STABLE SOLITARY WAVES WITH PRESCRIBED L^2-MASS FOR THE CUBIC SCHRÖDINGER SYSTEM WITH TRAPPING POTENTIALS

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Abstract. For the cubic Schrödinger system with trapping potentials in $\mathbb{R}^N$, $N \leq 3$, or in bounded domains, we investigate the existence and the orbital stability of standing waves having components with prescribed L^2-mass. We provide a variational characterization of such solutions, which gives information on the stability through of a condition of Grillakis-Shatah-Strauss type. As an application, we show existence of conditionally orbitally stable solitary waves when: a) the masses are small, for almost every scattering lengths, and b) in the defocusing, weakly interacting case, for any masses.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \leq 3$, be either the whole space, or a bounded Lipschitz domain, and let us consider two trapping potentials V_1, V_2, satisfying

$$V_i \in \mathcal{C}(\overline{\Omega}), \quad V_i \geq 0, \quad \lim_{|x|\to\infty} V_i(x) = +\infty, \tag{TraPot}$$

for $i = 1, 2$ (the latter holding, of course, only for $\Omega = \mathbb{R}^N$). In this paper we deal with solitary wave solutions to the following system of coupled Gross-Pitaevskii equations:

$$\begin{cases} \mathrm{i}\partial_t \Phi_1 + \Delta\Phi_1 - V_1(x)\Phi_1 + (\mu_1|\Phi_1|^2 + \beta|\Phi_2|^2)\Phi_1 = 0 \\ \mathrm{i}\partial_t \Phi_2 + \Delta\Phi_2 - V_2(x)\Phi_2 + (\mu_2|\Phi_2|^2 + \beta|\Phi_1|^2)\Phi_2 = 0 \\ \text{on } \Omega \times \mathbb{R}, \text{ with zero Dirichlet b.c. if } \Omega \text{ is bounded,} \end{cases} \tag{1.1}$$

aiming at extending to systems part of the results that we obtained in a previous paper concerning the single NLS [26]. Cubic Schrödinger systems like (1.1) appear as a relevant model in different physical contexts, such as nonlinear optics, fluid mechanics and Bose-Einstein condensation (see for instance [10, 31] and the references provided there). Their solutions show different qualitative behaviors depending on the sign of the *scattering lengths* μ_1, μ_2, β: when μ_i is positive (resp. negative), then the corresponding equation is said to be *focusing* (resp. *defocusing*); when β is positive (resp. negative), then the system is said to be *cooperative* (resp. *competitive*). Here we will deal with almost any of these choices, apart from a few degenerate cases. More precisely, we will assume that $(\mu_1, \mu_2, \beta) \in \mathbb{R}^3$ satisfies one of the following conditions:

$$\begin{array}{lll} \mu_1 \cdot \mu_2 < 0 & \text{and} & \beta \in \mathbb{R}; \\ \mu_1, \mu_2 \geq 0, \text{ not both zero,} & \text{and} & \beta \neq -\sqrt{\mu_1\mu_2}; \\ \mu_1, \mu_2 \leq 0, \text{ not both zero,} & \text{and} & \beta \neq \sqrt{\mu_1\mu_2} \end{array} \tag{NonDeg}$$

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(although partial results can be obtained also in certain complementary cases, see some of the remarks along this paper).

We will seek solutions to system (1.1) among functions which belong, at each fixed time, to the energy space

$$\mathcal{H}_{\mathbb{C}} = \left\{ (\Phi_1, \Phi_2) : \Phi_i \in H^1_0(\Omega, \mathbb{C}),\ \int_\Omega (|\nabla \Phi_i|^2 + V_i(x)\Phi_i^2)\,dx < \infty,\ i = 1,2 \right\},$$

endowed with its natural norm

$$\|(\Phi_1, \Phi_2)\|^2_{\mathcal{H}} = \sum_{i=1}^{2} \int_\Omega (|\nabla \Phi_i|^2 + V_i(x)|\Phi_i|^2)\,dx.$$

In such context, the system preserves, at least formally, both the masses

$$\mathcal{Q}(\Phi_i) = \int_\Omega |\Phi_i|^2\,dx, \qquad i = 1,2,$$

and the energy

$$\mathcal{E}(\Phi_1, \Phi_2) = \frac{1}{2}\|(\Phi_1, \Phi_2)\|^2_{\mathcal{H}} - F(\Phi_1, \Phi_2),$$

where, for shorter notation, we let

$$F(\Phi_1, \Phi_2) = \frac{1}{4}\int_\Omega (\mu_1|\Phi_1|^4 + 2\beta|\Phi_1|^2|\Phi_2|^2 + \mu_2|\Phi_2|^4)\,dx.$$

Since we work in dimension $N \leq 3$, we have that the nonlinearity is energy subcritical; furthermore, assumption (TraPot) implies that the embedding

$$\mathcal{H}_{\mathbb{C}} \hookrightarrow L^p(\Omega; \mathbb{C}^2) \text{ is compact}$$

for every $p < 2^* = 2N/(N-2)$ (for every p if $N \leq 2$), and hence, in particular, for $p = 2, 4$. On the other hand, when $N = 2$ the nonlinearity is L^2-critical, while when $N = 3$ it is L^2-supercritical. Indeed, we recall that the L^2-critical exponent is $1+4/N$, so that cubic nonlinearities are L^2-subcritical only in dimension $N = 1$. In general, the behavior of the nonlinearity with respect to the L^2-critical exponent has strong influence on the dynamics, at least in the focusing case (or in the cooperative one), see for instance the book [9].

Letting $\Phi_i(x,t) = e^{\mathrm{i}\omega_i t}U_i(x)$, where $(\omega_1, \omega_2) \in \mathbb{R}^2$ and (U_1, U_2) belongs to

$$\mathcal{H} = \mathcal{H}_{\mathbb{R}} := \left\{ (u_1, u_2) : u_i \in H^1_0(\Omega; \mathbb{R}),\ \int_\Omega (|\nabla u_i|^2 + V_i(x)u_i^2)\,dx < \infty,\ i = 1,2 \right\},$$

we have that solitary waves for (1.1) can be obtained by solving the elliptic system

$$\begin{cases} -\Delta U_1 + (V_1(x) + \omega_1)U_1 = \mu_1 U_1^3 + \beta U_1 U_2^2 \\ -\Delta U_2 + (V_2(x) + \omega_2)U_2 = \mu_2 U_2^3 + \beta U_2 U_1^2 \\ (u_1, u_2) \in \mathcal{H}. \end{cases}$$

In doing this, two different points of view are considered in the literature: on the one hand, one can consider the *chemical potentials* ω_i as given, and search for (U_1, U_2) as critical points of the action functional

$$\mathcal{A}_{(\omega_1,\omega_2)}(U_1, U_2) = \mathcal{E}(U_1, U_2) - \frac{\omega_1}{2}\mathcal{Q}(U_1) - \frac{\omega_2}{2}\mathcal{Q}(U_2);$$

on the other hand, one can take also the coefficients ω_i to be unknown. In this latter situation, it is natural to consider the masses $\mathcal{Q}(U_i)$ as given, so that ω_1, ω_2 can be understood as Lagrange multipliers when searching for critical points of

$$\mathcal{E}(U_1, U_2) \qquad \text{constrained to the manifold } \mathcal{M} := \{(U_1, U_2) : \mathcal{Q}(U_i) = m_i\},$$

$m_1, m_2 > 0$ (for further comments on this alternative, we refer to the discussion in the introduction of [26], and references therein).

Existence issues for the cubic elliptic system above (and for its autonomous counterpart) have attracted, in the last decade, a great interest, and a huge amount of related results is nowadays present in the literature. Most of them are concerned with the case of fixed chemical potentials; as a few example we quote here the papers [18, 20, 3, 7, 31, 19, 35, 6, 12, 24, 36, 11, 27, 32, 34], referring to their bibliography for an extensive list of references on this topic.

On the contrary, in this paper we consider the other point of view: given positive m_1, m_2,

$$\text{to find } (U_1, U_2, \omega_1, \omega_2) \in \mathcal{H} \times \mathbb{R}^2 \text{ s.t.} \begin{cases} -\Delta U_1 + (V_1(x) + \omega_1) U_1 = \mu_1 U_1^3 + \beta U_1 U_2^2 \\ -\Delta U_2 + (V_2(x) + \omega_2) U_2 = \mu_2 U_2^3 + \beta U_2 U_1^2 \\ \int_\Omega U_1^2 = m_1, \ \int_\Omega U_2^2 = m_2. \end{cases} \tag{1.2}$$

Up to our knowledge, only a few papers deal with the fixed masses approach: essentially [10, 25, 33, 29], all of which address the defocusing, competitive case. This case is particularly favorable, since the energy functional $\mathcal{E}$ is coercive (a non coercive case is considered in [16], even though for a quite different Schrödinger system). On the contrary, if at least one of the scattering lengths is positive, then $\mathcal{E}$ is no longer coercive, and the behavior of the nonlinearity with respect to the L^2-critical exponent becomes crucial. Indeed, in the subcritical case (i.e. in dimension $N = 1$), the constrained functional $\mathcal{E}|_{\mathcal{M}}$ is still coercive, and bounded below. But if $N = 2, 3$, then also $\mathcal{E}|_{\mathcal{M}}$ ceases to be coercive, and it becomes not bounded below. This is the main difficulty in searching for critical points of $\mathcal{E}|_{\mathcal{M}}$, indeed no "trivial" local minima for $\mathcal{E}|_{\mathcal{M}}$ can be identified, neither a Nehari-type manifold seems available.

Once solitary waves are obtained, a natural question regards their stability properties. The standard notion of stability, in this framework, is that of orbital stability, which we recall in Section 3 ahead. Orbital stability for power-type Schrödinger systems has been investigated in several papers, among which we mention [28, 21, 23, 22]. It is worth remarking that these papers are settled on the whole $\mathbb{R}^N$, without trapping potentials (i.e. without compact embeddings); for this reason, they are involved only in the L^2-subcritical case, in which the validity of a suitable Gagliardo-Nirenberg inequality can be exploited. We are not aware of any paper treating stability issues for nonlinear Schrödinger systems with L^2-critical or supercritical nonlinearity, except for some partial application in [15].

Our strategy to obtain solutions to problem (1.2) consists in introducing the following auxiliary maximization problem in $\mathcal{H}$:

$$M(\alpha, \rho_1, \rho_2) = \sup \left\{ F(u_1, u_2) : \|(u_1, u_2)\|_{\mathcal{H}}^2 = \alpha, \ \mathcal{Q}(u_1) = \rho_1, \ \mathcal{Q}(u_2) = \rho_2 \right\},$$

where the positive parameters α, ρ_1, ρ_2 are suitably fixed. Since both F and the constraints are even, possible maximum points can be chosen to have non negative components, as we will systematically (and often tacitly) do. As a matter of fact, the problem above leads to a new variational characterization of solutions to (1.2). In turn, such characterization contains information about the orbital stability of the corresponding solitary waves.

Coming to the detailed description of our results, let us recall that the compact embedding $\mathcal{H} \hookrightarrow L^2$ provides the existence of the principal eigenvalues λ_{V_i} of $-\Delta + V_i$, which are positive. Our first result reads as follows.

Theorem 1.1. *Let V_1, V_2 satisfy (TraPot) and μ_1, μ_2, β satisfy (NonDeg). If $\rho_1, \rho_2 > 0$ and $\alpha > \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$ then $M(\alpha, \rho_1, \rho_2)$ is achieved. Besides, for every maximum point (u_1, u_2) there exists $(\omega_1, \omega_2, \gamma) \in \mathbb{R}^3$, with $\gamma > 0$, such that*

$$(\sqrt{\gamma}u_1, \sqrt{\gamma}u_2, \omega_1, \omega_2) \quad \text{solves (1.2) with} \quad m_1 = \gamma\rho_1,\ m_2 = \gamma\rho_2. \tag{1.3}$$

In particular, by the maximum principle, u_1, u_2 can be chosen to be strictly positive in the interior of Ω. The proof of Theorem 1.1 is fully detailed in Section 2, where some further properties of M are also described, such as the continuity with respect to (α, ρ_1, ρ_2). In Section 3 we turn to stability issues in connection with M. We prove the following criterion for stability.

Theorem 1.2. *Under the assumptions of Theorem 1.1, let ρ_1, ρ_2 be fixed and suppose that, for some $\alpha_1 < \alpha_2$, there exists a C^1 curve*

$$(\alpha_1, \alpha_2) \ni \quad \alpha \mapsto (u_1(\alpha), u_2(\alpha), \omega_1(\alpha), \omega_2(\alpha), \gamma(\alpha)) \quad \in \mathcal{H} \times \mathbb{R}^3,$$

such that (1.3) holds, and $(u_1(\alpha), u_2(\alpha))$ achieves $M(\alpha, \rho_1, \rho_2)$ for every $\alpha \in (\alpha_1, \alpha_2)$. If furthermore

$$\alpha \mapsto \gamma(\alpha) \quad \text{is strictly increasing}$$

then the set of solitary wave solutions to (1.1) associated with $M(\alpha, \rho_1, \rho_2)$ (according to Theorem 1.1) is orbitally stable, for every $\alpha \in (\alpha_1, \alpha_2)$, among solutions which enjoy both local existence, uniformly in the $\mathcal{H}$ norm of the initial datum, and conservation of masses and energy.

In particular, if $M(\alpha, \rho_1, \rho_2)$ is uniquely achieved, then the corresponding solitary wave is conditionally stable, in the sense just explained.

The strict monotonicity of a parameter, as a condition for stability, is reminiscent of the abstract theory developed in [14, 15]. In fact, our proof is inspired by the classical paper by Shatah [30]. Observe that we only stated the *conditional* nonlinear orbital stability, where the condition is that the solution of system (1.1) corresponding to an initial datum (ϕ_1, ϕ_2) exists locally in time, with the time interval uniform in $\|(\phi_1, \phi_2)\|_{\mathcal{H}}$, and that the masses and the energy are preserved. In fact, these properties are known to be true for every initial datum in $\mathcal{H}$, at least when some further restrictions about V_i, μ_i, β are assumed, see for instance [9, Chapters 3 and 4]. However, being the field so vast, even a rough summary of well-posedness for Schrödinger systems with potential is far beyond the scopes of this paper. We refer the interested reader to the entry "NLS with potential" in the *DispersiveWiki* project webpage [13] (as well as to the entries "Cubic NLS on $\mathbb{R}^2$", "Cubic NLS on $\mathbb{R}^3$").

Finally, in Section 4 we provide two applications of Theorems 1.1, 1.2, proving, in some particular cases, existence of orbitally stable solitary wave solutions to (1.1) having prescribed masses.

Our first application deals with the case of small masses. In Section 4.1 we prove the following.

Theorem 1.3. *Let assumptions (TraPot), (NonDeg) hold. For every $k \geq 1$ there exists $\bar{m} > 0$, such that for every positive m_1, m_2 satisfying*

$$\frac{1}{k} \leq \frac{m_2}{m_1} \leq k, \qquad m_1 + m_2 \leq \bar{m},$$

there exists $(U_1, U_2, \omega_1, \omega_2) \in \mathcal{H} \times \mathbb{R}^2$, with U_i positive in Ω, solution to (1.2). Furthermore, the corresponding solitary wave

$$(\Phi_1(x,t), \Phi_2(x,t)) = (e^{\mathrm{i}\omega_1 t} U_1(x), e^{\mathrm{i}\omega_2 t} U_2(x))$$

is conditionally orbitally stable for system (1.1), in the sense of Theorem 1.2.

We remark that, apart from condition (NonDeg), no restriction about μ_1, μ_2 and β is required. In order to prove such theorem, we will exploit a parametric version of a classical result by Ambrosetti and Prodi [4] about the inversion of maps with singularities, see Theorem 4.1 below. In particular, we rely on the fact that, if m_1/m_2 is fixed, our problem can be reduced to an inversion of a map near an ordinary singular point, while this property is lost if one of the masses vanish. This is the reason for the restriction on m_1/m_2. On the other hand, when one mass vanishes, the system reduces to a single equation: since we already treated successfully this case in [26], it is presumable that the result should hold without such restriction.

As a last application, in Section 4.2 we deal with the case of defocusing, weakly interacting systems, meaning that μ_1, μ_2 are negative and $\beta^2 < \mu_1\mu_2$. In such case, Theorems 1.1 and 1.2 provide, for every choice of the masses m_1, m_2, the existence of a unique solitary wave, and its stability, see Theorem 4.9 below. As we mentioned, in this case $\mathcal{E}$ is coercive and bounded below, so that existence can be obtained also by the direct method, as already done in [10]. For the same reason, stability is somewhat expected, even though it can not be obtained directly, due to the lack of a suitable Gagliardo-Nirenberg inequality in dimension $N = 2, 3$.

As a final remark, let us mention that in the proofs of Theorems 1.1 and 1.2 we use the compact embedding $\mathcal{H} \hookrightarrow L^p$ just to pass from weak to strong convergence, for maximizing sequences associated to M. In the relevant case $\Omega = \mathbb{R}^N$, $V_i \equiv 1$, such compactness does not hold, but one could try to adapt the same strategy by using a concentration-compactness type argument. In conclusion, it is our belief that Theorems 1.1 and 1.2 should hold in a more general situation, however this falls out of the scopes of the present paper.

Notations and preliminaries. In the following, we will say that a pair (u_1, u_2) is positive (nonnegative) if both u_1 and u_2 are. We remark that, whenever $\mathcal{Q}(u_1)$, $\mathcal{Q}(u_2)$ are fixed to be positive, then both trivial and semitrivial pairs are excluded.

As we already noticed, the embedding $\mathcal{H} \hookrightarrow L^p$ is compact, for p Sobolev subcritical. In turn, the compact embedding implies the existence of a first eigenvalue. In the following we denote by φ_{V_i} the unique nonnegative function which achieves

$$\lambda_{V_i} = \inf \left\{ \int_\Omega \left(|\nabla \varphi|^2 + V_i(x)\varphi^2 \right) dx : \int_\Omega \varphi^2 \, dx = 1 \right\}.$$

We remark that $\lambda_{V_i} > 0$ by assumption (TraPot) (in fact, the positivity assumption there may be replaced by the requirement that V_i is bounded from below, by performing a change of gauge $\Phi_i \rightsquigarrow \Phi_i \exp[\mathrm{i}t \inf V_i]$). In such arguments, the compactness of the embedding is immediate if Ω is bounded; in case $\Omega = \mathbb{R}^N$, it can be obtained in a rather standard way, for instance mimicking the proof of [17, Proposition 6.1], which is performed in the particular case $V_i(x) = |x|^2$.

Throughout the paper, "i" indicates the imaginary unit, while i and j stand for indexes between 1 and 2, with $j \neq i$. Finally, we denote with C any positive constant we need not to specify, which may change its value even within the same expression.

2. A variational problem

Throughout this section, μ_1, μ_2, β satisfy assumption (NonDeg) while V_1, V_2 satisfy assumption (TraPot). For $(u_1, u_2) \in \mathcal{H}$, recall that

$$F(u_1, u_2) = \int_\Omega \left(\mu_1 \frac{u_1^4}{4} + \beta \frac{u_1^2 u_2^2}{2} + \mu_2 \frac{u_2^4}{4} \right) dx.$$

We consider the following maximization problem

$$M(\alpha, \rho_1, \rho_2) = \sup_{\mathcal{U}(\alpha,\rho_1,\rho_2)} F(u_1, u_2) \tag{2.1}$$

where, for $\rho_1, \rho_2 > 0$ and $\alpha \geq \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$, we define

$$\mathcal{U}(\alpha, \rho_1, \rho_2) = \left\{ (u_1, u_2) \in \mathcal{H} : \begin{array}{l} \|(u_1, u_2)\|^2_{\mathcal{H}} \leq \alpha, \\ \int_\Omega u_i^2 \, dx = \rho_i, \ i = 1, 2 \end{array} \right\}.$$

As we will see in a moment, under assumption (NonDeg), this definition of M is equivalent to the one given in the introduction.

Remark 2.1. For $\alpha = \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$ we have that

$$\mathcal{U}(\alpha, \rho_1, \rho_2) = \left\{ ((-1)^l \sqrt{\rho_1} \varphi_{V_1}, (-1)^m \sqrt{\rho_2} \varphi_{V_2}) : (l, m) \in \{0, 1\}^2 \right\},$$

thus F is constant in $\mathcal{U}$ and M is trivially achieved. Of course, if $\alpha < \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$ then $\mathcal{U}$ is empty.

Lemma 2.2. *For every $\alpha > \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$, the set*

$$\tilde{\mathcal{U}}(\alpha, \rho_1, \rho_2) = \left\{ (u_1, u_2) \in \mathcal{U}(\alpha, \rho_1, \rho_2) : \begin{array}{l} \|(u_1, u_2)\|^2_{\mathcal{H}} = \alpha, \\ \int_\Omega u_i \varphi_{V_i} \, dx \neq 0 \ i = 1, 2 \end{array} \right\}$$

is a submanifold of $\mathcal{H}$ of codimension 3.

Proof. It is easy to see that $\tilde{\mathcal{U}}$ is not empty. Letting

$$G(u_1, u_2) = \left(\int_\Omega u_1^2 \, dx - \rho_1, \int_\Omega u_2^2 \, dx - \rho_2, \|(u_1, u_2)\|^2_{\mathcal{H}} - \alpha \right),$$

it suffices to prove that for every $u \in \tilde{\mathcal{U}}(\alpha, \rho_1, \rho_2)$ the range of $G'(u_1, u_2)$ is $\mathbb{R}^3$. This can be checked by evaluating $G'(u_1, u_2)[\phi_1, \phi_2]$ with (ϕ_1, ϕ_2) equal to (u_1, u_2), $(\varphi_{V_1}, 0)$ and $(0, \varphi_{V_2})$ respectively, and recalling that $\alpha \neq \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$. □

Lemma 2.3. *For every $\alpha \geq \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$ (2.1) is achieved. Moreover, every maximum (u_1, u_2) belongs to $\tilde{\mathcal{U}}(\alpha, \rho_1, \rho_2)$, and there exist $\omega_1, \omega_2, \gamma \in \mathbb{R}$ such that*

$$-\Delta u_i + (V_i(x) + \omega_i) u_i = \gamma(\mu_i u_i^3 + \beta u_i u_j^2), \quad i = 1, 2, \ j \neq i. \tag{2.2}$$

Proof. If $\alpha = \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$ then by Remark 2.1 the result immediately follows by choosing $\omega_i = -\lambda_{V_i}$, $\gamma = 0$.

Otherwise, it is not difficult to see that $M(\alpha, \rho_1, \rho_2)$ is achieved by a couple $(u_1, u_2) \in \mathcal{U}(\alpha, \rho_1, \rho_2)$. Indeed, $\mathcal{U}(\alpha, \rho_1, \rho_2)$ is not empty and weakly compact in $\mathcal{H}$, $F(u_1, u_2)$ is weakly continuous and bounded in $\mathcal{U}(\alpha, \rho_1, \rho_2)$:

$$|F(u_1, u_2)| \leq C(|\mu_1| + |\mu_2| + |\beta|)\alpha^2.$$

By possibly taking $|u_i|$ we can suppose $u_i \geq 0$.

Suppose in view of a contradiction that the maximizer does not belong to $\tilde{\mathcal{U}}(\alpha, \rho_1, \rho_2)$, i.e. $\|(u_1, u_2)\|^2_{\mathcal{H}} < \alpha$. Then there exist two Lagrange multipliers ω_1, ω_2 such that almost everywhere we have

$$\mu_1 u_1^3 + \beta u_1 u_2^2 = \omega_1 u_1 \quad \text{and} \quad \beta u_1^2 u_2 + \mu_2 u_2^3 = \omega_2 u_2. \tag{2.3}$$

a) If $\beta^2 \neq \mu_1\mu_2$: this implies that the u_i are piecewise constant; since $u_i \in H^1_0(\Omega)$, $u_i \not\equiv 0$, we have reached a contradiction.
b) The remaining cases $\mu_1, \mu_2 > 0$, $\beta = \sqrt{\mu_1\mu_2}$ and $\mu_1, \mu_2 < 0$, $\beta = -\sqrt{\mu_1\mu_2}$ are much more delicate, and we will analyze them in detail during the remainder of the proof. First of all, we claim that $\omega_1 = \omega_2 = 0$. To start with, suppose that Ω is bounded, Consider the extension of u_i to the whole $\mathbb{R}^N$ by 0, denoting it also by u_i. With this notation, $u_i \in H^1(\mathbb{R}^N)$, hence by [37, Remark 3.3.5] we have that each u_i is *approximately continuous*, this meaning that for $\mathcal{H}^{N-1}$–a.e. $x_0 \in \mathbb{R}^N$ there exists a measurable set $A^i_{x_0}$ such that

$$\lim_{r\to 0} \frac{|A^i_{x_0} \cap B_r(x_0)|}{|B_r(x_0)|} = 1, \qquad u_i|_{A^i_{x_0}} \text{ is continuous at } x_0. \tag{2.4}$$

Observe that clearly $|A^1_{x_0} \cap A^2_{x_0} \cap B_r(x_0)|/|B_r(x_0)| \to 1$ as well. Thus, as Ω is Lipschitz (and hence $\mathcal{H}^{N-1}(\partial\Omega) > 0$) and $u_i = 0$ on $\partial\Omega$, there exist $x_0 \in \overline{\Omega}$ with $u_1(x_0) = u_2(x_0) = 0$, $A^i_{x_0}$ satisfying (2.4), and $x_n \in A^1_{x_0} \cap A^2_{x_0} \cap \Omega$ converging to x_0, such that either $u_1(x_n) \neq 0$ or $u_2(x_n) \neq 0$.

- If $u_1(x_n), u_2(x_n) \neq 0$, then (2.3) implies that
$$\mu_1 u_1^2(x_n) + \beta u_2^2(x_n) = \omega_1, \qquad \mu_2 u_2^2(x_n) + \beta u_1^2(x_n) = \omega_2,$$
and thus (by making $n \to \infty$) we have $\omega_1 = \omega_2 = 0$.
- If $u_1(x_n) \neq 0$ and $u_2(x_n) = 0$, then from (2.3) we have that $u_1^2(x_n) = \omega_1/\mu_1$, and thus $\omega_1 = 0$, a contradiction. Reasoning in an analogous way, the case $u_1(x_n) = 0$ and $u_2(x_n) \neq 0$ also leads to a contradiction.

Thus we have proved that $\omega_1 = \omega_2 = 0$ in the case Ω is bounded. If $\Omega = \mathbb{R}^N$ we can reason in a similar way. By (TraPot) we have that every $(u_1, u_2) \in \mathcal{H}$ satisfies

$$\lim_{R\to\infty} \int_{\mathbb{R}^N \setminus B_R} u_i^2 \, dx = 0, \qquad i = 1, 2. \tag{2.5}$$

Hence for every $\varepsilon > 0$ there exist x_ε with $0 < u_1(x_\varepsilon)^2 + u_2(x_\varepsilon)^2 \leq \varepsilon$ and $A^1_{x_\varepsilon}, A^2_{x_\varepsilon}$ satisfying (2.4). Proceeding as above, since ε is arbitrary, we obtain $\omega_1 = \omega_2 = 0$. Therefore we have proved that (2.3) writes as

$$\mu_i u_i^2 + \beta u_j^2 = 0 \quad i, j = 1, 2, \ j \neq i, \tag{2.6}$$

a.e. in Ω. This, in turn, implies $\mu_i\rho_i + \beta\rho_j = 0$, which provides a contradiction also in case b).

In conclusion, we have shown that the maximizer (u_1, u_2) belongs to $\tilde{\mathcal{U}}_\alpha$. By Lemma 2.2 the Lagrange multipliers theorem applies. Since we have shown in addition that (u_1, u_2) can not satisfy (2.3), we conclude that it satisfies (2.2). $\square$

Lemma 2.4. *Given $\alpha > \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$, let $(u_1, u_2) \in \tilde{\mathcal{U}}(\alpha, \rho_1, \rho_2)$ achieve (2.1). Then in (2.2) we have $\gamma > 0$.*

Proof. We proceed similarly to [26, Prop. 2.4]. For $i = 1, 2$ and $t \in \mathbb{R}$ close to 1, let

$$w_i(t) = tu_i + s_i(t)\sqrt{\rho_i}\varphi_{V_i},$$

where $s_i(t)$ are such that

$$\rho_i = \int_\Omega w_i(t)^2\,dx = t^2\rho_i + 2ts_i(t)\sqrt{\rho_i}\int_\Omega u_i\varphi_{V_i}\,dx + s_i(t)^2\rho_i, \quad s_i(1) = 0. \tag{2.7}$$

Since

$$\partial_{s_i}\left(t^2\rho_i + 2ts_i\sqrt{\rho_i}\int_\Omega u_i\varphi_{V_i}\,dx + s_i^2\rho_i\right)\bigg|_{(t,s)=(1,0)} = 2\sqrt{\rho_i}\int_\Omega u_i\varphi_{V_i}\,dx \neq 0,$$

the Implicit Function Theorem applies, providing that the maps $t \mapsto w_i(t)$ are of class C^1 in a neighborhood of $t = 1$. The first relation in (2.7) provides

$$0 = \int_\Omega u_i w_i'(1)\,dx = \rho_i + s_i'(1)\sqrt{\rho_i}\int_\Omega u_i\varphi_{V_i}\,dx.$$

Therefore $s_i'(1) = -\sqrt{\rho_i}/\int_\Omega u_i\varphi_{V_i}\,dx$ and $w_i'(1) = u_i - (\rho_i/\int_\Omega u_i\varphi_{V_i}\,dx)\varphi_{V_i}$. We use the last estimates to compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|\nabla w_i(t)|^2 + V_i(x)w_i(t)^2\right)\,dx\bigg|_{t=1} &= \int_\Omega \left(\nabla u_i\cdot\nabla w_i'(1) + V_i(x)u_iw_i'(1)\right)\,dx\\ &= \int_\Omega \left(|\nabla u_i|^2 + V_i(x)u_i^2\right)\,dx - \rho_i\lambda_{V_i},\end{aligned}$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|(w_1(t), w_2(t))\|^2_{\mathcal{H}}\bigg|_{t=1} = \alpha - (\lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2) > 0. \tag{2.8}$$

Thus there exists $\varepsilon > 0$ such that $(w_1(t), w_2(t)) \in \mathcal{U}(\alpha, \rho_1, \rho_2)$ for $t \in (1-\varepsilon, 1]$. Since $(w_1(1), w_2(1)) = (u_1, u_2)$ achieves the maximum of F in $\mathcal{U}(\alpha, \rho_1, \rho_2)$, we deduce

$$\frac{d}{dt}F(w_1(t), w_2(t))\bigg|_{t=1} \geq 0. \tag{2.9}$$

On the other hand, using (2.2) and the fact that $\int_\Omega u_i w_i'(1)\,dx = 0$, we have

$$\begin{aligned}&\gamma\frac{d}{dt}F(w_1(t), w_2(t))\bigg|_{t=1}\\ &= \int_\Omega \left[(-\Delta u_1 + V_1(x)u_1)w_1'(1) + (-\Delta u_2 + V_2(x)u_2)w_2'(1)\right]\,dx\\ &= \|(w_1(t), w_2(t))\|^2_{\mathcal{H}}\big|_{t=1}.\end{aligned}$$

By comparing the last relation with (2.8) and (2.9) we obtain the statement. $\square$

We are ready to prove our first main result.

Proof of Theorem 1.1. By Lemma 2.3, for any $(u_1, u_2) \in \arg\max M(\alpha, u_1, u_2)$ (which is not empty) there exists $(\omega_1, \omega_2, \gamma) \in \mathbb{R}^3$, such that (2.2) holds. Moreover, since by assumption $\alpha > \lambda_{V_1}\rho_1 + \lambda_{V_2}\rho_2$, Lemma 2.4 implies that $\gamma > 0$. The only thing that remains to prove is that (1.3) holds. This is a direct consequence of (2.2) since, setting $U_i = \sqrt{\gamma}u_i$, we obtain

$$\begin{aligned} -\Delta U_i + (V_i(x) + \omega_i)U_i &= \gamma^{1/2}\left(-\Delta u_i + (V_i(x) + \omega_i)u_i\right) \\ &= \gamma^{3/2}(\mu_i u_i^3 + \beta u_i u_j^2) = \mu_i U_i^3 + \beta U_i U_j^2. \quad \square \end{aligned}$$

In the remainder of this section we will prove some properties of M and of system (2.2) which we will use later on. A remarkable property is that M is a continuous function.

Lemma 2.5. *Let $(\alpha_n, \rho_{1,n}, \rho_{2,n}) \to (\bar\alpha, \bar\rho_1, \bar\rho_2)$, with $\alpha_n \geq \lambda_{V_1}\rho_{1,n} + \lambda_{V_2}\rho_{2,n}$. Then*

$$M(\alpha_n, \rho_{1,n}, \rho_{2,n}) \to M(\bar\alpha, \bar\rho_1, \bar\rho_2).$$

Proof. a) $\underline{\limsup M(\alpha_n, \rho_{1,n}, \rho_{2,n}) \leq M(\bar\alpha, \bar\rho_1, \bar\rho_2)}$. Indeed, let $(u_{1,n}, u_{2,n}) \in \tilde{\mathcal{U}}(\alpha_n, \rho_{1,n}, \rho_{2,n})$ achieve $M(\alpha_n, \rho_{1,n}, \rho_{2,n})$. Since α_n is bounded, we deduce that $(u_{1,n}, u_{2,n})$ converges (up to subsequences) weakly in $\mathcal{H}$ to some (u_1^*, u_2^*). By the compact embedding, $(u_1^*, u_2^*) \in \mathcal{U}(\bar\alpha, \bar\rho_1, \bar\rho_2)$ and

$$M(\alpha_n, \rho_{1,n}, \rho_{2,n}) \to F(u_1^*, u_2^*) \leq M(\bar\alpha, \bar\rho_1, \bar\rho_2).$$

b) $\underline{\liminf M(\alpha_n, \rho_{1,n}, \rho_{2,n}) \geq M(\bar\alpha, \bar\rho_1, \bar\rho_2)}$. We assume $\bar\alpha > \lambda_{V_1}\bar\rho_1 + \lambda_{V_2}\bar\rho_2$, the complementary case being an easy consequence of Remark 2.1. Let $(\bar u_1, \bar u_2) \in \tilde{\mathcal{U}}(\bar\alpha, \bar\rho_1, \bar\rho_2)$, with non negative components, achieve $M(\bar\alpha, \bar\rho_1, \bar\rho_2)$. To conclude, we will construct a sequence $(w_{1,n}, w_{2,n}) \in \tilde{\mathcal{U}}(\alpha_n, \rho_{1,n}, \rho_{2,n})$ in such a way that $(w_{1,n}, w_{2,n}) \to (\bar u_1, \bar u_2)$, strongly in $\mathcal{H}$. Indeed, this would imply

$$M(\alpha_n, \rho_{1,n}, \rho_{2,n}) \geq F(w_{1,n}, w_{2,n}) \to F(\bar u_1, \bar u_2) = M(\bar\alpha, \bar\rho_1, \bar\rho_2).$$

Since $\bar\alpha > \lambda_{V_1}\bar\rho_1 + \lambda_{V_2}\bar\rho_2$, we can assume without loss of generality that

$$\int_\Omega (|\nabla \bar u_1|^2 + V_1(x)\bar u_1^2)\,dx > \lambda_{V_1}\bar\rho_1. \tag{2.10}$$

Taking

$$w_1(a, b) = (1 + a)\bar u_1 + b\varphi_{V_1}, \qquad w_2(c) = (1 + c)\bar u_2$$

our task is reduced to apply the Inverse Function Theorem to the map

$$f(a, b, c) = \left(\|(w_1(a, b), w_2(c))\|^2_{\mathcal{H}}, \|w_1(a, b)\|^2_{L^2}, \|w_2(c)\|^2_{L^2}\right)$$

near $f(0, 0, 0) = (\bar\alpha, \bar\rho_1, \bar\rho_2)$. A direct calculation yields

$$\begin{aligned} \det f'(0,0,0) = 8\int_\Omega \bar u_2^2\,dx \bigg[&\int_\Omega (|\nabla \bar u_1|^2 + V_1(x)\bar u_1^2)\,dx \cdot \int_\Omega \bar u_1\varphi_{V_1}\,dx \\ &- \int_\Omega (\nabla \bar u_1 \cdot \nabla\varphi_{V_1} + V_1(x)\bar u_1\varphi_{V_1})\,dx \cdot \int_\Omega \bar u_1^2\,dx\bigg] \\ &= 8\bar\rho_2 \int_\Omega \bar u_1\varphi_{V_1}\,dx \left[\int_\Omega (|\nabla \bar u_1|^2 + V_1(x)\bar u_1^2)\,dx - \lambda_{V_1}\bar\rho_1\right], \end{aligned}$$

which is positive by (2.10). $\square$

Corollary 2.6. *Let* $(u_{1,n}, u_{2,n}) \in \tilde{\mathcal{U}}(\alpha_n, \rho_{1,n}, \rho_{2,n})$ *achieve* $M(\alpha_n, \rho_{1,n}, \rho_{2,n})$*. If* $(\alpha_n, \rho_{1,n}, \rho_{2,n}) \to (\bar\alpha, \bar\rho_1, \bar\rho_2)$ *then, up to subsequences,*

$$(u_{1,n}, u_{2,n}) \to (\bar u_1, \bar u_2) \qquad \textit{achieving } M(\bar\alpha, \bar\rho_1, \bar\rho_2),$$

the convergence being strong in $\mathcal{H}$*. Indeed, once weak convergence to a maximizer has been obtained, then Lemma 2.3 implies that* $\|(\bar u_1, \bar u_2)\|^2_{\mathcal{H}} = \bar\alpha = \lim_n \alpha_n = \lim_n \|(u_{1,n}, u_{2,n})\|^2_{\mathcal{H}}$.

As one may suspect, the convergence of the maxima and that of the maximizers implies the one of the Lagrange multipliers appearing in (2.2). As a matter of fact, this holds even in more general situations, as we show in the following lemma.

Lemma 2.7. *Take a sequence* $(u_{1,n}, u_{2,n}, \omega_{1,n}, \omega_{2,n}, \gamma_n)$ *such that*

$$\begin{cases} -\Delta u_{1,n} + (V_1(x) + \omega_{1,n})u_{1,n} = \gamma_n(\mu_1 u_{1,n}^3 + \beta u_{1,n} u_{2,n}^2) \\ -\Delta u_{2,n} + (V_2(x) + \omega_{2,n})u_{2,n} = \gamma_n(\mu_2 u_{2,n}^3 + \beta u_{2,n} u_{1,n}^2) \\ \int_\Omega u_{1,n}^2 \, dx = \rho_{1,n}, \quad \int_\Omega u_{2,n} \, dx = \rho_{2,n}, \end{cases}$$

and assume that

$$\rho_{1,n},\ \rho_{2,n} \quad \textit{and} \quad \|(u_{1,n}, u_{2,n})\|^2_{\mathcal{H}} =: \alpha_n \qquad \textit{are bounded}$$

both from above, and from below, away from zero. Then the sequences $\omega_{1,n}$*,* $\omega_{2,n}$*,* γ_n *are bounded.*

Proof. Take u_i such that $u_{i,n} \rightharpoonup u_i$ weakly in $\mathcal{H}$, strongly in $L^p(\Omega)$, $1 < p < 2^*$, and let $\int_\Omega u_i^2 \, dx =: \rho_i$.
a) $\omega_{i,n}$ are bounded. Suppose, in view of a contradiction, that $|\omega_{1,n}| \to \infty$. By multiplying the equation for $u_{1,n}$ by $u_{1,n}$ itself and dividing the result by $\omega_{1,n}$, we obtain

$$\frac{1}{\omega_{1,n}} \int_\Omega (|\nabla u_{1,n}|^2 + V_1(x) u_{1,n}^2) \, dx + \rho_{1,n} = \frac{\gamma_n}{\omega_{1,n}} \int_\Omega (\mu_1 u_{1,n}^4 + \beta u_{1,n}^2 u_{2,n}^2) \, dx.$$

As α_n is bounded, by taking the limit in n, it holds

$$\rho_1 = A \int_\Omega (\mu_1 u_1^4 + \beta u_1^2 u_2^2) \, dx$$

where $\lim_n \frac{\gamma_n}{\omega_{1,n}} =: A \neq 0$ (which also implies that $\gamma_n \to +\infty$). Going back to the first equation, multiplying it by an arbitrary test function ϕ, dividing the result by $\omega_{1,n}$, and passing to the limit, we see that

$$\int_\Omega u_1 \phi \, dx = A \int_\Omega (\mu_1 u_1^3 + \beta u_1 u_2^2) \phi \, dx,$$

and hence, since $u_1 > 0$ in Ω (by the maximum principle) we have the pointwise identity

$$1 = A(\mu_1 u_1^2 + \beta u_2^2).$$

As the trace of u_1 and u_2 is zero on $\partial\Omega$, we obtain a contradiction and thus $\omega_{1,n}$ is a bounded sequence. The case $\omega_{2,n}$ unbounded can be ruled out in an analogous way.

b) $\underline{\gamma_n \text{ is bounded.}}$ Assume by contradiction that $\gamma_n \to +\infty$. Multiplying the i–th equation by any test function ϕ, integrating by parts, dividing the result by γ_n and passing to the limit, at the end we deduce that

$$\mu_1 u_1^2 + \beta u_2^2 = 0, \qquad \mu_2 u_2^2 + \beta u_1^2 = 0.$$

Furthermore, the integration of these two equations yields the identities

$$\mu_1 \rho_1 + \beta \rho_2 = 0, \qquad \mu_2 \rho_2 + \beta \rho_1 = 0.$$

This clearly is a contradiction if (μ_1, μ_2, β) satisfies (NonDeg). □

To conclude this section, we give some hint of the kind of problems which arise in case assumption (NonDeg) does not hold.

Remark 2.8. When (NonDeg) does not hold there are specific conditions about ρ_1, ρ_2 which allow to develop the above theory in some cases. On the other hand, in general, degenerate situations may appear.

For instance, if $\mu_1, \mu_2 < 0$ and $\beta = \sqrt{\mu_1 \mu_2}$, then

$$F(u_1, u_2) = -\frac{|\mu_1|}{4} \int_\Omega \left(u_1^2 - \frac{\sqrt{|\mu_2|}}{\sqrt{|\mu_1|}} u_2^2 \right)^2 dx \leq 0;$$

if furthermore $\sqrt{|\mu_1|}\rho_1 = \sqrt{|\mu_2|}\rho_2$, then

$$F(\rho_1 \psi, \rho_2 \psi) = 0 \qquad \text{for every } \psi.$$

Choosing ψ as the eigenfunction achieving

$$\hat{\alpha} = \inf \left\{ \int_\Omega \left(|\nabla \psi|^2 + \frac{\rho_1 V_1(x) + \rho_2 V_2(x)}{\rho_1 + \rho_2} \psi^2 \right) dx : \int_\Omega \psi^2 \, dx = 1 \right\},$$

then $M_\alpha = 0$ is attained by $(\sqrt{\rho_1}\psi, \sqrt{\rho_2}\psi)$ for every $\alpha \geq \hat{\alpha}(\rho_1 + \rho_2)$, but it belongs to $\tilde{\mathcal{U}}_\alpha$ only for $\alpha = \hat{\alpha}$. Moreover, if $V_1 = V_2 = V$, then $\psi = \varphi_V$, and $(\sqrt{\rho_1}\varphi_V, \sqrt{\rho_2}\varphi_V, -\lambda_V, -\lambda_V, \gamma)$ is a solution of (2.2) for *every* $\gamma > 0$.

3. A general stability result

Let us fix $(\alpha^*, \rho_1^*, \rho_2^*)$ such that Theorem 1.1 holds. In this section we will show that, if for α near α^* the maximum points corresponding to $M(\alpha, \rho_1^*, \rho_2^*)$ are along a smooth curve, with the multiplier γ increasing with respect to α, then the corresponding solitary waves are conditionally orbitally stable for an associated Schrödinger system. As a byproduct, we will obtain the proof of Theorem 1.2.

To be precise, let us consider the following conditions:

(M1) $M(\alpha^*, \rho_1^*, \rho_2^*)$ is achieved by a unique positive pair (u_1^*, u_2^*).

(M2) There exists an interval (α_1, α_2) containing α^* and a C^1 curve

$$(\alpha_1, \alpha_2) \to \mathcal{H} \times \mathbb{R}^3, \qquad \alpha \mapsto (u_1(\alpha), u_2(\alpha), \omega_1(\alpha), \omega_2(\alpha), \gamma(\alpha))$$

such that $(u_1(\alpha^*), u_2(\alpha^*)) = (u_1^*, u_2^*)$ and

$$\begin{cases} (u_1(\alpha), u_2(\alpha)) \text{ achieves } M(\alpha, \rho_1^*, \rho_2^*), \\ -\Delta u_i + (V_i(x) + \omega_i) u_i = \gamma(\mu_i u_i^3 + \beta u_i u_j^2) \qquad i = 1, 2, \ j \neq i, \end{cases}$$

for every $\alpha \in (\alpha_1, \alpha_2)$ (recall Lemma 2.3).

(M3) The map

$$(\alpha_1,\alpha_2)\to\mathbb{R},\qquad \alpha\mapsto\gamma(\alpha)$$

is strictly increasing.

For easier notation, let us write $\omega_i^*=\omega_i(\alpha^*)$, $\gamma^*=\gamma(\alpha^*)$. Take the NLS system:

$$\begin{cases}\mathrm{i}\partial_t\Psi_i+\Delta\Psi_i-V_i(x)\Psi_i+\gamma^*(\mu_i|\Psi_i|^2+\beta|\Psi_j|^2)\Psi_i=0,\\ \Psi_i(0)=\psi_i,\ i=1,2,\qquad (\psi_1,\psi_2)\in\mathcal{H}_{\mathbb{C}}.\end{cases}\tag{3.1}$$

Associated to this system, we have the energy

$$\mathcal{E}_{\gamma^*}(\Psi_1,\Psi_2)=\frac{1}{2}\|(\Psi_1,\Psi_2)\|^2_{\mathcal{H}}-\gamma^*F(\Psi_1,\Psi_2)$$

and the masses $\mathcal{Q}(\Psi_i)=\int_\Omega|\Psi_i|^2\,dx$, $i=1,2$. For (3.1), we assume the following local well posedness property.

(LWP) We have local existence for (3.1), locally in time and uniformly in $\|(\psi_1,\psi_2)\|_{\mathcal{H}}$. Moreover, the energy and the masses are conserved along trajectories, that is

$$\mathcal{E}_{\gamma^*}(\Psi_1(t),\Psi_2(t))=\mathcal{E}_{\gamma^*}(\psi_1,\psi_2)\quad\text{ and }\quad\mathcal{Q}(\Psi_i(t))=\mathcal{Q}(\psi_i)\ \text{ for } i=1,2$$

for every existence time.

Let us recall the notion of orbital stability for the NLS system.

Definition 3.1. A standing wave solution $(e^{\mathrm{i}t\omega_1}u_1,e^{\mathrm{i}t\omega_2}u_2)$ is called *orbitally stable* for (3.1) if for each $\varepsilon>0$ there exists $\delta>0$ such that, whenever $(\psi_1,\psi_2)\in\mathcal{H}_{\mathbb{C}}$ satisfies $\|(\psi_1,\psi_2)-(u_1,u_2)\|_{\mathcal{H}}<\delta$ and $(\Psi_1(t,x),\Psi_2(t,x))$ solves (3.1) in some interval $[0,T_0)$, then

$$(\Psi_1(t),\Psi_2(t))\text{ can be continued to a solution in } 0\le t<\infty,\tag{3.2}$$

and

$$\sup_{0\le t<\infty}\inf_{s_1,s_2\in\mathbb{R}}\|(\Psi_1(t),\Psi_2(t))-(e^{\mathrm{i}s_1}u_1,e^{\mathrm{i}s_2}u_2)\|_{\mathcal{H}}<\varepsilon.\tag{3.3}$$

The purpose and main result of this section is to prove the following stability criterion.

Theorem 3.2. *Let μ_1,μ_2,β satisfy assumption (NonDeg) and V_1,V_2 satisfy assumption (TraPot). Under condition (LWP), take $(\alpha^*,\rho_1^*,\rho_2^*)$ for which (M1)–(M3) hold. Then*

$$(e^{\mathrm{i}t\omega_1^*}u_1^*,e^{\mathrm{i}t\omega_2^*}u_2^*)\quad\textit{is orbitally stable for (3.1).}$$

From now on we will work under the assumptions of Theorem 3.2. As we mentioned in the introduction, the proof is inspired by [30].

Let us first check the following consequence of the uniqueness property (M1).

Lemma 3.3. *Given $\alpha,\rho_1,\rho_2>0$, we have*

$$M(\alpha,\rho_1,\rho_2)=\sup\left\{F(w_1,w_2):\ (w_1,w_2)\in\mathcal{H}_{\mathbb{C}},\ (|w_1|,|w_2|)\in\mathcal{U}(\alpha,\rho_1,\rho_2)\right\}.\tag{3.4}$$

Moreover, if $(w_1,w_2)\in\mathcal{H}_{\mathbb{C}}$ achieves $M(\alpha^*,\rho_1^*,\rho_2^*)$, then

$$w_1=e^{\mathrm{i}s_1}u_1^*,\qquad w_2=e^{\mathrm{i}s_2}u_2^*$$

for some $s_1,s_2\in\mathbb{R}$.

Proof. Denote by $\widetilde{M}(\alpha,\rho_1,\rho_2)$ the right hand side of (3.4); clearly, $M(\alpha,\rho_1,\rho_2)\le \widetilde{M}(\alpha,\rho_1,\rho_2)$. On the other hand, given any $(w_1,w_2)\in\mathcal{H}_{\mathbb{C}}$ satisfying

$$\|(w_1,w_2)\|^2_{\mathcal{H}}\le\alpha,\qquad \int_\Omega |w_i|^2\,dx=\rho_i,$$

by the diamagnetic inequality[1] it is clear that $(|w_1|,|w_2|)\in\mathcal{U}(\alpha,\rho_1,\rho_2)$ with $F(|w_1|,|w_2|)=F(w_1,w_2)$. Thus equality (3.4) holds.

Let us now check the second statement of the lemma. Take $(w_1,w_2)\in\mathcal{H}_{\mathbb{C}}$ achieving $M(\alpha^*,\rho_1^*,\rho_2^*)$. By the considerations of the previous paragraph, we have that also $(|w_1|,|w_2|)$ achieves $M(\alpha^*,\rho_1^*,\rho_2^*)$, and in particular (cf. Lemma 2.3)

$$\int_\Omega |\nabla|w_i||^2\,dx=\int_\Omega|\nabla w_i|^2\,dx(=\alpha^*).$$

Thus there exists $(u_1,u_2)\in\mathcal{H}$ (real valued) and $k_i\in\mathbb{R}$ such that

$$w_i=u_i+\mathrm{i}k_iu_i=(1+k_i\mathrm{i})u_i=r_ie^{\mathrm{i}s_i}u_i$$

for some $r_1,r_2>0$, $s_1,s_2\in\mathbb{R}$. By (M1), we have that $(|w_1|,|w_2|)=(r_1|u_1|,r_2|u_2|)=(u_1^*,u_2^*)$, which ends the proof. □

Lemma 3.4. *Take $(\psi_1,\psi_2)\in\mathcal{H}_{\mathbb{C}}$ and assume that, for some $\bar\alpha\in(\alpha_1,\alpha_2)$, we have*

$$\mathcal{E}_{\gamma^*}(\psi_1,\psi_2)<\frac{\bar\alpha}{2}-\gamma^*M(\bar\alpha,\mathcal{Q}(\psi_1),\mathcal{Q}(\psi_2)).\tag{3.5}$$

Then

$$\|(\psi_1,\psi_2)\|^2_{\mathcal{H}}<\bar\alpha\quad\Rightarrow\quad\|(\Psi_1(t),\Psi_2(t))\|^2_{\mathcal{H}}<\bar\alpha\qquad\forall\,t\ \text{ in the existence interval}$$
$$\|(\psi_1,\psi_2)\|^2_{\mathcal{H}}>\bar\alpha\quad\Rightarrow\quad\|(\Psi_1(t),\Psi_2(t))\|^2_{\mathcal{H}}>\bar\alpha\qquad\forall\,t\ \text{ in the existence interval.}$$

Proof. Suppose, in view of a contradiction, that for some $\bar t$ we have

$$\|(\Psi_1(\bar t),\Psi_2(\bar t))\|^2_{\mathcal{H}}=\bar\alpha.$$

Then, by assumption (3.5) and the conservation of energy,

$$\begin{aligned}\frac{\bar\alpha}{2}-\gamma^*F(\Psi_1(\bar t),\Psi_2(\bar t))&=\mathcal{E}_{\gamma^*}(\Psi_1(\bar t),\Psi_2(\bar t))\\&=\mathcal{E}_{\gamma^*}(\psi_1,\psi_2)<\frac{\bar\alpha}{2}-\gamma^*M(\bar\alpha,\mathcal{Q}(\psi_1),\mathcal{Q}(\psi_2)),\end{aligned}$$

which yields

$$M(\bar\alpha,\mathcal{Q}(\psi_1),\mathcal{Q}(\psi_2))<F(\Psi_1(\bar t),\Psi_2(\bar t)).$$

On the other hand, by conservation of mass, we have $(\Psi_1(\bar t),\Psi_2(\bar t))\in\mathcal{U}(\bar\alpha,\mathcal{Q}(\psi_1),\mathcal{Q}(\psi_2))$, which provides a contradiction. □

Lemma 3.5. *The function*

$$\begin{aligned}e(\alpha):&=\frac{\alpha}{2}-\gamma^*M(\alpha,\rho_1^*,\rho_2^*)\\&=\mathcal{E}_{\gamma^*}(u_1(\alpha),u_2(\alpha))\end{aligned}$$

has a strict local minimum at $\alpha=\alpha^*$

[1]Take $w:\Omega\to\mathbb{C}$ such that $\int_\Omega|\nabla w|^2\,dx<\infty$. Then $\int_\Omega|\nabla|w||^2\,dx\le\int_\Omega|\nabla w|^2\,dx$. Moreover, equality holds if and only if the real and imaginary parts of w are proportional functions. See e.g. [Lieb-Loss, Analysis, Theorem 7.21].

Proof. Step 1. Let
$$\frac{d}{d\alpha}(u_1(\alpha), u_2(\alpha)) =: (v_1(\alpha), v_2(\alpha)).$$
Differentiating the identities
$$\sum_{i=1}^{2}\int_\Omega (|\nabla u_i|^2 + V_i(x)u_i^2)\,dx = \alpha, \qquad \int_\Omega u_i^2\,dx = \rho_i^*\ (i=1,2)$$
with respect to α, we obtain
$$\sum_{i=1}^{2}\int_\Omega (\nabla u_i\cdot\nabla v_i + V_i(x)u_iv_i)\,dx = \frac{1}{2}, \qquad \int_\Omega u_iv_i\,dx = 0\ (i=1,2). \tag{3.6}$$
Test the equation for u_i:
$$-\Delta u_i + (V_i(x)+\omega_i)u_i = \gamma(\mu_i u_i^3 + \beta u_i u_j^2)$$
by v_i; combining the result with (3.6), we obtain:
$$\gamma\int_\Omega \left(\mu_1u_1^3v_1 + \mu_2u_2^3v_2 + \beta u_1u_2^2v_1 + \beta u_1^2u_2v_2\right)\,dx = \frac{1}{2}. \tag{3.7}$$
Step 2. As
$$e(\alpha) = \frac{\alpha}{2} - \gamma^*\int_\Omega\left(\frac{\mu_1u_1^4}{4} + \frac{\mu_2u_2^4}{4} + \frac{\beta u_1^2u_2^2}{2}\right)\,dx,$$
taking the derivative in α we see that, by step 1,
$$\begin{aligned} e'(\alpha) &= \frac{1}{2} - \gamma^*\int_\Omega\left(\mu_1u_1^3v_1 + \mu_2u_2^3v_2 + \beta u_1u_2^2v_1 + \beta u_1^2u_2v_2\right)\,dx \\ &= \frac{1}{2}\left(1 - \frac{\gamma^*}{\gamma}\right). \end{aligned}$$
As $\gamma(\alpha)$ is strictly increasing in a neighborhood of α^* (cf. assumption (M3)), the result follows. □

Proof of Theorem 3.2.
a) Proof of property (3.2). Fix a small ε so that $\alpha^*\pm\varepsilon \in (\alpha_1,\alpha_2)$ and
$$e(\alpha^*) < e(\alpha^*\pm\varepsilon)$$
(recall Lemma 3.5). Moreover, take $\eta = \eta(\varepsilon)$ so that
$$e(\alpha^*) < e(\alpha^*\pm\varepsilon) - \eta,$$
which we can rewrite as
$$\mathcal{E}_{\gamma^*}(u_1^*, u_2^*) < \frac{\alpha^*\pm\varepsilon}{2} - \gamma^* M(\alpha^*\pm\varepsilon, \rho_1^*, \rho_2^*) - \eta.$$
Then, for $\delta>0$ sufficiently small and $\|(\psi_1,\psi_2)-(u_1^*,u_2^*)\|^2_{\mathcal{H}} < \delta$, we have
$$\mathcal{E}_{\gamma^*}(\psi_1,\psi_2) < \frac{\alpha^*\pm\varepsilon}{2} - \gamma^* M(\alpha^*\pm\varepsilon, \mathcal{Q}(\psi_1), \mathcal{Q}(\psi_2)),$$
where we have used the $\mathcal{H}$–continuity of $\mathcal{E}_{\gamma^*}$ and $\mathcal{Q}$, as well as Lemma 2.5. Moreover, since we have (for δ small)
$$\alpha^* - \delta < \|(\psi_1,\psi_2)\|^2_{\mathcal{H}} < \alpha^* + \delta,$$
then Lemma 3.4 applied with $\bar\alpha = \alpha^*\pm\varepsilon$ implies that
$$\alpha^* - \varepsilon < \|(\Psi_1(t),\Psi_2(t))\|^2_{\mathcal{H}} < \alpha^* + \varepsilon$$

and in particular $(\Psi_1(t), \Psi_2(t))$ is defined for all $t \geq 0$ (as the existence interval in time is uniform with respect to the norm of the initial data, cf. (LWP)).

b) Proof of property (3.3). If (3.3) does not hold, then we can find initial data $(\psi_{1n}, \psi_{2n}) \to (u_1^*, u_2^*)$ in $\mathcal{H}_{\mathbb{C}}$, a sequence $(t_n)_n$, and $\eta > 0$ such that

$$\inf_{s_1,s_2\in\mathbb{R}} \|(\Psi_{1n}(t_n), \Psi_{2n}(t_n)) - (e^{\mathrm{i}s_1}u_1^*, e^{\mathrm{i}s_2}u_2^*)\|_{\mathcal{H}} \geq \eta \tag{3.8}$$

(here, of course, (Ψ_{1n}, Ψ_{2n}) is the solution to (3.1) corresponding to the initial datum (ψ_{1n}, ψ_{2n})). By a), we can suppose without loss of generality that the sequences satisfy

$$\mathcal{E}_{\gamma^*}(\psi_{1n}, \psi_{2n}) < \frac{1}{2}\left(\alpha^* \pm \frac{1}{n}\right) - \gamma^* M\left(\alpha^* \pm \frac{1}{n}, \rho_1^*, \rho_2^*\right) \tag{3.9}$$

and

$$\alpha^* - \frac{1}{n} < \|(\Psi_{1n}(t_n), \Psi_{2n}(t_n))\|_{\mathcal{H}}^2 < \alpha^* + \frac{1}{n}. \tag{3.10}$$

Moreover, by the conservation of mass along trajectories,

$$\int_\Omega |\Psi_{in}(t_n)|^2\, dx = \int_\Omega |\psi_{in}|^2 \to \rho_i^* \qquad i = 1, 2.$$

In particular, $(\Psi_{1n}(t_n), \Psi_{2n}(t_n))$ is bounded in $\mathcal{H}$, hence up to a subsequence we have weak convergence in $\mathcal{H}_{\mathbb{C}}$ to (w_1, w_2), strongly in $L^2 \cap L^4$. The limiting configuration then satisfies

$$\|(w_1, w_2)\|_{\mathcal{H}}^2 \leq \alpha^*, \qquad \int_\Omega w_i^2\, dx = \rho_i \ \ i = 1, 2 \tag{3.11}$$

so that $F(w_1, w_2) \leq M(\alpha^*, \rho_1^*, \rho_2^*)$. On the other hand, we have

$$\begin{aligned}\alpha^* - \frac{1}{n} - \gamma^* F(\Psi_{1n}(t_n), \Psi_{2n}(t_n)) &< \mathcal{E}_{\gamma^*}(\Psi_{1n}(t_n), \Psi_{2n}(t_n)) = \mathcal{E}_{\gamma^*}(\psi_{1n}, \psi_{2n}) \\ &< \frac{1}{2}\left(\alpha^* - \frac{1}{n}\right) - \gamma^* M(\alpha^* - \frac{1}{n}, \rho_1^*, \rho_2^*),\end{aligned}$$

where the first inequality is due to (3.10) and the second one to (3.9). Hence

$$M\left(\alpha^* - \frac{1}{n}, \rho_1^*, \rho_2^*\right) < F(\Psi_{1n}(t_n), \Psi_{2n}(t_n))$$

and (by (M2) and again by strong L^4 convergence)

$$M(\alpha^*, \rho_1^*, \rho_2^*) \leq F(w_1, w_2). \tag{3.12}$$

Thus (w_1, w_2) achieves $M(\alpha^*, \rho_1^*, \rho_2^*)$ and $(\Psi_{1n}(t_n), \Psi_{2n}(t_n)) \to (w_1, w_2)$ strongly in $\mathcal{H}_{\mathbb{C}}$. Finally, we obtain a contradiction by combining (3.8) with Lemma 3.3. □

End of the proof of Theorem 1.2. In the assumptions of the theorem, let us fix any $\alpha \in (\alpha_1, \alpha_2)$, and relabel the triplet (α, ρ_1, ρ_2) as $(\alpha^*, \rho_1^*, \rho_2^*)$. If $M(\alpha^*, \rho_1^*, \rho_2^*)$ is achieved by a unique pair, then the proof follows from Theorem 3.2, once one notices that (Ψ_1, Ψ_2) solves (3.1), if and only if $(\Phi_1, \Phi_2) = \sqrt{\gamma^*}(\Psi_1, \Psi_2)$ solves (1.1). Without the uniqueness assumption, one may repeat the proof with minor changes, observing that, by Corollary 2.6, the set of pairs (u_1, u_2) achieving $M(\alpha^*, \rho_1^*, \rho_2^*)$ is compact in $\mathcal{H}$. □

4. Applications

4.1. **The case of small masses.** To prove Theorem 1.3, we will use the following parametric version of a well known result due to Ambrosetti and Prodi [5]. In the following, Ker and Range denote respectively the kernel and the range of a linear operator.

Theorem 4.1. *Let X, Y be Banach spaces, $U \subset X$ an open set, $I \subset \mathbb{R}$ an open interval, and $\Phi \in C^2(U \times I, Y)$.*

Take $(x^*, \vartheta^*) \in U \times I$ such that:

1. *there exists a continuous curve $\bar{x} : I \to U$, with $\bar{x}(\vartheta^*) = x^*$, and*
$$\begin{cases} \Phi(x, \vartheta) = 0 \\ (x, \vartheta) \in U \times I \end{cases} \qquad \Longleftrightarrow \qquad x = \bar{x}(\vartheta), \ \ \vartheta \in I;$$
2. *there exists $\varphi^* \in X$, non trivial, such that*
$$\mathrm{Ker}(\Phi_x(x^*, \vartheta^*)) = \mathrm{span}\{\varphi^*\};$$
3. *there exists a nontrivial $\Psi \in Y^*$ (independent of ϑ) such that*
$$\mathrm{Range}(\Phi_x(\bar{x}(\vartheta), \vartheta)) = \mathrm{Ker}\,\Psi \qquad \text{for every } \vartheta \in I;$$
4. $\langle \Psi, \Phi_{xx}(x^*, \vartheta^*)[\varphi^*, \varphi^*] \rangle > 0.$

Finally, let $z \in Y$ be such that $\langle \Psi, z \rangle = 1$.

Then there exist $\bar{\delta} > 0$, $\bar{\varepsilon} > 0$ such that, for every $|\vartheta - \vartheta^*| < \bar{\delta}$ the equation
$$\Phi(x, \vartheta) = \varepsilon z, \qquad x \in B_{\bar{\delta}}(x^*) \tag{4.1}$$
has no solutions when $-\bar{\varepsilon} \le \varepsilon < 0$, while for each $0 < \varepsilon \le \bar{\varepsilon}$ it has exactly two solutions
$$x = x_+(\varepsilon, \vartheta), \qquad x = x_-(\varepsilon, \vartheta).$$
Furthermore, the maps $x_\pm : (0, \bar{\varepsilon}] \times (\vartheta^* - \bar{\delta}, \vartheta^* + \bar{\delta}) \to B_{\bar{\delta}}(x^*)$ are of class C^2 and continuous up to $\varepsilon = 0^+$. More precisely,
$$x_\pm(\varepsilon, \vartheta) = \bar{x}(\vartheta) + [\varphi^* + \eta_\pm(\varepsilon, \vartheta)]\, t_\pm(\varepsilon, \vartheta), \tag{4.2}$$
where the maps $\eta_\pm$ are $C^1([0, \bar{\varepsilon}] \times (\vartheta^* - \bar{\delta}, \vartheta^* + \bar{\delta}))$ with $\eta(0, \vartheta^*) = 0$, while the functionals $t_\pm$ are C^2 for $\varepsilon > 0$, continuous (and vanishing) up to $\varepsilon = 0^+$, and
$$\pm t_\pm(\varepsilon, \vartheta) > 0, \quad \pm \frac{\partial t_\pm(\varepsilon, \vartheta)}{\partial \varepsilon} \ge \frac{C}{\sqrt{\varepsilon}}, \qquad \text{for } \varepsilon \in (0, \bar{\varepsilon}],$$
for a suitable $C > 0$ (related to the positive number appearing in assumption (4)).

The proof of such theorem follows very closely the one of the original Ambrosetti-Prodi result [5, Section 3.2, Lemma 2.5], taking however into account the dependence on the parameter ϑ, which is not present in the latter. For the reader's convenience, here we summarize the proof, enlightening the main differences.

Proof. To start with, we apply a Lyapunov-Schmidt reduction to equation (4.1). To this aim, let $L := \Phi_x(x^*, \vartheta^*) \in \mathcal{L}(X, Y)$, and let $W \subset X$ denote a topological complement of $\mathrm{Ker}\, L$. Then, for every $x \in X$, we can write
$$x = x^* + t\varphi^* + w, \qquad \text{for unique } t \in \mathbb{R},\ w \in W.$$

Analogously, since $Y = \mathrm{span}\{z\} \oplus \mathrm{Range}\, L$, for every $y \in Y$ we can uniquely write

$$y =: \underbrace{Py}_{\in \mathrm{span}\{z\}} + \underbrace{Qy}_{\in \mathrm{Range}\, L}, \qquad \text{where } Py = \langle \Psi, y \rangle z$$

(Ψ appearing in assumption (3)). Using such decompositions, equation (4.1) writes

$$\begin{cases} P\Phi(x,\vartheta) = \langle \Psi, \Phi(x^* + t\varphi^* + w, \vartheta) \rangle z = \varepsilon z \\ Q\Phi(x,\vartheta) = Q\Phi(x^* + t\varphi^* + w, \vartheta) = 0. \end{cases} \tag{4.3}$$

Now, by construction, we can apply the Implicit Function Theorem to the second equation in order to solve for w near $(t, \vartheta, w) = (0, \vartheta^*, 0)$ (indeed, in such point, the partial derivative of the l.h.s. with respect to w is $L : W \to \mathrm{Range}\, L$, which is invertible). As a consequence, for some positive δ,

$$\begin{cases} Q\Phi(x^* + t\varphi^* + w, \vartheta) = 0 \\ (t, \vartheta - \vartheta^*, w) \in [-\delta, \delta]^2 \times B'_\delta(0) \end{cases} \iff w = w(t, \vartheta),\ (t, \vartheta - \vartheta^*) \in [-\delta, \delta]^2$$

(here B' denotes the ball in W). For future reference, we notice that, possibly decreasing δ, the C^2 function w satisfies

$$w(0, \vartheta) = \bar{x}(\vartheta) - x^* \qquad \text{for every } \vartheta - \vartheta^* \in [-\delta, \delta]. \tag{4.4}$$

Indeed, since $\bar{x}(\vartheta^*) = x^*$, this follows from the fact that w is the unique solution of the above equation near $(t, \vartheta, w) = (0, \vartheta^*, 0)$, together with the fact that

$$Q\Phi(x^* + 0 \cdot \varphi^* + (\bar{x}(\vartheta) - x^*), \vartheta) = Q\Phi(\bar{x}(\vartheta), \vartheta) = 0$$

by assumption (1). Furthermore, we also have that

$$w_t(0, \vartheta^*) = 0. \tag{4.5}$$

Indeed, taking the partial derivative of the second equation in (4.3) with respect to t, we obtain

$$Q\Phi_x(x^* + t\varphi^* + w(t,\vartheta), \vartheta)[\varphi^* + w_t(t,\vartheta)] = 0;$$

for $(t, \vartheta) = (0, \vartheta^*)$, this yields

$$0 = Q\Phi_x(x^*, \vartheta^*)[\varphi^* + w_t(0, \vartheta^*)] = Lw_t(0, \vartheta^*),$$

so that $w_t(0, \vartheta^*) \in \mathrm{Ker}\, L$. Since $w_t(0, \vartheta^*) \in W$ by definition, (4.5) follows.

Substituting $w = w(t, \vartheta)$ in the first equation in (4.3) we obtain the bifurcation equation

$$\text{find } (t, \vartheta - \vartheta^*) \in [-\delta, \delta]^2 \text{ s.t.} \qquad \chi(t, \vartheta) := \langle \Psi, \Phi(x^* + t\varphi^* + w(t,\vartheta), \vartheta) \rangle = \varepsilon, \tag{4.6}$$

which is locally equivalent to (4.1). Equation (4.4) implies that, for every $\vartheta - \vartheta^* \in [-\delta, \delta]$,

$$\chi(0, \vartheta) = \langle \Psi, \Phi(\bar{x}(\vartheta), \vartheta) \rangle = 0.$$

On the other hand, direct calculations yield

$$\begin{aligned} \chi_t(t,\vartheta) &= \langle \Psi, \Phi_x(x^* + t\varphi^* + w(t,\vartheta), \vartheta)[\varphi^* + w_t(t,\vartheta)] \rangle \\ \chi_{tt}(t,\vartheta) &= \langle \Psi, \Phi_{xx}(x^* + t\varphi^* + w(t,\vartheta), \vartheta)[\varphi^* + w_t(t,\vartheta)]^2 \\ &\qquad + \Phi_x(x^* + t\varphi^* + w(t,\vartheta), \vartheta)[w_{tt}(t,\vartheta)] \rangle. \end{aligned}$$

Using assumption (3) and equations (4.4), (4.5), we infer

$$\begin{aligned} \chi_t(0,\vartheta) &= 0 \\ \chi_{tt}(0,\vartheta^*) &= \langle \Psi, \Phi_{xx}(x^*, \vartheta^*)[\varphi^*, \varphi^*] \rangle > 0 \end{aligned} \tag{4.7}$$

by assumption (4). Since χ is C^2, we can find positive constants C_1, C_2 such that

$$\begin{cases} 2C_1 \le \chi_{tt}(t,\vartheta) \le 2C_2 \\ 2C_1|t| \le \operatorname{sign}(t)\chi_t(t,\vartheta) \le 2C_2|t| \\ C_1 t^2 \le \chi(t,\vartheta) \le C_2 t^2 \end{cases} \qquad \text{for every } (t,\vartheta-\vartheta^*) \in [-\bar\delta,\bar\delta]^2,$$

for some suitable $\bar\delta \le \delta$. As a first consequence, (4.6) is not solvable for $\varepsilon < 0$. Furthermore, defining

$$\bar\varepsilon := \min_{|\vartheta-\vartheta^*|\le\bar\delta} \chi(\pm\bar\delta,\vartheta) > 0$$

we deduce that, for every $\vartheta-\vartheta^* \in [-\bar\delta,\bar\delta]$ and $\varepsilon \in (0,\bar\varepsilon]$, there exist $-\bar\delta \le t_-(\varepsilon,\vartheta) < 0 < t_+(\varepsilon,\vartheta) \le \bar\delta$ such that

$$\begin{cases} \chi(t,\vartheta) = \varepsilon \\ (t,\vartheta-\vartheta^*,\varepsilon) \in [-\bar\delta,\bar\delta]^2 \times (0,\bar\varepsilon] \end{cases} \iff t = t_\pm(\varepsilon,\vartheta),\ (\vartheta-\vartheta^*,\varepsilon) \in [-\bar\delta,\bar\delta] \times (0,\bar\varepsilon].$$

Clearly $t_\pm(0^+,\vartheta) = 0$, uniformly in ϑ. Moreover, since $\chi_t(t,\vartheta) \ne 0$ for $t \ne 0$, the Implicit Function Theorem implies that the maps $t_\pm$ are C^2 for $\varepsilon > 0$, with

$$\pm\frac{\partial t_\pm(\varepsilon,\vartheta)}{\partial\varepsilon} = \frac{\pm 1}{\chi_t(t_\pm(\varepsilon,\vartheta),\vartheta)} \ge \frac{1}{2C_2|t_\pm(\varepsilon,\vartheta)|} \ge \frac{1}{2C_2}\sqrt{\frac{C_1}{\chi(t_\pm(\varepsilon,\vartheta),\vartheta)}} = \frac{\sqrt{C_1}}{2C_2}\frac{1}{\sqrt{\varepsilon}}.$$

Setting

$$\begin{aligned} x_\pm(\varepsilon,\vartheta) &= x^* + t_\pm(\varepsilon,\vartheta)\varphi^* + w(t_\pm(\varepsilon,\vartheta),\vartheta) \\ &= \bar x(\vartheta) + t_\pm(\varepsilon,\vartheta)\varphi^* + w(t_\pm(\varepsilon,\vartheta),\vartheta) - (\bar x(\vartheta) - x^*) \\ &= \bar x(\vartheta) + \left[\varphi^* + \frac{w(t_\pm(\varepsilon,\vartheta),\vartheta) - w(0,\vartheta)}{t_\pm(\varepsilon,\vartheta)}\right] t_\pm(\varepsilon,\vartheta), \end{aligned}$$

one can complete the proof by recalling that the maps

$$\eta_\pm(\varepsilon,\vartheta) := \begin{cases} [w(t_\pm(\varepsilon,\vartheta),\vartheta) - w(0,\vartheta)]/t_\pm(\varepsilon,\vartheta) & \varepsilon \ne 0 \\ w_t(0,\vartheta) & \varepsilon = 0 \end{cases}$$

are C^1 up to $\varepsilon = 0$, and that $\eta(0,\vartheta^*) = 0$ by equation (4.5). □

Remark 4.2. The following uniform in ϑ limit:

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \pm\frac{t_\pm(\varepsilon,\vartheta)}{\sqrt{\varepsilon}} &= \lim_{t\to 0^+} \frac{t}{\sqrt{\chi(\pm t,\vartheta)}} = \lim_{t\to 0^+} \frac{t}{\sqrt{\chi_{tt}(\xi_\pm,\vartheta)t^2/2}} \\ \text{(for suitable } -t < \xi_- < 0 < \xi_+ < t) \qquad &= \sqrt{\frac{2}{\chi_{tt}(0,\vartheta)}} =: a(\vartheta), \end{aligned}$$

$$\lim_{\varepsilon\to 0^+} \frac{w(t_\pm(\varepsilon,\vartheta),\vartheta) - w(0,\vartheta)}{t_\pm(\varepsilon,\vartheta)} = w_t(0,\vartheta) = \eta(0,\vartheta),$$

implies that, as $\varepsilon \to 0^+$,

$$x_\pm(\varepsilon,\vartheta) = \bar x(\vartheta) \pm a(\vartheta)\,[\varphi^* + \eta(0,\vartheta)]\,\sqrt{\varepsilon} + o(\sqrt{\varepsilon}), \qquad \text{uniformly in } \vartheta,$$

where

$$a(\vartheta^*) = \sqrt{\frac{2}{\langle \Psi, \Phi_{xx}(x^*,\vartheta^*)[\varphi^*,\varphi^*]\rangle}} > 0.$$

Remark 4.3. A point (x^*, ϑ^*) satisfying assumptions (2), (3) and (4) in Theorem 4.1 (the latter ones for $\vartheta = \vartheta^*$) is said to be *ordinary singular* for Φ. As a matter of fact, assumption (1) insures not only that (x^*, ϑ^*) is ordinary singular for Φ, but also that $(\bar{x}(\vartheta), \vartheta)$ exhibits an ordinary singular type geometry, at least for $|\vartheta - \vartheta^*|$ small.

In order to apply the previous abstract result, let $X = \mathcal{H} \times \mathbb{R}^3$, $Y = \mathcal{H}^* \times \mathbb{R}^3$ and take the C^2 map $\Phi : X \times \mathbb{R} \to Y$ defined by

$$\Phi(u_1, u_2, \omega_1, \omega_2, \gamma, \vartheta) = \begin{pmatrix} \Delta u_1 - (V_1(x) + \omega_1)u_1 + \gamma(\mu_1 u_1^3 + \beta u_1 u_2^2) \\ \Delta u_2 - (V_2(x) + \omega_2)u_2 + \gamma(\mu_2 u_2^3 + \beta u_1^2 u_2) \\ \int_\Omega u_1^2\,dx - (\cos^2\vartheta)/\lambda_{V_1} \\ \int_\Omega u_2^2\,dx - (\sin^2\vartheta)/\lambda_{V_2} \\ \sum_{i=1}^2 \int_\Omega \left(|\nabla u_i|^2 + V_i(x)u_i^2\right)\,dx - 1 \end{pmatrix}. \tag{4.8}$$

Remark 4.4. Note that, recalling the definition of $\tilde{\mathcal{U}}(\alpha, \rho_1^*, \rho_2^*)$ from Section 2, we have that $\Phi(u_1, u_2, \omega_1, \omega_2, \gamma, \vartheta) = (0, 0, 0, 0, \varepsilon)$ if and only if

$$(u_1, u_2) \in \tilde{\mathcal{U}}\left(1 + \varepsilon, \frac{\cos^2\vartheta}{\lambda_{V_1}}, \frac{\sin^2\vartheta}{\lambda_{V_2}}\right) \text{ and equation (2.2) holds.}$$

Finally, we define $\bar{x} : \mathbb{R} \to X$ as

$$\begin{aligned} \bar{x}(\vartheta) &:= (\bar{u}_1(\vartheta), \bar{u}_2(\vartheta), \bar{\omega}_1(\vartheta), \bar{\omega}_2(\vartheta), \bar{\gamma}(\vartheta)) \\ &= \left(\frac{\cos\vartheta}{\sqrt{\lambda_{V_1}}}\varphi_{V_1}, \frac{\sin\vartheta}{\sqrt{\lambda_{V_2}}}\varphi_{V_2}, -\lambda_{V_1}, -\lambda_{V_2}, 0\right), \text{ and} \\ (\bar{\rho}_1(\vartheta), \bar{\rho}_2(\vartheta)) &:= \left(\frac{\cos^2\vartheta}{\lambda_{V_1}}, \frac{\sin^2\vartheta}{\lambda_{V_2}}\right). \end{aligned} \tag{4.9}$$

In the following, we will systematically adopt the above notation, possibly dropping the explicit dependence on ϑ when no confusion may arise.

We start with the following lemma, which will ensure that assumption (1) in Theorem 4.1 holds for $\bar{x}(\cdot)$ in $I = (0, \pi/2)$ (and suitable U).

Lemma 4.5. *Take $\vartheta \notin \mathbb{Z}\pi/2$ and $\varepsilon_n \to 0^+$, and suppose that*

$$\Phi(u_{1,n}, u_{2,n}, \omega_{1,n}, \omega_{2,n}, \gamma_n, \vartheta) = (0, 0, 0, 0, \varepsilon_n).$$

Then, up to subsequences, $\omega_{i,n} \to \bar{\omega}_i(\vartheta)$ $(i = 1, 2)$, $\gamma_n \to 0$, and

$$u_{1,n} \to (-1)^l \bar{u}_1(\vartheta), \quad u_{2,n} \to (-1)^m \bar{u}_2(\vartheta), \qquad \text{strongly in } \mathcal{H},$$

for some $(l, m) \in \{0, 1\}^2$.

In particular, for $\vartheta \in (0, \pi/2)$, and $\tilde{U} \subset \mathcal{H}$ open, containing the above possible limits only for $l = m = 0$, we have that

$$\begin{gathered} \Phi(u_1, u_2, \omega_1, \omega_2, \gamma, \vartheta) = (0, 0, 0, 0, 0), \qquad (u_1, u_2) \in \tilde{U} \\ \Updownarrow \\ (u_1, u_2, \omega_1, \omega_2, \gamma) = \bar{x}(\vartheta). \end{gathered}$$

Proof. As ε_n is a bounded sequence, then $(u_{1,n}, u_{2,n})$ is bounded in $\mathcal{H}$ and, up to a subsequence, $(u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2)$ weakly in $\mathcal{H}$, with u_i being nontrivial functions satisfying $\int_\Omega u_i^2\,dx = \bar\rho_i(\vartheta)$, $i=1,2$. By definition of λ_{V_i}, we have

$$1 \le \sum_{i=1}^2 \int_\Omega (|\nabla u_i|^2 + V_i(x)u_i^2)\,dx \le \liminf_n \sum_{i=1}^2 \int_\Omega (|\nabla u_{i,n}|^2 + V_i(x)u_{i,n}^2)\,dx$$
$$\le \liminf_n (1+\varepsilon_n) = 1.$$

Thus the convergence is strong, and u_1, u_2 are normalized eigenfunctions. On the other hand, from Lemma 2.7 we have that $\omega_{i,n} \to \omega_i$, $\gamma_n \to \gamma$ for some constants ω_i, γ. These must satisfy (recall that u_1, u_2 are nontrivial)

$$\lambda_{V_1} + \omega_1 = \gamma(\mu_1 u_1^2 + \beta u_2^2), \qquad \lambda_{V_2} + \omega_2 = \gamma(\mu_2 u_2^2 + \beta u_1^2).$$

As $u_1 = u_2 = 0$ on $\partial\Omega$ if Ω is bounded, or u_1, u_2 satisfy (2.5) in case $\Omega = \mathbb{R}^N$, we deduce that $\omega_i = -\lambda_{V_i} = \bar\omega_i$, $i=1,2$. In turn, by assumption (NonDeg), $\gamma = 0$. □

Remark 4.6. The lemma above is false for $\vartheta \in \mathbb{Z}\pi/2$. Indeed, for instance,

$$\Phi\left(\frac{1}{\sqrt{\lambda_{V_1}}}\varphi_{V_1}, 0, -\lambda_{V_1}, \omega_2, 0\right) = (0,0,0,0,0)$$

for every ω_2 (and not only for $\omega_2 = -\lambda_{V_2}$).

A direct computation shows that the partial derivative of Φ with respect to the variables $x := (u_1, u_2, \omega_1, \omega_2, \gamma)$,

$$\Phi_x(x,\vartheta) : \mathcal{H} \times \mathbb{R}^3 \to \mathcal{H}^* \times \mathbb{R}^3,$$

computed at $h = (v_1, v_2, o_1, o_2, g)$, yields, for $i = 1,2$ and $j \ne i$:

$$(\Phi_x(x,\vartheta)[h])_i = \Delta v_i - (V_i(x)+\omega_i)v_i - o_i u_i + g(\mu_i u_i^3 + \beta u_i u_j^2)$$
$$+ \gamma(3\mu_i u_i^2 v_i + \beta u_j^2 v_i + 2\beta u_1 u_2 v_j),$$
$$(\Phi_x(x,\vartheta)[h])_{i+2} = 2\int_\Omega u_i v_i\,dx$$
$$(\Phi_x(x,\vartheta)[h])_5 = 2\sum_{i=1}^2 \int_\Omega (\nabla u_i \cdot \nabla v_i + V_i(x) u_i v_i)\,dx.$$

Lemma 4.7. *Given $\vartheta \in (0, \pi/2)$, denote*

$$L_\vartheta := \Phi_x(\bar x(\vartheta), \vartheta).$$

Then:

a) Ker L_ϑ *has dimension one, being spanned by the vector*

$$\varphi^* := (\psi_1(\vartheta), \psi_2(\vartheta), o_1(\vartheta), o_2(\vartheta), 1),$$

where

$$o_i(\vartheta) = \frac{1}{\bar\rho_i}\int_\Omega (\mu_i \bar u_i^2 + \beta \bar u_j^2)\bar u_i^2\,dx,$$

and $\psi_i(\vartheta)$ is the unique solution of

$$-\Delta\psi_i + V_i(x)\psi_i - \lambda_{V_i}\psi_i = \mu_i \bar u_i^3 + \beta \bar u_i \bar u_j^2 - o_i \bar u_i \quad \textit{with} \int_\Omega \psi_i \bar u_i\,dx = 0;$$

b) Range L_ϑ = Ker Ψ, *where $\Psi : \mathcal{H}^* \times \mathcal{H}^* \times \mathbb{R}^3 \to \mathbb{R}$ is defined by*

$$\Psi(\xi_1, \xi_2, h_1, h_2, k) = k - (\lambda_{V_1} h_1 + \lambda_{V_2} h_2);$$

c) $\Psi\left(\Phi_{xx}(\bar x(\vartheta), \vartheta)[\varphi^*, \varphi^*]\right) > 0$.

Proof. *a)* Let for the moment $\zeta_i(\vartheta) = \mu_i \bar u_i^3(\vartheta) + \beta \bar u_i(\vartheta) \bar u_j^2(\vartheta)$, $i = 1, 2,\ j \neq i$. The equation $L_\vartheta[v_1, v_2, o_1, o_2, g] = (0, 0, 0, 0, 0)$ is equivalent to ($i = 1, 2,\ j \neq i$):

$$-\Delta v_i + V_i(x) v_i - \lambda_{V_i} v_i = -o_i \bar u_i + g\zeta_i,$$

$$\int_\Omega \bar u_i v_i \, dx = \int_\Omega (\nabla \bar u_i \cdot \nabla v_i + V_i(x) \bar u_i v_i) \, dx = 0.$$

Testing the i–th equation by $\bar u_i$, one obtains each o_i in function of g:

$$o_i = \frac{g}{\bar\rho_i} \int_\Omega \zeta_i \bar u_i \, dx = g o_i(\vartheta).$$

Therefore one has to solve

$$-\Delta \psi_i + V_i(x)\psi_i - \lambda_{V_i}\psi_i = g\left[\zeta_i - \frac{\bar u_i}{\bar\rho_i}\int_\Omega \zeta_i \bar u_i \, dx\right] \quad \text{ with } \int_\Omega \psi_i \bar u_i \, dx = 0,$$

and this can be uniquely done by choosing $v_i = g\psi_i(\vartheta)$, by Fredholm's Alternative.

b) Take $(\xi_1, \xi_2, h_1, h_2, k) = L_\vartheta(v_1, v_2, o_1, o_2, g) \in \operatorname{Range} L_\vartheta$. Then from the last three equations of this identity, and the fact that $\bar u_i$ are eigenfunctions, we deduce that

$$\begin{aligned} k &= 2\sum_{i=1}^2 \int_\Omega (\nabla \bar u_i \cdot \nabla v_i + V_i(x)\bar u_i v_i) \, dx \\ &= 2\sum_{i=1}^2 \lambda_{V_i} \int_\Omega \bar u_i v_i \, dx = \sum_{i=1}^2 \lambda_{V_i} h_i. \end{aligned}$$

which shows that $\operatorname{Range} L_\vartheta \subset \operatorname{Ker}\Psi$. Reciprocally, given $(\xi_1, \xi_2, h_1, h_2, k) \in \mathcal{H}^* \times \mathbb{R}^3$ with $k = \lambda_{V_1} h_1 + \lambda_{V_2} h_2$, let w_i (for $i = 1, 2$) be the solution to

$$-\Delta w_i + V_i(x) w_i - \lambda_{V_i} w_i = \varphi_{V_i}\langle \xi_i, \varphi_{V_i}\rangle - \xi_i, \qquad \int_\Omega w_i \varphi_{V_i} \, dx = 0,$$

which exists, unique, by Fredholm's Alternative. Then

$$\begin{aligned} L_\vartheta &\left[\frac{h_1}{2\sqrt{\bar\rho_1}}\varphi_{V_1} + w_1, \frac{h_2}{2\sqrt{\bar\rho_2}}\varphi_{V_2} + w_2, \frac{1}{\sqrt{\bar\rho_1}}\langle \xi_1, \varphi_{V_1}\rangle, \frac{1}{\sqrt{\bar\rho_2}}\langle \xi_2, \varphi_{V_2}\rangle, 0\right] \\ &= (\xi_1, \xi_2, h_1, h_2, k). \end{aligned}$$

c) One can check directly that

$$\Phi_{xx}(\bar x(\vartheta), \vartheta)[\psi_1(\vartheta), \psi_2(\vartheta), o_1(\vartheta), o_2(\vartheta), 1]^2$$

is given by

$$\begin{pmatrix} -2o_1\psi_1 + 2(3\mu_1 \bar u_1^2 \psi_1 + \beta \bar u_2^2 \psi_1 + 2\beta \bar u_1 \bar u_2 \psi_2) \\ -2o_2\psi_2 + 2(3\mu_2 \bar u_2^2 \psi_2 + \beta \bar u_1^2 \psi_2 + 2\beta \bar u_1 \bar u_2 \psi_1) \\ 2\int_\Omega \psi_1^2 \, dx \\ 2\int_\Omega \psi_2^2 \, dx \\ 2\sum_{i=1}^2 \int_\Omega (|\nabla \psi_i|^2 + V_i(x)\psi_i^2) \, dx \end{pmatrix}.$$

Its image through Ψ is

$$2\sum_{i=1}^2 \left(\int_\Omega (|\nabla \psi_i|^2 + V_i(x)\psi_i^2) \, dx - \lambda_{V_i}\int_\Omega \psi_i^2 \, dx\right),$$

which is strictly positive since $\psi_i \not\equiv 0$ and $\int_\Omega \psi_i \varphi_{V_i} \, dx = 0$. □

Now we are in position to apply Theorem 4.1.

Lemma 4.8. *For every $\vartheta^* \in (0,\pi/2)$ there exist $\bar\delta, \bar\varepsilon$ such that for every $\vartheta \in (\vartheta^* - \bar\delta, \vartheta^* + \bar\delta)$ the problem*

$$\begin{cases} \Phi(u_1,u_2,\omega_1,\omega_2,\gamma,\vartheta) = (0,0,0,0,\varepsilon), \\ (u_1,u_2,\omega_1,\omega_2,\gamma) \in X \end{cases}$$

has exactly two positive solutions $x_\pm = x_\pm(\varepsilon,\vartheta)$ for each $0 < \varepsilon \le \bar\varepsilon$, and no solution for $\varepsilon < 0$. Moreover, $\gamma_- < 0 < \gamma_+$,

$$(u_{1+}(\varepsilon,\vartheta), u_{2+}(\varepsilon,\vartheta)) \text{ achieves } M\left(1+\varepsilon, \bar\rho_1(\vartheta), \bar\rho_1(\vartheta)\right)$$

(uniquely among positive solutions) and

$$\frac{\partial \gamma_+(\varepsilon,\vartheta)}{\partial\varepsilon} \ge C > 0 \qquad \text{for every } (\varepsilon,\vartheta) \in (0,\bar\varepsilon] \times (\vartheta^* - \bar\delta, \vartheta^* + \bar\delta).$$

Proof. In view of Lemmas 4.5 and 4.7, most part of the statement is a direct consequence of Theorem 4.1. Indeed, under the above notation, by choosing $z = (0,0,0,0,1)$, we have $\Psi(z) = 1$ and thus the existence of $\bar\delta, \bar\varepsilon$ and $x_\pm$. One can use again Lemma 4.5 to insure that $x_\pm$ are the only two solutions not only locally, but also among all positive solutions. Now, the last component of equation (4.2) writes

$$\gamma_\pm(\varepsilon,\vartheta) = \left[1 + \tilde\eta_\pm(\varepsilon,\vartheta)\right] t_\pm(\varepsilon,\vartheta),$$

where the functions $t_\pm$ satisfy

$$\pm t_\pm(\varepsilon,\vartheta) > 0, \quad t_\pm(0^+,\vartheta) = 0, \quad \pm\frac{\partial t_\pm(\varepsilon,\vartheta)}{\partial\varepsilon} \ge \frac{C}{\sqrt{\varepsilon}}, \qquad \text{for } \varepsilon \in (0,\bar\varepsilon),$$

while the functions $\tilde\eta_\pm$ are C^1 up to $\varepsilon = 0$, with $\tilde\eta_\pm(0,\vartheta^*) = 0$. In particular, by taking possibly smaller values of $\bar\delta, \bar\varepsilon$, we can assume that

$$|\tilde\eta_\pm(\varepsilon,\vartheta)| \le \frac{1}{2}, \qquad |\partial_\varepsilon\tilde\eta_+(\varepsilon,\vartheta) t_+(\varepsilon,\vartheta)| \ge \frac{C}{3\sqrt{\bar\varepsilon}}.$$

This is sufficient to insure that $\gamma_+ > 0$ and $\gamma_- < 0$ so that only (u_{1+}, u_{2+}) achieves M (Lemmas 2.3, 2.4 and Remark 4.4). Furthermore, this also implies that

$$\frac{\partial\gamma_+}{\partial\varepsilon} = [1+\tilde\eta_+]\frac{\partial t_+}{\partial\varepsilon} + \frac{\partial\tilde\eta_+}{\partial\varepsilon} t_+ \ge \frac{1}{2}\frac{C}{\sqrt{\varepsilon}} - \frac{1}{3}\frac{C}{\sqrt{\bar\varepsilon}} > 0 \qquad \text{for } \varepsilon \in (0,\bar\varepsilon). \qquad \square$$

Proof of Theorem 1.3. As usual, recall that pairs (u_1,u_2) achieving $M(\alpha,\rho_1,\rho_2)$ correspond to pairs $(U_1,U_2) = \sqrt{\gamma}(u_1,u_2)$ which solve system (1.2) with $m_i = \gamma\rho_i$. Let $k \ge 1$ be fixed. Choosing $\rho_i = \bar\rho_i(\vartheta)$, we have that

$$\frac{1}{k} \le \frac{m_2}{m_1} = \frac{\rho_2}{\rho_1} \le k \quad \iff \quad \vartheta_- := \arctan\sqrt{\frac{\lambda_{V_2}}{k\lambda_{V_1}}} \le \vartheta \le \arctan\sqrt{\frac{k\lambda_{V_2}}{\lambda_{V_1}}} =: \vartheta_+.$$

Now, since $[\vartheta_-,\vartheta_+] \subset (0,\pi/2)$, for every $\vartheta^* \in [\vartheta_-,\vartheta_+]$ we can apply Lemma 4.8. By compactness, we end up with a uniform $\bar\varepsilon > 0$ such that, writing $\alpha = 1+\varepsilon$ and

$$x_+(\alpha-1,\vartheta) = (u_1(\alpha,\vartheta), u_2(\alpha,\vartheta), \omega_1(\alpha,\vartheta), \omega_2(\alpha,\vartheta), \gamma(\alpha,\vartheta)),$$

we have that $(u_1(\alpha,\vartheta), u_2(\alpha,\vartheta))$ achieves $M(\alpha, \bar\rho_1(\vartheta), \bar\rho_2(\vartheta))$, uniquely among positive pairs, for every $\alpha \in (1, 1+\bar\varepsilon]$, $\vartheta \in [\vartheta_-,\vartheta_+]$. As a consequence, Theorem 1.1 provides the existence of the corresponding solitary waves. Furthermore, x^+ is C^1 and

$$\frac{\partial\gamma_+(\alpha,\vartheta)}{\partial\alpha} > 0, \qquad \text{for every } \alpha \in (1, 1+\bar\varepsilon],\ \vartheta \in [\vartheta_-,\vartheta_+].$$

Applying Theorem 1.2, for ϑ fixed, by uniqueness we obtain that the solitary waves are stable. Recalling that $\gamma(1^+,\vartheta)\equiv 0$, the theorem follows by choosing

$$\bar m := \min_{\vartheta\in[\vartheta_-,\vartheta_+]} \left[\bar\rho_1(\vartheta)+\bar\rho_2(\vartheta)\right]\sqrt{\gamma(\bar\varepsilon,\vartheta)}>0. \qquad \square$$

4.2. **Defocusing system with weak interaction.** The purpose of this section is to prove the following.

Theorem 4.9. *Let $\mu_1,\mu_2<0$ and $\beta^2<\mu_1\mu_2$. Let V_1,V_2 satisfy (TraPot).*
For every $\rho_1,\rho_2>0$ the set

$$\mathcal{S}=\left\{(u_1,u_2,\omega_1,\omega_2,\gamma,\alpha)\in\mathcal{H}\times\mathbb{R}^4:\ \begin{array}{c}\gamma>0,\ \alpha>\lambda_{V_1}\rho_1+\lambda_{V_2}\rho_2,\\ \|(u_1,u_2)\|^2_{\mathcal{H}}=\alpha,\ \int_\Omega u_i^2\,dx=\rho_i,\\ u_i>0,\quad \text{system (2.2) holds}\end{array}\right\}$$

is a smooth curve which can be parameterized by a unique map

$$\alpha\mapsto(u_1(\alpha),u_2(\alpha),\omega_1(\alpha),\omega_2(\alpha),\gamma(\alpha)),$$

so that $(u_1(\alpha),u_2(\alpha))$ achieves $M(\alpha,\rho_1,\rho_2)$. Moreover, if $(u_1,u_2,\omega_1,\omega_2,\gamma,\cdot)\in\mathcal{S}$, then γ is increasing to $+\infty$. As a consequence, the standing wave $(e^{\mathrm{i}t\omega_1}\sqrt{\gamma}u_1,e^{\mathrm{i}t\omega_2}\sqrt{\gamma}u_2)$ is the only positive solution to (1.2) corresponding to $m_i=\gamma\rho_i$. Furthermore, it is conditionally orbitally stable for (1.1), in the sense of Theorem 1.2.

In the rest of the section, we assume that $\mu_1,\mu_2<0$, $\beta^2<\mu_1\mu_2$, and V_1,V_2 satisfy (TraPot). In particular, (2.1) rewrites as

$$-M(\alpha,\rho_1,\rho_2)=\inf_{\tilde{\mathcal{U}}(\alpha,\rho_1,\rho_2)}\int_\Omega\left(\frac{|\mu_1|}{4}u_1^4-\frac{\beta}{2}u_1^2u_2^2+\frac{|\mu_2|}{4}u_2^4\right)dx.$$

Since $\beta^2<\mu_1\mu_2$, we are minimizing a functional which is positive and coercive. Moreover, a convexity property holds in the following sense:

$$|m_1|u_1^4-2bu_1^2u_2^2+|m_2|u_2^4=G(u_1^2,u_2^2),\quad\text{with } G \text{ convex}.$$

This is sufficient to ensure the following uniqueness result.

Lemma 4.10. *For fixed $\omega_1,\omega_2\in\mathbb{R}$ and $\gamma>0$ there is at most one positive solution $(u_1,u_2)\in\mathcal{H}$ of system (2.2).*

Proof. In the case Ω bounded, the result is proved in [2, Theorem 4.1]. If $\Omega=\mathbb{R}^N$, $\beta<0$ and $V_1(x)=V_2(x)=|x|^2$, the uniqueness is shown in [1]. Let us give a sketch of such proof. Take two couples of solutions (u_1,u_2) and (v_1,v_2) and let $w_i=u_i/v_i$. Then

$$-\nabla\cdot(v_i^2\nabla w_i)=u_i\Delta v_i-v_i\Delta u_i=\gamma w_iv_i^2\left(\mu_iv_i^2(w_i^2-1)+\beta v_j^2(w_j^2-1)\right).$$

We test by $(w_i^2-1)/w_i$ in a ball of radius R to obtain

$$\begin{aligned}\int_{B_R}v_i^2|\nabla w_i|^2\left(1+\frac{1}{w_i^2}\right)dx&=\gamma\int_{B_R}v_i^2(w_i^2-1)\left(\mu_iv_i^2(w_i^2-1)+\beta v_j^2(w_j^2-1)\right)\\&\quad+\int_{\partial B_R}\left[\left(u_i-\frac{v_i^2}{u_i}\right)\nabla u_i-\left(\frac{u_i^2}{v_i}-v_i\right)\nabla v_i\right]\cdot\nu\,d\sigma.\end{aligned}$$

Since $\mu_i<0$ and $\beta^2<\mu_1\mu_2$, there exists $\kappa>0$ such that

$$|\beta|\le\sqrt{|\mu_1|-\kappa}\sqrt{|\mu_2|-\kappa}, \tag{4.10}$$

so that the previous equality implies

$$\sum_{i=1}^{2}\int_{B_R}\left[v_i^2|\nabla w_i|^2\left(1+\frac{1}{w_i^2}\right)+\gamma k v_i^4(w_i^2-1)^2\right]dx$$
$$\leq \sum_{i=1}^{2}\int_{\partial B_R}\left[\left(u_i-\frac{v_i^2}{u_i}\right)\nabla u_i-\left(\frac{u_i^2}{v_i}-v_i\right)\nabla v_i\right]\cdot\nu\, d\sigma.$$

In [1, Proposition 2.3], suitable a priori estimates are obtained, which yield the existence of a sequence $R_k\to\infty$ such that

$$\sum_{i=1}^{2}\int_{\partial B_{R_k}}\left[\left(u_i-\frac{v_i^2}{u_i}\right)\nabla u_i-\left(\frac{u_i^2}{v_i}-v_i\right)\nabla v_i\right]\cdot\nu\, d\sigma\to 0 \tag{4.11}$$

as $R_k\to\infty$, which provides $u_i=v_i$ for $i=1,2$.

The same scheme can be applied also in the case of more general potentials satisfying (TraPot), exploiting the following a priori estimates. □

Lemma 4.11. *Let V_1,V_2 satisfy (TraPot), $\mu_1,\mu_2<0$, and $\beta<\sqrt{\mu_1\mu_2}$. There exist constants $C,c_0,R_0>0$, depending only on $\mu_1,\mu_2,\beta,V_1,V_2,\omega_1,\omega_2$, such that every positive solution $(u_1,u_2)\in\mathcal{H}$ of*

$$\begin{cases} -\Delta u_1+(V_1(x)+\omega_1)u_1=\mu_1u_1^3+\beta u_1u_2^2 & \mathbb{R}^N\\ -\Delta u_2+(V_2(x)+\omega_2)u_2=\mu_2u_2^3+\beta u_2u_1^2 & \mathbb{R}^N\end{cases}$$

satisfies

$$\|u_i\|_{L^\infty(\mathbb{R}^N)}\leq C,\qquad u_i(x)\leq Ce^{-\sqrt{c_0}(|x|-R_0)},\ |x|\geq R_0,\qquad i=1,2.$$

Proof. Uniform bounds. Let $m=(\mu_2/\mu_1)^{1/4}$. Proceeding as in [12, Theorem 2.1], we see that there exists $\delta>0$ such that

$$m(\mu_1u_1^3+\beta u_1u_2^2)+\mu_2u_2^3+\beta u_1^2u_2\leq-\delta(mu_1+u_2)^3, \tag{4.12}$$

for every $u_1,u_2>0$. Let

$$M=\max_{x\in\mathbb{R}^N}(-V_1(x)-\omega_1,-V_2(x)-\omega_2).$$

Notice that $M>0$ since

$$\int_{\mathbb{R}^N}[(V_1(x)+\omega_1)u_1^2+(V_2(x)+\omega_2)u_2^2]\,dx=$$
$$=\int_{\mathbb{R}^N}[-|\nabla u_1|^2-|\nabla u_2|^2+\mu_1u_1^4+\mu_2u_2^4+2\beta u_1^2u_2^2]\,dx<0,$$

and hence we can define

$$z=\sqrt{\delta}(mu_1+u_2)-\sqrt{M}.$$

By applying in turn Kato's inequality, the definition of M and (4.12), we have (here χ_U is the characteristic function of the set $U\subset\mathbb{R}^N$)

$$\begin{aligned}\Delta z^+\geq \chi_{\{z\geq0\}}\sqrt{\delta}[&mu_1(V_1(x)+\omega_1)-m(\mu_1u_1^3+\beta u_1u_2^2)\\ &+u_2(V_2(x)+\omega_2)-(\mu_2u_2^3+\beta u_1^2u_2)]\\ \geq{}& \chi_{\{z\geq0\}}\sqrt{\delta}(mu_1+u_2)[-M+\delta(mu_1+u_2)^2].\end{aligned}$$

We replace the definition of z in the right hand side above to obtain

$$\Delta z^+\geq\chi_{\{z\geq0\}}(z+\sqrt{M})[-M+(z+\sqrt{M})^2]\geq(z^+)^3,$$

which implies $z^+ \equiv 0$ by the non-existence result [8, Lemma 2], and hence the L^∞-bounds.

Decay at infinity. By the previous step and by (TraPot), there exist $c_0, R_0 > 0$ such that

$$V_i(x) + \omega_i - \beta u_j^2 \geq V_i(x) + \omega_i - |\beta| \|u_j\|^2_{L^\infty(\mathbb{R}^N)} \geq c_0, \quad |x| \geq R_0,$$

$i = 1, 2$, $j \neq i$. Then

$$-\Delta u_i + c_0 u_i \leq \mu_i u_i^3 < 0, \quad |x| \geq R_0,$$

$i = 1, 2$. Let

$$W_i(r) = \|u_i\|_{L^\infty(\mathbb{R}^N)} e^{-\sqrt{c_0}(r-R_0)}, \quad r = |x| \geq R_0,$$

then we have $W_i(R_0) \geq \max_{\partial B_{R_0}} u_i$ and

$$-\Delta W_i + c_0 W_i \geq 0, \quad r \geq R_0.$$

By the maximum principle (which applies thanks to (2.5)), we deduce that $u_i(x) \leq W_i(|x|)$ for $|x| \geq R_0$. □

We define a map $\Phi : \mathcal{H} \times \mathbb{R}^4 \to \mathcal{H}^* \times \mathbb{R}^3$ acting on $(u_1, u_2, \omega_1, \omega_2, \gamma, \alpha) \in \mathcal{H} \times \mathbb{R}^4$ as follows

$$\begin{aligned} &\text{for } i = 1, 2 \qquad \Phi_i = \Delta u_i - (V_i(x) + \omega_i) u_i + \gamma u_i (\mu_i u_i^2 + \beta u_j^2), \quad j \neq i \\ &\text{for } i = 3, 4 \qquad \Phi_i = \int_\Omega u_i^2 \, dx - \rho_i, \\ &\qquad\qquad\qquad \Phi_5 = \|(u_1, u_2)\|^2_{\mathcal{H}} - \alpha. \end{aligned} \tag{4.13}$$

Lemma 4.12. *If $(u_1, u_2, \omega_1, \omega_2, \gamma, \alpha) \in \mathcal{S}$, then the linear bounded operator*

$$L = \Phi_{(u_1,u_2,\omega_1,\omega_2,\gamma)}(u_1, u_2, \omega_1, \omega_2, \gamma, \alpha) : \mathcal{H} \times \mathbb{R}^3 \to \mathcal{H}^* \times \mathbb{R}^3$$

is invertible.

Proof. By Fredholm's Alternative, it will be enough to prove that L is injective. L acts on (v_1, v_2, o_1, o_2, g) as follows:

$$L_i = \Delta v_i - (V_i(x) + \omega_i) v_i - o_i u_i + g u_i (\mu_i u_i^2 + \beta u_j^2) + \gamma (3\mu_i u_i^2 v_i + \beta u_j^2 v_i + 2\beta u_1 u_2 v_j),$$

for $i = 1, 2$, $L_i = 2\int_\Omega u_i v_i \, dx$ for $i = 3, 4$, and

$$L_5 = 2 \int_\Omega (\nabla u_1 \cdot \nabla v_1 + V_1(x) u_1 v_1 + \nabla u_2 \cdot \nabla v_2 + V_2(x) u_2 v_2) \, dx.$$

Suppose that $L(v_1, v_2, o_1, o_2, g) = 0$. Testing the equation for u_i by v_i, taking the sum for $i = 1, 2$ and using $L_3 = L_4 = L_5 = 0$ we find

$$\sum_{\substack{i=1 \\ j \neq i}}^{2} \int_\Omega u_i v_i (\mu_i u_i^2 + \beta u_j^2) \, dx = 0. \tag{4.14}$$

Testing the equation $L_i = 0$ by v_i for $i = 1, 2$, taking the sum and using the previous equality, we obtain

$$\sum_{\substack{i=1 \\ j \neq i}}^{2} \int_\Omega \left\{ |\nabla v_i|^2 + (V_i(x) + \omega_i) v_i^2 - \gamma v_i^2 (3\mu_i u_i^2 + \beta u_j^2) \right\} dx - 4\gamma\beta \int_\Omega u_1 u_2 v_1 v_2 \, dx = 0. \tag{4.15}$$

On the other hand, testing the equation for u_i by v_i^2/u_i leads to (the boundary term vanishes as in (4.11))

$$\begin{aligned}\int_\Omega \left\{-(V_i(x)+\omega_i)+\gamma(\mu_i u_i^2+\beta u_j^2)\right\} v_i^2\,dx &= \int_\Omega \nabla u_i\cdot\nabla\left(\frac{v_i^2}{u_i}\right)dx\\ &= -\int_\Omega\left|\frac{v_i}{u_i}\nabla u_i-\nabla v_i\right|^2 dx+\int_\Omega|\nabla v_i|^2\,dx\le\int_\Omega|\nabla v_i|^2\,dx.\end{aligned}\tag{4.16}$$

Taking the sum for $i=1,2$ and exploiting (4.15) we obtain

$$\sum_{\substack{i=1\\ j\neq i}}^2\int_\Omega \mu_i u_i^2 v_i^2\,dx+2\beta\int_\Omega u_1u_2v_1v_2\,dx\ge 0.$$

This, together with (4.10), implies $-\kappa\int_\Omega(u_1^2v_1^2+u_2^2v_2^2)\,dx\ge0$, and hence $v_1\equiv v_2\equiv 0$. In turn, the equations $L_1=L_2=0$ become $g(\mu_i u_i^2+\beta u_j^2)=o_i$, $i=1,2$, which provides $g=o_1=o_2=0$. □

Reasoning as in the previous proof, we also have the following related result regarding non degeneracy.

Lemma 4.13. *Given $\omega_1,\omega_2\in\mathbb{R}$ and $\gamma>0$, the positive solution $(u_1,u_2)\in\mathcal{H}$ of system (2.2) is non degenerate as a critical point of the action functional*

$$\mathcal{A}_{\gamma,\omega_1,\omega_2}(u_1,u_2)=\frac12\|(u_1,u_2)\|^2_{\mathcal{H}}-\gamma F(u_1,u_2)+\frac{\omega_1}{2}\mathcal{Q}_1(u_1)+\frac{\omega_2}{2}\mathcal{Q}_2(u_2).\tag{4.17}$$

Proof. Take $(v_1,v_2)\in\mathcal{H}$ such that

$$-\Delta v_i+(V_i(x)+\omega_i)v_i=\gamma(3\mu_i u_i^2 v_i+\beta u_j^2 v_i+2\beta u_1u_2v_j)\qquad\text{for } i,j=1,2,\ j\neq i.$$

By testing the equation of v_i by v_i itself, integrating by parts and summing up, we are lead to (4.15). Following the previous proof, we can then obtain once again that $-\kappa\int_\Omega(u_1^2v_1^2+u_2^2v_2^2)\,dx\ge0$, and hence $v_1\equiv v_2\equiv0$, which proves the claim. □

Lemma 4.14. *If $(u_1,u_2,\omega_1,\omega_2,\gamma,\alpha)\in\mathcal{S}$ then $\gamma'(\alpha)>0$ for every $\alpha>\lambda_{V_1}\rho_1+\lambda_{V_2}\rho_2$.*

Proof. Fix α and consider the corresponding $(u_1,u_2,\omega_1,\omega_2,\gamma)$. Observe that, due to the assumptions on β,μ_1,μ_2, the functional $\mathcal{A}_{\gamma,\omega_1,\omega_2}$ admits a global minimum in $\mathcal{H}$. From the uniqueness result of Lemma 4.10, we deduce that actually

$$\min_{\mathcal{H}}\mathcal{A}_{\gamma,\omega_1,\omega_2}=\mathcal{A}_{\gamma,\omega_1,\omega_2}(u_1,u_2).$$

By combining this with the non degeneracy result of Lemma 4.13, we have that

$$\mathcal{A}''_{\gamma,\omega_1,\omega_2}(u_1,u_2)[(\phi_1,\phi_2),(\phi_1,\phi_2)]>0\qquad\forall\,(\phi_1,\phi_2)\neq(0,0).\tag{4.18}$$

Thanks to Lemma 4.12, we can locally differentiate the elements of $\mathcal{S}$ with respect to α. Let

$$\frac{d}{d\alpha}(u_1(\alpha),u_2(\alpha))=:(v_1(\alpha),v_2(\alpha)).$$

Then for $i,j=1,2$, $j\neq i$, we have

$$-\Delta v_i+(V_i(x)+\omega_i)v_i+\omega_i'u_i=\gamma(3\mu_iu_i^2v_i+\beta v_iu_j^2+2\beta u_1u_2v_j)+\gamma'u_i(\mu_iu_i^2+\beta u_j^2)\tag{4.19}$$

and identity (3.7) hold. By taking $(\phi_1, \phi_2) = (v_1, v_2)$ in (4.18), and using (4.19), (3.7), we deduce

$$\begin{aligned}\mathcal{A}''_{\gamma,\omega_1,\omega_2}(u_1,u_2)[(v_1,v_2),(v_1,v_2)] &= \sum_{i=1}^{2}\int_\Omega (|\nabla v_i|^2 + (V_i(x)+\omega_i)v_i^2 - 3\gamma\mu_i u_i^2 v_i^2)\,dx \\ &\quad -\gamma\beta\int_\Omega (v_1^2u_2^2 + 4u_1u_2v_1v_2 + u_1^2v_2^2)\,dx \\ &= \gamma'\int_\Omega (\mu_1 u_1^3 v_1 + \mu_2 u_2^3 v_2 + \beta u_1u_2(v_1u_2+u_1v_2))\,dx = \frac{\gamma'}{2\gamma} > 0,\end{aligned}$$

which yields $\gamma' > 0$. ☐

Remark 4.15. In the assumptions of the previous lemma, proceeding very similarly to [26, Lemma 5.6], it is also possible to prove that $\omega_1'(\alpha)\rho_1 + \omega_2'(\alpha)\rho_2 < 0$.

End of the proof of Theorem 4.9. Combining Lemma 4.12 with Lemma 4.8, and proceeding as in [26, Proposition 5.4] we obtain that $\mathcal{S}$ is a smooth curve which can be parameterized by a unique map in α. Theorem 1.1 and 1.2 apply, providing the existence (and uniqueness) of the corresponding family of standing waves, which are stable by Lemma 4.14. Finally, by minimizing the energy

$$\mathcal{E}_\gamma(u_1,u_2) = \frac{1}{2}\|(u_1,u_2)\|^2_{\mathcal{H}} - \gamma F(u_1,u_2)$$

with $\mathcal{Q}(u_i) = \rho_i$, we obtain existence of elements of $\mathcal{S}$ for every $\gamma > 0$ ☐

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