On the energy inequality for weak solutions to the Navier–Stokes equations of compressible fluids on unbounded domains

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1. Introduction

Let $T > 0$ be fixed and let $\Omega \subset \mathbb{R}^3$ be an exterior domain (an unbounded domain with compact Lipschitz boundary $\partial\Omega$). We consider the Navier–Stokes equations of a compressible isentropic viscous fluid

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \varrho^\gamma = \mathrm{div}_x \mathbb{S}, \tag{1.2}$$

in the unknown variables

$$\varrho = \varrho(x,t) : \Omega \times (0,T) \to \mathbb{R} \quad \text{and} \quad \mathbf{u} = \mathbf{u}(x,t) : \Omega \times (0,T) \to \mathbb{R}^3$$

representing the density and velocity of the fluid, respectively. The adiabatic constant γ is subjected to the technical constraint

$$\gamma > \frac{3}{2},$$

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while the viscous stress tensor $\mathbb{S}$ fulfills *Newton's rheological law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I},$$

with shear viscosity coefficient μ and bulk viscosity coefficient η satisfying

$$\mu > 0 \quad \text{and} \quad \eta \geq 0.$$

Accordingly, we may write

$$\mathrm{div}_x \mathbb{S} = \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla_x \mathrm{div}_x \mathbf{u},$$

where

$$\lambda = \eta - \frac{2}{3}\mu.$$

As the underlying physical domain is unbounded, the system is supplemented with the far field conditions

$$\lim_{|x|\to\infty} \varrho(x,t) = 0, \qquad \lim_{|x|\to\infty} \mathbf{u}(x,t) = 0,$$

and the no-slip boundary condition

$$\mathbf{u}(x,t)_{|x\in\partial\Omega} = 0.$$

Finally, we prescribe the initial conditions

$$\varrho(0) = \varrho_0 \quad \text{and} \quad (\varrho\mathbf{u})(0) = \boldsymbol{q}_0$$

where ϱ_0 and $\boldsymbol{q}_0$ are given functions, complying with the following assumptions:

- $\varrho_0 \in L^1(\Omega) \cap L^\gamma(\Omega)$ and $\varrho_0 \geq 0$ almost everywhere.
- $\boldsymbol{q}_0 \in L_{\mathrm{loc}}^{\frac{2\gamma}{\gamma+1}}(\overline{\Omega}; \mathbb{R}^3)$ is such that $\boldsymbol{q}_0(x) = 0$ whenever $\varrho_0(x) = 0$. Moreover

$$\frac{|\boldsymbol{q}_0|^2}{\varrho_0} \in L^1(\Omega).$$

Under the above general assumptions on the structural coefficients and the initial data, the compressible Navier–Stokes equations (1.1)–(1.2) are known to admit at least one globally defined weak solution[1] $(\varrho, \mathbf{u})$ (see [5,12,14]). In particular, for the energy functional

$$E(t) = \int_\Omega \Big[\frac{1}{2}\varrho|\mathbf{u}|^2(t) + \frac{1}{\gamma-1}\varrho^\gamma(t)\Big]\, \mathrm{d}x,$$

the weak solutions can be constructed to fulfill the *energy inequality in the integral form*

$$E(t) + \int_0^t \int_\Omega \big[\mu|\nabla_x \mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}_x \mathbf{u}|^2\big]\, \mathrm{d}x\, \mathrm{d}r \leq E_0 \tag{1.3}$$

for almost any $t \in [0,T]$, where

$$E_0 = \int_\Omega \Big[\frac{1}{2}\frac{|\boldsymbol{q}_0|^2}{\varrho_0} + \frac{1}{\gamma-1}\varrho_0^\gamma\Big]\, \mathrm{d}x.$$

[1] See Section 3 for the precise definition.

In this paper, we show the existence of globally defined weak solutions with associated energy $E(t)$ satisfying the *energy inequality in the differential form*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} E + \int_\Omega \left[\mu|\nabla_x \mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}_x \mathbf{u}|^2\right] \mathrm{d}x \le 0, \quad \text{in } \mathcal{D}'(0,T), \\ \operatorname{ess}\limsup_{t\to 0^+} E(t) \le E_0. \end{cases} \tag{1.4}$$

Although the existence of such solutions on a bounded domain is relatively easy to show (cf. Kukučka [10] for a rather general class of bounded domains with rough boundaries), the case of unbounded domains is not obvious and, as a matter of fact, an open problem (see the discussion in [14]). We also note that some results on the long-time behavior of solutions to the compressible Navier–Stokes system require (1.4) rather than (1.3), see [6]. In the incompressible situation, analogous results have been obtained by Leray [11] when $\Omega = \mathbb{R}^3$ and by Galdi, Maremonti, Miyakawa and Sohr [8,13] when Ω is an exterior domain. Our proof relies on some refined pressure estimates on bounded domains (see Lemma 3) and the fact that the total mass of the fluid is finite and conserved (see Definition 1). When the mass of the fluid is infinite, namely, when

$$\lim_{|x|\to\infty} \varrho(x,t) > 0,$$

the existence of globally defined weak solutions satisfying (1.4) remains an open question.

Remark. It is immediate to see that inequality (1.4) is equivalent to

$$E(t) + \int_s^t \int_\Omega \left[\mu|\nabla_x \mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}_x \mathbf{u}|^2\right] \mathrm{d}x\, \mathrm{d}r \le E(s),$$

for almost every $t, s \in [0,T]$ with $t \ge s$, the case $s = 0$ included.

Plan of the paper

In Section 2 we introduce the notation and we recall some basic tools needed in the analysis. In Section 3 we state the main result, whose proof is carried out in Section 4. Section 5 is devoted to final remarks.

2. Preliminaries and notation

For $p \in [1,\infty]$ and $k \in \mathbb{N}$, the symbols $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{W}_0^{k,p}(\Omega)$ will denote the usual Lebesgue and Sobolev spaces of vector or tensor valued functions on Ω. We introduce the homogeneous Sobolev space $D_0^{1,2}(\Omega)$, defined as the completion of $C^\infty_{\mathrm{cpt}}(\Omega)$ with respect to the norm

$$\|f\|_{D_0^{1,2}(\Omega)} = \left(\int_\Omega |\nabla f|^2 \,\mathrm{d}x\right)^{1/2}.$$

Accordingly, we set

$$\boldsymbol{D}_0^{1,2}(\Omega) = D_0^{1,2}(\Omega;\mathbb{R}^3).$$

Given a bounded Lipschitz domain $\Pi \subset \mathbb{R}^3$, we have the *Sobolev inequality*

$$\|f\|_{L^{\frac{3p}{3-p}}(\Pi)} \le c(p)\|\nabla f\|_{L^p(\Pi)}, \quad 1 \le p < 3, \tag{2.1}$$

for every $f \in W_0^{1,p}(\Pi)$, with a constant $c(p) > 0$ independent of Π (see e.g. [9]). In particular, when $p = 2$,

$$\|f\|_{L^6(\Pi)} \le c(p)\|\nabla f\|_{L^2(\Pi)}. \tag{2.2}$$

We will also need the inequality

$$\Big\| f - \frac{1}{|\varPi|}\int_{\varPi} f \,\mathrm{d}x \Big\|_{L^p(\varPi)} \le 2\|f\|_{L^p(\varPi)}, \tag{2.3}$$

where $|\varPi|$ denotes the measure of $\varPi$. Finally, for $1 < q, p < \infty$, we introduce the Banach space

$$\boldsymbol{E}^{q,p}(\varPi) = \big\{ f \in \boldsymbol{L}^q(\varPi) : \operatorname{div} f \in L^p(\varPi) \big\}$$

endowed with the norm

$$\|f\|_{\boldsymbol{E}^{q,p}(\varPi)} = \|f\|_{\boldsymbol{L}^q(\varPi)} + \|\operatorname{div} f\|_{L^p(\varPi)}.$$

We will also encounter the space $\boldsymbol{E}_0^{q,p}(\varPi)$, defined as the completion of $C^\infty_{\mathrm{cpt}}(\varPi)$ with respect to the above norm.

Remark. The space $D_0^{1,2}(\varPi)$ coincides with $W_0^{1,2}(\varPi)$ as long as $\varPi$ is bounded Lipschitz.

3. The main result

We begin by recalling the definition of weak solution to Navier–Stokes system (1.1)–(1.2).

Definition 1. A couple $(\varrho, \mathbf{u})$ is called *weak solution* of the Navier–Stokes equations (1.1) and (1.2) with initial data ϱ_0 and $\boldsymbol{q}_0$ if the following conditions are satisfied:

- The density ϱ is nonnegative and such that

$$\varrho \in L^\infty(0,T; L^1(\varOmega) \cap L^\gamma(\varOmega)).$$

- The velocity $\mathbf{u}$ is such that

$$\mathbf{u} \in L^2(0,T; \boldsymbol{D}_0^{1,2}(\varOmega)).$$

- The total mass of the fluid is conserved, namely,

$$\int_{\varOmega} \varrho(t)\,\mathrm{d}x = \int_{\varOmega} \varrho_0 \,\mathrm{d}x = M_0, \quad \forall t \in [0,T]. \tag{3.1}$$

- The integral identity

$$\int_0^T \int_{\varOmega} \big[\varrho B(\varrho)(\partial_t \varphi + \mathbf{u}\cdot\nabla_x \varphi) - b(\varrho)\operatorname{div}_x \mathbf{u}\,\varphi\big] \,\mathrm{d}x\,\mathrm{d}t = -\int_{\varOmega} \varrho_0 B(\varrho_0)\varphi(0)\,\mathrm{d}x$$

  holds for every $\varphi \in C^\infty_{\mathrm{cpt}}([0,T) \times \overline{\varOmega})$ and every function $B : [0,\infty) \to \mathbb{R}$ such that

$$B(r) = B(1) + \int_1^r \frac{b(s)}{s^2}\,\mathrm{d}s$$

  for some bounded $b \in C[0,\infty)$.

- The integral identity

$$\int_0^T \int_\Omega \left[\varrho\mathbf{u}\cdot\partial_t\varphi + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\varphi + \varrho^\gamma \mathrm{div}_x\varphi\right] \mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mathbb{S} : \nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \boldsymbol{q}_0\varphi(0)\,\mathrm{d}x$$

holds for every $\varphi \in C^\infty_{\mathrm{cpt}}([0,T)\times\Omega;\mathbb{R}^3)$.
- The energy

$$E(t) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2(t) + \frac{1}{\gamma-1}\varrho^\gamma(t)\right] \mathrm{d}x$$

satisfies inequality (1.3) for almost any $t \in [0,T]$.

Remark. Observe that, due to (1.3), the function $\varrho|\mathbf{u}|^2$ belongs to $L^\infty(0,T;L^1(\Omega))$.

The main result of the paper reads as follows.

Theorem 2. *Within our general assumptions on the structural coefficients and the initial data, the compressible Navier–Stokes equations (1.1)–(1.2) admit at least one weak solution whose associated energy satisfies the inequality in differential form (1.4).*

4. Proof of Theorem 2

Along the paper, B_R will denote the three-dimensional open ball of radius R centered at 0. Without loss of generality, we suppose that

$$\mathbb{R}^3 \setminus \Omega \subset B_1 \tag{4.1}$$

and, for every $n \in \mathbb{N}$, we set

$$\Omega_n = \Omega \cap B_n.$$

We also denote by ϱ_{0n} and $\boldsymbol{q}_{0n}$ the restriction of ϱ_0 and $\boldsymbol{q}_0$ to Ω_n.

In light of the results [2,4,12], for all $n \in \mathbb{N}$ there exists a globally defined weak solution $(\varrho_n, \mathbf{u}_n)$ to the Navier–Stokes equations (1.1)–(1.2) in $\Omega_n \times (0,T)$ with initial datum $(\varrho_{0n}, \boldsymbol{q}_{0n})$ and no-slip boundary conditions

$$\mathbf{u}_{n|\partial\Omega_n} = 0,$$

whose associated energy functional

$$E_n(t) = \int_{\Omega_n} \left[\frac{1}{2}\varrho_n|\mathbf{u}_n|^2(t) + \frac{1}{\gamma-1}\varrho_n^\gamma(t)\right] \mathrm{d}x$$

satisfies the energy inequality in differential form

$$\frac{\mathrm{d}}{\mathrm{d}t}E_n + \int_{\Omega_n} \left[\mu|\nabla_x\mathbf{u}_n|^2 + (\lambda+\mu)|\mathrm{div}_x\mathbf{u}_n|^2\right] \mathrm{d}x \le 0 \tag{4.2}$$

in $\mathcal{D}'(0,T)$. We extend $\varrho_n, \mathbf{u}_n$ to be zero outside $\Omega_n\times(0,T)$. We also know that the density ϱ_n is nonnegative and such that

$$\int_{\Omega_n} \varrho_n(t)\,\mathrm{d}x = \int_{\Omega_n} \varrho_{0n}\,\mathrm{d}x \le \int_\Omega \varrho_0\,\mathrm{d}x = M_0, \quad \forall t\in[0,T]. \tag{4.3}$$

Remark. Note that the density ϱ_n as a function of the time t is continuous with values in $L^1(\Omega_n)$, see e.g. [12].

Moreover, the following estimates hold:

$$\sup_{t\in[0,T]} \|\varrho_n(t)\|_{L^\gamma(\Omega_n)} \leq c(E_0), \tag{4.4}$$

$$\operatorname{ess}\sup_{t\in[0,T]} \|\sqrt{\varrho_n}\mathbf{u}_n(t)\|_{\boldsymbol{L}^2(\Omega_n)} \leq c(E_0), \tag{4.5}$$

$$\int_0^T \int_{\Omega_n} |\nabla_x \mathbf{u}_n|^2 \,\mathrm{d}x \,\mathrm{d}t \leq c(E_0), \tag{4.6}$$

where the constant $c(E_0) > 0$ depends on the initial energy E_0, but is independent on n. In particular, from (4.6),

$$\|\mathbb{S}(\nabla_x \mathbf{u}_n)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega_n))} \leq c(E_0).$$

In addition, exploiting (4.3), (4.4) and interpolation,

$$\sup_{t\in[0,T]} \|\varrho_n(t)\|_{L^p(\Omega_n)} \leq c(M_0, E_0), \quad 1 \leq p \leq \gamma, \tag{4.7}$$

for some constant $c(M_0, E_0) > 0$ depending on M_0 and E_0, but independent of n. All these uniform estimates, as well as the Sobolev embeddings (2.1) and (2.2), will be used several times in what follows, often without explicit mention.

Refined pressure estimates

Our goal is to prove that the sequence ϱ_n is uniformly bounded in $L^{\gamma+\theta}(\Omega_n \times (0,T))$ for some $\theta = \theta(\gamma) > 0$.

Lemma 3. *There exists $\theta = \theta(\gamma) > 0$ such that the estimate*

$$\int_0^T \int_{\Omega_n} \varrho_n^{\gamma+\theta} \,\mathrm{d}x \,\mathrm{d}t \leq c(T, M_0, E_0)$$

holds for some constant $c(T, M_0, E_0) > 0$ depending on T, the initial mass M_0 and the initial energy E_0, but independent on n.

We need a preliminary result [1,7,14] concerning the existence of the so-called Bogovskii's operator $\mathcal{B}_\Pi$. In particular, for the statement (iv), see e.g. Lemmas 3.17 and 3.18 in [14].

Lemma 4. *Let $\Pi \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then, there exists a bounded linear operator*

$$\mathcal{B}_\Pi : \left\{ f \in L^p(\Pi) : \int_\Pi f \,\mathrm{d}x = 0 \right\} \to \boldsymbol{W}_0^{1,p}(\Pi), \quad 1 < p < \infty,$$

such that:

(i) *the function $v = \mathcal{B}_\Pi(f)$ solves the problem*

$$\operatorname{div} v = f$$

almost everywhere in Π.

(ii) *the inequality*

$$\|\nabla \mathcal{B}_{\Pi}(f)\|_{\boldsymbol{L}^p(\Pi)} \leq c_1(p,\Pi)\|f\|_{L^p(\Pi)}$$

holds with a constant $c_1(p,\Pi) > 0$ *depending on* p *and* Π.

(iii) *if* $f = \operatorname{div} g$ *for some* $g \in \boldsymbol{E}_0^{q,p}(\Pi)$ *and* $1 < q < \infty$, *then the inequality*

$$\|\mathcal{B}_{\Pi}(f)\|_{\boldsymbol{L}^q(\Pi)} \leq c_2(q,\Pi)\|g\|_{\boldsymbol{L}^q(\Pi)}$$

holds with a constant $c_2(q,\Pi) > 0$ *depending on* q *and* Π.

(iv) *when* $\Pi = B_R$, *the constants* c_1 *and* c_2 *appearing in the estimates above are independent of the radius* R, *namely,*

$$c_1(p,B_R) = c_1(p) \quad \textit{and} \quad c_2(q,B_R) = c_2(q).$$

We are now in a position to prove Lemma 3. Along the proof, $c(T,M_0,E_0) > 0$ will denote a *generic* constant depending on T, M_0 and E_0, but independent on n.

For a fixed

$$0 < \theta < \frac{\gamma}{3}$$

to be suitably chosen later, we compute, with help of (4.3),

$$\begin{aligned}\int_0^T \int_{\Omega_n} \varrho_n^{\gamma+\theta} \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\{x\in\Omega_n:\varrho_n(x,t)\leq 1\}} \varrho_n^{\gamma+\theta} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\{x\in\Omega_n:\varrho_n(x,t)>1\}} \varrho_n^{\gamma+\theta} \,\mathrm{d}x\,\mathrm{d}t \\ &\leq M_0 T + \int_0^T \int_{\Omega_1} \varrho_n^{\gamma+\theta} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\{x\in\Omega_n\setminus\Omega_1:\varrho_n(x,t)>1\}} \varrho_n^{\gamma+\theta} \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Exploiting statements (i)–(iii) of Lemma 4 and performing similar computations as in [5,14], the second integral above can be estimated as

$$\int_0^T \int_{\Omega_1} \varrho_n^{\gamma+\theta} \,\mathrm{d}x\,\mathrm{d}t \leq c(T,E_0,\Omega_1),$$

for some positive constant $c(T,E_0,\Omega_1)$ depending on T, E_0 and the domain Ω_1, but independent on n. Therefore, we only need to control the last integral

$$\int_0^T \int_{\{x\in\Omega_n\setminus\Omega_1:\varrho_n(x,t)>1\}} \varrho_n^{\gamma+\theta} \,\mathrm{d}x\,\mathrm{d}t,$$

uniformly with respect to n. To this end, let $b \in C^2[0,\infty)$ be such that

$$b(r) = \begin{cases} 0 \quad \text{for } r \leq \frac{1}{2}, \\ b'(r) \geq 0 \quad \text{for } r \in \left(\frac{1}{2}, 1\right), \\ b(r) = r^\theta \quad \text{for } r \geq 1. \end{cases}$$

Note that, in view of (4.4),

$$\sup_{t\in[0,T]} \|b(\varrho_n)(t)\|_{L^p(\Omega_n)} \leq c(M_0,E_0), \quad 1 \leq p \leq \frac{\gamma}{\theta}. \tag{4.8}$$

With reference to Lemma 4, let now

$$\mathcal{B}_n = \mathcal{B}_{B_n}$$

be the Bogovskii operator associated to the ball B_n. Setting

$$\overline{b(\varrho_n)} = b(\varrho_n) - \frac{1}{|B_n|}\int_{B_n} b(\varrho_n)\,\mathrm{d}x$$

and exploiting properties (ii) and (iv) of Lemma 4, together with inequalities (2.3) and (4.8), we infer that

$$\sup_{t\in[0,T]} \|\nabla_x \mathcal{B}_n\left(\overline{b(\varrho_n)}\right)\|_{\boldsymbol{L}^p(B_n)} \le c(M_0,E_0), \quad 1 < p < \frac{\gamma}{\theta}. \tag{4.9}$$

By the same token, due to the Sobolev embedding (2.1) and inequality (2.3),

$$\sup_{t\in[0,T]} \|\mathcal{B}_n\left(\overline{b(\varrho_n)}\right)\|_{\boldsymbol{L}^q(B_n)} \le c(M_0,E_0), \quad \frac{3}{2} < q \le \infty. \tag{4.10}$$

At this point, taking $\psi \in C^\infty_{\mathrm{cpt}}(0,T)$, we test[2] the momentum equation (1.2) by

$$\psi(t)\phi(x)\mathcal{B}_n\left(\overline{b(\varrho_n)}\right)$$

where $\phi \in C^\infty(\mathbb{R}^3)$ is such that

$$\begin{cases} 0 \le \phi \le 1, \\ \phi \equiv 1 \quad \text{on } \mathbb{R}^3 \setminus \Omega_1, \\ \phi \equiv 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

We obtain

$$\int_0^T \psi \int_{\Omega_n} \phi \varrho_n^\gamma b(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t = \sum_{j=1}^{9} \mathfrak{I}_j$$

where

$$\mathfrak{I}_1 = \frac{1}{|B_n|}\int_0^T \psi \int_{B_n} b(\varrho_n)\,\mathrm{d}x \int_{B_n} \phi\varrho_n^\gamma\,\mathrm{d}x\,\mathrm{d}t$$

$$\mathfrak{I}_2 = -\int_0^T \psi \int_{\Omega_n} \varrho_n^\gamma \nabla_x\phi\cdot\mathcal{B}_n\left(\overline{b(\varrho_n)}\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$\mathfrak{I}_3 = \int_0^T \psi \int_{\Omega_n} \phi\,\mathbb{S}(\nabla_x\mathbf{u}_n) : \nabla_x\mathcal{B}_n\left(\overline{b(\varrho_n)}\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$\mathfrak{I}_4 = \int_0^T \psi \int_{\Omega_n} \mathbb{S}(\nabla_x\mathbf{u}_n)\cdot\nabla_x\phi\cdot\mathcal{B}_n\left(\overline{b(\varrho_n)}\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$\mathfrak{I}_5 = -\int_0^T \psi \int_{\Omega_n} \phi(\varrho_n\mathbf{u}_n\otimes\mathbf{u}_n) : \nabla_x\mathcal{B}_n\left(\overline{b(\varrho_n)}\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$\mathfrak{I}_6 = -\int_0^T \psi \int_{\Omega_n} (\varrho_n\mathbf{u}_n\otimes\mathbf{u}_n)\cdot\nabla_x\phi\cdot\mathcal{B}_n\left(\overline{b(\varrho_n)}\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$\mathfrak{I}_7 = -\int_0^T \psi' \int_{\Omega_n} \phi\varrho_n\mathbf{u}_n\cdot\mathcal{B}_n\left(\overline{b(\varrho_n)}\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$\mathfrak{I}_8 = \int_0^T \psi \int_{\Omega_n} \phi\varrho_n\mathbf{u}_n\cdot\mathcal{B}_n\left(\mathrm{div}_x(b(\varrho_n)\mathbf{u}_n)\right)\,\mathrm{d}x\,\mathrm{d}t$$

[2] The multiplication can be justified by means of a standard regularization procedure (see e.g. [3,14]).

$$\mathfrak{I}_9 = \int_0^T \psi \int_{\Omega_n} \phi \varrho_n \mathbf{u}_n \cdot \mathcal{B}_n \left(\overline{h(\varrho_n)} \right) \, \mathrm{d}x \, \mathrm{d}t$$

having set

$$h(\varrho_n) = (b'(\varrho_n)\varrho_n - b(\varrho_n)) \operatorname{div}_x \mathbf{u}_n$$

and

$$\overline{h(\varrho_n)} = h(\varrho_n) - \frac{1}{|B_n|} \int_{B_n} h(\varrho_n) \, \mathrm{d}x.$$

Our aim is to estimate the nine integrals above making use of (4.8)–(4.10). First,

$$\begin{aligned}
|\mathfrak{I}_1| &\leq \|\psi\|_{L^\infty(0,T)} \|\phi \varrho_n^\gamma\|_{L^\infty(0,T;L^1(\Omega_n))} \|b(\varrho_n)\|_{L^1((0,T)\times\Omega_n)} \\
&\leq c(T, M_0, E_0) \|\psi\|_{L^\infty(0,T)}, \\
|\mathfrak{I}_2| &\leq T \|\psi\|_{L^\infty(0,T)} \|\varrho_n^\gamma \nabla_x \phi\|_{L^\infty(0,T;\boldsymbol{L}^1(\Omega_n))} \|\mathcal{B}_n \left(\overline{b(\varrho_n)} \right) \|_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega_n))} \\
&\leq c(T, M_0, E_0) \|\psi\|_{L^\infty(0,T)} \|\nabla_x \phi\|_{\boldsymbol{L}^\infty(\mathbb{R}^3)}.
\end{aligned}$$

Then,

$$\begin{aligned}
|\mathfrak{I}_3| &\leq \|\psi\|_{L^\infty(0,T)} \|\phi \, \mathbb{S}(\nabla_x \mathbf{u}_n)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega_n))} \|\nabla_x \mathcal{B}_n \left(\overline{b(\varrho_n)} \right) \|_{L^2(0,T;\boldsymbol{L}^2(\Omega_n))} \\
&\leq c(T, M_0, E_0) \|\psi\|_{L^\infty(0,T)}, \\
|\mathfrak{I}_4| &\leq \|\psi\|_{L^\infty(0,T)} \|\mathbb{S}(\nabla_x \mathbf{u}_n) \cdot \nabla_x \phi\|_{L^2(0,T;\boldsymbol{L}^2(\Omega_n))} \|\mathcal{B}_n \left(\overline{b(\varrho_n)} \right) \|_{L^2(0,T;\boldsymbol{L}^2(\Omega_n))} \\
&\leq c(T, M_0, E_0) \|\psi\|_{L^\infty(0,T)} \|\nabla_x \phi\|_{\boldsymbol{L}^\infty(\mathbb{R}^3)}.
\end{aligned}$$

Moreover, taking

$$0 < \theta < \frac{2\gamma - 3}{3}, \tag{4.12}$$

we can also draw the controls

$$\begin{aligned}
|\mathfrak{I}_5| &\leq \|\psi\|_{L^\infty(0,T)} \|\phi \varrho_n\|_{L^\infty(0,T;L^\gamma(\Omega_n))} \|\nabla \mathbf{u}_n\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega_n))} \|\nabla_x \mathcal{B}_n \left(\overline{b(\varrho_n)} \right) \|_{L^\infty\left(0,T;\boldsymbol{L}^{\frac{3\gamma}{2\gamma-3}}(\Omega_n)\right)} \\
&\leq c(T, M_0, E_0) \|\psi\|_{L^\infty(0,T)}, \\
|\mathfrak{I}_6| &\leq \|\psi\|_{L^\infty(0,T)} \|\varrho_n \nabla_x \phi\|_{L^\infty(0,T;\boldsymbol{L}^\gamma(\Omega_n))} \|\nabla \mathbf{u}_n\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega_n))} \|\mathcal{B}_n \left(\overline{b(\varrho_n)} \right) \|_{L^\infty\left(0,T;\boldsymbol{L}^{\frac{3\gamma}{2\gamma-3}}(\Omega_n)\right)} \\
&\leq c(T, M_0, E_0) \|\psi\|_{L^\infty(0,T)} \|\nabla_x \phi\|_{\boldsymbol{L}^\infty(\mathbb{R}^3)}, \\
|\mathfrak{I}_7| &\leq \|\psi'\|_{L^1(0,T)} \|\phi \sqrt{\varrho_n}\|_{L^\infty(0,T;L^{2\gamma}(\Omega_n))} \|\sqrt{\varrho_n} \mathbf{u}_n\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega_n))} \|\mathcal{B}_n \left(\overline{b(\varrho_n)} \right) \|_{L^\infty\left(0,T;\boldsymbol{L}^{\frac{2\gamma}{\gamma-1}}(\Omega_n)\right)} \\
&\leq c(T, M_0, E_0) \|\psi'\|_{L^1(0,T)}.
\end{aligned}$$

Exploiting now properties (iii) and (iv) of Lemma 4,

$$|\mathfrak{I}_8| \leq c_2 \|\psi\|_{L^\infty(0,T)} \|\phi \varrho_n\|_{L^\infty(0,T;L^\gamma(\Omega_n))} \int_0^T \|\mathbf{u}_n(t)\|_{\boldsymbol{L}^6(\Omega_n)} \|b(\varrho_n)\mathbf{u}_n(t)\|_{\boldsymbol{L}^{\frac{6\gamma}{5\gamma-6}}(\Omega_n)} \, \mathrm{d}t.$$

Since

$$\|b(\varrho_n)\mathbf{u}_n(t)\|_{\boldsymbol{L}^{\frac{6\gamma}{5\gamma-6}}(\Omega_n)} \leq \|\mathbf{u}_n(t)\|_{\boldsymbol{L}^6(\Omega_n)} \|b(\varrho_n)(t)\|_{L^{\frac{3\gamma}{2\gamma-3}}(\Omega_n)},$$

in light of (4.8) and (4.12), we conclude that

$$|\mathfrak{I}_8| \le c_2 \|\psi\|_{L^\infty(0,T)} \|\phi\varrho_n\|_{L^\infty(0,T;L^\gamma(\Omega_n))} \|\nabla_x \mathbf{u}_n\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega_n))} \|b(\varrho_n)\|_{L^\infty\left(0,T;\boldsymbol{L}^{\frac{3\gamma}{2\gamma-3}}(\Omega_n)\right)}$$
$$\le c(T, M_0, E_0)\|\psi\|_{L^\infty(0,T)}.$$

In order to control the last integral $\mathfrak{I}_9$, we shall distinguish two cases. When $\gamma < 6$, we estimate

$$|\mathfrak{I}_9| \le \|\psi\|_{L^\infty(0,T)} \|\phi\varrho_n\|_{L^\infty(0,T;L^\gamma(\Omega_n))} \int_0^T \|\mathbf{u}_n(t)\|_{\boldsymbol{L}^6(\Omega_n)} \|\mathcal{B}_n\left(\overline{h(\varrho_n)}\right)(t)\|_{\boldsymbol{L}^{\frac{6\gamma}{5\gamma-6}}(\Omega_n)}.$$

In addition,

$$\|\mathcal{B}_n\left(\overline{h(\varrho_n)}\right)(t)\|_{\boldsymbol{L}^{\frac{6\gamma}{5\gamma-6}}(\Omega_n)} \le \|h(\varrho_n)(t)\|_{L^{\frac{6\gamma}{7\gamma-6}}(\Omega_n)}$$
$$\le \|\nabla_x \mathbf{u}_n(t)\|_{\boldsymbol{L}^2(\Omega_n)} \|b'(\varrho_n)\varrho_n - b(\varrho_n)(t)\|_{L^{\frac{3\gamma}{2\gamma-3}}(\Omega_n)},$$

yielding

$$|\mathfrak{I}_9| \le \|\psi\|_{L^\infty(0,T)} \|\phi\varrho_n\|_{L^\infty(0,T;L^\gamma(\Omega_n))} \|\nabla_x \mathbf{u}_n\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega_n))} \cdot \|b'(\varrho_n)\varrho_n - b(\varrho_n)\|_{L^\infty\left(0,T;L^{\frac{3\gamma}{2\gamma-3}}(\Omega_n)\right)}$$
$$\le c(T, M_0, E_0)\|\psi\|_{L^\infty(0,T)}.$$

On the other hand, when $\gamma \ge 6$,

$$|\mathfrak{I}_9| \le \|\psi\|_{L^\infty(0,T)} \|\phi\varrho_n\|_{L^\infty(0,T;L^3(\Omega_n))} \int_0^T \|\mathbf{u}_n(t)\|_{\boldsymbol{L}^6(\Omega_n)} \|\mathcal{B}_n\left(\overline{h(\varrho_n)}\right)(t)\|_{\boldsymbol{L}^2(\Omega_n)},$$

and, analogously to the previous case,

$$\|\mathcal{B}_n\left(\overline{h(\varrho_n)}\right)(t)\|_{\boldsymbol{L}^2(\Omega_n)} \le \|\nabla_x \mathbf{u}_n(t)\|_{\boldsymbol{L}^2(\Omega_n)} \|\varrho_n b'(\varrho_n) - b(\varrho_n)(t)\|_{L^3(\Omega_n)}.$$

Therefore, again,

$$|\mathfrak{I}_9| \le c(T, M_0, E_0)\|\psi\|_{L^\infty(0,T)}.$$

Collecting the estimates above on the integrals $\mathfrak{I}_1, \dots, \mathfrak{I}_9$,

$$\int_0^T \psi \int_{\Omega_n} \phi\varrho_n^\gamma b(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t \le c(T, M_0, E_0)\big[\|\psi\|_{L^\infty(0,T)} + \|\psi'\|_{L^1(0,T)} + \|\psi\|_{L^\infty(0,T)}\|\nabla_x\phi\|_{\boldsymbol{L}^\infty(\mathbb{R}^3)}\big].$$

Using a standard approximation argument (see e.g. [14]) we obtain

$$\int_0^T \int_{\Omega_n} \phi\varrho_n^\gamma b(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t \le c(T, M_0, E_0)(1 + \|\nabla_x\phi\|_{\boldsymbol{L}^\infty(\mathbb{R}^3)}).$$

Observing that

$$\int_0^T \int_{\Omega_n} \phi\varrho_n^\gamma b(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T \int_{\{x\in\Omega_n:\varrho_n(x,t)>1\}} \phi\varrho_n^\gamma b(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t$$

we end up with

$$\int_0^T \int_{\{x\in\Omega_n:\varrho_n(x,t)>1\}} \phi\varrho_n^{\gamma+\theta}\,\mathrm{d}x\,\mathrm{d}t \le c(T, M_0, E_0)(1 + \|\nabla_x\phi\|_{\boldsymbol{L}^\infty(\mathbb{R}^3)}).$$

Finally, an exploitation of (4.11) entails

$$\int_0^T \int_{\{x\in\Omega_n\setminus\Omega_1:\varrho_n(x,t)>1\}} \varrho_n^{\gamma+\theta}\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \int_{\{x\in\Omega_n:\varrho_n(x,t)>1\}} \phi\varrho_n^{\gamma+\theta}\,\mathrm{d}x\,\mathrm{d}t$$
$$\le c(T, M_0, E_0)\big(1 + \|\nabla_x\phi\|_{\boldsymbol{L}^\infty(\mathbb{R}^3)}\big).$$

The proof of Lemma 3 is finished.

Conclusion of the proof of Theorem 2

In light of the results [5,14], the sequence $(\varrho_n, \mathbf{u}_n)$ admits a subsequence converging to a weak solution $(\varrho, \mathbf{u})$ of the Navier–Stokes equations (1.1)–(1.2) in $\Omega \times (0,T)$,

$$\nabla_x \mathbf{u}_n \to \nabla_x \mathbf{u} \quad \text{weakly in } \boldsymbol{L}^2(\Omega \times (0,T)) \tag{4.13}$$

and, for every fixed ball B_R,

$$\varrho_n |\mathbf{u}_n|^2 \to \varrho |\mathbf{u}|^2 \quad \text{weakly in } L^2\left(0,T; L^{\frac{6\gamma}{4\gamma+3}}(B_R)\right), \tag{4.14}$$

$$\varrho_n^\gamma \to \varrho^\gamma \quad \text{weakly in } L^{\frac{5\gamma-3}{3\gamma}}(B_R \times (0,T)). \tag{4.15}$$

In addition, the total mass is conserved, namely, (3.1) holds (see [5, Proposition 2.1]).

Our plan is to pass to the limit as $n \to \infty$ in the differential inequality (4.2). Making use of (4.13), one can immediately pass to the limit in the viscous term

$$\int_{\Omega_n} \left[\mu |\nabla_x \mathbf{u}_n|^2 + (\lambda + \mu) |\mathrm{div}_x \mathbf{u}_n|^2\right] \mathrm{d}x$$

by means of lower weak semicontinuity of convex functionals, as

$$\int_0^T \psi \int_\Omega |\nabla_x \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \le \liminf_{n\to\infty} \int_0^T \psi \int_\Omega |\nabla_x \mathbf{u}_n|^2 \,\mathrm{d}x\,\mathrm{d}t,$$
$$\int_0^T \psi \int_\Omega |\mathrm{div}_x \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \le \liminf_{n\to\infty} \int_0^T \psi \int_\Omega |\mathrm{div}_x \mathbf{u}_n|^2 \,\mathrm{d}x\,\mathrm{d}t$$

for every nonnegative $\psi \in \mathcal{D}(0,T)$. The next step is to show that, for every $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ independent of n such that

$$\int_0^T \int_{\{x\in\Omega:|x|\ge R\}} \varrho_n^\gamma \,\mathrm{d}x\,\mathrm{d}t < \varepsilon \tag{4.16}$$

and

$$\int_0^T \int_{\{x\in\Omega:|x|\ge R\}} \varrho_n |\mathbf{u}_n|^2 \,\mathrm{d}x\,\mathrm{d}t < \varepsilon. \tag{4.17}$$

In order to prove the two estimates above, we need the following result.

Lemma 5. *The convergence*

$$\varrho_n \to \varrho \quad \text{in } L^1(\Omega \times (0,T))$$

holds up to a subsequence.

Proof. Recalling that $\varrho_n \equiv 0$ outside Ω_n and collecting (3.1) and (4.3), we infer that

$$\int_0^T \int_\Omega (\varrho - \varrho_n) \,\mathrm{d}x\,\mathrm{d}t = T \int_\Omega (\varrho_0 - \varrho_{0n}) \,\mathrm{d}x\,\mathrm{d}t.$$

Due the Dominated Convergence theorem, the right-hand side of the equality above tends to zero, yielding

$$\|\varrho_n\|_{L^1(\Omega\times(0,T))} \to \|\varrho\|_{L^1(\Omega\times(0,T))}. \tag{4.18}$$

Next, from [5,14], there exists a subsequence of ϱ_n, which we shall denote by $\varrho_{1,k}$, which converges to ϱ pointwise almost everywhere in

$$\overline{\Omega}_1 \times (0,T)$$

as $k \to \infty$. Then, for every $m \in \mathbb{N}$, we construct by induction a subsequence $\varrho_{m+1,k}$ of $\varrho_{m,k}$ which converges to ϱ pointwise almost everywhere in

$$\overline{\Omega}_{m+1} \times (0,T)$$

as $k \to \infty$. Hence, the "diagonal" sequence

$$\varrho_n = \varrho_{n,n}$$

converges to ϱ pointwise almost everywhere in $\Omega \times (0,T)$. In light of (4.18), we are done.

Collecting Lemmas 3 and 5, by interpolation we conclude that

$$\varrho_n \to \varrho \quad \text{in } L^\gamma(\Omega \times (0,T))$$

up to a subsequence, yielding (4.16). Then, for every fixed $R > 0$, due to (2.2), interpolation, estimates (4.4)–(4.6) and convergence (4.18),

$$\begin{aligned}
\int_0^T \int_{\{x\in\Omega:|x|\geq R\}} \varrho_n |\mathbf{u}_n|^2 \,\mathrm{d}x\,\mathrm{d}t &\leq c(E_0) \int_0^T \|\mathbf{u}_n(t)\|_{\boldsymbol{L}^6(\Omega)} \Big(\int_{\{x\in\Omega:|x|\geq R\}} \varrho_n^{3/2}(t)\,\mathrm{d}x \Big)^{1/3} \mathrm{d}t \\
&\leq c(E_0) \left[\int_0^T \Big(\int_{\{x\in\Omega:|x|\geq R\}} \varrho_n^{3/2}(t)\,\mathrm{d}x \Big)^{2/3} \mathrm{d}t \right]^{1/2} \\
&\leq c(T, M_0, E_0) \left[\int_0^T \int_{\{x\in\Omega:|x|\geq R\}} \varrho_n^{\gamma} \,\mathrm{d}x\,\mathrm{d}t \right]^{\alpha/2\gamma}
\end{aligned}$$

for some $\alpha \in (0,1)$. Hence, an exploitation of (4.16) provides the desired control (4.17). Finally, in light of (4.14)–(4.18), we can pass to the limit as $n \to \infty$ in the remaining terms of inequality (4.2).

5. Final remarks

I. In light of the results [10], we expect that the regularity of the boundary $\partial\Omega$ can be weakened to cover domains that may contain cusps.

II. Theorem 2 can be proved in the same way even if the motion of the fluid is driven by a bounded external force, namely, if the momentum equation (1.2) is replaced by

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla\varrho^\gamma = \mathrm{div}_x\mathbb{S} + \varrho\,\mathbf{f}$$

with

$$\mathbf{f} \in \boldsymbol{L}^\infty(\Omega \times (0,T)) \cap L^\infty(0,T;\boldsymbol{L}^1(\Omega)).$$

In addition, when $\mathbf{f}$ is the gradient of a scalar potential $F = F(x)$, the solution $(\varrho, \mathbf{u})$ converges to a fixed stationary state as time goes to infinity, provided that F is bounded and Lipschitz continuous and the upper level sets

$$[F > k] = \{x \in \Omega : F(x) > k\}$$

are connected in Ω for every k (see [6]).

III. The results contained in this paper hold also in two space dimension, for any $\gamma > 1$.

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