} \cdot [1 + \chi_{c1}(t, t_1) \cdot \varphi_{c1}(t, t_1)] = 0
\end{aligned} \tag{13}$$

that is:

$$\Delta X_I^{p1}(t)\left\{\delta_{11}^{p1}+\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot[1+\chi_{c1}(t,t_1)\cdot\varphi_{c1}(t,t_1)]\right\} = \bar{\varepsilon}_{p1}-\Delta X_I^{p1}(t_1)\cdot\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot\varphi_{c1}(t,t_1)[1-\chi_{c1}(t,t_1)] \tag{14}$$

Term $\Delta X_I^{p1}(t_1)$ can be easily computed by setting $t = t_1$ (i.e. $\varphi_1(t_1,t_1) = 0$) in Eq. (14):

$$\Delta X_I^{p1}(t_1)\cdot\delta_{11}^{p1}-\bar{\varepsilon}_{p1}+\Delta X_I^{p1}(t_1)\cdot\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}=0 \tag{15}$$

that is:

$$\Delta X_I^{p1}(t_1)=\frac{\bar{\varepsilon}_{p1}}{\delta_{11}^{p1}+\delta_{11}^{c1}\frac{E_{c1}}{E_{c1}(t_1)}} \tag{16}$$

Note that Eq. (16) is independent of the prestressing method adopted (it is a matter of fact that concrete behaviour after time t_1 is independent of the way $\bar{\varepsilon}_{p1}$ was applied). It is interesting to observe that when replacing Eq. (9) into Eq. (16), that is when dealing with post-tensioning, we get:

$$\Delta X_I^{p1}(t_1)=X_{p1}|_{z=z*}\cdot(1-r) \tag{17}$$

whereas when dealing with pre-tensioning (i.e. making use of Eq. (8) instead of Eq. (9)) we get:

$$\Delta X_I^{p1}(t_1)=\frac{\delta_{11}^{p1}}{\delta_{11}^{p1}+\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}}X_{p1}(1-r) \tag{18}$$

$$\Delta X_I^{p1}(t_1)-X_{p1}(1-r)=\left[\frac{\delta_{11}^{p1}}{\delta_{11}^{p1}+\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}}-1\right]X_{p1}(1-r) = -\frac{\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}}{\delta_{11}^{p1}+\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}}\cdot X_{p1}(1-r) \tag{19}$$

Whatever the prestressing method, when replacing Eq. (16) into Eq. (14) we get:

$$\Delta X_I^{p1}(t)=\frac{\bar{\varepsilon}_{p1}}{\delta_{11}^{p1}+\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot[1+\chi_{c1}(t,t_1)\cdot\varphi_{c1}(t,t_1)]} \left\{1-\frac{\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}}{\delta_{11}^{p1}+\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}}\varphi_{c1}(t,t_1)[1-\chi_{c1}(t,t_1)]\right\} \tag{20}$$

In this time interval $\Delta X_{II}^{p1}(t)=\Delta X_{III}^{p1}(t)=\Delta X_{IV}^{p1}(t)=0$.

3.1.2. $t_3 \leqslant t < t_4$

Compatibility equations:

$$\begin{cases}\varepsilon_{p1}(t)-\varepsilon_{p1}(t_1)+\varepsilon_{c1}(t,y_{p1})-\varepsilon_{c1}(t_1,y_{p1})=0\\ \varepsilon_{c1}(t,y_{c1})-\varepsilon_{c1}(t_3,y_{c1})+\varepsilon_{c2}(T,y_{c2})=0\\ 1/r_{c1}(t)-1/r_{c1}(t_3)+1/r_{c2}(T)=0\end{cases} \tag{21}$$

The first of Eq. (21) enforces compatibility between tendon 1 and the surrounding concrete, whereas the last two equations enforce strain and curvature compatibility on the contact surface between piece of concrete 1 and piece of concrete 2.

When making use of Eqs. (1)–(3) and considering that $\Delta X_{II}^{p1}(t_3)=\Delta X_{III}^{p1}(t_3)=0$, Eq. (21) becomes:

$$\begin{cases}\Delta X_I^{p1}(t)\cdot\left\{\delta_{11}^{p1}+\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot[1+\chi_1(t,t_1)\cdot\varphi_1(t,t_1)]\right\}+\Delta X_{II}^{p1}(t)\cdot\delta_{12}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_3)}\cdot[1+\chi_1(t,t_3)\cdot\varphi_1(t,t_3)]\\ \quad+\Delta X_{III}^{p1}(t)\cdot\delta_{13}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_3)}\cdot[1+\chi_1(t,t_3)\cdot\varphi_1(t,t_3)]=\bar{\varepsilon}_{p1}-\Delta X_I^{p1}(t_1)\cdot\delta_{11}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot\varphi_1(t,t_1)[1-\chi_1(t,t_1)]\\ \Delta X_I^{p1}(t)\cdot\delta_{21}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot[1+\chi_1(t,t_1)\cdot\varphi_1(t,t_1)]\\ \quad+\Delta X_{II}^{p1}(t)\cdot\left\{\delta_{22}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_3)}\cdot[1+\chi_1(t,t_3)\cdot\varphi_1(t,t_3)]+\delta_{22}^{c2}\cdot\frac{E_{c2}}{E_{c2}(T_1)}\cdot[1+\chi_2(T,T_1)\cdot\varphi_2(T,T_1)]\right\}\\ \quad+\Delta X_{III}^{p1}(t)\cdot\left\{\delta_{23}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_3)}[1+\chi_1(t,t_3)\cdot\varphi_1(t,t_3)]+\delta_{23}^{c2}\cdot\frac{E_{c2}}{E_{c2}(T_1)}\cdot[1+\chi_2(T,T_1)\cdot\varphi_2(T,T_1)]\right\}\\ \quad=-\Delta X_I^{p1}(t_1)\cdot\delta_{21}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot\varphi_1(t,t_1)[1-\chi_1(t,t_1)]+\Delta X_I^{p1}(t_3)\cdot\delta_{21}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot[1+\chi_1(t_3,t_1)\cdot\varphi_1(t_3,t_1)]\\ \quad+\Delta X_I^{p1}(t_1)\cdot\delta_{21}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot\varphi_1(t_3,t_1)[1-\chi_1(t_3,t_1)]\\ \Delta X_I^{p1}(t)\cdot\delta_{31}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot[1+\chi_1(t,t_1)\cdot\varphi_1(t,t_1)]\\ \quad+\Delta X_{II}^{p1}(t)\cdot\left\{\delta_{32}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_3)}\cdot[1+\chi_1(t,t_3)\cdot\varphi_1(t,t_3)]+\delta_{32}^{c2}\cdot\frac{E_{c2}}{E_{c2}(T_1)}\cdot[1+\chi_2(T,T_1)\cdot\varphi_2(T,T_1)]\right\}\\ \quad+\Delta X_{III}^{p1}(t)\cdot\left\{\delta_{33}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_3)}\cdot[1+\chi_1(t,t_3)\cdot\varphi_1(t,t_3)]+\delta_{33}^{c2}\cdot\frac{E_{c2}}{E_{c2}(T_1)}\cdot[1+\chi_2(T,T_1)\cdot\varphi_2(T,T_1)]\right\}\\ \quad=-\Delta X_I^{p1}(t_1)\cdot\delta_{31}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot\varphi_1(t,t_1)[1-\chi_1(t,t_1)]+\Delta X_I^{p1}(t_3)\cdot\delta_{31}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot[1+\chi_1(t_3,t_1)\cdot\varphi_1(t_3,t_1)]\\ \quad+\Delta X_I^{p1}(t_1)\cdot\delta_{31}^{c1}\cdot\frac{E_{c1}}{E_{c1}(t_1)}\cdot\varphi_1(t_3,t_1)[1-\chi_1(t_3,t_1)]\end{cases} \tag{22}$$

Eq. (17) shows that in post-tensioning the tensile stress resultant in the tendon at time t_1 is equal to the prestressing force applied by the jack at the anchorage minus the friction loss up to the cross section z^* (i.e. $X_{p1}|_{z=z^*}$), reduced by relaxation r of the tendon. On the contrary, when dealing with pre-tensioning Eq. (18) states that the stress resultant in the tendon decreases immediately at tendon release from the prestressing bed. This is the so-called "elastic" loss of prestress that is caused by concrete elastic shortening:

Note that the terms at the right of the equality sign in the second and third of Eq. (22) account for the strains that developed before bonding of the two pieces of concrete. For the sake of simplicity, by assuming that:

$$\begin{aligned} f_{ij}^{\text{sec}}(t_k,t_m)&=\delta_{ij}^{\text{sec}}\cdot\frac{E_{c\ \text{sec}}}{E_{c\ \text{sec}}(t_m)}\cdot[1+\chi_{\text{sec}}(t_k,t_m)\cdot\varphi_{\text{sec}}(t_k,t_m)]\\ g_{ij}^{\text{sec}}(t_k,t_m)&=\delta_{ij}^{\text{sec}}\cdot\frac{E_{c\ \text{sec}}}{E_{c\ \text{sec}}(t_m)}\cdot\varphi_{\text{sec}}(t_k,t_m)\cdot[1-\chi_{\text{sec}}(t_k,t_m)]\end{aligned} \tag{23}$$

Eq. (22) become:

$$\begin{cases} \Delta X_I^{p1}(t)\cdot\left[\delta_{11}^{p1}+f_{11}^{c1}(t,t_1)\right]+\Delta X_{II}^{p1}(t)\cdot f_{12}^{c1}(t,t_3)+\Delta X_{III}^{p1}(t)\cdot f_{13}^{c1}(t,t_3) \\ \quad =\bar{\varepsilon}_{p1}-\Delta X_I^{p1}(t_1)\cdot g_{11}^{c1}(t,t_1) \\ \Delta X_I^{p1}(t)\cdot f_{21}^{c1}(t,t_1)+\Delta X_{II}^{p1}(t)\cdot\left[f_{22}^{c1}(t,t_3)+f_{22}^{c2}(T,T_1)\right]+\Delta X_{III}^{p1}(t)\cdot\left[f_{23}^{c1}(t,t_3)+f_{23}^{c2}(T,T_1)\right] \\ \quad =-\Delta X_I^{p1}(t_1)\cdot g_{21}^{c1}(t,t_1)+\Delta X_I^{p1}(t_3)\cdot f_{21}^{c1}(t_3,t_1)+\Delta X_I^{p1}(t_1)\cdot g_{21}^{c1}(t_3,t_1) \\ \Delta X_I^{p1}(t)\cdot f_{31}^{c1}(t,t_1)+\Delta X_{II}^{p1}(t)\cdot\left[f_{32}^{c1}(t,t_3)+f_{32}^{c2}(T,T_1)\right]+\Delta X_{III}^{p1}(t)\cdot\left[f_{33}^{c1}(t,t_3)+f_{33}^{c2}(T,T_1)\right] \\ \quad =-\Delta X_I^{p1}(t_1)\cdot g_{31}^{c1}(t,t_1)+\Delta X_I^{p1}(t_3)\cdot f_{31}^{c1}(t_3,t_1)+\Delta X_I^{p1}(t_1)\cdot g_{31}^{c1}(t_3,t_1) \end{cases} \tag{24}$$

In this time interval $\Delta X_{IV}^{p1}(t)=0$.

Note that when setting $t=t_3$ in Eq. (24) we get $\Delta X_{II}^{p1}(t_3)=\Delta X_{III}^{p1}(t_3)=0$. Moreover, the value of $\Delta X_I^{p1}(t_3)$ is exactly the same as that computed by means of Eq. (20). This implies that the stress resultant in tendon 1 immediately before bonding between the two pieces of concrete is equal to the one computed immediately after it (because bond itself does not give rise to any sudden instantaneous elastic change in $\Delta X_I^{p1}(t)$).

3.1.3. $t \geqslant t_4$

Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t)-\varepsilon_{p1}(t_1)+\varepsilon_{c1}(t,y_{p1})-\varepsilon_{c1}(t_1,y_{p1})=0 \\ \varepsilon_{c1}(t,y_{c1})-\varepsilon_{c1}(t_3,y_{c1})+\varepsilon_{c2}(T,y_{c2})=0 \\ 1/r_{c1}(t)-1/r_{c1}(t_3)+1/r_{c2}(T)=0 \\ \varepsilon_{p2}(T)+\varepsilon_{c2}(T,y_{p2})-\varepsilon_{c2}(T_2,y_{p2})=0 \end{cases} \tag{25}$$

The last of Eq. (25) enforces bond between tendon 2 and the surrounding concrete.

When making use of Eqs. (1)–(3) and taking into account that $\Delta X_{II}^{p1}(t_3)=\Delta X_{III}^{p1}(t_3)=\Delta X_{IV}^{p1}(t_4)=0$, Eq. (25) becomes:

$$\begin{cases} \Delta X_I^{p1}(t)\cdot\left[\delta_{11}^{p1}+f_{11}^{c1}(t,t_1)\right]+\Delta X_{II}^{p1}(t)\cdot f_{12}^{c1}(t,t_3)+\Delta X_{III}^{p1}(t)\cdot f_{13}^{c1}(t,t_3) \\ \quad =\bar{\varepsilon}_{p1}-\Delta X_I^{p1}(t_1)\cdot g_{11}^{c1}(t,t_1) \\ \Delta X_I^{p1}(t)\cdot f_{21}^{c1}(t,t_1)+\Delta X_{II}^{p1}(t)\cdot\left[f_{22}^{c1}(t,t_3)+f_{22}^{c2}(T,T_1)\right] \\ \quad +\Delta X_{III}^{p1}(t)\cdot\left[f_{23}^{c1}(t,t_3)+f_{23}^{c2}(T,T_1)\right]+\Delta X_{IV}^{p1}(t)\cdot f_{24}^{c2}(T,T_2) \\ \quad =-\Delta X_I^{p1}(t_1)\cdot g_{21}^{c1}(t,t_1)+\Delta X_I^{p1}(t_3)\cdot f_{21}^{c1}(t_3,t_1)+\Delta X_I^{p1}(t_1)\cdot g_{21}^{c1}(t_3,t_1) \\ \Delta X_I^{p1}(t)\cdot f_{31}^{c1}(t,t_1)+\Delta X_{II}^{p1}(t)\cdot\left[f_{32}^{c1}(t,t_3)+f_{32}^{c2}(T,T_1)\right] \\ \quad +\Delta X_{III}^{p1}(t)\cdot\left[f_{33}^{c1}(t,t_3)+f_{33}^{c2}(T,T_1)\right]+\Delta X_{IV}^{p1}(t)\cdot f_{34}^{c2}(T,T_2) \\ \quad =-\Delta X_I^{p1}(t_1)\cdot g_{31}^{c1}(t,t_1)+\Delta X_I^{p1}(t_3)\cdot f_{31}^{c1}(t_3,t_1)+\Delta X_I^{p1}(t_1)\cdot g_{31}^{c1}(t_3,t_1) \\ \Delta X_{II}^{p1}(t)\cdot f_{42}^{c2}(T,T_1)+\Delta X_{III}^{p1}(t)\cdot f_{43}^{c2}(T,T_1)+\Delta X_{IV}^{p1}(t)\cdot\left[\delta_{44}^{p2}+f_{44}^{c2}(T,T_2)\right] \\ \quad =\Delta X_{II}^{p1}(t_4)\cdot f_{42}^{c2}(T_2,T_1)+\Delta X_{III}^{p1}(t_4)\cdot f_{43}^{c2}(T_2,T_1) \end{cases} \tag{26}$$

Again, when setting $t=t_4$ in Eq. (24) and in (26) we get the same values for $\Delta X_{II}^{p1}(t_4)$ and $\Delta X_{III}^{p1}(t_4)$. The stress resultants mutually transferred between the two pieces of concrete immediately before grouting of tendon 2 are equal to those computed immediately after grouting (i.e. grouting itself does not give rise to any sudden elastic change in $\Delta X_{II}^{p1}(t)$ and $\Delta X_{III}^{p1}(t)$).

3.2. Prestressing of tendon 2

The unknowns that refer to prestressing of tendon 2 will be indicated with superscript p2.

Tendon 2 is stressed at time t_4.

3.2.1. $t \geqslant t_4$

Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t)+\varepsilon_{c1}(t,y_{p1})=0 \\ \varepsilon_{c1}(t,y_{c1})+\varepsilon_{c2}(T,y_{c2})=0 \\ 1/r_{c1}(t)+1/r_{c2}(T)=0 \\ \varepsilon_{p2}(T)-\bar{\varepsilon}_{p2}+\varepsilon_{c2}(T,y_{p2})=0 \end{cases} \tag{27}$$

Once more the first of Eq. (27) enforces compatibility between tendon 1 and the surrounding concrete, the following two equations enforce strain and curvature compatibility on the contact surface between the two pieces of concrete, and the last enforces compatibility between tendon 2 and the surrounding concrete.

When making use of Eqs. (1)–(3) Eq. (27) becomes:

$$\begin{cases} \Delta X_I^{p2}(t)\cdot\left[\delta_{11}^{p1}+f_{11}^{c1}(t,t_4)\right]+\Delta X_{II}^{p2}(t)\cdot f_{12}^{c1}(t,t_4)+\Delta X_{III}^{p2}(t)\cdot f_{13}^{c1}(t,t_4) \\ \quad =-\Delta X_I^{p2}(t_4)\cdot g_{11}^{c1}(t,t_4)-\Delta X_{II}^{p2}(t_4)\cdot g_{12}^{c1}(t,t_4)-\Delta X_{III}^{p2}(t_4)\cdot g_{13}^{c1}(t,t_4) \\ \Delta X_I^{p2}(t)\cdot f_{21}^{c1}(t,t_4)+\Delta X_{II}^{p2}(t)\cdot\left[f_{22}^{c1}(t,t_4)+f_{22}^{c2}(T,T_2)\right] \\ \quad +\Delta X_{III}^{p2}(t)\cdot\left[f_{23}^{c1}(t,t_4)+f_{23}^{c2}(T,T_2)\right]+\Delta X_{IV}^{p2}(t)\cdot f_{24}^{c2}(T,T_2) \\ \quad =-\Delta X_I^{p2}(t_4)\cdot g_{21}^{c1}(t,t_4)-\Delta X_{II}^{p2}(t_4)\cdot\left[g_{22}^{c1}(t,t_4)+g_{22}^{c2}(T,T_2)\right] \\ \quad -\Delta X_{III}^{p2}(t_4)\cdot\left[g_{23}^{c1}(t,t_4)+g_{23}^{c2}(T,T_2)\right]-\Delta X_{IV}^{p2}(t_4)\cdot g_{24}^{c2}(T,T_2) \\ \Delta X_I^{p2}(t)\cdot f_{31}^{c1}(t,t_4)+\Delta X_{II}^{p2}(t)\cdot\left[f_{32}^{c1}(t,t_4)+f_{32}^{c2}(T,T_2)\right] \\ \quad +\Delta X_{III}^{p2}(t)\cdot\left[f_{33}^{c1}(t,t_4)+f_{33}^{c2}(T,T_2)\right]+\Delta X_{IV}^{p2}(t)\cdot f_{34}^{c2}(T,T_2) \\ \quad =-\Delta X_I^{p2}(t_4)\cdot g_{31}^{c1}(t,t_4)-\Delta X_{II}^{p2}(t_4)\cdot\left[g_{32}^{c1}(t,t_4)+g_{32}^{c2}(T,T_2)\right] \\ \quad -\Delta X_{III}^{p2}(t_4)\cdot\left[g_{33}^{c1}(t,t_4)+g_{33}^{c2}(T,T_2)\right]-\Delta X_{IV}^{p2}(t_4)\cdot g_{34}^{c2}(T,T_2) \\ \Delta X_{II}^{p2}(t)\cdot f_{42}^{c2}(T,T_2)+\Delta X_{III}^{p2}(t)\cdot f_{43}^{c2}(T,T_2)+\Delta X_{IV}^{p2}(t)\cdot\left[\delta_{44}^{p2}+f_{44}^{c2}(T,T_2)\right] \\ \quad =\bar{\varepsilon}_{p2}-\Delta X_{II}^{p2}(t_4)\cdot g_{42}^{c2}(T,T_2)-\Delta X_{III}^{p2}(t_4)\cdot g_{43}^{c2}(T,T_2)-\Delta X_{IV}^{p2}(t_4)\cdot g_{44}^{c2}(T,T_2) \end{cases} \tag{28}$$

where terms $\Delta X_I^{p2}(t_4), \Delta X_{II}^{p2}(t_4)$, $\Delta X_{III}^{p2}(t_4)$ and $X_{III}^{p2}(t_4)$ can be computed by setting $t=t_4$ in Eq. (28) (they account for the instantaneous elastic response of the whole section to stressing of tendon 2).

3.3. Shrinkage of piece of concrete 1

The unknowns that refer to shrinkage of piece of concrete 1 will be indicated with superscript sh1.

When dealing with post-tensioning, bonding begins to act at time t_1 (i.e. after grouting of tendon 1). The cross section is homogeneous before this time and therefore shrinkage of concrete does not give rise to any self-equilibrated stress distribution in the time interval $t_0 \leqslant t_1$.

When dealing with prestressing, bonding begins at time $t_0 \leqslant t_1$, (i.e. immediately after concrete hardening). Nevertheless the time interval between concrete hardening and the release of tendon 1 from the stressing bed corresponds to concrete curing. During this short time interval, the stiffness of concrete part 1 is rather low. In this phase concrete is constrained by friction with the mould and bond to tendon 1 (still anchored to the stressing bed). In short, in common practice the effect of shrinkage in the time interval $t_0 t \leqslant t_1$ is usually neglected.

3.3.1. $t_1 \leqslant t < t_3$

Compatibility between tendon 1 and the surrounding concrete (concrete strain ε_{c1} is the sum of the viscoelastic plus the shrinkage strain and $\Delta X_I^{sh1}(t_1)=0$):

$$\varepsilon_{p1}(t)+\varepsilon_{c1}(t,y_{p1})=0 \tag{29}$$

that by means of Eqs. (1), (2), (3), (4) and (23) becomes:

$$\Delta X_I^{sh1}(t)\cdot\left[\delta_{11}^{p1}+f_{11}^{c1}(t,t_1)\right]=-\delta_{10}^{sh1}(t) \tag{30}$$

In this time interval $\Delta X_{II}^{sh1}(t)=\Delta X_{III}^{sh1}(t)=\Delta X_{IV}^{sh1}(t)=0$.

3.3.2. $t_3 \leqslant t < t_4$

Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t)+\varepsilon_{c1}(t,y_{p1})=0 \\ \varepsilon_{c1}(t,y_{c1})-\varepsilon_{c1}(t_3,y_{c1})+\varepsilon_{c2}(T,y_{c2})=0 \\ 1/r_{c1}(t)-1/r_{c1}(t_3)+1/r_{c2}(T)=0 \end{cases} \tag{31}$$

where again the first of Eq. (31) enforces compatibility between tendon 1 and the surrounding concrete, and the last two equations enforce strain and curvature compatibility on the contact surface between piece of concrete 1 and piece of concrete 2.

When replacing Eqs. (1), (2), (3), (4) and (23) into Eq. (31), and taking into account that $\Delta X_{I}^{sh1}(t_1) = \Delta X_{II}^{sh1}(t_3) = \Delta X_{III}^{sh1}(t_3) = 0$, $\delta_{30} = \delta_{40} = 0$ we get:

$$\begin{cases} \Delta X_{I}^{sh1}(t)\cdot\left[\delta_{11}^{p1}+f_{11}^{c1}(t,t_1)\right]+\Delta X_{II}^{sh1}(t)\cdot f_{12}^{c1}(t,t_3)+\Delta X_{III}^{sh1}(t)\cdot f_{13}^{c1}(t,t_3)=-\delta_{10}^{sh1}(t) \\ \Delta X_{I}^{sh1}(t)\cdot f_{21}^{c1}(t,t_1)+\Delta X_{II}^{sh1}(t)\cdot\left[f_{22}^{c1}(t,t_3)+f_{22}^{c2}(T,T_1)\right] \\ \quad +\Delta X_{III}^{sh1}(t)\cdot\left[f_{23}^{c1}(t,t_3)+f_{23}^{c2}(T,T_1)\right] \\ \quad =-\delta_{20}^{sh1}(t)+\Delta X_{I}^{sh1}(t_3)\cdot f_{21}^{c1}(t_3,t_1) \\ \Delta X_{I}^{sh1}(t)\cdot f_{31}^{c1}(t,t_1)+\Delta X_{II}^{sh1}(t)\cdot\left[f_{32}^{c1}(t,t_3)+f_{32}^{c2}(T,T_1)\right] \\ \quad +\Delta X_{III}^{sh1}(t)\cdot\left[f_{33}^{c1}(t,t_3)+f_{33}^{c2}(T,T_1)\right] \\ \quad =\Delta X_{I}^{sh1}(t_3)\cdot f_{31}^{c1}(t_3,t_1) \end{cases} \tag{32}$$

In this time interval $\Delta X_{IV}^{sh1}(t) = 0$. Once more $\Delta X_{I}^{sh1}(t_3)$ can be computed by setting $t = t_3$ in Eq. (30) or in the first of Eq. (32) (they are identical, considering that $\Delta X_{II}^{sh1}(t_3) = \Delta X_{III}^{sh1}(t_3) = 0$, or bond between the two pieces of concrete does not give rise to any sudden elastic change in $\Delta X_{I}^{sh1}(t)$).

3.3.3. $t \geqslant t_4$

Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t)+\varepsilon_{c1}(t,y_{p1})=0 \\ \varepsilon_{c1}(t,y_{c1})-\varepsilon_{c1}(t_3,y_{c1})+\varepsilon_{c2}(T,y_{c2})=0 \\ 1/r_{c1}(t)-1/r_{c1}(t_3)+1/r_{c2}(T)=0 \\ \varepsilon_{p2}(T)+\varepsilon_{c2}(T,y_{p2})-\varepsilon_{c2}(T_2,y_{p2})=0 \end{cases} \tag{33}$$

that is $\left(\Delta X_{I}^{sh1}(t_1) = \Delta X_{II}^{sh1}(t_3) = \Delta X_{III}^{sh1}(t_3) = \Delta X_{IV}^{sh1}(T_2) = 0,\ \delta_{14} = \delta_{41} = \delta_{30}^{sh1} = \delta_{40}^{sh1} = 0\right)$:

$$\begin{cases} \Delta X_{I}^{sh1}(t)\cdot\left[\delta_{11}^{p1}+f_{11}^{c1}(t,t_1)\right]+\Delta X_{II}^{sh1}(t)\cdot f_{12}^{c1}(t,t_3)+\Delta X_{III}^{sh1}(t)\cdot f_{13}^{c1}(t,t_3)=-\delta_{10}^{sh1}(t) \\ \Delta X_{I}^{sh1}(t)\cdot f_{21}^{c1}(t,t_1)+\Delta X_{II}^{sh1}(t)\cdot\left[f_{22}^{c1}(t,t_3)+f_{22}^{c2}(T,T_1)\right] \\ \quad +\Delta X_{III}^{sh1}(t)\cdot\left[f_{23}^{c1}(t,t_3)+f_{23}^{c2}(T,T_1)\right]+\Delta X_{IV}^{sh1}(t)\cdot f_{24}^{c2}(T,T_2) \\ \quad =-\delta_{20}^{sh1}(t)+\Delta X_{I}^{sh1}(t_3)\cdot f_{21}^{c1}(t_3,t_1) \\ \Delta X_{I}^{sh1}(t)\cdot f_{31}^{c1}(t,t_1)+\Delta X_{II}^{sh1}(t)\cdot\left[f_{32}^{c1}(t,t_3)+f_{32}^{c2}(T,T_1)\right] \\ \quad +\Delta X_{III}^{sh1}(t)\cdot\left[f_{33}^{c1}(t,t_3)+f_{33}^{c2}(T,T_1)\right]+\Delta X_{IV}^{sh1}(t)\cdot f_{34}^{c2}(T,T_2) \\ \quad =\Delta X_{I}^{sh1}(t_3)\cdot f_{31}^{c1}(t_3,t_1) \\ \Delta X_{II}^{sh1}(t)\cdot f_{42}^{c2}(T,T_1)+\Delta X_{III}^{sh1}(t)\cdot f_{43}^{c2}(T,T_1)+\Delta X_{IV}^{sh1}(t)\cdot\left[\delta_{44}^{p2}+f_{44}^{c2}(T,T_2)\right] \\ \quad =\Delta X_{II}^{sh1}(t_4)\cdot f_{42}^{c2}(T_2,T_1)+\Delta X_{III}^{sh1}(t_4)\cdot f_{43}^{c2}(T_2,T_1) \end{cases} \tag{34}$$

where $\Delta X_{II}^{sh1}(t_4)$ and $\Delta X_{III}^{sh1}(t_4)$ can be computed by setting $t = t_4$ in Eq. (34) (or in Eq. (32)).

3.4. Shrinkage of piece of concrete 2

The unknowns that refer to shrinkage of piece of concrete 2 will be indicated with superscript sh2.

3.4.1. $t_3 \leqslant t < t_4$

Compatibility equations (concrete strain ε_{c2} is the sum of the viscoelastic and the shrinkage strain):

$$\begin{cases} \varepsilon_{p1}(t)+\varepsilon_{c1}(t,y_{p1})=0 \\ \varepsilon_{c1}(t,y_{c1})+\varepsilon_{c2}(T,y_{c2})=0 \\ 1/r_{c1}(t)+1/r_{c2}(T)=0 \end{cases} \tag{35}$$

that is $\left(\Delta X_{I}^{sh2}(t_3) = \Delta X_{II}^{sh2}(t_3) = \Delta X_{III}^{sh2}(t_3) = 0,\ \delta_{10}^{sh2} = \delta_{30}^{sh2} = 0\right)$:

$$\begin{cases} \Delta X_{I}^{sh2}(t)\cdot\left[\delta_{11}^{p1}+f_{11}^{c1}(t,t_3)\right]+\Delta X_{II}^{sh2}(t)\cdot f_{12}^{c1}(t,t_3)+\Delta X_{III}^{sh2}(t)\cdot f_{13}^{c1}(t,t_3)=0 \\ \Delta X_{I}^{sh2}(t)\cdot f_{21}^{c1}(t,t_3)+\Delta X_{II}^{sh2}(t)\cdot\left[f_{22}^{c1}(t,t_3)+f_{22}^{c2}(T,T_1)\right] \\ \quad +\Delta X_{III}^{sh2}(t)\cdot\left[f_{23}^{c1}(t,t_3)+f_{23}^{c2}(T,T_1)\right]=-\delta_{20}^{sh2}(t) \\ \Delta X_{I}^{sh2}(t)\cdot f_{31}^{c1}(t,t_3)+\Delta X_{II}^{sh2}(t)\cdot\left[f_{32}^{c1}(t,t_3)+f_{32}^{c2}(T,T_1)\right]+ \\ \quad \Delta X_{III}^{sh2}(t)\cdot\left[f_{33}^{c1}(t,t_3)+f_{33}^{c2}(T,T_1)\right]=0 \end{cases} \tag{36}$$

In this time interval $\Delta X_{IV}^{sh2}(t) = 0$.

3.4.2. $t \geqslant t_4$

Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t)+\varepsilon_{c1}(t,y_{p1})=0 \\ \varepsilon_{c1}(t,y_{c1})+\varepsilon_{c2}(T,y_{c2})=0 \\ 1/r_{c1}(t)+1/r_{c2}(T)=0 \\ \varepsilon_{p2}(T)+\varepsilon_{c2}(T,y_{p1})-\varepsilon_{c2}(T_2,y_{p1})=0 \end{cases} \tag{37}$$

that is $\left(\Delta X_{I}^{sh2}(t_3) = \Delta X_{II}^{sh2}(t_3) = \Delta X_{III}^{sh2}(t_3) = \Delta X_{IV}^{sh2}(T_2) = 0,\ \delta_{14} = \delta_{41} = \delta_{10}^{sh2} = \delta_{30}^{sh2} = 0\right)$:

$$\begin{cases} \Delta X_{I}^{sh2}(t)\cdot\left[\delta_{11}^{p1}+f_{11}^{c1}(t,t_3)\right]+\Delta X_{II}^{sh2}(t)\cdot f_{12}^{c1}(t,t_3)+\Delta X_{III}^{sh2}(t)\cdot f_{13}^{c1}(t,t_3)=0 \\ \Delta X_{I}^{sh2}(t)\cdot f_{21}^{c1}(t,t_3)+\Delta X_{II}^{sh2}(t)\cdot\left[f_{22}^{c1}(t,t_3)+f_{22}^{c2}(T,T_1)\right] \\ \quad +\Delta X_{III}^{sh2}(t)\cdot\left[f_{23}^{c1}(t,t_3)+f_{23}^{c2}(T,T_1)\right]+\Delta X_{IV}^{sh2}(t)\cdot f_{24}^{c2}(T,T_2)=-\delta_{20}^{sh2}(t) \\ \quad \Delta X_{I}^{sh2}(t)\cdot f_{31}^{c1}(t,t_3)+\Delta X_{II}^{sh2}(t)\cdot\left[f_{32}^{c1}(t,t_3)+f_{32}^{c2}(T,T_1)\right] \\ \quad +\Delta X_{III}^{sh2}(t)\cdot\left[f_{33}^{c1}(t,t_3)+f_{33}^{c2}(T,T_1)\right]+\Delta X_{IV}^{sh2}(t)\cdot f_{34}^{c2}(T,T_2)=0 \\ \Delta X_{II}^{sh2}(t)\cdot f_{42}^{c2}(T,T_1)+\Delta X_{III}^{sh2}(t)\cdot f_{43}^{c2}(T,T_1)+\Delta X_{IV}^{sh2}(t)\cdot\left[\delta_{44}^{p2}+f_{44}^{c2}(T,T_2)\right] \\ \quad =-\delta_{40}^{sh2}(t)+\Delta X_{II}^{sh2}(t_4)\cdot f_{42}^{c2}(T_2,T_1)+\Delta X_{III}^{sh2}(t_4)\cdot f_{43}^{c2}(T_2,T_1) \end{cases} \tag{38}$$

where $\Delta X_{II}^{sh2}(t_4)$ and $\Delta X_{III}^{sh2}(t_4)$ can be computed by setting $t = t_4$ in Eq. (38) (or in Eq. (36)).

3.5. External long term load

The unknowns that refer to an external long term load are indicated by superscript g.

In this case it is necessary to distinguish between the loads that start acting before grouting of tendon 1 (i.e. when tendon 1 is unbounded) and those that start acting when tendon 1 is bonded to piece of concrete 1.

3.5.1. i-Th long term load that starts acting at time $t_i^g \leqslant t_1^-$ (prior to grouting of tendon 1)

Post-tensioning usually induces an upward bending of the beam (that is the beam detaches from the mould). In this case the dead load of piece of concrete 1 starts acting at post-tensioning of tendon 1 (i.e. before grouting).

3.5.1.1. $t_i^g \leqslant t \leqslant t_1^-$. Bond between tendon 1 and the surrounding concrete does not act and therefore the first theorem of linear viscoelasticity applies to piece of concrete 1 (the stress in concrete 1 is constant in time and equal to its elastic value, while the stress in tendon 1 is zero).

3.5.1.2. $t_1 \leqslant t < t_3$. Compatibility equation:

$$\varepsilon_{p1}(t)+\varepsilon_{c1}(t,y_{p1})-\varepsilon_{c1}(t_1,y_{p1})=0 \tag{39}$$

that is $\left(\Delta X_{I,i}^{g}(t_1) = 0\right)$:

$$\Delta X_{I,i}^{g}(t)\cdot\left[\delta_{11}^{p1}+f_{11}^{c1}(t,t_1)\right]+\delta_{10,i}^{g}\cdot\left[\varphi_{c1}(t,t_i^g)-\varphi_{c1}(t_1,t_i^g)\right]=0 \tag{40}$$

and therefore:

$$\Delta X_{I,i}^{g}(t)=-\frac{\delta_{10,i}^{g}\cdot\left[\varphi_{c1}(t,t_i^g)-\varphi_{c1}(t_1,t_i^g)\right]}{\delta_{11}^{p1}+f_{11}^{c1}(t,t_1)} \tag{41}$$

In this time interval $\Delta X^g_{II,i}(t) = \Delta X^g_{III,i}(t) = \Delta X^g_{IV,i}(t) = 0$.

3.5.1.3. $t_3 \leqslant t < t_4$. Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t) + \varepsilon_{c1}(t, y_{p1}) - \varepsilon_{c1}(t_1, y_{p1}) = 0 \\ \varepsilon_{c1}(t, y_{c1}) - \varepsilon_{c1}(t_3, y_{c1}) + \varepsilon_{c2}(T, y_{c2}) = 0 \\ 1/r_{c1}(t) - 1/r_{c1}(t_3) + 1/r_{c2}(T) = 0 \end{cases} \tag{42}$$

Once stated that the external load acts on piece of concrete 1, i.e. an homogeneous viscoelastic cross section, and that $\Delta X^g_{I,i}(t_1) = \Delta X^g_{II,i}(t_3) = \Delta X^g_{III,i}(t_3) = 0, \Delta X^g_{I,i}(t_3) \neq 0$, Eq. (42) become:

$\Delta X^g_{II,i}(t_4)$ and $\Delta X^g_{III,i}(t_4)$ can be easily computed by setting $t = t_4$ in Eq. (45).

3.5.2. i-Th long term load that starts acting at time $t_1^+ \leqslant t_i^g < t_3$ (after grouting of tendon 1)

3.5.2.1. $t_1 \leqslant t < t_3$. Compatibility equations:

$$\varepsilon_{p1}(t) + \varepsilon_{c1}(t, y_{p1}) = 0 \tag{46}$$

that is ($\Delta X^g_{I,i}(t_i^g) \neq 0$; the load acts on a homogeneous viscoelastic cross section, which is that of piece of concrete 1):

$$\begin{cases} \Delta X^g_{I,i}(t) \cdot \left[\delta^{p1}_{11} + f^{c1}_{11}(t, t_1)\right] + \Delta X^g_{II,i}(t) \cdot f^{c1}_{12}(t, t_3) + \Delta X^g_{III,i}(t) \cdot f^{c1}_{13}(t, t_3) \\ \qquad = -\delta^g_{10,i} \cdot \left[\varphi_{c1}(t, t_i^g) - \varphi_{c1}(t_1, t_i^g)\right] \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{21}(t, t_1) + \Delta X^g_{II,i}(t) \cdot \left[f^{c1}_{22}(t, t_3) + f^{c2}_{22}(T, T_1)\right] \\ \qquad + \Delta X^g_{III,i}(t) \cdot \left[f^{c1}_{23}(t, t_3) + f^{c2}_{23}(T, T_1)\right] = -\delta^g_{20,i} \cdot \left[\varphi_{c1}(t, t_i^g) - \varphi_{c1}(t_3, t_i^g)\right] + \Delta X^g_{I,i}(t_3) \cdot f^{c1}_{21}(t_3, t_1) \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{31}(t, t_1) + \Delta X^g_{II,i}(t) \cdot \left[f^{c1}_{32}(t, t_3) + f^{c2}_{32}(T, T_1)\right] \\ \qquad + \Delta X^g_{III,i}(t) \cdot \left[f^{c1}_{33}(t, t_3) + f^{c2}_{33}(T, T_1)\right] = -\delta^g_{30,i} \cdot \left[\varphi_{c1}(t, t_i^g) - \varphi_{c1}(t_3, t_i^g)\right] + \Delta X^g_{I,i}(t_3) \cdot f^{c1}_{31}(t_3, t_1) \end{cases} \tag{43}$$

In this time interval $\Delta X^g_{IV,i}(t) = 0$.

Term $\Delta X^g_{I,i}(t_3)$ can be computed by setting $t = t_3$ in Eq. (41) or in the first of Eq. (43) (taking into account that $\Delta X^g_{II,i}(t_3) = \Delta X^g_{III,i}(t_3) = 0$). The result indicates that the stress resultant in tendon 1 immediately before bonding between the two pieces of concrete is equal to the one measured immediately after it (i.e. bond itself does not give rise to any sudden change in $\Delta X^g_{I,i}(t)$).

3.5.1.4. $t \geqslant t_4$. Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t) + \varepsilon_{c1}(t, y_{p1}) - \varepsilon_{c1}(t_1, y_{p1}) = 0 \\ \varepsilon_{c1}(t, y_{c1}) - \varepsilon_{c1}(t_1, y_{c1}) + \varepsilon_{c2}(T, y_{c2}) = 0 \\ 1/r_{c1}(t) - 1/r_{c1}(t) + 1/r_{c2}(T) = 0 \\ \varepsilon_{p2}(T) + \varepsilon_{c2}(T, y_{p1}) - \varepsilon_{c2}(T_2, y_{p1}) = 0 \end{cases} \tag{44}$$

Once more, the external load still acts on piece of concrete 1 and $\Delta X^g_{I,i}(t_1) = \Delta X^g_{II,i}(t_3) = \Delta X^g_{III,i}(t_3) = \Delta X^g_{IV,i}(t_4) = 0$, $\delta^g_{40,i} = 0, \Delta X^g_{I,i}(t_1) \neq 0$, $\Delta X^g_{II,i}(t_4) \neq 0, \Delta X^g_{III,i}(t_4) \neq 0$). Therefore, Eq. (44) became:

$$\begin{cases} \Delta X^g_{I,i}(t) \cdot \left[\delta^{p1}_{11} + f^{c1}_{11}(t, t_1)\right] + \Delta X^g_{II,i}(t) \cdot f^{c1}_{12}(t, t_3) + \Delta X^g_{III,i}(t) \cdot f^{c1}_{13}(t, t_3) \\ \quad = -\delta^g_{10,i} \cdot \left[\varphi_{c1}(t, t_i^g) - \varphi_{c1}(t_1, t_i^g)\right] \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{21}(t, t_1) + \Delta X^g_{II,i}(t) \cdot \left[f^{c1}_{22}(t, t_3) + f^{c2}_{22}(T, T_1)\right] \\ \quad + \Delta X^g_{III,i}(t) \cdot \left[f^{c1}_{23}(t, t_3) + f^{c2}_{23}(T, T_1)\right] + \Delta X^g_{IV,i}(t) \cdot f^{c2}_{24}(T, T_2) \\ \quad = -\delta^g_{20,i} \cdot \left[\varphi_{c1}(t, t_i^g) - \varphi_{c1}(t_3, t_i^g)\right] + \Delta X^g_{I,i}(t_3) \cdot f^{c1}_{21}(t_3, t_1) \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{31}(t, t_1) + \Delta X^g_{II,i}(t) \cdot \left[f^{c1}_{32}(t, t_3) + f^{c2}_{32}(T, T_1)\right] \\ \quad + \Delta X^g_{III,i}(t) \cdot \left[f^{c1}_{33}(t, t_3) + f^{c2}_{33}(T, T_1)\right] + \Delta X^g_{IV,i}(t) \cdot f^{c2}_{34}(T, T_2) \\ \quad = -\delta^g_{30,i} \cdot \left[\varphi_{c1}(t, t_i^g) - \varphi_{c1}(t_3, t_i^g)\right] + \Delta X^g_{I,i}(t_3) \cdot f^{c1}_{31}(t_3, t_1) \\ \Delta X^g_{II,i}(t) \cdot f^{c2}_{42}(T, T_1) + \Delta X^g_{III,i}(t) \cdot f^{c2}_{43}(T, T_1) + \Delta X^g_{IV}(t) \cdot \left[\delta^{p2}_{44} + f^{c2}_{44}(T, T_2)\right] \\ \quad = \Delta X^g_{II,i}(t_4) \cdot f^{c2}_{42}(T_2, T_1) + \Delta X^g_{III,i}(t_4) \cdot f^{c2}_{43}(T_2, T_1) \end{cases} \tag{45}$$

$$\Delta X^g_{I,i}(t) \cdot \left[\delta^{p1}_{11} + f^{c1}_{11}(t, t_i^g)\right] + \Delta X^g_{I,i}(t_i^g) \cdot g^{c1}_{11}(t, t_i^g) + \delta^g_{10,i} \cdot \left[1 + \varphi_{c1}(t, t_i^g)\right] = 0 \tag{47}$$

When setting $t = t_i^g$ in Eq. (47) we get:

$$\Delta X^g_{I,i}(t_i^g) = -\frac{\delta^g_{10,i}}{\delta^{p1}_{11} + \delta^{c1}_{11} \cdot \frac{E_{c1}}{E_{c1}(t_i^g)}} \tag{48}$$

and finally:

$$\Delta X^g_{I,i}(t) = -\frac{\Delta X^g_{I,i}(t_i^g) \cdot g^{c1}_{11}(t, t_i^g) + \delta^g_{10,i} \cdot \left[1 + \varphi_{c1}(t, t_i^g)\right]}{\delta^{p1}_{11} + f^{c1}_{11}(t, t_i^g)} \tag{49}$$

In this time interval $\Delta X^g_{II,i}(t) = \Delta X^g_{III,i}(t) = \Delta X^g_{IV,i}(t) = 0$.

3.5.2.2. $t_3 \leqslant t < t_4$. Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t) + \varepsilon_{c1}(t, y_{p1}) = 0 \\ \varepsilon_{c1}(t, y_{c1}) - \varepsilon_{c1}(t_3, y_{c1}) + \varepsilon_{c2}(T, y_{c2}) = 0 \\ 1/r_{c1}(t) - 1/r_{c1}(t_3) + 1/r_{c2}(T) = 0 \end{cases} \tag{50}$$

or $\left(\Delta X^g_{I,i}(t_i^g) \neq 0, \Delta X^g_{I,i}(t_3) \neq 0, \Delta X^g_{II,i}(t_3) = \Delta X^g_{III,i}(t_3) = 0\right.$, the load acts on a homogeneous viscoelastic cross section):

$$\begin{cases} \Delta X^g_{I,i}(t) \cdot \left[\delta^{p1}_{11} + f^{c1}_{11}(t, t_i^g)\right] + \Delta X^g_{II,i}(t) \cdot f^{c1}_{12}(t, t_3) + \Delta X^g_{III,i}(t) \cdot f^{c1}_{13}(t, t_3) \\ \quad = -\delta^g_{10,i} \cdot \left[1 + \varphi_{c1}(t, t_i^g)\right] - \Delta X^g_{I,i}(t_i^g) \cdot g^{c1}_{11}(t, t_i^g) \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{21}(t, t_i^g) + \Delta X^g_{II,i}(t) \cdot \left[f^{c1}_{22}(t, t_3) + f^{c2}_{22}(T, T_1)\right] \\ \quad + \Delta X^g_{III,i}(t) \cdot \left[f^{c1}_{23}(t, t_3) + f^{c2}_{23}(T, T_1)\right] = -\delta^g_{20,i} \cdot \left[\varphi_{c1}(t, t_i^g) - \varphi_{c1}(t_3, t_i^g)\right] \\ \quad - \Delta X^g_{I,i}(t_i^g) \cdot \left[g^{c1}_{21}(t, t_i^g) - g^{c1}_{21}(t_3, t_i^g)\right] + \Delta X^g_{I,i}(t_3) \cdot f^{c1}_{21}(t_3, t_i^g) \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{31}(t, t_i^g) + \Delta X^g_{II,i}(t) \cdot \left[f^{c1}_{32}(t, t_3) + f^{c2}_{32}(T, T_1)\right] \\ \quad + \Delta X^g_{III,i}(t) \cdot \left[f^{c1}_{33}(t, t_3) + f^{c2}_{33}(T, T_1)\right] = -\delta^g_{30,i} \cdot \left[\varphi_{c1}(t, t_i^g) - \varphi_{c1}(t_3, t_i^g)\right] \\ \quad - \Delta X^g_{I,i}(t_i^g) \cdot \left[g^{c1}_{31}(t, t_i^g) - g^{c1}_{31}(t_3, t_i^g)\right] + \Delta X^g_{I,i}(t_3) \cdot f^{c1}_{31}(t_3, t_i^g) \end{cases} \tag{51}$$

In this time interval $\Delta X^g_{IV,i}(t) = 0$.

Term $\Delta X^g_{I,i}(t_3)$ can be computed by setting $t = t_3$ in Eq. (41) or in the first of Eq. (43).

3.5.2.3. $t \geqslant t_4$. Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t) + \varepsilon_{c1}(t, y_{p1}) = 0 \\ \varepsilon_{c1}(t, y_{c1}) - \varepsilon_{c1}(t_3, y_{c1}) + \varepsilon_{c2}(T, y_{c2}) = 0 \\ 1/r_{c1}(t) - 1/r_{c1}(t_3) + 1/r_{c2}(T) = 0 \\ \varepsilon_{p2}(T) + \varepsilon_{c2}(T, y_{p1}) - \varepsilon_{c2}(T_2, y_{p1}) = 0 \end{cases} \tag{52}$$

that is (the load acts on a homogeneous viscoelastic cross section and moreover $\Delta X^g_{I,i}(t^g_i) \neq 0, \Delta X^g_{I,i}(t_3) \neq 0, \Delta X^g_{II,i}(t_3) = \Delta X^g_{III,i}(t_3) = \Delta X^g_{IV,i}(t_4) = 0, \delta^g_{40,i} = 0, \Delta X^g_{II,i}(t_4) \neq 0, \Delta X^g_{III,i}(t_4) \neq 0$):

$$\begin{cases} \Delta X^g_{I,i}(t) \cdot \left[\delta^{p1}_{11} + f^{c1}_{11}(t, t^g_i)\right] + \Delta X^g_{II,i}(t) \cdot f^{c1}_{12}(t, t_3) + \Delta X^g_{III,i}(t) \cdot f^{c1}_{13}(t, t_3) \\ \quad = -\delta^g_{10,i} \cdot [1 + \varphi_{c1}(t, t^g_i)] - \Delta X^g_{I,i}(t^g_i) \cdot g^{c1}_{11}(t, t^g_i) \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{21}(t, t^g_i) + \Delta X^g_{II,i}(t) \cdot [f^{c1}_{22}(t, t_3) + f^{c2}_{22}(T, T_1)] \\ \quad + \Delta X^g_{III,i}(t) \cdot [f^{c1}_{23}(t, t_3) + f^{c2}_{23}(T, T_1)] + \Delta X^g_{IV,i}(t) \cdot f^{c2}_{24}(T, T_2) = -\delta^g_{20,i} \cdot [\varphi_{c1}(t, t^g_i) - \varphi_{c1}(t, t_3)] \\ \quad - \Delta X^g_{I,i}(t^g_i) \cdot [g^{c1}_{21}(t, t^g_i) - g^{c1}_{21}(t_3, t^g_i)] + \Delta X^g_{I,i}(t_3) \cdot f^{c1}_{21}(t_3, t^g_i) \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{31}(t, t^g_i) + \Delta X^g_{II,i}(t) \cdot [f^{c1}_{32}(t, t_3) + f^{c2}_{32}(T, T_1)] \\ \quad + \Delta X^g_{III,i}(t) \cdot [f^{c1}_{33}(t, t_3) + f^{c2}_{33}(T, T_1)] + \Delta X^g_{IV,i}(t) \cdot f^{c2}_{34}(T, T_2) = -\delta^g_{30,i} \cdot [\varphi_{c1}(t, t^g_i) - \varphi_{c1}(t, t_3)] \\ \quad - \Delta X^g_{I,i}(t^g_i) \cdot [g^{c1}_{31}(t, t^g_i) - g^{c1}_{31}(t_3, t^g_i)] + \Delta X^g_{I,i}(t_3) \cdot f^{c1}_{31}(t_3, t^g_i) \\ \Delta X^g_{II,i}(t) \cdot f^{c2}_{42}(T, T_1) + \Delta X^g_{III,i}(t) \cdot f^{c2}_{43}(T, T_1) + \Delta X^{sh2}_{IV}(t) \cdot \left[\delta^{p2}_{44} + f^{c2}_{44}(T, T_2)\right] \\ \quad = \Delta X^g_{II,i}(t_4) \cdot f^{c2}_{42}(T_2, T_1) + \Delta X^g_{III,i}(t_4) \cdot f^{c2}_{43}(T_2, T_1) \end{cases} \tag{53}$$

where terms $\Delta X^g_{II,i}(t_4)$ and $\Delta X^g_{III,i}(t_4)$ can be computed by setting $t = t_4$ either in Eq. (53) or in Eq. (51).

3.5.3. i-Th long term load that starts acting at time $t_3 \leqslant t^g_i < t_4 (\Rightarrow T^g_i = t^g_i - t_3 + T_1)$ (after bond between the two pieces of concrete)

3.5.3.1. $t_3 \leqslant t < t_4$. Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t) + \varepsilon_{c1}(t, y_{p1}) = 0 \\ \varepsilon_{c1}(t, y_{c1}) + \varepsilon_{c2}(T, y_{c2}) = 0 \\ 1/r_{c1}(t) + 1/r_{c2}(T) = 0 \end{cases} \tag{54}$$

that is $\left(\Delta X^g_{I,i}(t^g_i) \neq 0, \Delta X^g_{II,i}(t^g_i) \neq 0, \Delta X^g_{III,i}(t^g_i) \neq 0\right.$, the load acts on a homogeneous viscoelastic cross section):

$$\begin{cases} \Delta X^g_{I,i}(t) \cdot \left[\delta^{p1}_{11} + f^{c1}_{11}(t, t^g_i)\right] + \Delta X^g_{II,i}(t) \cdot f^{c1}_{12}(t, t^g_i) + \Delta X^g_{III,i}(t) \cdot f^{c1}_{13}(t, t^g_i) \\ \quad = -\delta^g_{10,i} \cdot [1 + \varphi_{c1}(t, t^g_i)] - \Delta X^g_{I,i}(t^g_i) \cdot g^{c1}_{11}(t, t^g_i) - \Delta X^g_{II,i}(t^g_i) \cdot g^{c1}_{12}(t, t^g_i) - \Delta X^g_{III,i}(t^g_i) \cdot g^{c1}_{13}(t, t^g_i) \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{21}(t, t^g_i) + \Delta X^g_{II,i}(t) \cdot [f^{c1}_{22}(t, t_3) + f^{c2}_{22}(T, T^g_i)] \\ \quad + \Delta X^g_{III,i}(t) \cdot [f^{c1}_{23}(t, t_3) + f^{c2}_{23}(T, T^g_i)] = -\delta^g_{20,i} \cdot [1 + \varphi_{c1}(t, t^g_i)] \\ \quad - \Delta X^g_{I,i}(t^g_i) \cdot g^{c1}_{21}(t, t^g_i) - \Delta X^g_{II,i}(t^g_i) \cdot [g^{c1}_{22}(t, t_3) + g^{c2}_{22}(T, T^g_i)] - \Delta X^g_{III,i}(t^g_i) \cdot [g^{c1}_{23}(t, t_3) + g^{c2}_{23}(T, T^g_i)] \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{31}(t, t^g_i) + \Delta X^g_{II,i}(t) \cdot [f^{c1}_{32}(t, t_3) + f^{c2}_{32}(T, T^g_i)] \\ \quad + \Delta X^g_{III,i}(t) \cdot [f^{c1}_{33}(t, t_3) + f^{c2}_{33}(T, T^g_i)] = -\delta^g_{30,i} \cdot [1 + \varphi_{c1}(t, t^g_i)] \\ \quad - \Delta X^g_{I,i}(t^g_i) \cdot g^{c1}_{31}(t, t^g_i) - \Delta X^g_{II,i}(t^g_i) \cdot [g^{c1}_{32}(t, t_3) + g^{c2}_{32}(T, T^g_i)] - \Delta X^g_{III,i}(t^g_i) \cdot [g^{c1}_{33}(t, t_3) + g^{c2}_{33}(T, T^g_i)] \end{cases} \tag{55}$$

Terms $\Delta X^g_{I,i}(t^g_i), \Delta X^g_{II,i}(t^g_i)$ and $\Delta X^g_{III,i}(t^g_i)$ can be easily computed by setting $t = t^g_i$ in Eq. (55).

In this time interval $\Delta X^g_{IV,i}(t) = 0$.

3.5.3.2. $t \geqslant t_4$. Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t) + \varepsilon_{c1}(t, y_{p1}) = 0 \\ \varepsilon_{c1}(t, y_{c1}) + \varepsilon_{c2}(T, y_{c2}) = 0 \\ 1/r_{c1}(t) + 1/r_{c2}(T) = 0 \\ \varepsilon_{p2}(T) + \varepsilon_{c2}(T, y_{p1}) - \varepsilon_{c2}(T_2, y_{p1}) = 0 \end{cases} \tag{56}$$

that is $\left(\Delta X^g_{I,i}(t^g_i) \neq 0, \Delta X^g_{II,i}(t^g_i) \neq 0, \Delta X^g_{III,i}(t^g_i) \neq 0, \Delta X^g_{IV,i}(t_4) = 0, \delta^g_{40,i} = 0\right.$, the load acts on a homogeneous viscoelastic cross section):

$$\begin{cases} \Delta X^g_{I,i}(t) \cdot \left[\delta^{p1}_{11} + f^{c1}_{11}(t, t^g_i)\right] + \Delta X^g_{II,i}(t) \cdot f^{c1}_{12}(t, t^g_i) + \Delta X^g_{III,i}(t) \cdot f^{c1}_{13}(t, t^g_i) \\ \quad = -\delta^g_{10,i} \cdot [1 + \varphi_{c1}(t, t^g_i)] - \Delta X^g_{I,i}(t^g_i) \cdot g^{c1}_{11}(t, t^g_i) - \Delta X^g_{II,i}(t^g_i) \cdot g^{c1}_{12}(t, t^g_i) - \Delta X^g_{III,i}(t^g_i) \cdot g^{c1}_{13}(t, t^g_i) \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{21}(t, t^g_i) + \Delta X^g_{II,i}(t) \cdot [f^{c1}_{22}(t, t^g_i) + f^{c2}_{22}(T, T^g_i)] \\ \quad + \Delta X^g_{III,i}(t) \cdot [f^{c1}_{23}(t, t^g_i) + f^{c2}_{23}(T, T^g_i)] + \Delta X^g_{IV,i}(t) \cdot f^{c2}_{24}(T, T_2) = -\delta^g_{20,i} \cdot [1 + \varphi_{c1}(t, t^g_i)] \\ \quad - \Delta X^g_{I,i}(t^g_i) \cdot g^{c1}_{21}(t, t^g_i) - \Delta X^g_{II,i}(t^g_i) \cdot [g^{c1}_{22}(t, t^g_i) + g^{c2}_{22}(T, T^g_i)] - \Delta X^g_{III,i}(t^g_i) \cdot [g^{c1}_{23}(t, t^g_i) + g^{c2}_{23}(T, T^g_i)] \\ \Delta X^g_{I,i}(t) \cdot f^{c1}_{31}(t, t^g_i) + \Delta X^g_{II,i}(t) \cdot [f^{c1}_{32}(t, t^g_i) + f^{c2}_{32}(T, T^g_i)] \\ \quad + \Delta X^g_{III,i}(t) \cdot [f^{c1}_{33}(t, t^g_i) + f^{c2}_{33}(T, T^g_i)] + \Delta X^g_{IV,i}(t) \cdot f^{c2}_{34}(T, T_2) = -\delta^g_{30,i} \cdot [1 + \varphi_{c1}(t, t^g_i)] \\ \quad - \Delta X^g_{I,i}(t^g_i) \cdot g^{c1}_{31}(t, t^g_i) - \Delta X^g_{II,i}(t^g_i) \cdot [g^{c1}_{32}(t, t^g_i) + g^{c2}_{32}(T, T^g_i)] - \Delta X^g_{III,i}(t^g_i) \cdot [g^{c1}_{33}(t, t^g_i) + g^{c2}_{33}(T, T^g_i)] \\ \Delta X^g_{II,i}(t) \cdot f^{c2}_{42}(T, T^g_i) + \Delta X^g_{III,i}(t) \cdot f^{c2}_{43}(T, T^g_i) + \Delta X^g_{IV,i}(t) \cdot \left[\delta^{p2}_{44} + f^{c2}_{44}(T, T_2)\right] \\ \quad = \Delta X^g_{II,i}(t_4) \cdot f^{c2}_{42}(T_2, T^g_i) + \Delta X^g_{II,i}(t^g_i) \cdot [g^{c2}_{42}(T_2, T^g_i) - g^{c1}_{42}(t, t^g_i)] + \Delta X^g_{III,i}(t_4) \cdot f^{c2}_{43}(T_2, T^g_i) \\ \quad + \Delta X^g_{III,i}(t^g_i) \cdot [g^{c2}_{43}(T_2, T^g_i) - g^{c1}_{43}(t, t^g_i)] \end{cases} \tag{57}$$

3.5.4. i-Th long term load that starts acting at time $t_i^g > t_4 (\Rightarrow T_i^g = t_i^g - t_3 + T_1)$ (after grouting of tendon 2)

Compatibility equations:

$$\begin{cases} \varepsilon_{p1}(t) + \varepsilon_{c1}(t, y_{p1}) = 0 \\ \varepsilon_{c1}(t, y_{c1}) + \varepsilon_{c2}(T, y_{c2}) = 0 \\ 1/r_{c1}(t) + 1/r_{c2}(T) = 0 \\ \varepsilon_{p2}(T) + \varepsilon_{c2}(T_2, y_{p1}) = 0 \end{cases} \tag{58}$$

that is $(\Delta X_{I,i}^g(t_i^g) \neq 0, \Delta X_{II,i}^g(t_i^g) \neq 0, \Delta X_{III,i}^g(t_i^g) \neq 0, \Delta X_{IV,i}^g(t_i^g) \neq 0,$ $\delta_{40,i}^g = 0$, the load acts on a homogeneous viscoelastic cross section):

$$\begin{cases} \Delta X_{I,i}^g(t) \cdot \left[\delta_{11}^{p1} + f_{11}^{c1}(t, t_i^g)\right] + \Delta X_{II,i}^g(t) \cdot f_{12}^{c1}(t, t_i^g) + \Delta X_{III,i}^g(t) \cdot f_{13}^{c1}(t, t_i^g) \\ \quad = -\delta_{10,i}^g \cdot [1 + \varphi_{c1}(t, t_i^g)] - \Delta X_{I,i}^g(t_i^g) \cdot g_{11}^{c1}(t, t_i^g) - \Delta X_{II,i}^g(t_i^g) \cdot g_{12}^{c1}(t, t_i^g) - \Delta X_{III,i}^g(t_i^g) \cdot g_{13}^{c1}(t, t_i^g) \\ \Delta X_{I,i}^g(t) \cdot f_{21}^{c1}(t, t_i^g) + \Delta X_{II,i}^g(t) \cdot [f_{22}^{c1}(t, t_3) + f_{22}^{c2}(T, T_i^g)] \\ \quad + \Delta X_{III,i}^g(t) \cdot [f_{23}^{c1}(t, t_3) + f_{23}^{c2}(T, T_i^g)] + \Delta X_{IV,i}^g(t) \cdot f_{24}^{c2}(T, T_2) = -\delta_{20,i}^g \cdot [1 + \varphi_{c1}(t, t_i^g)] \\ \quad - \Delta X_{I,i}^g(t_i^g) \cdot g_{21}^{c1}(t, t_i^g) - \Delta X_{II,i}^g(t_i^g) \cdot [g_{22}^{c1}(t, t_3) + g_{22}^{c2}(T, T_i^g)] \\ \quad - \Delta X_{III,i}^g(t_i^g) \cdot [g_{23}^{c1}(t, t_3) + g_{23}^{c2}(T, T_i^g)] - \Delta X_{IV,i}^g(t_i^g) \cdot g_{24}^{c2}(T, T_i^g) \\ \Delta X_{I,i}^g(t) \cdot f_{31}^{c1}(t, t_i^g) + \Delta X_{II,i}^g(t) \cdot [f_{32}^{c1}(t, t_3) + f_{32}^{c2}(T, T_i^g)] \\ \quad + \Delta X_{III,i}^g(t) \cdot [f_{33}^{c1}(t, t_3) + f_{33}^{c2}(T, T_i^g)] + \Delta X_{IV,i}^g(t) \cdot f_{34}^{c2}(T, T_2) = -\delta_{30,i}^g \cdot [1 + \varphi_{c1}(t, t_i^g)] \\ \quad - \Delta X_{I,i}^g(t_i^g) \cdot g_{31}^{c1}(t, t_i^g) - \Delta X_{II,i}^g(t_i^g) \cdot [g_{32}^{c1}(t, t_3) + g_{32}^{c2}(T, T_i^g)] \\ \quad - \Delta X_{III,i}^g(t_i^g) \cdot [g_{33}^{c1}(t, t_3) + g_{33}^{c2}(T, T_i^g)] - \Delta X_{IV,i}^g(t_i^g) \cdot g_{34}^{c2}(T, T_i^g) \\ \Delta X_{II,i}^g(t) \cdot f_{42}^{c2}(T, T_i^g) + \Delta X_{III,i}^g(t) \cdot f_{43}^{c2}(T, T_i^g) + \Delta X_{IV,i}^g(t) \cdot \left[\delta_{44}^{p2} + f_{44}^{c2}(T, T_2)\right] \\ \quad = -\Delta X_{II,i}^g(t_i^g) \cdot g_{42}^{c2}(T, T_i^g) - \Delta X_{III,i}^g(t_i^g) \cdot g_{43}^{c2}(T, T_i^g) - \Delta X_{IV,i}^g(t_i^g) \cdot g_{44}^{c2}(T, T_i^g) \end{cases} \tag{59}$$

Terms $\Delta X_{I,i}^g(t_i^g), \Delta X_{II,i}^g(t_i^g), \Delta X_{III,i}^g(t_i^g)$ and $\Delta X_{IV,i}^g(t_i^g)$ can be easily computed by setting $t = t_i^g$ in Eq. (59).

4. Superposition of the redundant unknowns

According to McHenry's superposition principle [16]:

$$X_n(t) = \Delta X_n^{sh1}(t) + \Delta X_n^{p1}(t) + \Delta X_n^{sh2}(t) + \Delta X_n^{p2}(t) + \sum_i \Delta X_{n,i}^g(t) \; \forall \; t \geqslant t_0 \tag{60}$$

where n = I, II, III or IV and $\Delta X_{n,i}^g(t) \neq 0$ when $t \geqslant t_i^g$.

According to the assumption (Bernoulli–Navier) that plane sections remain plane after deformation, the stresses in the four pieces of the cross-section at time t are:

$$\begin{aligned} &\sigma_{p1}(t) = \frac{X_I(t)}{A_{p1}} \text{ (tension is positive)} \\ &\sigma_{c1}(y_1, t) = \frac{X_I(t) - X_{II}(t) + \sum_i \Delta N_i}{A_{c1}} + \frac{X_I(t) \cdot y_{p1} + X_{II}(t) \cdot y_{1,\sup} + X_{III}(t) - \sum_i \Delta M_i}{J_{c1}} y_1 \text{ (compression is positive)} \\ &\sigma_{c2}(y_1, t) = \frac{X_{II}(t) + X_{IV}(t)}{A_{c2}} + \frac{X_{II}(t) \cdot y_{2,\inf} - X_{III}(t) + X_{IV}(t) \cdot y_{p2}}{J_{c2}} y_2 \text{ (compression is positive)} \\ &\sigma_{p2}(t) = \frac{X_{IV}(t)}{A_{p2}} \text{ (tension is positive)} \end{aligned} \tag{61}$$

where, once again, $\Delta N_i \neq 0$ and $\Delta M_i \neq 0$ only if $t \geqslant t_i^g$.

Verification at the ultimate limit state calls for the pre-strain (i.e. the strain in the tendon when the surrounding concrete is at zero stress, resulting from suitable instantaneous variations of the internal forces), that is:

$$\begin{aligned} \bar{\bar{\varepsilon}}_{p1}(t) &= \frac{\sigma_{p1}(t)}{E_{p1}} + \frac{\sigma_{c1}(y_{p1}, t)}{E_{c1}(t)} \\ \bar{\bar{\varepsilon}}_{p2}(t) &= \frac{\sigma_{p2}(t)}{E_{p2}} + \frac{\sigma_{c2}(y_{p2}, T)}{E_{c2}(T)} \end{aligned} \tag{62}$$

The approach that replaces the non-compatible strains (i.e. $\bar{\varepsilon}_{p1}$ and $\bar{\varepsilon}_{p2}$) with the pre-strains (i.e. $\bar{\bar{\varepsilon}}_{p1}(t)$ and $\bar{\bar{\varepsilon}}_{p2}(t)$) in the analyses made in the service states and at ultimate under both short and long term loads is wrong from a physical point of view. This statement is the consequence of the observation that the pre-strain is the sum of the non-compatible strain of the cable and a compatible strain due to the delayed behaviour of concrete. Nevertheless, the mathematical outcome is exact whatever the load condition, the level of prestressing and the instantaneous constitutive laws of the materials (see [17] for the demonstration).

5. Numerical example

The example refers to a girder bridge, made with precast, pre-tensioned beams and a cast-in-situ top slab (Fig. 3). The top slab reinforcing bars are not taken into account.

The time history of the structure is the following:

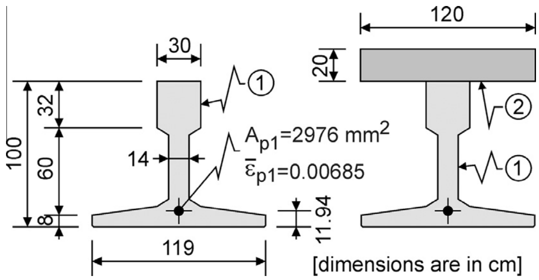

Fig. 3. Girder bridge cross section.

- t_0 = 3 days (end of curing of the girder, onset of shrinkage strains)
- t_1 = 7 days (prestressing of the girder)
- $t_2 = t_3 = t_4$ = 30 days (beginning of shrinkage in the slab; activation of the connection)
- t_5 = 30,000 days (≈82 years)

Note that since the beams are steam-cured immediately after casting to speed up their removal from the prestressing bed, ages t_0 and t_1 are to be regarded not as actual times, but rather as times of normal curing "equivalent" to shorter times of accelerated curing.

The deck is assumed to be simply supported, over a span of 20 m. In the following, reference is made to the mid-span section.

Therefore, the following values of bending moment are assumed in the calculations:

- M_1 = 400 kNm (self-weight of the girder, applied at 7 days)
- M_2 = 300 kNm (self-weight of the slab, applied at 27 days)
- M_3 = 180 kNm (permanent loads, equal to 3.00 kN/m^2, applied at 55 days)

Assuming that the stress applied by the jack in the prestressing bed is equal to 1375 MPa, and allowing for a 3% reduction of the stress because of short-time relaxation in the prestressing bed, the non-compatible strain is equal to $\bar{\varepsilon}_{p1} = 1375 \cdot (1 - 0.03)/195{,}000 \approx 6.84‰$.

The constitutive viscoelastic law used is the one described in [18].

The beams are characterised by an average compressive strength equal to f_{cm} = 48 MPa, while that of the top slab is equal to 33 MPa. The cement is assumed to have a normal hardening (s = 0.25), and the relative humidity, RH, of 50%. Shrinkage of the two pieces of concrete is taken into account in the calculations.

The time evolution of the stresses in the prestressing strands, and at significant points of the beam and slab are shown in Fig. 4. They were computed under three different assumptions, concerning the level of approximation used in modelling the visco-elastic behaviour of the structure:

1. General Method, that implies the numerical solution of Volterra integral equations.

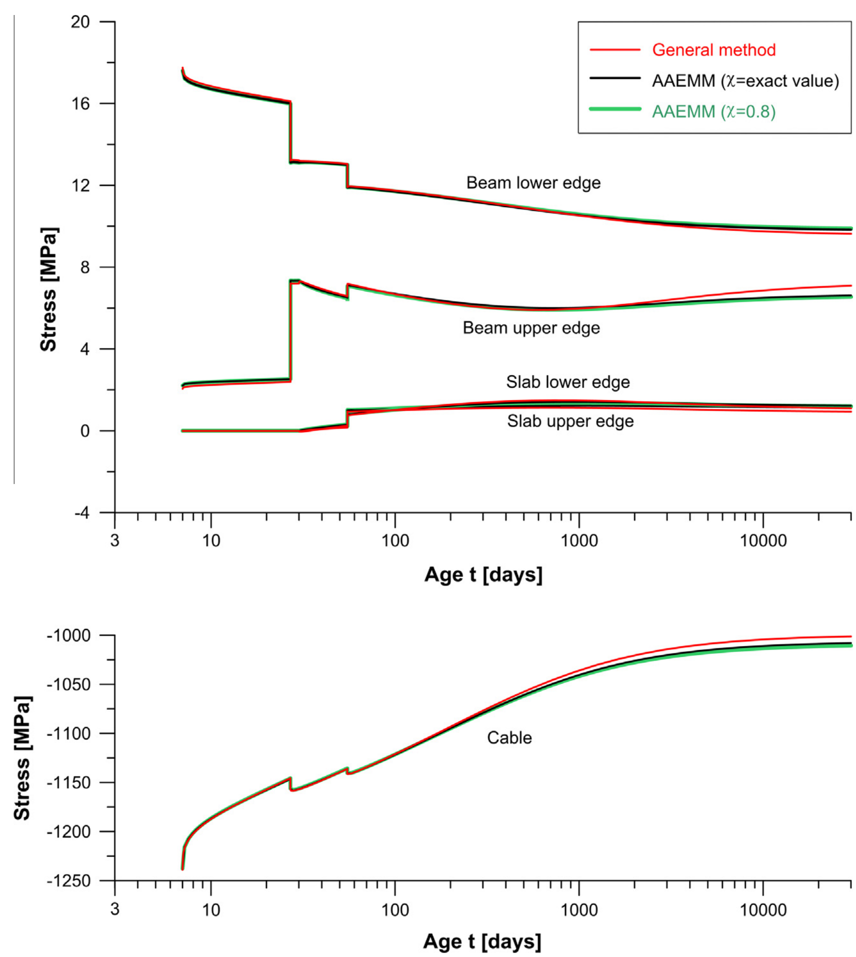

Fig. 4. Girder bridge midspan cross section stress evolution.

2. Age-Adjusted Effective Modulus Method, using the classic approach, where the aging coefficient χ is evaluated on the basis of the compliance and relaxation functions.
3. Age-Adjusted Effective Modulus Method, with the aging coefficient constant and equal to 0.80.

When looking at these diagrams it can be stated that even when dealing with the worst approximation (i.e. aging coefficient equal to 0.80) the solution is accurate, in spite of the complexity of the problem. This outcome suggest the opportunity of verifying the extent of the error made by this method in other cases of practical relevance.

6. Conclusions

The stress–strain time evolution computation problem of concrete compact cross-sections cast and prestressed in consecutive stages under long term loading has been solved by means of an approximate algebraic method.

The method is lengthy, but its formulas are simple and therefore can be easily implemented in a spreadsheet. The general method, presented in a previous paper [7], is complicated and cumbersome as the problem itself is complex. The system of Volterra integral equations, which represents the mathematical synthesis of the physics of this problem, cannot be solved in closed-form because of the complexity of the creep function usually adopted for concrete. Therefore, the solution has to be achieved by means of a refined step-by-step time integration method. The problem of time integration is overcome when dealing with algebraic methods, as long as the aging coefficient, $\chi(t, t_0^*)$, is known or is set by means of a suitable approximate formula.

Note that Eq. (1) still applies when dealing with an elastic material (that is when $\varphi(t, t_0^*) = 0$ and $E(t_0^*) = E$). This approach is therefore able to solve many modern structural problems such as the service behaviour of new structural elements made of a precast, prestressed main load bearing element combined with a cast-in-place concrete cross-section (even if post-tensioned after gaining the final shape), composite steel-concrete sections and reinforced or prestressed concrete elements rehabilitated or strengthened by means of concrete jacketing or steel or carbon fibre reinforced polymer plate bonding. Moreover this method makes it possible to determine the pre-strain of the tendons, and therefore, to take into account creep effects in the cross section analyses at the ultimate limit state of bending and compression of the composite cross section.

In the last decades construction phases, driven by new technologies and by the need to reduce costs, have become increasingly complicated. Approximate but reliable solutions are needed in common practice during the design phase (to speed it up) and during the structural verification phase (to guarantee the reliability of the outputs of more refined solutions, or to predict the long-term behaviour of the structural element if the expanse of a refined solution is not justified by the economic value of the job). From this point of view, algebraic methods are a good solution. It would be anyway interesting to verify the error ensuing from the adoption of approximate formulas for the aging coefficient in selected cases of practical interest. This is the reason why in a following paper the output of a computer program written according to the general method will be compared with the outcomes of the algebraic methods to verify the accuracy of the latter; especially when the aging coefficient $\chi(t, t_0^*)$ is evaluated by means of approximate formulas.

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