Electronic Journal of Probability

Electron. J. Probab. **20** (2015), no. 89, 1–34.
ISSN: 1083-6489 DOI: 10.1214/EJP.v20-3531

Infinite energy solutions to inelastic homogeneous Boltzmann equations

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Abstract

This paper is concerned with the existence, shape and dynamical stability of infinite-energy equilibria for a class of spatially homogeneous kinetic equations in space dimensions $d \geq 2$. Our results cover in particular Bobylev's model for inelastic Maxwell molecules. First, we show under certain conditions on the collision kernel, that there exists an index $\alpha \in (0, 2)$ such that the equation possesses a nontrivial stationary solution, which is a scale mixture of radially symmetric α-stable laws. We also characterize the mixing distribution as the fixed point of a smoothing transformation. Second, we prove that any transient solution that emerges from the NDA of some (not necessarily radial symmetric) α-stable distribution converges to an equilibrium. The key element of the convergence proof is an application of the central limit theorem to a representation of the transient solution as a weighted sum of projections of randomly rotated i.i.d. random vectors.

Keywords: Central Limit Theorems; Inelastic Boltzmann Equation; Infinite Energy Solutions; Normal Domain of Attraction; Multidimensional Stable Laws.
AMS MSC 2010: Primary 60F05, Secondary 60K40; 82C40.
Submitted to EJP on May 16, 2014, final version accepted on August 15, 2015.
Supersedes arXiv:arXiv:1309.7217.

1 Introduction

1.1 The equation

In this paper, we analyze the long-time asymptotics of the velocity distribution in kinetic models for spatially homogeneous inelastic Maxwellian molecules [13]. The dynamics is governed by a simplified version of the Boltzmann equation, see (1.1) below. It is known since the works of McKean [39, 40] that in the homogeneous Maxwell case, the central limit theorem provides a powerful tool to describe the convergence to equilibrium, at least qualitatively. Here we deal with an extension of McKean's idea to a situation that has not been analyzed in general before (a brief review of related works in special cases is provided below): we study the regime in which infinite kinetic energy and inelastic collisions balance in such a way that the Boltzmann equation admits

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non degenerate stationary states. To prove equilibration of solutions under presumably minimal hypotheses on the initial conditions, we develop a non-trivial extension of our techniques [4, 5] from one to multiple space dimensions. Like in the one-dimensional situation, our main tool is a description of particle collisions by weighted sums of random variables. However, the related machinery is much more complex in space dimensions $d \geq 2$, due to the highly non-trivial properties of the rotation group in $\mathbb{R}^d$. The probabilistic representation that we develop here has been inspired by the one used in a related work [28] for the particular case of fully elastic Maxwell molecules in $d = 3$.

We assume that the space dimension d is at least two, with the physical situation $d = 3$ being the most interesting choice. Under the *cut off assumption* and after proper normalization of the collision frequency, the evolution equation for the time-dependent velocity distribution $\mu : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}^d)$ is given by

$$\begin{cases} \partial_t \mu(t) + \mu(t) = Q_+(\mu(t), \mu(t)) \qquad (t > 0) \\ \mu(0) = \mu_0 \end{cases} \tag{1.1}$$

where the *collisional gain operator* Q_+ can be described by means of the weak formulation [13] as follows: for every $\varphi \in C_b^0(\mathbb{R}^d)$,

$$\begin{aligned} &\int_{\mathbb{R}^d} \varphi(v) Q_+(\mu,\mu)\,\mathrm{d}v \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \Big[\frac{\varphi(v')+\varphi(v'_*)}{2}\Big]\mathfrak{b}\Big(\sigma\cdot\frac{(v-v_*)}{|v-v_*|}\Big)\mathfrak{u}_{\mathbb{S}}(\mathrm{d}\sigma)\mu(\,\mathrm{d}v)\mu(\,\mathrm{d}v_*), \end{aligned} \tag{1.2}$$

where the *post-collisional velocities* v', v'_* are functions of the *pre-collisional velocities* v, v_* and of the unit vector σ,

$$\begin{aligned} v' = v'(v,v_*,\sigma) &= \frac{1}{2}(v+v_*) + \Big(\frac{\delta}{2}(v-v_*) + \frac{1-\delta}{2}|v-v_*|\sigma\Big), \\ v'_* = v'_*(v,v_*,\sigma) &= \frac{1}{2}(v+v_*) - \Big(\frac{\delta}{2}(v-v_*) + \frac{1-\delta}{2}|v-v_*|\sigma\Big); \end{aligned} \tag{1.3}$$

$\delta \in (0, 1/2)$ is the modulus of inelasticity; $\mathfrak{u}_{\mathbb{S}}$ is the uniform probability (normalized surface measure) on the $(d-1)$–dimensional sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$; and the *cross section* $\mathfrak{b} \in L^1(-1, 1)$ is a properly normalized (see (1.7) below) symmetric density function.

The characteristic property of Maxwellian molecules – in contrast to more general ideal gases – is that the cross section does *not* explicitly depend on the norm $|v - v_*|$ of the relative velocity. This property allows to restate (1.1) as an evolution equation for the characteristic function $\hat{\mu}(t;\xi) = \int \exp(i\xi\cdot v)\mu(t;\mathrm{d}v)$ of $\mu(t)$,

$$\partial_t \hat{\mu}(t) + \hat{\mu}(t) = \widehat{Q_+}[\hat{\mu}(t), \hat{\mu}(t)]. \tag{1.4}$$

The Fourier transformed collision operator $\widehat{Q_+}$ possesses an explicit integral representation [13], recalled in (3.1)-(3.2). Starting from there, we prove that there are non-negative random variables $r^\pm$ and random rotations $R^\pm$ in $\mathrm{SO}(d)$, such that

$$\widehat{Q_+}[\hat{\mu},\hat{\mu}](\rho\mathcal{O}\mathbf{e}_d) = \mathbb{E}\big[\hat{\mu}(\rho r^+\mathcal{O}R^+\mathbf{e}_d)\hat{\mu}(\rho r^-\mathcal{O}R^-\mathbf{e}_d)\big] \tag{1.5}$$

holds for all $\rho \in \mathbb{R}_+$ and all $\mathcal{O} \in \mathrm{SO}(d)$. This new representation of $\widehat{Q}_+$ is of crucial importance for our analysis of (1.4) by probabilistic methods. The existence of such a representation (1.5) is by no means obvious. A similar expression has been given in the situation of fully elastic Maxwell molecules in [28], which is (1.3) with $\delta = 0$. For inelastic molecules, it is proven in Proposition 3.2 below.

Notice:

In the following, we assume that the reader is familiar with basic notions of the central limit theorem, in particular with the Lévy representation of multi-dimensional α-stable distributions and their *normal domain of attraction* (NDA). A brief introduction to this topic is included in Appendix A.

1.2 Related results

In the rich literature on long-time asymptotics for (1.1), both solutions with finite (kinetic) energy, that is,

$$\int_{\mathbb{R}^d} |v|^2 \mu(t; \mathrm{d}v) < \infty \quad \text{for all } t > 0,$$

and with infinite energy have been studied. In order to relate our own results to the existing literature, we briefly recall a small selection of results on convergence to equilibrium for elastic and inelastic Maxwell molecules; the following summary is focussed on weak convergence results under minimal hypotheses on the initial conditions.

- *Finite energy solutions for fully elastic collisions.* The only stationary solutions of finite energy to the fully elastic Maxwell model [12] are Gaussians, and these attract *all* solutions of finite energy. This is known as Tanaka's theorem [45]. Various simple proofs are available, see e.g. [46].
- *Infinite energy solutions for fully elastic collisions.* The elastic Maxwell model does not admit stationary solutions of infinite energy [21]. However, Bobylev and Cercignani [14] have identified for every $\alpha \in (0, 2)$ a family of *self-similar* solutions for which the αth moment is marginally divergent. These self-similar solutions converge vaguely to zero as time goes to infinity, i.e., the velocities concentrate at infinity. It has been shown recently [20] that the self-similar solutions for a given α attract all transient solutions (of infinite energy) whose initial condition's characteristic function $\hat{\mu}_0$ satisfies

$$\lim_{|\xi| \to 0} \frac{\hat{\mu}_0(\xi) - 1}{|\xi|^\alpha} = K \quad \text{for some } K < 0. \tag{1.6}$$

- *Finite energy solutions for inelastic collisions.* Inelastic Maxwellian molecules lose kinetic energy in every collision. If the energy is finite initially, then it converges to zero exponentially fast in time [13]. As was conjectured by Ernst and Brito [30], this collapse happens in a self-similar way. More precisely, there is a time-dependent rescaling of the velocity variable such that the rescaled Boltzmann equation possesses a family of non-trivial stationary solutions, the so-called homogeneous cooling states. It has further been proven [11, 15, 17, 19] that any solution of finite energy to the rescaled equation eventually converges towards one of these cooling states.
- *Infinite energy solutions for inelastic collisions.* This case has received less attention than the aforementioned situations. Some results are available for the *inelastic Kac model* [44], which is a one-dimensional caricature of inelastic Maxwell molecules: for each inelastic Kac model, there is precisely one $\alpha \in (0, 2)$, such that the symmetric α-stable laws are stationary solutions and attract all transient solutions that start in their respective NDA [6]. A generalization of this result has been obtained by the authors [5] for Kac-type models with more complicated collisions and a richer class of stationary states. A related generalization [16, 18] also covers the case of radially symmetric solutions to the inelastic Kac model in

multiple space dimensions. The existence of a family of self-similar solutions is proven; the αth moment of these solutions is marginally divergent, for a model specific $\alpha \in (0,2)$. It has further been shown that the self-similar solutions attract all radially symmetric solutions whose initial condition satisfies a condition that is slightly more restrictive[1] than (1.6) as above.

Various of these fundamental weak convergence results have been made quantitative (e.g. in terms of estimates on convergence *rates*) and improved qualitatively (by proving e.g. convergence in *strong* topologies). Naturally, such improvements require additional hypotheses on the initial data (like higher moments or finite entropy) and are not of interest here. We refer the reader to the reviews [23, 47], and to the more recent results on self-similar asymptotics for inelastic Maxwell molecules [22] and for inelastic hard spheres [42].

1.3 Results and Method

In the present paper, we study the long time asymptotics of infinite energy solutions to inelastic Boltzmann equations. In particular, we show the existence of a family of stationary solutions of (1.1) and we represent them as scale mixtures of radially symmetric α-stable laws. Our main result is the dynamic stability of stationary solutions under assumptions on the initial conditions that we expect to be minimal, as stated in the next theorem.

Theorem 1.1. *Consider equation* (1.1) *with collision operator* (1.2)-(1.3)*, where* $\delta \in (0,1/2)$ *and the cross section* $\mathfrak{b}$ *is a symmetric function such that*

$$\int_{-1}^{1} \mathfrak{b}(z)\sqrt{(1-z^2)^{d-3}}\,\mathrm{d}z = \int_0^1 \sqrt{z^{-1}(1-z)^{d-3}}\,\mathrm{d}z. \tag{1.7}$$

Then there are a unique exponent $\alpha \in (0,2)$ *and a probability measure* $\mathfrak{m}$ *on* $\mathbb{R}_+$ *– both computable from* δ *and* $\mathfrak{b}$ *in principle – such that the following is true.*

A one-parameter family $(\mu_\infty^c)_{c>0}$ *of stationary solutions to* (1.1) *is given in terms of their characteristic functions* $\hat{\mu}_\infty^c$ *by*

$$\hat{\mu}_\infty^c(\xi) = \int_{\mathbb{R}_+} \exp\big(-cu|\xi|^\alpha\big)\mathfrak{m}(\mathrm{d}u) \quad \textit{for all } \xi \in \mathbb{R}^d.$$

If μ_0 *belongs to the NDA of a full* α*-stable distribution* (*centered, if* $\alpha > 1$*, and an additional condition is needed if* $\alpha = 1$ *— see* (2.9) *in* Section 2.2)*, then the corresponding solution* μ *to* (1.1) *converges weakly to a stationary solution* μ_∞^c*. In particular, the* μ_∞^c *are the only stationary solutions that belong to the NDA of some* α*-stable distribution on* $\mathbb{R}^d$*.*

Apparently, these are the first results on the stability of stationary solutions in the inelastic Maxwell model *without* the assumption of radial symmetry. Indeed, it seems that the approach to derive long-time asymptotics directly from contraction estimates on the Fourier transform of the transient solutions — like in [18] or [20] — needs an hypothesis on the initial datum of the form (1.6). This hypothesis is significantly stronger than ours, as can be seen from the characterization of NDAs by means of characteristic functions, see e.g. [1]. For instances, (1.6) implies that μ_0 belongs to the NDA of a *radially symmetric* α-stable law, which further implies that μ_0 is radially symmetric "asymptotically" on the complement of large balls. We only require that μ_0 belongs to the NDA of *some* full α-stable law. In fact, we expect that the NDA is a *sharp* characterization of the basin of attraction for the kinetic equation in the sense that all other transient

[1] Using the results from [4], it is easy to verify that condition (1.6) is actually sufficient.

solutions either concentrate at the origin or vaguely converge to zero as time tends to infinity.

The key element in our proof is a probabilistic representation of the solution to (1.4), which is derived by combining the Wild series expansion of the solution with the particular reformulation (1.5) of the Fourier transformed collision operator. We then use a combination of contraction estimates and techniques related to the central limit theorem on that probabilistic representation to prove its convergence to the Fourier transform of a mixture of radially symmetric stable laws in the long time limit. This core part of the proof of our main result is carried out in a more general abstract framework: we study solutions to integro-differential equations of the form (1.4)-(1.5) in which the characteristic function $\hat{\mu}$ is replaced by an general continuous and bounded function $\mathcal{U} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$. Our abstract result on long-time asymptotics stated in Theorem 2.3 — which contains Theorem 1.1 as a special case — could be of interest by itself, and is independent of whether $\mathcal{U}(t, \cdot)$ is a characteristic function or not at finite times $t > 0$. The latter property depends on the law of $r^{\pm}$ and $R^{\pm}$ in (1.5). It holds for the inelastic Maxwell model, and also for more general collisional kernels; an example is given in Section 3.4.

The general idea of a probabilistic representation of Boltzmann like equations goes back essentially to McKean [39, 40], who applied it to the Kac equation. The idea has since then been extended and refined, for instance in [25, 26, 33, 34] (for the Kac equation) and [4, 5, 6, 7, 8, 43] (for various one-dimensional Kac-type kinetic equations).

The extension to dimension $d > 1$ is by no means straightforward. Only in the recent paper [28], Dolera and Regazzini derived a suitable probabilistic representation of the solution of the homogeneous Boltzmann equation in dimension $d = 3$, using particular coordinates on $\mathbb{R}^3$ and its rotation group. Here, we extend the Dolera-Regazzini probabilistic representation to equation (1.4) with kernels of the form (1.5), in arbitrary dimensions $d \geq 2$. Our probabilistic representation is summarized in Proposition 2.5, which should be an interesting result in itself.

1.4 Plan of the paper

In Section 2 below we introduce the abstract framework for studying long-time asymptotics of homogeneous kinetic models, formulate our hypotheses, and state the general Theorem 2.3, which eventually implies our main result in Theorem 1.1. The central element here is the derivation of the probabilistic representation of solutions in Subsection 2.3. In Section 3, we verify that the model (1.1)–(1.3) for inelastic Maxwell molecules indeed fits into the provided framework. Sections 4 and 5 contain the proof of the abstract Theorem 2.3, which is naturally divided into two parts: Section 4 is concerned with contraction estimates on a random walk in the rotation group, which is induced by our probabilistic representation. In Section 5, we apply the machinery of the central limit theorem to our probabilistic representation to obtain the long-time asymptotics of transient solutions to the generalized kinetic models. The Appendix contains a summary of various results on α-stable distributions that are relevant to our proofs.

2 An abstract Boltzmann-like equation

Below, we introduce our abstract generalization of (1.4)-(1.5) and derive the probabilistic representation of its solutions. It will be shown in Section 3 that inelastic Maxwell molecules fall into the considered model class. Thus, our abstract result in Theorem 2.3 below implies the main Theorem 1.1.

2.1 Notations

Rotation group. Denote by $\mathrm{SO}(d)$ the usual orientation-preserving rotation group in $\mathbb{R}^d$ and by $\mathrm{SO}^*(d)$ its subgroup that acts on $\mathbb{R}^{d-1} \subset \mathbb{R}^d$ only, i.e. that leaves the "last" unit vector $\mathbf{e}_d := (0, \dots, 0, 1) \in \mathbb{R}^d$ invariant. More explicitly, a matrix U belongs to $\mathrm{SO}^*(d)$ if

$$U = \begin{pmatrix} U^* & 0_{d-1} \\ 0_{d-1}^T & 1 \end{pmatrix} \tag{2.1}$$

where U^* is a matrix in $\mathrm{SO}(d-1)$, and 0_{d-1} denotes the $(d-1)$-dimensional null column vector. Clearly $\mathrm{SO}^*(d)$ is isomorphic to $\mathrm{SO}(d-1)$. Further, let $\mathfrak{H}$ be the Haar measure on $\mathrm{SO}(d)$.

Convolution. For two probability measures $\mathfrak{B}$ and $\mathfrak{B}'$ on $\mathrm{SO}(d)$, define their *convolution* $\mathfrak{B} \star \mathfrak{B}'$ as the probability measure

$$\mathfrak{B} \star \mathfrak{B}'(A) := \int_{\mathrm{SO}(d)} \mathfrak{B}(R^T A) \, \mathrm{d}\mathfrak{B}'(R) \quad \text{for every measurable set } A \text{ in } \mathrm{SO}(d).$$

Accordingly, we define powers $\mathfrak{B}^{\star 2} = \mathfrak{B} \star \mathfrak{B}$ etc.

Complex valued functions. $C_b^0(\mathbb{R}^d; \mathbb{C})$ is the Banach space of complex valued bounded and continuous functions $\mathcal{U} : \mathbb{R}^d \to \mathbb{C}$, equipped with the supremum norm $\| \cdot \|_\infty$. We designate the closed convex subset $E = \{f \in C_b^0(\mathbb{R}^d; \mathbb{C}) \, : \, \|f\|_\infty \le 1\}$.

2.2 Main assumptions and results

Let (r^-, r^+, R^-, R^+) be a random element defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathrm{SO}(d) \times \mathrm{SO}(d)$ and denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. Our assumptions on the law of (r^-, r^+, R^-, R^+) are the following

(H1) *For any rotations $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{SO}(d)$ such that $\mathcal{O}_1 \mathbf{e}_d = \mathcal{O}_2 \mathbf{e}_d$*, we have

$$(\mathcal{O}_1 r^- R^- \mathbf{e}_d, \mathcal{O}_1 r^+ R^+ \mathbf{e}_d) \stackrel{\mathcal{L}}{=} (\mathcal{O}_2 r^- R^- \mathbf{e}_d, \mathcal{O}_2 r^+ R^+ \mathbf{e}_d).$$

(H2) *There are an $\alpha \in (0, 2)$ and a $\gamma \in (1, 2]$* such that

$$\mathbb{E}[(r^-)^\alpha + (r^+)^\alpha] = 1, \quad \text{and} \quad \mathbb{E}[(r^-)^{\alpha\gamma} + (r^+)^{\alpha\gamma}] < 1,$$

and, in addition, $\mathbb{P}\{r^- > 0\} + \mathbb{P}\{r^+ > 0\} > 1$.

For later reference, we introduce the (convex) function $\mathcal{S} : [0, \infty) \to [-1; \infty]$ by

$$\mathcal{S}(s) := \mathbb{E}\big[(r^+)^s + (r^-)^s\big] - 1. \tag{2.2}$$

Then (H2) can be rephrased in the form that

$$\mathcal{S}(\alpha) = 0 \quad \text{and} \quad \mathcal{S}(\alpha\gamma) < 0 \quad \text{for some } \alpha \in (0, 2) \text{ and } \gamma \in (1, 2].$$

Under hypothesis (H2), the following defines probability measures $\mathfrak{B}^+$ and $\mathfrak{B}^-$ on $\mathrm{SO}(d)$:

$$\int_{\mathrm{SO}(d)} f(R) \mathfrak{B}^\pm(\mathrm{d}R) = \frac{\mathbb{E}\big[(r^\pm)^\alpha f\big(R^\pm\big)\big]}{\mathbb{E}[(r^\pm)^\alpha]} \quad \text{for all } f \in C_b^0(\mathrm{SO}(d)). \tag{2.3}$$

In addition to (H1)-(H2), we shall assume further:

(H3) *The probability measures $\mathfrak{B}^\pm$ are non-singular with respect to the Haar measure, i.e. they have a non-trivial absolutely continuous component with respect to $\mathfrak{H}$.*

Before stating the general form of our main result, we briefly comment on the role of assumptions (H2) and (H3). Assumption (H2) is a classical hypothesis which guarantees the existence of a (unique up to scaling) fixed point of the *smoothing transformation* associated with (r^-, r^+). The respective result is the following.

Proposition 2.1 (see [2, 29])**.** *Under assumption* (H2) *there is a unique probability measure* $\mathfrak{m}$ *on* $[0, +\infty)$ *with* $\int u\mathfrak{m}(\mathrm{d}u) = 1$ *whose characteristic function* $\hat{\mathfrak{m}}$ *satisfies*

$$\hat{\mathfrak{m}}(y) = \mathbb{E}[\hat{\mathfrak{m}}((r^-)^\alpha y)\hat{\mathfrak{m}}((r^+)^\alpha y)] \quad \textit{for all } y \in \mathbb{R}. \tag{2.4}$$

Moreover, for every $p > 1$*,* $\int u^p\mathfrak{m}(\mathrm{d}u) < +\infty$ *if and only if* $\mathcal{S}(p\alpha) < 0$*.*

Assumption (H3) entails the convergence of the n-fold convolution $(\mathfrak{B}^\pm)^{\star n}$ to the Haar measure $\mathfrak{H}$. See e.g. [9] for a proof of exponentially fast convergence in total variation. We only need a corollary of that result, which is formulated in Proposition 4.8.

With the notations and preliminary results at hand, we can formulate our abstract problem and the general result. Given a random element (r^-, r^+, R^-, R^+) satisfying hypotheses (H1), consider the bilinear operator $\mathcal{Q}_+ : E \times E \to E$ defined by

$$\mathcal{Q}_+[\phi_1, \phi_2](\xi) := \mathbb{E}\big[\phi_1(\rho r^+\mathcal{O}R^+\mathbf{e}_d)\phi_2(\rho r^-\mathcal{O}R^-\mathbf{e}_d)\big] \tag{2.5}$$

for $\xi = \rho\mathcal{O}\mathbf{e}_d$. Note that this operator is well-defined: Thanks to (H1), the right-hand side of (2.5) does not depend on the particular choice of $\mathcal{O}$ in $\xi = \rho\mathcal{O}\mathbf{e}_d$. Boundedness in modulus by one is a consequence of Jensen's inequality: $\|\mathcal{Q}_+[\phi_1, \phi_2]\|_\infty \leq \|\phi_1\|_\infty\|\phi_2\|_\infty$. Continuity of $\xi \mapsto \mathcal{Q}_+[\phi_1, \phi_2](\xi)$ at $\xi = 0$ is a consequence of the dominated convergence theorem, performing the limit $\rho \downarrow 0$ for each (r^-, r^+, R^-, R^+) inside the expectation. At an arbitrary $\xi = \rho\mathbf{e}^*$ with $\rho > 0$ and $\mathbf{e}^* \in \mathbb{S}^{d-1}$, one uses that there is an open neighborhood $\mathcal{A}$ of $\mathbf{e}^*$ and a continuous function $B : \mathcal{A} \to \mathrm{SO}(d)$ such that $B(\mathbf{e})\mathbf{e}_d = \mathbf{e}$ for every $\mathbf{e} \in \mathcal{A}$; continuity at ξ again follows from the dominated convergence theorem.

Our abstract initial value problem is the following:

$$\begin{aligned} &\frac{d}{dt}\mathcal{U}(t) = \mathcal{Q}_+[\mathcal{U}(t), \mathcal{U}(t)] - \mathcal{U}(t) \\ &\mathcal{U}(0) = \mathcal{U}_0. \end{aligned} \tag{2.6}$$

Proposition 2.2. *Let* (H1) *be in force. For every initial condition* $\mathcal{U}_0 \in E$*, there exists a unique solution* $\mathcal{U} \in C^0\big([0, +\infty); E\big) \cap C^1\big((0, +\infty); C_b^0(\mathbb{R}^d, \mathbb{C})\big)$ *of* (2.6)*. It possesses the following series — or* Wild sum *— representation*

$$\mathcal{U}(t, \xi) = \sum_{n=0}^\infty e^{-t}(1 - e^{-t})^n\mathcal{U}_n(\xi), \tag{2.7}$$

where $\mathcal{U}_n \in E$ *are defined inductively from the initial condition* $\mathcal{U}_0$ *as follows:*

$$\mathcal{U}_{n+1} = \frac{1}{n+1}\sum_{k=0}^n \mathcal{Q}_+[\mathcal{U}_k, \mathcal{U}_{n-k}] \quad \textit{for all } n = 0, 1, 2, \ldots \tag{2.8}$$

Proof. Existence and uniqueness of a solution are guaranteed by the Picard-Lindelöf theorem. Indeed, the right-hand side of the differential equation in (2.6) is a Lipschitz continuous mapping from E to $C_b^0(\mathbb{R}^d, \mathbb{C})$:

$$\begin{aligned} \big\|\mathcal{Q}_+[\phi_1, \phi_1] - \mathcal{Q}_+[\phi_2, \phi_2]\big\|_\infty &\leq \|\phi_1 - \phi_2\|_\infty\|\phi_2\|_\infty + \|\phi_1\|_\infty\|\phi_1 - \phi_2\|_\infty \\ &\leq 2\|\phi_1 - \phi_2\|_\infty, \end{aligned}$$

for arbitrary $\phi_1, \phi_2 \in E$, thanks to Jensen's inequality. For proving (2.7), we start by observing that $\|\mathcal{U}_n\|_\infty \leq 1$ for all n, and that consequently, the power series with respect to $(1-e^{-t})$ is absolutely convergent in $C_b^0(\mathbb{R}^d;\mathbb{C})$. Interchanging summation and differentiation with respect to $t>0$, and using the bilinear structure of $\mathcal{Q}_+$, one obtains that

$$\begin{aligned}\frac{d}{dt}\mathcal{U}(t) &= -\mathcal{U}(t)+\sum_{n=0}^{\infty}e^{-2t}(1-e^{-t})^n(n+1)\mathcal{U}_{n+1}\\ &= -\mathcal{U}(t)+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathcal{Q}_+\big[e^{-t}(1-e^{-t})^k\mathcal{U}_k, e^{-t}(1-e^{-t})^{n-k}\mathcal{U}_{n-k}\big].\end{aligned}$$

Rearranging the double sum, and using bilinearity again, one concludes that the differential equation in (2.6) is satisfied. Clearly, the initial condition is satisfied as well. □

We are primarily interested in the application of (2.6) in the setting of (1.4)-(1.5), where the solution $\mathcal{U}$ is the characteristic function $\hat{\mu}$ of a time-dependent probability measure μ. However, even if $\mathcal{U}_0$ is a characteristic function, it does apparently not follow without further assumptions on the law of (r^-, r^+, R^-, R^+) in (2.5) that the same is true for $\mathcal{U}(t)$ at some $t>0$. It would be clearly sufficient to ask in addition for

(H4) *for every couple of characteristic functions $\hat{\mu}_1$ and $\hat{\mu}_2$ the function $\mathcal{Q}_+(\hat{\mu}_1,\hat{\mu}_2)$ is a characteristic function.*

However, the following result, which implies our main Theorem 1.1, does not need hypothesis (H4).

Theorem 2.3. *For a given random element (r^-, r^+, R^-, R^+), let $\mathcal{Q}_+$ be the bilinear operator defined by* (2.5)*. Assume that there is an $\alpha\in(0,2)$ such that hypotheses* (H1)-(H3) *hold. Consider the initial value problem* (2.6) *with an initial condition $\mathcal{U}_0(\xi)=\hat{\mu}_0(\xi)$, where μ_0 is a probability measure that belongs to the NDA of a full α-stable distribution with Lévy measure ϕ. If $\alpha>1$, assume further that μ_0 is centered, while if $\alpha=1$, assume that there is some $\gamma_0\in\mathbb{R}^d$ with*

$$\lim_{R\to+\infty}\sup_{\sigma\in\mathbb{S}^{d-1}}\Big|\int_{-R<\sigma\cdot v\le R}\sigma\cdot v\,\mu_0(\mathrm{d}v)-\sigma\cdot\gamma_0\Big|=0. \tag{2.9}$$

Then the unique solution $\mathcal{U}(t)$ to (2.6) *converges for every $\xi\in\mathbb{R}^d$ to*

$$\hat{\mu}^c_\infty(\xi)=\int_{[0,\infty)}\exp\big(-cu|\xi|^\alpha\big)\mathfrak{m}(\mathrm{d}u).$$

$\hat{\mu}^c_\infty$ is the characteristic function of a mixture of radially symmetric α-stable laws, with the mixing distribution $\mathfrak{m}$ that is determined by Proposition 2.1*, and*

$$c=\frac{1}{\Gamma(\alpha)\sin(\pi\alpha/2)}\int_{\mathbb{S}^{d-1}}\int_{\{y:y\cdot\sigma>1\}}\phi(\mathrm{d}y)\mathfrak{u}_{\mathbb{S}}(\mathrm{d}\sigma).$$

In particular, the $\hat{\mu}^c_\infty$ are the only stationary solutions of (2.6) *in the class of the characteristic functions of probability measures in the NDA of some full α-stable distribution on $\mathbb{R}^d$.*

The proof of Theorem 2.3 is given in Section 5.

2.3 A probabilistic representation

As already mentioned in the introduction, the key element in our proof of Theorem 2.3 is a suitable stochastic representation of $\mathcal{U}(t)$. This probabilistic representation enables us to study the long-time asymptotics of $\mathcal{U}(t)$ by methods related to the central limit theorem. In the rest of this section, $\mathcal{U}$ is a solution to the initial value problem (2.6), for a given initial value $\mathcal{U}_0\in E$.

The setup is the following. On a sufficiently large probability space $(\Omega,\mathcal{F},\mathbb{P})$ consider:

- a sequence of independent random variables $(\ell_n)_{n\geq 1}$ such that each ℓ_n is uniformly distributed on $\{1,\dots,n\}$;
- a sequence of i.i.d. random elements $(r_n^-, r_n^+, R_n^-, R_n^+)_{n\geq 1}$ with the same law of (r^-, r^+, R^-, R^+) defined in Section 2.2.

Assume also that $(\ell_n)_{n\geq 1}$ and $(r_n^-, r_n^+, R_n^-, R_n^+)_{n\geq 1}$ are stochastically independent. Define recursively the random array $(\beta_{j,n}, O_{j,n})_{\left\{ {n\geq 0 \atop 1\leq j\leq n+1} \right\}}$ by setting

$$
\begin{aligned}
&O_{1,0} := \mathbf{1}_d, \beta_{1,0} := 1, \quad \text{and for all } n\geq 1: \\
&(O_{1,n},\dots,O_{n+1,n}) = \big(O_{1,n-1},\dots,O_{\ell_n-1,n-1}, O_{\ell_n,n-1}R_n^-, O_{\ell_n,n-1}R_n^+, \\
&\qquad\qquad\qquad\qquad\qquad O_{\ell_n+1,n-1},\dots,O_{n,n-1}\big) \\
&(\beta_{1,n},\dots,\beta_{n+1,n}) = \big(\beta_{1,n-1},\dots,\beta_{\ell_n-1,n-1}, \beta_{\ell_n,n-1}r_n^-, \beta_{\ell_n,n-1}r_n^+, \\
&\qquad\qquad\qquad\qquad\qquad \beta_{\ell_n+1,n-1},\dots,\beta_{n,n-1}\big).
\end{aligned}
$$

This construction extends the one given in [5], where a class of one-dimensional generalized Kac equations is considered. For given $n\geq 1$, one should think of the quantities $\beta_{n,j}$ and $O_{n,j}$ as attached to the $n+1$ leaves of a binary tree (whose shape is determined by ℓ_1 to ℓ_n) with n internal nodes. In the context of the Kac model, these binary trees are commonly referred to as *McKean trees*.

Proposition 2.4. *For every $n\geq 0$, every $\rho\in\mathbb{R}_+$ and every $\mathcal{O}\in \mathrm{SO}(d)$ one has*

$$
\mathcal{U}_n(\rho\mathcal{O}\mathbf{e}_d) = \mathbb{E}\Big[\prod_{j=1}^{n+1} \mathcal{U}_0\big(\rho\beta_{j,n}\mathcal{O}O_{j,n}\mathbf{e}_d\big)\Big]. \tag{2.10}
$$

Proof. For $n=0$ there is nothing to prove. For $n=1$ the statement reduces to the definition of $\mathcal{Q}_+$ in (2.5). We proceed by induction on n.

Fix $n\geq 1$ and assume that (2.10) is true for all $k=0,\dots,n-1$ in place of n. By construction, $\beta_{1,1}=r_1^-$, $\beta_{1,2}=r_1^+$, and $O_{1,1}=R_1^-$, $O_{1,2}=R_1^+$. Consequently, we can write

$$
\begin{aligned}
\beta_{j,n} &= r_1^-\beta'_{j,n}, \quad O_{j,n} = R_1^- O'_{j,n} \qquad \text{for } j=1,\dots,J, \\
\beta_{j,n} &= r_1^+\beta''_{j,n}, \quad O_{j,n} = R_1^+ O''_{j,n} \qquad \text{for } j=J+1,\dots,n+1,
\end{aligned} \tag{2.11}
$$

with a random index $J\in\{1,\dots,n\}$ depending on ℓ_1 to ℓ_n. The factorization (2.11) corresponds to splitting the nth binary tree at the root into a left tree (with J leaves) and a right tree (with $n+1-J$ leaves). It is easy to see that J is uniformly distributed on $\{1,\dots,n\}$, see e.g. [5]. It is further easy to see that, given $(J, r_1^-, r_1^+, R_1^-, R_1^+)$, the random elements $(\beta'_{j,n}, O'_{j,n})_{j=1,\dots,J}$ and $(\beta''_{j,n}, O''_{j,n})_{j=J+1,\dots,n+1}$ are conditionally independent. Their conditional distribution, given the event $\{J=k\}$, satisfies

$$
\begin{aligned}
\big(\beta'_{j,n}, O'_{j,n}\big)_{j=1,\dots,k} &\stackrel{\mathcal{L}}{=} \big(\beta_{j,k-1}, O_{j,k-1}\big)_{j=1,\dots,k}, \\
\big(\beta''_{j,n}, O''_{j,n}\big)_{j=k+1,\dots,n+1} &\stackrel{\mathcal{L}}{=} \big(\beta_{j,n-k}, O_{j,n-k}\big)_{j=1,\dots,n+1-k}.
\end{aligned}
$$

Thus, if (r^-, r^+, R^-, R^+) is defined as above and it is assumed independent of all the rest, using the induction hypothesis, one can write

$$
\begin{aligned}
\mathbb{E}\Big[\prod_{j=1}^{n+1}\mathcal{U}_0\big(\rho\beta_{j,n}\mathcal{O}O_{j,n}\mathbf{e}_d\big)\Big]
&= \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\Big[\mathbb{E}\Big[\prod_{j=1}^{k}\mathcal{U}_0\big(\rho r^-\mathcal{O}R^-\beta_{j,k-1}O_{j,k-1}\mathbf{e}_d\big)\Big|r^-,r^+,R^-,R^+\Big]\\
&\quad\cdot\mathbb{E}\Big[\prod_{j=1}^{n+1-k}\mathcal{U}_0\big(\rho r^+\mathcal{O}R^+\beta_{j,n-k}O_{j,n-k}\mathbf{e}_d\big)\Big|r^-,r^+,R^-,R^+\Big]\Big]\\
&= \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\big[\mathcal{U}_{k-1}\big(\rho r^-\mathcal{O}R^-\mathbf{e}_d\big)\mathcal{U}_{n-k}\big(\rho r^+\mathcal{O}R^+\mathbf{e}_d\big)\big],
\end{aligned}
$$

which, by (2.5) and (2.8), equals to $\mathcal{U}_n$. □

Formula (2.10) is *almost* the desired probabilistic representation. In order to establish the connection to the central limit theorem, we now assume in addition that $\mathcal{U}_0(\xi)=\hat{\mu}_0(\xi)$ is the characteristic function of a given probability measure μ_0 on $\mathbb{R}^d$. And we consider a sequence of i.i.d. random vectors $(X_j)_{j\geq 1}$ with distribution μ_0 that is independent of $(\ell_n)_{n\geq 1}$ and $(R_n^-,R_n^+,r_n^-,r_n^+)_{n\geq 1}$. This allows to re-interprete the right-hand side of (2.10) as a randomly weighted sum of projections of the randomly rotated vectors X_j.

Proposition 2.5. *Assume that* $\mathcal{U}_0(\xi)=\hat{\mu}_0(\xi)$ *is the characteristic function of a given probability measure* μ_0 *on* $\mathbb{R}^d$*. Then, for every* $n\geq 0$*, every* $\rho\in\mathbb{R}_+$ *and every* $\mathcal{O}\in\mathrm{SO}(d)$ *one has*

$$
\mathcal{U}_n(\rho\mathcal{O}\mathbf{e}_d)=\mathbb{E}\Bigg[\exp\Bigg(i\rho\sum_{k=1}^{n+1}\big(\beta_{k,n}\mathcal{O}O_{k,n}\mathbf{e}_d\big)\cdot X_k\Bigg)\Bigg].
$$

If in addition (H4) *is true, then the previous equality holds for every* ρ *in* $\mathbb{R}$*.*

Remark 2.6. The first representation of this type has been derived in [28] for the *fully elastic* Boltzmann equation in $\mathbb{R}^3$.

Proof. Observe that

$$
\begin{aligned}
\mathbb{E}\Bigg[\exp\Bigg(i\rho\sum_{k=1}^{n+1}\big(\beta_{k,n}\mathcal{O}O_{k,n}\mathbf{e}_d\big)\cdot X_k\Bigg)\Bigg]
&= \mathbb{E}\Bigg[\mathbb{E}\Bigg[\prod_{k=1}^{n+1}\exp\Big(i\big(\rho\beta_{k,n}\mathcal{O}O_{k,n}\mathbf{e}_d\big)\cdot X_k\Big)\Bigg|(\beta_{j,n},O_{j,n})_{1\leq j\leq n+1}\Bigg]\Bigg]\\
&= \mathbb{E}\Bigg[\prod_{k=1}^{n+1}\hat{\mu}_0\big(\rho\beta_{k,n}\mathcal{O}O_{k,n}\mathbf{e}_d\big)\Bigg]=\mathcal{U}_n(\rho\mathcal{O}\mathbf{e}_d)=\mathcal{U}_n(\rho\mathbf{e}),
\end{aligned}
$$

where we have used (2.10). If (H4) holds $\mathcal{U}_n$ is a characteristic function. Hence the last statement follows recalling that two characteristic functions that coincide on the positive real axis are equal. □

3 The inelastic Maxwell models as a special case

The aim of this section is to show that the homogeneous Boltzmann equation with collision rules (1.3) is indeed a special case of the abstract equation considered in the previous section. Theorem 1.1 then follows as a corollary of Theorem 2.3.

Our starting point is the equation in its Fourier representation (1.4), which has been derived in [13], with the collision kernel

$$\widehat{Q_+}[\hat{\mu}, \hat{\mu}](\xi) = \mathbb{E}[\hat{\mu}(Y_\xi^+)\hat{\mu}(Y_\xi^-)], \tag{3.1}$$

where, for any $\xi \in \mathbb{R}^d$, the two random vectors Y_ξ^- and Y_ξ^+ in $\mathbb{R}^d$ are given by

$$Y_\xi^- := \frac{1-\delta}{2}(\xi - |\xi|\mathbf{n}), \quad Y_\xi^+ := \frac{1+\delta}{2}\xi + \frac{1-\delta}{2}|\xi|\mathbf{n}, \tag{3.2}$$
with a random unit vector $\mathbf{n}$ which has law $\mathfrak{b}\big(\sigma \cdot \xi/|\xi|\big)\mathfrak{u}_{\mathbb{S}}(\,\mathrm{d}\sigma)$.

Below, we rewrite (3.1) in the form (1.5), with suitable random quantities $r^\pm$ and $R^\pm$ satisfying (H1)-(H4).

3.1 Preliminaries on rotation groups

We start by recalling some well-known facts about the Haar probability measure. By definition, a random matrix O has Haar distribution on $\mathrm{SO}(k)$ if

$$GO \stackrel{\mathcal{L}}{=} O^T \stackrel{\mathcal{L}}{=} O$$

for every orthogonal matrix $G \in \mathrm{SO}(k)$. By elementary considerations, it follows that

$$O\mathbf{e} \quad \text{is uniformly distributed on } \mathbb{S}^{k-1}, \tag{3.3}$$

for any $\mathbf{e} \in \mathbb{S}^{k-1}$. We say that a random matrix U in $\mathrm{SO}(d)$ is uniformly distributed on $\mathrm{SO}^*(d)$ if it can be written in the form (2.1), where U^* is a random matrix in $\mathrm{SO}(d-1)$ with respective Haar distribution.

We call a measure λ on $\mathbb{S}^{d-1}$ *invariant under* $\mathrm{SO}^*(d)$, if $\lambda(\mathcal{O}B) = \lambda(B)$ for every $\mathcal{O}$ in $\mathrm{SO}^*(d)$ and for all measurable sets $B \subseteq \mathbb{S}^{d-1}$. Since $\mathrm{SO}^*(d)$ acts transitively on each of the $(d-2)$-dimensional spheres $\{y \in \mathbb{S}^{d-1} | \mathbf{e}_d \cdot y = z\}$ with $z \in (-1,1)$, the invariant measure λ is uniquely determined by its *projected measure* $\Pi\lambda$ on $[-1,1]$, given by $\Pi\lambda(J) = \lambda\big(\{y \in \mathbb{S}^{d-1} | \mathbf{e}_d \cdot y \in J\}\big)$ for all measurable $J \subseteq [-1,1]$. In the particular case that $\lambda(\mathrm{d}\sigma) = f(\sigma \cdot \mathbf{e}_d)\mathfrak{u}_{\mathbb{S}}(\mathrm{d}\sigma)$ with $f : [-1,1] \to \mathbb{R}$, the projected measure $\Pi\lambda$ has a density (w.r.t. Lebesuge measure) $\Pi f : [-1,1] \to \mathbb{R}$ with

$$\Pi f(z) := \frac{\mathrm{d}(\Pi\lambda)}{\mathrm{d}z} = \frac{1}{B_d} f(z)(1-z^2)^{\frac{d-3}{2}} \tag{3.4}$$

where

$$B_d = \int_0^1 z^{-\frac{1}{2}}(1-z)^{\frac{d-3}{2}}\,\mathrm{d}z,$$

which is easily verified by the change of variables formula.

Finally, we denote by $Z^{i,j}(\psi) \in \mathrm{SO}(d)$ the matrix of the rotation about the angle ψ in the $\mathbf{e}_i - \mathbf{e}_j$-plane: the only nonzero elements of $Z^{i,j}(\psi)$ are

$$\begin{aligned} &Z^{i,j}_{kk}(\psi) = 1, \qquad k = 1, \dots, d, \qquad k \neq i, j \\ &Z^{i,j}_{ii}(\psi) = \cos(\psi), \qquad Z^{i,j}_{ij} = \sin(\psi), \\ &Z^{i,j}_{ji}(\psi) = -\sin(\psi), \qquad Z^{i,j}_{jj} = \cos(\psi). \end{aligned}$$

Since rotations in the $\mathbf{e}_1 - \mathbf{e}_d$-plane play a distinguished role in the following, we write

$$Z_\psi := Z^{1,d}(\psi)$$

for brevity.

The following probabilistic interpretation of Hurwitz's [36] representation of the Haar measure will be of importance.

Theorem 3.1. *There are random rotations U_1, U_2 in $\mathrm{SO}(d)$ and a random angle ψ_* in $[0,\pi]$ such that*

- *U_1, U_2 and ψ_* are independent,*
- *U_1 is uniformly distributed on $\mathrm{SO}^*(d)$, and $U_2 \in \mathrm{SO}^*(d)$ a.s.,*
- *ψ_* has a continuous probability density function that is positive on $(0,\pi)$,*
- *the law of $U_1 Z_{\psi_*} U_2$ is the Haar measure on $\mathrm{SO}(d)$.*

Sketch of the proof. In [36] it is shown that an arbitrary rotation matrix $\mathcal{O} \in \mathrm{SO}(d)$ may be written as a product of $d(d-1)/2$ rotations in two-dimensional subspaces. Specifically:

$$\mathcal{O} = F_1 F_2 \dots F_{d-1}$$

where F_i is a concatenation of $d-i$ rotations,

$$F_i = Z^{d-i,d-i+1}(\psi_{i-1,i}) Z^{d-i+1,d-i+2}(\psi_{i-2,i}) \dots Z^{d-1,d}(\psi_{0,i}).$$

The Haar distribution on $\mathrm{SO}(d)$ is obtained by choosing the $\psi_{r,s}$ independent of each other, and such that $\psi_{r,s}$ is absolutely continuous with density $\sin(\psi)^r \mathbb{1}_{[0,\pi)}(\psi)$ for $r = 1,\dots,s-1$ and $s = 1,\dots,d$, whereas the $\psi_{0,s}$ are uniformly distributed on $[0,2\pi)$. As a consequence of the above representation, by simple geometric considerations one obtains the result. □

3.2 Definition of the probabilistic representation

Given the cross section $\mathfrak{b}$ on $(-1,1)$, define the projected density $\Pi\mathfrak{b}$ according to (3.4). Since $\mathfrak{b}$ is normalized as stated in (1.7), $\Pi\mathfrak{b}$ is a probability density. Let ψ be a random angle in $(0,\pi)$ such that $\cos\psi$ has $\Pi\mathfrak{b}$ as density, which is equivalent to saying that ψ itself is distributed with law

$$\mathfrak{b}(\cos\eta)\sin^{d-2}\eta \, \mathrm{d}\eta. \tag{3.5}$$

Further, let U_1, U_2 be random rotations taking values in $\mathrm{SO}^*(d)$ — independent of each other and independent of ψ — with the properties from Theorem 3.1. In particular, U_1 is uniformly distributed on $\mathrm{SO}^*(d)$. From that, define two further random angles $\psi^\pm$ in $(0,\pi)$ implicitly by

$$\cos\psi^- = 2^{-1/2}\sqrt{1-\cos\psi}, \quad \cos\psi^+ = 2^{-1/2}\frac{(1+\delta)+(1-\delta)\cos\psi}{\sqrt{(1+\delta^2)+(1-\delta^2)\cos\psi}}. \tag{3.6}$$

Now set

$$r^- := 2^{-1/2}(1-\delta)\sqrt{1-\cos\psi}, \qquad R^- := U_1 Z_{\psi^-} U_2, \tag{3.7}$$

$$r^+ := 2^{-1/2}\sqrt{(1+\delta^2)+(1-\delta^2)\cos\psi}, \qquad R^+ := U_1 Z_{\psi^+} U_2. \tag{3.8}$$

Finally, recall the definition of $Y_\xi^\pm$ from (3.2).

Proposition 3.2. *For every vector ξ and every $\mathcal{O} \in \mathrm{SO}(d)$ such that $\xi = |\xi|\mathcal{O}\mathbf{e}_d$ one has*

$$\left(Y_\xi^-, Y_\xi^+\right) \stackrel{\mathcal{L}}{=} \left(|\xi| r^- \mathcal{O} R^- \mathbf{e}_d, |\xi| r^+ \mathcal{O} R^+ \mathbf{e}_d\right). \tag{3.9}$$

The essential ingredient of the proof is the following.

Lemma 3.3. *For $\xi = \mathbf{e}_d$, the random unit vector $\mathbf{n}$ in (3.2) admits the representation $\mathbf{n} \stackrel{\mathcal{L}}{=} U_1 Z_\psi U_2 \mathbf{e}_d$.*

Proof. We need to show that the law λ of $U_1 Z_\psi U_2 \mathbf{e}_d$ is the same as the law λ' of $\mathbf{n}$. Both λ and λ' are invariant under $\mathrm{SO}^*(d)$: for λ', this is clear by definition in (3.2). For λ, this follows since U_1, ψ and U_2 are independent, and $GU_1 \stackrel{\mathcal{L}}{=} U_1$ for every $G \in \mathrm{SO}^*(d)$. By our considerations on $\mathrm{SO}^*(d)$-invariant measures above, it therefore suffices to show that the projected measures are equal, $\Pi\lambda = \Pi\lambda'$.

For λ', we obtain from the definition of $\mathbf{n}$ and formula (3.4) that $\mathbf{n} \cdot \mathbf{e}_d$ has law $\Pi\mathfrak{b}$. Concerning λ, recall that U_1 and U_2 take values in $\mathrm{SO}^*(d)$ a.s., which implies that

$$\mathbf{e}_d \cdot U_1 Z_\psi U_2 \mathbf{e}_d = (U_1^T \mathbf{e}_d) \cdot (Z_\psi U_2 \mathbf{e}_d) = \mathbf{e}_d \cdot (Z_\psi \mathbf{e}_d) = \cos\psi,$$

using the definition of Z_ψ. The claim now follows since $\cos\psi$ has law $\Pi\mathfrak{b}$ by definition. □

Proof of Proposition 3.2. Let $\xi = |\xi|\mathcal{O}\mathbf{e}_d$ be given. For any bounded continuos function f

$$\begin{aligned}\mathbb{E}[f(Y_\xi^-)] &= \int_{\mathbb{S}^{d-1}} f\Big(\frac{1-\delta}{2}|\xi|\mathcal{O}(\mathbf{e}_d - \mathcal{O}^T\sigma)\Big)\mathfrak{b}(\mathcal{O}^T\sigma\cdot\mathbf{e}_d)\mathfrak{u}_{\mathbb{S}}(d\sigma)\\ &= \int_{\mathbb{S}^{d-1}} f\Big(|\xi|\mathcal{O}\frac{1-\delta}{2}(\mathbf{e}_d - \sigma)\Big)\mathfrak{b}(\sigma\cdot\mathbf{e}_d)\mathfrak{u}_{\mathbb{S}}(d\sigma) = \mathbb{E}[f(|\xi|\mathcal{O}Y_{\mathbf{e}_d}^-)],\end{aligned}$$

where we have used (3.2) and a change of variables in the integral. Hence $Y_\xi^- \stackrel{\mathcal{L}}{=} |\xi|\mathcal{O}Y_{\mathbf{e}_d}^-$. Since $Y_\xi^- + Y_\xi^+ = \xi$ and $Y_{\mathbf{e}_d}^+ + Y_{\mathbf{e}_d}^- = \mathbf{e}_d$, it follows further that

$$(Y_\xi^-, Y_\xi^+) \stackrel{\mathcal{L}}{=} (|\xi|\mathcal{O}Y_{\mathbf{e}_d}^-, |\xi|\mathcal{O}Y_{\mathbf{e}_d}^+).$$

It is thus sufficient to prove the claim for $\xi = \mathbf{e}_d$ and $\mathcal{O} = \mathbf{1}_d$. By Lemma 3.3, we have

$$\begin{aligned}(Y_{\mathbf{e}_d}^-, Y_{\mathbf{e}_d}^+) &\stackrel{\mathcal{L}}{=} \Big(\frac{1-\delta}{2}(\mathbf{e}_d - U_1 Z_\psi U_2 \mathbf{e}_d), \frac{1+\delta}{2}\mathbf{e}_d + \frac{1-\delta}{2}U_1 Z_\psi U_2 \mathbf{e}_d\Big)\\ &= \Big(U_1\Big[\frac{1-\delta}{2}(\mathbf{1}_d - Z_\psi)\Big]U_2\mathbf{e}_d, U_1\Big[\frac{1+\delta}{2}\mathbf{1}_d + \frac{1-\delta}{2}Z_\psi\Big]U_2\mathbf{e}_d\Big).\end{aligned}$$

To finish the proof, observe that we have

$$\begin{aligned}U_1\Big[\frac{1-\delta}{2}(\mathbf{1}_d - Z_\psi)\Big]U_2\mathbf{e}_d &= r^- R^- \mathbf{e}_d,\\ U_1\Big[\frac{1+\delta}{2}\mathbf{1}_d + \frac{1-\delta}{2}Z_\psi\Big]U_2\mathbf{e}_d &= r^+ R^+ \mathbf{e}_d,\end{aligned}$$

which easily follows from our definitions of $r^\pm$ and $R^\pm$ by elementary geometric considerations. □

3.3 Verification of (H1)–(H4)

It remains to verify that the random quantities defined in (3.7)-(3.8) satisfy the hypotheses (H1)–(H4). Condition (H1) is a direct consequence of Proposition 3.2, since with $\xi := \mathcal{O}_1\mathbf{e}_d = \mathcal{O}_2\mathbf{e}_d$, one has that

$$\big(\mathcal{O}_1 r^- R^- \mathbf{e}_d, \mathcal{O}_1 r^+ R^+ \mathbf{e}_d\big) \stackrel{\mathcal{L}}{=} (Y_\xi^-, Y_\xi^+) \stackrel{\mathcal{L}}{=} \big(\mathcal{O}_2 r^- R^- \mathbf{e}_d, \mathcal{O}_2 r^+ R^+ \mathbf{e}_d\big).$$

The validity of condition (H4) is a classical fact, see e.g. [13]: $\hat{Q}_+$ in (3.1) is the Fourier transform of the inelastic Maxwell operator Q_+ that maps probability measures to probability measures, hence $\hat{Q}_+$ maps characteristic functions to characteristic functions. A detailed proof in the case of elastic Maxwell molecules can be found in [27, Propositions 2.2 and 2.3].

The validity of (H2) is a consequence of the following.

Lemma 3.4. *There is a unique* $\alpha \in (0,2)$ *such that* $\mathbb{E}[(r^+)^\alpha + (r^-)^\alpha] = 1$*, and* $\mathbb{E}[(r^+)^{\alpha\gamma} + (r^-)^{\alpha\gamma}] < 1$ *for every* $\gamma > 1$*.*

Proof. The convex function $\mathcal{S}(s) = \mathbb{E}[(r^+)^s + (r^-)^s] - 1$ defined in (2.2) becomes

$$\mathcal{S}(s) = \mathbb{E}\Big[\Big((1-\delta)^2 \frac{1-\cos\psi}{2}\Big)^{s/2}\Big] + \mathbb{E}\Big[\Big(\frac{1+\delta^2}{2} + \frac{1-\delta^2}{2}\cos\psi\Big)^{s/2}\Big] - 1.$$

On one hand, $\mathcal{S}(0) = 1$, because ψ is an absolutely continuous random variable. On the other hand, since $0 < r^\pm < 1$ almost surely, it follows that $\lim_{s\to+\infty} \mathcal{S}(s) = -1$. Finally, at $s = 2$, we have

$$\mathcal{S}(2) = \mathbb{E}\big[(r^+)^2 + (r^-)^2\big] - 1 = \delta(\delta-1)\mathbb{E}[1-\cos\psi] < 0.$$

By convexity of $\mathcal{S}$, this proves the claim. □

Having verified hypothesis (H2), we define probability measures $\mathfrak{B}^\pm$ according to (2.3). The next lemma shows that (a strong version of) conditions (H3) is satisfied as well.

Lemma 3.5. $\mathfrak{B}^\pm$ *are absolutely continuous with respect to the Haar measure.*

Proof. Recall Theorem 3.1, and let U_1, U_2 and ψ_*, ψ be chosen as indicated above. Further, observe that, since U_1, U_2, ψ are independent, and since the law of ψ is given in (3.5), one can write, for every $f \in C_0^b(\mathrm{SO}(d))$,

$$\begin{aligned}\int_{\mathrm{SO}(d)} f(R)\mathfrak{B}^\pm(\mathrm{d}R) &= \frac{\mathbb{E}\big[(r^\pm)^\alpha f\big(U_1 Z_{\psi^\pm} U_2\big)\big]}{\mathbb{E}[(r^\pm)^\alpha]} \\ &= \frac{\mathbb{E}\big[\int_{(0,\pi)} (r^\pm(\eta))^\alpha f\big(U_1 Z_{\psi^\pm(\eta)} U_2\big)\mathfrak{b}(\cos\eta)\sin^{d-2}\eta\,\mathrm{d}\eta\big]}{\int_{(0,\pi)} (r^\pm(\eta))^\alpha \mathfrak{b}(\cos\eta)\sin^{d-2}\eta\,\mathrm{d}\eta}\end{aligned}$$

where $\psi^\pm(\eta)$ and $r^\pm(\eta)$ are defined as functions of η via (3.6)–(3.8) using η in place ψ. Hence

$$\int_{\mathrm{SO}(d)} f(R)\mathfrak{B}^\pm(\mathrm{d}R) = \mathbb{E}\big[f\big(U_1 Z_{\tilde\psi^\pm} U_2\big)\big],$$

where $\tilde\psi^\pm$ are defined via (3.6) from a random angle $\tilde\psi$ — being independent of U_1 and U_2 — in $(0,\pi)$ with law

$$\frac{\big(r^\pm(\eta)\big)^\alpha \mathfrak{b}(\cos\eta)\sin^{d-2}\eta\,\mathrm{d}\eta}{\int_{(0,\pi)} (r^\pm(u))^\alpha \mathfrak{b}(\cos u)\sin^{d-2} u\,\mathrm{d}u}.$$

It thus suffices to show that the laws of the random rotations $U_1 Z_{\tilde\psi^\pm} U_2$ are absolutely continuous with respect to the law of $U_1 Z_{\psi_*} U_2$. Since $\tilde\psi$ has a density on $(0,\pi)$, also $\cos\tilde\psi^\pm$ given via (3.6) have densities on $(-1,1)$, and thus $\tilde\psi^\pm$ themselves have densities on $(0,\pi)$, all with respect to the Lebesgue measure on the respective intervals. Since further the density of ψ_* is positive on $(0,\pi)$, it follows that the laws of $\tilde\psi^\pm$ are absolutely continuous with respect to that of ψ_*. Then also the law of the triple $(U_1, \tilde\psi_\pm, U_2)$ is absolutely continuous with respect to the law of (U_1, ψ_*, U_2) on $\mathrm{SO}^*(d) \times (0,\pi) \times \mathrm{SO}^*(d)$. And the respective images in $\mathrm{SO}(d)$ under the continuous map $(G_1, \theta, G_2) \mapsto G_1 Z_\theta G_2$ inherit the absolute continuity. □

3.4 Extension of the model

In the first place, our ansatz (2.5) for the collisional kernel and hypotheses (H1)-(H4) have been chosen to accomodate the example of inelastic Maxwell molecules. However, there are other reasonable models which fit into that framework. To indicate the applicability of our results to such models, we briefly discuss one particular example.

In the collision rules (1.3) for Maxwellian molecules, the modulus of inelasticity $\delta \in (0, 1/2)$ is a constant. Thus, in each individual collision of two particles, a certain fixed fraction of kinetic energy (in the direction of impact) is transferred to the background heat bath. It seems reasonable to refine this model, replacing the constant δ by a random quantity — independent of σ — with an a priori given distribution on $(0, 1/2)$.

Clearly, the resulting model is again of the type (1.5). Choosing the random quantities $r^\pm$ and $R^\pm$ in analogy to (3.7)-(3.8), the verification of hypotheses (H1)-(H4) can be performed along the same lines as above, working "conditionally on δ". Consequently, the abstract Theorem 2.3 holds for that model. This implies in particular that also Theorem 1.1 carries over verbatim, with the interpretation that the dependence on δ means dependence on its distribution.

4 Study of an instrumental process on $C^0(\mathrm{SO}(d))$

We continue to assume that a random quadruple (r^+, r^-, R^+, R^-) satisfying the hypotheses (H1)–(H3) is given. In particular, $\alpha \in (0, 2)$ and $\gamma \in (1, 2]$ are such that (H2) holds. Recall the definition of the associated random array $(\beta_{j,n}, O_{j,n})_{\left\{\substack{n\geq 0\\ 1\leq j\leq n+1}\right\}}$ from the probabilistic representation developed in Section 2.3.

This section is devoted to the proof of convergence of the following auxiliary random processes $(\Psi_n)_{n\geq 0}$ taking values in $C^0(\mathrm{SO}(d))$. Given a continuous function $\Psi_0 \in C^0(\mathrm{SO}(d))$, define for all $n \geq 1$:

$$\Psi_n(\mathcal{O}) := \sum_{j=1}^{n+1} \beta_{j,n}^\alpha \Psi_0(\mathcal{O} O_{j,n}). \tag{4.1}$$

The ultimate goal is to show convergence of Ψ_n to a (random) constant function in the sense made precise in Proposition 4.2 below. In order to characterize the limit, we start with an auxiliary result.

Lemma 4.1. *The random quantities*

$$M_n^{(\alpha)} := \sum_{j=1}^{n+1} \beta_{j,n}^\alpha \quad \text{and} \quad \beta_{(n)} := \max_{j=1,\dots,n+1} \beta_{j,n}, \tag{4.2}$$

have the following properties:

(i) $\mathbb{E}[M_n^{(\alpha)}] = 1$ *for every* n.

(ii) $M_n^{(\alpha)}$ *converges almost surely to a random variable* $M_\infty^{(\alpha)}$ *as* $n \to +\infty$. *The characteristic function of* $M_\infty^{(\alpha)}$ *satisfies equation* (2.4) *and* $\mathbb{E}[(M_\infty^{(\alpha)})] = 1$. *Moreover,* $\mathbb{E}[(M_\infty^{(\alpha)})^p] < +\infty$ *if and only if* $\mathbb{E}[(r^+)^{\alpha p} + (r^-)^{\alpha p}] < 1$.

(iii) $\beta_{(n)}$ *converges to zero in probability.*

Proof. Claims (i) and (ii) are contained in Proposition 2 of [5], while claim (iii) is Lemma 3 in [5]. □

The main result of this section is:

Proposition 4.2. *For $n \to +\infty$, Ψ_n converges weakly to $m_0 M_\infty^{(\alpha)}$ in $C^0(\mathrm{SO}(d))$, where*

$$m_0 := \int_{\mathrm{SO}(d)} \Psi_0(\mathcal{O})\mathfrak{H}(\mathrm{d}\mathcal{O}). \tag{4.3}$$

Hence, for every fixed $\mathcal{O} \in \mathrm{SO}(d)$, the sums $\sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \Psi_0(\mathcal{O}O_{j,n})$ converge weakly to $m_0 M_\infty^{(\alpha)}$.

For the sake of simplicity the proof of Proposition 4.2 is split into several steps. Some of them use techniques developed in [8].

4.1 Basic properties of Ψ_n

Introduce the L^p-norms with respect to the Haar measure $\mathfrak{H}$ on measurable functions $f : \mathrm{SO}(d) \to \mathbb{R}$ as usual:

$$\|f\|_{L^p} := \left(\int_{\mathrm{SO}(d)} |f(\mathcal{O})|^p \mathfrak{H}(\mathrm{d}\mathcal{O}) \right)^{1/p} \quad \text{for all } p \geq 1,$$
$$\|f\|_{L^\infty} := \operatorname{ess\,sup}_{\mathcal{O}\in\mathrm{SO}(d)} |f(\mathcal{O})|.$$

Lemma 4.3. *For every $n \geq 0$,*

$$\mathbb{E}\Big[\int_{\mathrm{SO}(d)} \Psi_n(\mathcal{O})\mathfrak{H}(\mathrm{d}\mathcal{O}) \Big] = m_0, \tag{4.4}$$

where m_0 is given in (4.3)*, and*

$$\mathbb{E}\big[\|\Psi_n\|_{L^p}\big] \leq \|\Psi_0\|_{L^\infty}. \tag{4.5}$$

Proof. Since $\mathfrak{H}$ is right invariant,

$$\begin{aligned}\int_{\mathrm{SO}(d)} \Psi_n(\mathcal{O})\mathfrak{H}(\mathrm{d}\mathcal{O}) &= \int_{\mathrm{SO}(d)} \sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \Psi_0(\mathcal{O}O_{j,n})\mathfrak{H}(\mathrm{d}\mathcal{O}) \\ &= \sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \int_{\mathrm{SO}(d)} \Psi_0(\mathcal{O})\mathfrak{H}(\mathrm{d}\mathcal{O}).\end{aligned}$$

Now (4.4) follows by means of (i) in Lemma 4.1. Another application of that property yields (4.5):

$$\begin{aligned}\mathbb{E}\big[\|\Psi_n\|_{L^p}\big] &= \mathbb{E}\Big[\Big(\int_{\mathrm{SO}(d)} \Big(\sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \Psi_0(\mathcal{O}O_{j,n}) \Big)^p \mathfrak{H}(\mathrm{d}\mathcal{O}) \Big)^{1/p} \Big] \\ &\leq \|\Psi_0\|_{L^\infty} \mathbb{E}\Big[\sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \Big] = \|\Psi_0\|_{L^\infty}.\end{aligned}$$

□

Lemma 4.4. *The laws of Ψ_n form a tight sequence of probability measures on $C^0(\mathrm{SO}(d))$ and hence they are relatively sequentially compact.*

Proof. By the classical tightness criterion for sequences of random continuous functions, see e.g. Theorem 16.5 in [38], it suffices to show that

$$w(\Psi_n, \delta) := \sup\big\{ |\Psi_n(\mathcal{O}_1) - \Psi_n(\mathcal{O}_2)| \,\big|\, \|\mathcal{O}_1 - \mathcal{O}_2\|_* \leq \delta \big\},$$

where $\|\cdot\|_*$ is the matrix (operator) norm induced by the euclidean norm on $\mathbb{R}^d$, satisfies

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{E}[w(\Psi_n,\delta)] = 0. \tag{4.6}$$

Observe that for arbitrary $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{SO}(d)$,

$$\begin{aligned} |\Psi_n(\mathcal{O}_1) - \Psi_n(\mathcal{O}_2)| &= \Big|\sum_{j=1}^{n+1}\beta_{j,n}^\alpha[\Psi_0(\mathcal{O}_1 O_{j,n}) - \Psi_0(\mathcal{O}_2 O_{j,n})]\Big| \\ &\le \sum_{j=1}^{n+1}\beta_{j,n}^\alpha|\Psi_0(\mathcal{O}_1') - \Psi_0(\mathcal{O}_2')|, \end{aligned}$$

with $\mathcal{O}_i' = \mathcal{O}_i O_{j,n}$ for $i = 1, 2$. Since $O_{j,n}$ is a rotation matrix,

$$\|\mathcal{O}_1' - \mathcal{O}_2'\|_* = \|(\mathcal{O}_1 - \mathcal{O}_2)O_{j,n}\|_* = \|\mathcal{O}_1 - \mathcal{O}_2\|_*.$$

It follows that

$$\mathbb{E}[w(\Psi_n,\delta)] \le \mathbb{E}\Big[\sum_{j=1}^{n+1}\beta_{j,n}^\alpha\Big]\sup\big\{|\Psi_0(\mathcal{O}_1) - \Psi_0(\mathcal{O}_2)|\,\big|\,\|\mathcal{O}_1 - \mathcal{O}_2\|_* \le \delta\big\}.$$

The expectation value on the right-hand side equals to one, independently of n, by Lemma 4.1 (i). The supremum, which is also independent of n, tends to zero for $\delta \downarrow 0$, since the continuous function Ψ_0 on the compact manifold $\mathrm{SO}(d)$ is automatically *uniformly* continuous. Since $C^0(\mathrm{SO}(d))$ is a Polish space, the last part of the statement follows from Prohorov's Theorem, see e.g. Thm. 17, Chapter 18 in [32]. □

4.2 Definition of the recursion operator

Given $A \in \mathrm{SO}(d)$ and a function f on $\mathrm{SO}(d)$, we denote by $A^\# f$ and $A_\# f$ the functions given by

$$A^\# f(\mathcal{O}) = f(\mathcal{O}A) \quad \text{and} \quad A_\# f(\mathcal{O}) = f(\mathcal{O}A^T) \quad \text{for all } \mathcal{O} \in \mathrm{SO}(d). \tag{4.7}$$

Observe that $A^\#(B^\# f) = (AB)^\# f$ for arbitrary $A, B \in \mathcal{O}$, since

$$A^\#(B^\# f)(\mathcal{O}) = B^\# f(\mathcal{O}A) = f(\mathcal{O}AB).$$

With these notations,

$$\Psi_n(\mathcal{O}) = \sum_{j=1}^{n+1}\beta_{j,n}^\alpha\Psi_0(\mathcal{O}O_{j,n}) = \sum_{j=1}^{n+1}\beta_{j,n}^\alpha O_{j,n}^\#\Psi_0(\mathcal{O}).$$

Introduce a sequence (ν_n) of probability measures on $C^0(\mathrm{SO}(d))$ by

$$\nu_0 := \delta_{\Psi_0}, \quad \text{and for every } n \ge 1, \quad \nu_n := \mathrm{Law}\Big(\sum_{j=1}^{n+1}\beta_{j,n}^\alpha O_{j,n}^\#\Psi_0\Big). \tag{4.8}$$

Next, define a recursion operator T on the set $\mathcal{P}(C^0(\mathrm{SO}(d)))$ of all probability measures on $C^0(\mathrm{SO}(d))$ as follows. Given $\nu', \nu'' \in \mathcal{P}(C^0(\mathrm{SO}(d)))$, let Ψ' and Ψ'' be two independent random functions with distributions ν' and ν'', respectively, which are also independent of (r^-, r^+, R^-, R^+). Then define

$$T[\nu', \nu''] := \mathrm{Law}\big((r^-)^\alpha(R^-)^\#\Psi' + (r^+)^\alpha(R^+)^\#\Psi''\big). \tag{4.9}$$

T has a fixed point: set $\Psi_\infty := m_0 M_\infty^{(\alpha)}$ and $\nu_\infty := \mathrm{Law}(\Psi_\infty)$. Using Lemma 4.1, it is easy to see that

$$\nu_\infty = T[\nu_\infty, \nu_\infty]. \tag{4.10}$$

In the following, we shall show that this fixed point is attractive in a suitable metric.

Lemma 4.5. *For each $n \geq 1$, the following recursion relation holds:*

$$\nu_n = \frac{1}{n} \sum_{k=1}^{n} T[\nu_{k-1}, \nu_{n-k}]. \tag{4.11}$$

Proof. The proof is similar to the one of Proposition 2.4. With the notations (2.11), we can write

$$\begin{aligned}\Psi_n &= \sum_{j=1}^{J} \big(r_1^- \beta'_{j,n}\big)^\alpha (R_1^- O'_{j,n})^\# \Psi_0 + \sum_{j=J+1}^{n+1} \big(r_1^+ \beta''_{j,n}\big)^\alpha (R_1^+ O''_{j,n})^\# \Psi_0 \\ &= (r_1^-)^\alpha (R_1^-)^\# \Big(\sum_{j=1}^{J} (\beta'_{j,n})^\alpha (O'_{j,n})^\# \Psi_0 \Big) + (r_1^+)^\alpha (R_1^+)^\# \Big(\sum_{j=J+1}^{n+1} (\beta''_{j,n})^\alpha (O''_{j,n})^\# \Psi_0 \Big),\end{aligned}$$

using the rule $A^\#(B^\# f) = (AB)^\# f$ discussed above. To conclude, observe that — conditionally on $\{J = k\}$ —

$$\sum_{j=1}^{J} (\beta'_{j,n})^\alpha (O'_{j,n})^\# \Psi_0 \stackrel{\mathcal{L}}{=} \Psi_{k-1}, \qquad \sum_{j=J+1}^{n+1} (\beta''_{j,n})^\alpha (O''_{j,n})^\# \Psi_0 \stackrel{\mathcal{L}}{=} \Psi_{n-k}. \qquad \square$$

The goal for the rest of this section is to show that the map T is a contractive in an appropriate metric. Once this is shown, the proof of Proposition 4.2 follows easily.

4.3 Contraction in Fourier distance

Recall that $L^2(\mathrm{SO}(d), \mathfrak{H})$ is a real Hilbert space with respect to the scalar product

$$\langle g, f \rangle_{L^2} = \int_{\mathrm{SO}(d)} g(\mathcal{O}) f(\mathcal{O}) \, \mathrm{d}\mathfrak{H}(\mathcal{O}).$$

For a probability measure ν on $C^0(\mathrm{SO}(d))$, define its L^2-characteristic functional (or Fourier transform) $\hat{\nu} : L^2(\mathrm{SO}(d), \mathfrak{H}) \to \mathbb{C}$ by

$$\hat{\nu}(g) := \int_{C^0(\mathrm{SO}(d))} \exp\big(i \langle g, f \rangle_{L^2}\big) \, \nu(\mathrm{d}f) = \mathbb{E}\big[\exp\big(i \langle g, \Psi \rangle_{L^2}\big)\big],$$

where Ψ is a random function with law ν. Further, let $\mathcal{P}_{L^2;\gamma}(C^0(\mathrm{SO}(d)))$ be the set of the measures ν on $C^0(\mathrm{SO}(d))$ such that

$$\int_{C^0(\mathrm{SO}(d))} \|f\|_{L^2}^\gamma \nu(\mathrm{d}f) < +\infty;$$

recall that $\gamma \in (1, 2]$ was chosen such that (H2) is satisfied. Now introduce the *Fourier distance* between any $\nu', \nu'' \in \mathcal{P}_{L^2;\gamma}(C^0(\mathrm{SO}(d)))$ by

$$\begin{aligned} d_\gamma(\nu', \nu'') &:= \sup_{0 \neq g \in L^2} \frac{\big|\mathbb{E}\big[\exp(i\langle g, \Psi' \rangle_{L^2}) - \exp(i \langle g, \Psi'' \rangle_{L^2}) - i \langle g, \Psi' - \Psi'' \rangle_{L^2}\big]\big|}{\|g\|_{L^2}^\gamma} \\ &= \sup_{0 \neq g \in L^2} \frac{\big|\hat{\nu}'(g) - \hat{\nu}''(g) - i \langle g, \Delta \rangle_{L^2}\big|}{\|g\|_{L^2}^\gamma} \quad \text{with} \quad \Delta := \mathbb{E}[\Psi' - \Psi''], \end{aligned}$$

where Ψ' and Ψ'' are two random functions distributed according to ν' and ν''. This definition is inspired by the *Fourier metric*, which was first introduced in the context of kinetic equations in [35], and has since then been generalized in manifold ways, see e.g. [8] for another application to measures on matrices. Note that d_γ is well-defined and finite on $\mathcal{P}_{L^2;\gamma}(C^0(\mathrm{SO}(d)))$. Indeed, if Ψ has law $\nu \in \mathcal{P}_{L^2;\gamma}(C^0(\mathrm{SO}(d)))$, then $\mathbb{E}[\Psi]$ is a well-defined function in $C^0(\mathrm{SO}(d))$, and moreover, using the inequality $|e^{ix} - 1 - ix| \leq C_\gamma |x|^\gamma$ (see e.g. Lemma 1, section 8.4 in [24]), one can write

$$\begin{aligned} d_\gamma(\nu', \nu'') &\leq C_\gamma \Big\{ \sup_{g : \|g\|_{L^2} \neq 0} \frac{1}{\|g\|_{L^2}^\gamma} \mathbb{E}\Big[|\langle g, \Psi' \rangle_{L^2}|^\gamma + |\langle g, \Psi'' \rangle_{L^2}|^\gamma \Big] \Big\} \\ &\leq C_\gamma \Big\{ \int_{C^0(\mathrm{SO}(d))} \|f\|_{L^2}^\gamma \nu'(\mathrm{d}f) + \int_{C^0(\mathrm{SO}(d))} \|f\|_{L^2}^\gamma \nu''(\mathrm{d}f) \Big\}. \end{aligned} \tag{4.12}$$

The last quantity is finite since Ψ's distribution belongs to $\mathcal{P}_{L^2;\gamma}(C^0(\mathrm{SO}(d)))$. Notice further that ν' and ν'' might be "close" with respect to d_γ even if their expectation values differ significantly.

Lemma 4.6. *Given $\gamma \in [1, 2]$ satisfying (H2), let ν_1', ν_2' and ν_1'', ν_2'' be probability measures in $\mathcal{P}_{L^2;\gamma}(C^0(\mathrm{SO}(d)))$ and let Ψ_1', Ψ_1'', Ψ_2', Ψ_2'' be random functions with respective laws, that are independent of (r^+, r^-, R^+, R^-). Then*

$$\begin{aligned} & d_\gamma\big(T[\nu_1', \nu_1''], T[\nu_2', \nu_2'']\big) \leq \mathbb{E}[(r^-)^{\alpha\gamma}] d_\gamma(\nu_1', \nu_2') + \mathbb{E}[(r^+)^{\alpha\gamma}] d_\gamma(\nu_1'', \nu_2'') \\ & \quad + \max\big(2, \mathbb{E}[\|\Psi_1''\|_{L^2}], \mathbb{E}[\|\Psi_2'\|_{L^2}]\big) \mathbb{E}[(r^-)^{\alpha\gamma} + (r^+)^{\alpha\gamma}] \\ & \qquad \cdot \big(\|\mathbb{E}[\Psi_1' - \Psi_2']\|_{L^2} + \|\mathbb{E}[\Psi_1'' - \Psi_2'']\|_{L^2} \big). \end{aligned}$$

Proof. We proceed in analogy to the proof of Lemma 6 in [8]. Set $\nu_j := T[\nu_j', \nu_j'']$ and let Ψ_j be distributed with laws ν_j, respectively. Recalling the definition of $R^\#$ and $R_\#$ from (4.7), we obtain

$$\begin{aligned} \hat{\nu}_j(g) &= \mathbb{E}\big[\exp\big(i \langle g, (r^-)^\alpha (R^-)^\# \Psi_j' + (r^+)^\alpha (R^+)^\# \Psi_j'' \rangle_{L^2} \big) \big] \\ &= \mathbb{E}\big[\hat{\nu}_j'\big((r^-)^\alpha (R^-)_\# g \big) \hat{\nu}_j''\big((r^+)^\alpha (R^+)_\# g \big) \big] \\ &= \mathbb{E}\big[\hat{\nu}_j'(g') \hat{\nu}_j''(g'') \big], \end{aligned}$$

with $g' := (r^-)^\alpha (R^-)_\# g$ and $g'' := (r^+)^\alpha (R^+)_\# g$. Next, define $\Delta' := \mathbb{E}[\Psi_1' - \Psi_2']$, $\Delta'' := \mathbb{E}[\Psi_1'' - \Psi_2'']$, and observe that

$$\Delta := \mathbb{E}[\Psi_1 - \Psi_2] = \mathbb{E}[(r^-)^\alpha (R^-)^\# \Delta' + (r^+)^\alpha (R^+)^\# \Delta''],$$

so that $\langle g, \Delta \rangle_{L^2} = \mathbb{E}\big[\langle g', \Delta' \rangle_{L^2} + \langle g'', \Delta'' \rangle_{L^2} \big]$. We thus have

$$\begin{aligned} \delta(g) &:= \hat{\nu}_1(g) - \hat{\nu}_2(g) - i \langle g, \Delta \rangle_{L^2} \\ &= \mathbb{E}\big[\hat{\nu}_1'(g') \hat{\nu}_1''(g'') - \hat{\nu}_2'(g') \hat{\nu}_2''(g'') - i \langle g', \Delta' \rangle_{L^2} - i \langle g'', \Delta'' \rangle_{L^2} \big] \\ &= \mathbb{E}\big[\big(\hat{\nu}_1'(g') - \hat{\nu}_2'(g') - i \langle g', \Delta' \rangle_{L^2} \big) \hat{\nu}_1''(g'') - i \big(1 - \hat{\nu}_1''(g'') \big) \langle g', \Delta' \rangle_{L^2} \big] \\ &+ \mathbb{E}\big[\hat{\nu}_2'(g') \big(\hat{\nu}_1''(g'') - \hat{\nu}_2''(g'') - i \langle g'', \Delta'' \rangle_{L^2} \big) - i \big(1 - \hat{\nu}_2'(g') \big) \langle g'', \Delta'' \rangle_{L^2} \big]. \end{aligned}$$

Hence, we find

$$\begin{aligned} |\delta(g)| &\leq \mathbb{E}\big[\|g'\|_{L^2}^\gamma \big]\, d_\gamma(\nu_1', \nu_2') + \mathbb{E}\big[\|g''\|_{L^2}^\gamma \big]\, d_\gamma(\nu_1'', \nu_2'') \\ &\quad + \mathbb{E}\big[\big| 1 - \hat{\nu}_1''(g'') \big| \|g'\|_{L^2} \big] \|\Delta'\|_{L^2} + \mathbb{E}\big[\big| 1 - \hat{\nu}_2'(g') \big| \|g''\|_{L^2} \big] \|\Delta''\|_{L^2}. \end{aligned}$$

Since R^- and R^+ are orthogonal matrices, we have

$$\|g'\|_{L^2} = (r^-)^\alpha \|g\|_{L^2}, \quad \|g''\|_{L^2} = (r^+)^\alpha \|g\|_{L^2},$$

and so

$$\begin{aligned}
\|g\|_{L^2}^{-\gamma}|\delta(g)| &\le \mathbb{E}\big[(r^-)^{\alpha\gamma}\big]\, d_\gamma(\nu_1',\nu_2') + \mathbb{E}\big[(r^+)^{\alpha\gamma}\big]\, d_\gamma(\nu_1'',\nu_2'') \\
&\quad + \mathbb{E}\Big[\frac{|1-\hat\nu_1''(g'')|}{\|g\|_{L^2}^{\gamma-1}}(r^-)^\alpha\Big]\|\Delta'\|_{L^2} + \mathbb{E}\Big[\frac{|1-\hat\nu_2'(g')|}{\|g\|_{L^2}^{\gamma-1}}(r^+)^\alpha\Big]\|\Delta''\|_{L^2} \\
&\le \mathbb{E}\big[(r^-)^{\alpha\gamma}\big]\, d_\gamma(\nu_1',\nu_2') + \mathbb{E}\big[(r^+)^{\alpha\gamma}\big]\, d_\gamma(\nu_1'',\nu_2'') \\
&\quad + \mathbb{E}\Big[(r^-)^\alpha (r^+)^{\alpha(\gamma-1)}\Big] \sup_{h\neq 0}\Big(\frac{|1-\hat\nu_1''(h)|}{\|h\|_{L^2}^{\gamma-1}}\Big)\|\Delta'\|_{L^2} \\
&\quad + \mathbb{E}\Big[(r^-)^{\alpha(\gamma-1)} (r^+)^{\alpha}\Big] \sup_{h\neq 0}\Big(\frac{|1-\hat\nu_2'(h)|}{\|h\|_{L^2}^{\gamma-1}}\Big)\|\Delta''\|_{L^2}.
\end{aligned}$$

By definition of the Fourier transform, and since $|1-e^{ix}|\le|x|$, it follows that

$$\big|1-\hat\nu_1''(h)\big| \le \|h\|_{L^2}\mathbb{E}\big[\|\Psi_1''\|_{L^2}\big].$$

Since also $|1-\hat\nu_1''(h)|\le 2$ for all h, we have that

$$\sup_{h\neq 0}\Big(\frac{|1-\nu_1''(h)|}{\|h\|_{L^2}^{\gamma-1}}\Big) \le \max\{2,\mathbb{E}\big[\|\Psi_1''\|_{L^2}\big]\},$$

and similarly for the other supremum. To finish the proof, observe that by Young's inequality

$$\mathbb{E}\big[(r^-)^\alpha(r^+)^{\alpha(\gamma-1)} + (r^-)^{\alpha(\gamma-1)}(r^+)^\alpha\big] \le \mathbb{E}[(r^-)^{\alpha\gamma}+(r^+)^{\alpha\gamma}]. \qquad \square$$

4.4 Contraction of means

Lemma 4.6 almost yields contractivity of T in the Fourier distance d_γ for some appropriate $\gamma>1$. Below, we provide a control on the remainder term, given by the L^2-distance of the expectation values of the argument measures.

Proposition 4.7. *There are constants $\kappa^- < \mathbb{E}[(r^-)^\alpha]$ and $\kappa^+ < \mathbb{E}[(r^+)^\alpha]$ such that*

$$\|\mathbb{E}[(r^\pm)^\alpha (R^\pm)^\# f\|_{L^2} \le \kappa^\pm\|f\|_{L^2}$$

for every $f\in C^0(\mathrm{SO}(d))$ with $\int_{\mathrm{SO}(d)} f(\mathcal{O})\mathfrak{H}(\mathrm{d}\mathcal{O})=0$.

To prove Proposition 4.7 we need some preliminary results. Recalling the definition of $\mathfrak{B}^\pm$ from (2.3), introduce continuous linear operators $L^\pm$ on $L^2(\mathrm{SO}(d),\mathfrak{H})$ by

$$(L^\pm f)(\mathcal{O}) := \int_{\mathrm{SO}(d)} f(\mathcal{O}R)\mathfrak{B}^\pm(\mathrm{d}R).$$

Since for every $f,g\in L^2(\mathrm{SO}(d),\mathfrak{H})$, we have that

$$\begin{aligned}
\langle L^\pm f, g\rangle_{L^2} &= \int_{\mathrm{SO}(d)^2} f(\mathcal{O}R)g(\mathcal{O})\mathfrak{B}^\pm(\mathrm{d}R)\mathfrak{H}(\mathrm{d}\mathcal{O}) \\
&= \int_{\mathrm{SO}(d)^2} f(\mathcal{O}')g(\mathcal{O}'R^T)\mathfrak{B}^\pm(\mathrm{d}R)\mathfrak{H}(\mathrm{d}\mathcal{O}'),
\end{aligned}$$

it follows that the adjoint operator $(L^\pm)^*$ of $L^\pm$ is given by

$$((L^\pm)^* f)(\mathcal{O}) = \int_{\mathrm{SO}(d)} f(\mathcal{O}R^T)\mathfrak{B}^\pm(\mathrm{d}R).$$

Consider the symmetric operator $(L^{\pm})^* L^{\pm}$ on $L^2(\mathrm{SO}(d), \mathfrak{H})$, which can be written as

$$((L^{\pm})^* L^{\pm} f)(\mathcal{O}) = \int_{\mathrm{SO}(d)^2} f(\mathcal{O} R_2^T R_1) \mathfrak{B}^{\pm}(\mathrm{d}R_1) \mathfrak{B}^{\pm}(dR_2)$$
$$= \int_{\mathrm{SO}(d)} f(\mathcal{O}B) \tilde{\mathfrak{B}}^{\pm}(\mathrm{d}B),$$

where we define $\tilde{\mathfrak{B}}^{\pm}$ as the law of the random rotation $R_2^T R_1$ for independent R_1, R_2 with distribution $\mathfrak{B}^{\pm}$ each. It is easy to see that the powers of $(L^{\pm})^* L^{\pm}$ admit the representations

$$[(L^{\pm})^* L^{\pm}]^n f(\mathcal{O}) = \int_{\mathrm{SO}(d)} f\big(\mathcal{O}B\big) (\tilde{\mathfrak{B}}^{\pm})^{\star n}(\mathrm{d}B),$$

where $^{\star n}$ denotes the n-fold convolution of a measure. The following result is essential for the proof of Proposition 4.7.

Proposition 4.8 (Bhattacharya)**.** *Let G be a compact, connected, Hausdorff group and let β be a probability measure on G such that β has a nonzero absolutely continuous component with respect to the normalized Haar measure $\mathfrak{H}$ on G. Then there is $n \geq 1$ and $0 < c \leq 1$ such that*

$$\beta^{\star 2^n}(B) \geq c\mathfrak{H}(B) \tag{4.13}$$

for every measurable $B \subset G$.

Actually, in the proof of Theorem 3 in [9] it is shown that there are a set $A \subseteq G$ of positive Haar measure, a positive number $\bar{c} > 0$ and an index $N_0 \in \mathbb{N}$ such that, for every g in G,

$$(h\mathbb{1}_A)^{\star 2N_0}(g) \geq \bar{c}$$

where h denotes the density of the absolutely continuous component of β, and $\star$ is the convolution of functions. Here clearly N_0 can be replaced by any power of two that is larger or equal, at the possible expense of diminishing $\bar{c}$ to another (still positive) constant c. This obviously implies our assertion (4.13).

Lemma 4.9. *There are $\tilde{\kappa}^{\pm} < 1$ and $n \geq 1$ such that*

$$\|[(L^{\pm})^* L^{\pm}]^{2^n} f\|_{L^2} \leq \tilde{\kappa}^{\pm} \|f\|_{L^2}$$

for every $f \in L^2(\mathrm{SO}(d))$ with $\int_{\mathrm{SO}(d)} f(\mathcal{O}) \mathfrak{H}(\mathrm{d}\mathcal{O}) = 0$.

Proof. We follow the lines of the proof of Theorem 2 in [9]. Assumption (H3) implies that the probability measures $\tilde{\mathfrak{B}}^{\pm}$s have nonzero absolutely continuous component with respect to the Haar measure. Hence we can apply Proposition 4.8. If $(\tilde{\mathfrak{B}}^{\pm})^{\star 2^n} = \mathfrak{H}$ then $\|[(L^{\pm})^* L^{\pm})]^{2^n} f\|_{L^2} = 0$, and there is nothing to be proved. If instead $(\tilde{\mathfrak{B}}^{\pm})^{\star 2^n} \neq \mathfrak{H}$, then $c < 1$ in (4.13), and hence one can write

$$(\tilde{\mathfrak{B}}^{\pm})^{\star 2^n} = [(1-c)\Gamma + c\mathfrak{H}],$$

where $\Gamma = (1-c)^{-1}((\tilde{\mathfrak{B}}^{\pm})^{\star 2^n} - c\mathfrak{H})$ is a probability measure on $\mathrm{SO}(d)$. Since f is such that $\int f(\mathcal{O})\mathfrak{H}(d\mathcal{O}) = 0$, then, using also Jensen inequality,

$$\|[(L^{\pm})^* L^{\pm})]^{2^n} f\|_{L^2}^2 = \int \Big(\int f(\mathcal{O}B) (\tilde{\mathfrak{B}}^{\pm})^{\star 2^n}(\mathrm{d}B) \Big)^2 \mathfrak{H}(\mathrm{d}\mathcal{O})$$
$$= (1-c)^2 \int \Big(\int f(\mathcal{O}B) \Gamma(\mathrm{d}B) \Big)^2 \mathfrak{H}(\mathrm{d}\mathcal{O})$$
$$\leq (1-c)^2 \int \int f(\mathcal{O}B)^2 \mathfrak{H}(\mathrm{d}\mathcal{O}) \Gamma(\mathrm{d}B) = (1-c)^2 \|f\|_{L^2}^2.$$

This shows the desired inequality, with $\tilde{\kappa}^{\pm} = (1-c) < 1$. □

Proof of Propostion 4.7. Observe that

$$\|(L^\pm)^* L^\pm f\|_{L^2}^2 = \langle [(L^\pm)^* L^\pm]^2 f, f\rangle_{L^2} \le \|[(L^\pm)^* L^\pm]^2 f\|_{L^2} \|f\|_{L^2}$$

by the symmetry of $(L^\pm)^* L^\pm$. Similarly, for every $m \ge 0$, we have

$$\left\|[(L^\pm)^* L^\pm]^{2^m} f\right\|_{L^2}^2 \le \left\|[(L^\pm)^* L^\pm]^{2^{m+1}} f\right\|_{L^2}^2 \|f\|_{L^2},$$

and iteration of these estimates leads to

$$\left\|(L^\pm)^* L^\pm f\right\|_{L^2}^{2^n} \le \left\|[(L^\pm)^* L^\pm]^{2^n} f\right\|_{L^2} \|f\|_{L^2}^{2^n-1}$$

for arbitrary $n \ge 0$. We combine this estimate with

$$\|L^\pm f\|_{L^2}^2 = \langle (L^\pm)^* L^\pm f, f\rangle_{L^2} \le \|(L^\pm)^* L^\pm f\|_{L^2} \|f\|_{L^2}$$

to obtain

$$\|L^\pm f\|_{L^2}^{2^{n+1}} \le \|(L^\pm)^* L^\pm f\|_{L^2}^{2^n} \|f\|_{L^2}^{2^n} \le \|[(L^\pm)^* L^\pm]^{2^n} f\|_{L^2} \|f\|_{L^2}^{2^n-1} \|f\|_{L^2}^{2^n}.$$

Thus, by Lemma 4.9, we arrive at

$$\|L^\pm f\|_{L^2}^{2^{n+1}} \le \tilde{\kappa}^\pm \|f\|_{L^2}^{2^{n+1}}.$$

Taking the 2^{n+1}th root, the hypothesis follows with $\kappa^\pm := (\tilde{\kappa}^\pm)^{1/2^{n+1}} < 1$. □

4.5 Convergence of Fourier transforms

On basis of the Fourier distance d_γ, we define yet another distance on $\mathcal{P}_{L^2;\gamma}(C^0(\mathrm{SO}(d)))$ by

$$D_{\gamma,a}(\nu', \nu'') := d_\gamma(\nu', \nu'') + a\|\mathbb{E}[\Psi' - \Psi'']\|_{L^2}$$

where Ψ', Ψ'' have law ν', ν'', respectively. Here a is a positive constant to be determined later. Clearly, this distance satisfies the convexity inequality

$$D_{\gamma,a}\Big(\frac{1}{n}\sum_{i=1}^n \mu_i', \frac{1}{n}\sum_{i=1}^n \mu_i''\Big) \le \frac{1}{n}\sum_{i=1}^n D_{\gamma,a}(\mu_i', \mu_i''). \tag{4.14}$$

Proposition 4.10. *$D_{\gamma,a}(\nu_n, \nu_\infty) \to 0$ as $n \to \infty$ for $\gamma \in (0, 2]$ satisfying* (H2) *and an appropriate choice of $a > 0$.*

Proof. We are going to show that

$$D_{\gamma,a}(\nu_n, \nu_\infty) \le \frac{\lambda}{n}\sum_{k=1}^n D_{\gamma,a}(\nu_{k-1}, \nu_\infty) \tag{4.15}$$

with some $\lambda \in (0, 1)$, uniformly in $n \ge 1$. This implies that $D_{\gamma,a}(\nu_n, \nu_\infty) \le \lambda^n D_{\gamma,a}(\nu_0, \nu_\infty)$, and thus converges to zero provided that

$$D_{\gamma,a}(\nu_0, \nu_\infty) < +\infty. \tag{4.16}$$

To verify (4.16), first observe that

$$\|\mathbb{E}[\Psi_0 - \Psi_\infty]\|_{L^2} \le \|\Psi_0\|_\infty + m_0,$$

thanks to (i) in Lemma 4.1. Moreover, recalling (4.12),

$$d_\gamma(\nu_0, \nu_\infty) \le C_\gamma\Big[\|\Psi_0\|_{L^2}^\gamma + \mathbb{E}[\|\Psi_\infty\|_{L^2}^\gamma\Big] = C_\gamma\|\Psi_0\|_{L^2}^\gamma + C_\gamma m_0^\gamma \mathbb{E}[(M_\infty^{(\alpha)})^\gamma].$$

The last term is finite by (ii) of Lemma 4.1.

For the proof of (4.15), substitute (4.11) and (4.10) into (4.14) to obtain

$$D_{\gamma,a}(\nu_n,\nu_\infty) \le \frac{1}{n}\sum_{k=1}^{n} D_{\gamma,a}(T[\nu_{k-1},\nu_{n-k}],T[\nu_\infty,\nu_\infty]).$$

Using the definitions of T and of $D_{\gamma,a}$, the terms on the right-hand side can be estimated as follows:

$$\begin{aligned} D_{\gamma,a}(T[\nu_{k-1},\nu_{n-k}],T[\nu_\infty,\nu_\infty]) &\le d_\gamma(T[\nu_{k-1},\nu_{n-k}],T[\nu_\infty,\nu_\infty]) \\ &+ a\Big(\|\mathbb{E}[(r^-)^\alpha(R^-)^\star\Delta_{k-1}\|_{L^2} + \|\mathbb{E}[(r^+)^\alpha(R^+)^\star\Delta_{n-k}\|_{L^2}\Big) \end{aligned} \tag{4.17}$$

where $\Delta_k := \mathbb{E}[\Psi_k - \Psi_\infty] \in L^2(\mathrm{SO}(d);\mathfrak{H})$ satisfies

$$\int_{\mathrm{SO}(d)} \Delta_k(\mathcal{O})\mathfrak{H}(\mathrm{d}\mathcal{O}) = 0 \quad \text{for every } k,$$

thanks to (4.4). Hence Proposition 4.7 is applicable to estimate the last term on the right-hand side in (4.17). In combination with an estimate of the first term by means of Lemma 4.6 – which applies because of (4.5) – we arrive at

$$D_{\gamma,a}(\nu_n,\nu_\infty) \le \frac{1}{n}\sum_{k=1}^{n}\Big[\lambda_\gamma d_\gamma(\nu_{k-1},\nu_\infty) + [a(\kappa^- + \kappa^+) + 2C']\|\mathbb{E}[\Psi_{k-1} - \Psi_\infty]\|_{L^2}\Big]$$

with $\lambda_\gamma := \mathbb{E}[(r^-)^{\alpha\gamma}] + \mathbb{E}[(r^+)^{\alpha\gamma}] < 1$ and $C' := \max\{2, \|\Psi_0\|_\infty\}\mathbb{E}[(r^-)^{\alpha\gamma} + (r^+)^{\alpha\gamma}]$. Further, recalling that $\kappa^- + \kappa^+ < \mathbb{E}[(r^-)^\alpha] + \mathbb{E}[(r^+)^\alpha] = 1$, we can choose $a > 0$ such that $a(\kappa^- + \kappa^+) + 2C' < a$. Thus we have shown (4.15), with

$$\lambda := \max\{\lambda_\gamma, \kappa^- + \kappa^+ + 2C'/a\} < 1.$$ □

4.6 Proof of Proposition 4.2

By Proposition 4.10 one gets

$$\hat{\nu}_n(g) \to \hat{\nu}_\infty(g) \tag{4.18}$$

for every g in $L^2(\mathrm{SO}(d);\mathfrak{H})$. According to Lemma 4.4, $(\Psi_n)_n$ is a tight sequence in $C^0(\mathrm{SO}(d))$. Assume that a subsequence $\Psi_{n'}$ converges weakly in $C^0(\mathrm{SO}(d))$ to a limit Y. Since $f \mapsto \exp\{i\langle g, f\rangle_{L^2}\}$ is a continuous function on $C^0(\mathrm{SO}(d))$ for any g in L^2, one gets that $\hat{\nu}_{n'}(g) \to \mathbb{E}[e^{i\langle g,Y\rangle_{L^2}}]$, and hence

$$\mathbb{E}[e^{i\langle g,\Psi_\infty\rangle_{L^2}}] = \mathbb{E}[e^{i\langle g,Y\rangle_{L^2}}]$$

for every g in L^2. Using the previous identity it is easy to see that the finite dimensional law of Y and Ψ_∞ are the same and hence they have the same distribution as processes (see, e.g., Proposition 3.2 [38]). The last part of the proof follows by the continuous mapping theorem, since point evaluation is a continuous functional on $C^0[\mathrm{SO}(d)]$.

5 Proof of the main theorem

5.1 Preliminary weak convergence results

Recall that we deal with initial conditions μ_0 belonging to the NDA of a (full) α-stable law with Lévy measure ϕ. Let X_0 be a random variable with probability distribution μ_0. For every $x, u \in \mathbb{R}^d$, set $F_0(x,u) = \mathbb{P}\{u \cdot X_0 \le x\}$, $F_0(x^-,u) = \lim_{y\uparrow x} F_0(y,u)$ and

$$B_x = \{y \in \mathbb{R}^d : x \cdot y > 1\}.$$

Let $\mathcal{B}_n$ denote the σ–field generate by the $\beta_{j,n}$'s and $O_{j,n}$, i.e.

$$\mathcal{B}_n = \sigma(O_{j,n}, \beta_{j,n} : j = 1, \dots, n+1).$$

Moreover, given any $\mathcal{O} \in \mathrm{SO}(d)$, write

$$\varpi_{j,n} := \mathcal{O} O_{j,n} \mathbf{e}_d \qquad j = 1, \dots, n+1$$

and, for every $y > 0$, define

$$Q_{1,n}(y) := \frac{1}{y^\alpha} \sum_{j=1}^{n+1} \mathbb{P}\Big\{\beta_{j,n}\varpi_{j,n} \cdot X_0 \geq 1/y \Big| \mathcal{B}_n\Big\} = \frac{1}{y^\alpha} \sum_{j=1}^{n+1} \Big[1 - F_0\Big(\Big(\frac{1}{y\beta_{j,n}}\Big)^{-}, \varpi_{j,n}\Big)\Big]$$
$$Q_{2,n}(y) := \frac{1}{y^\alpha} \sum_{j=1}^{n+1} \mathbb{P}\Big\{\beta_{j,n}\varpi_{j,n} \cdot X_0 \leq -1/y \Big| \mathcal{B}_n\Big\} = \frac{1}{y^\alpha} \sum_{j=1}^{n+1} F_0\Big(-\frac{1}{y\beta_{j,n}}, \varpi_{j,n}\Big).$$

Observe that by Lemma A.3 in Appendix it follows that

$$\lim_{y\downarrow 0} Q_{1,n}(y) = \sum_{j=1}^{n+1} \beta_{j,n}^\alpha \phi(B_{\varpi_{j,n}}) \quad \text{and} \quad \lim_{y\downarrow 0} Q_{2,n}(y) = \sum_{j=1}^{n+1} \beta_{j,n}^\alpha \phi(B_{-\varpi_{j,n}}).$$

Hence setting

$$(Q_{1,n}(0), Q_{2,n}(0)) := (\sum_{j=1}^{n+1} \beta_{j,n}^\alpha \phi(B_{\varpi_{j,n}}), \sum_{j=1}^{n+1} \beta_{j,n}^\alpha \phi(B_{-\varpi_{j,n}})),$$

the random function $y \mapsto (Q_{1,n}(y), Q_{2,n}(y))$ is a *càdlàg* (i.e. right continuos with left-hand limits) function from $[0, +\infty)$ to $\mathbb{R}^2$. Since, clearly, all the finite dimensional components are measurable, $(Q_{1,n}, Q_{2,n})$ can be seen as process taking values in the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$ of càdlàg functions with the Skorohod topology (see, e.g., [37] and Thm. 4.5 in [10]). Furthermore, given any $\gamma_0 \in \mathbb{R}^d$ and $\mathcal{O} \in \mathrm{SO}(d)$, define

$$Q_{3,n} := \sum_{j=1}^{n+1} \beta_{j,n}^\alpha \mathcal{O} O_{j,n} \mathbf{e}_d \cdot \gamma_0.$$

Then $(M_n^{(\alpha)}, Q_{1,n}, Q_{2,n}, Q_{3,n})$ is a process taking values in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^4)$, since $Q_{3,n}$ and $M_n^{(\alpha)}$ can be seen as constant random functions (w.r.t. y).

Proposition 5.1. *Assume* (H1)-(H3). *The sequence of processes* $(M_n^{(\alpha)}$, $Q_{1,n}, Q_{2,n}$, $Q_{3,n})_{n\geq 1}$ *converges in law in* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^4)$ *to the constant process* $(M_\infty^{(\alpha)}, cM_\infty^{(\alpha)}$, $cM_\infty^{(\alpha)}, 0)$ *where*

$$c := \int_{\mathbb{S}^{d-1}} \int_{\{y : y\cdot s > 1\}} \phi(dy) \mathfrak{u}_{\mathbb{S}}(ds). \tag{5.1}$$

Proof. First of all note that the functions $\mathcal{O} \mapsto \int_{\{y:y\cdot\mathcal{O}\mathbf{e}_d > 1\}} \phi(dy)$ and $\mathcal{O} \mapsto \int_{\{y:y\cdot\mathcal{O}\mathbf{e}_d < -1\}} \phi(dy)$ are uniformly continuous on $\mathrm{SO}(d)$. Indeed, we know from Lemma A.2 in Appendix that $x \mapsto \int_{\{y:y\cdot x>1\}} \phi(dy)$ is continuous in $\mathbb{R}^d \setminus \{0\}$. Hence it is uniformly continuous on $\mathbb{S}^{d-1}$ and the continuity of $\mathcal{O} \mapsto \mathcal{O}\mathbf{e}_d$ and $\mathcal{O} \mapsto -\mathcal{O}\mathbf{e}_d$ entails the claim.

Now write for $i = 1, 2$

$$Q_{i,n}(y) = Q_{i,n}(0) + R_{i,n}(y),$$

and observe that, for $0 < y \leq \delta$, one has

$$|R_{1,n}(y)| \leq \sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \sup_u |(1/y\beta_{j,n})^{\alpha}(1 - F_0((1/y\beta_{j,n})^-, u)) - \phi(B_u)|$$
$$\leq M_n^{(\alpha)} \sup_{z \leq \delta\beta_{(n)}} \sup_u |z^{-\alpha}(1 - F_0((1/z)^-, u)) - \phi(B_u)|.$$

Hence

$$\sup_{0 \leq y \leq \delta} |R_{1,n}(y)| \leq M_n^{(\alpha)} \sup_{0<y<\delta\beta_{(n)}} \sup_u |y^{-\alpha}(1 - F_0((1/y)^-, u)) - \phi(B_u)|.$$

Analogously

$$\sup_{0 \leq y \leq \delta} |R_{2,n}(y)| \leq M_n^{(\alpha)} \sup_{0<y<\delta\beta_{(n)}} \sup_u |y^{-\alpha} F_0(-1/y, u)) - \phi(B_{-u})|.$$

Since $\beta_{(n)} \to 0$ in probability by (iii) of Lemma 4.1, using Lemma A.3 one obtains that for every $\delta > 0$ and every $\epsilon > 0$

$$\lim_{n \to +\infty} \mathbb{P}\{ \sup_{0 \leq y \leq \delta} [|R_{1,n}(y)| + |R_{2,n}(y)|] > \epsilon \} = 0. \tag{5.2}$$

Now let $(t_0, t_1, t_2, t_3) \in \mathbb{R}^4$ and consider

$$\Psi_0(\mathcal{O}) := t_0 + t_1 \int_{\{y: y\cdot\mathcal{O}\mathbf{e}_d > 1\}} \phi(dy) + t_2 \int_{\{y: y\cdot\mathcal{O}\mathbf{e}_d < -1\}} \phi(dy) + t_3 \mathcal{O}\mathbf{e}_d \cdot \gamma_0$$

which is a continuous function on $\mathrm{SO}(d)$ by the considerations above. Then the corresponding Ψ_n, defined in (4.1), satisfies

$$\Psi_n(\mathcal{O}) = t_0 M_n^{(\alpha)} + t_1 \sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \phi(B_{\varpi_{j,n}}) + t_2 \sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \phi(B_{-\varpi_{j,n}}) + t_3 Q_{3,n}$$

since $\varpi_{j,n} = \mathcal{O} O_{j,n} \mathbf{e}_d$. At this stage observe that

$$\int_{\mathrm{SO}(d)} \mathcal{O}\mathbf{e}_d \cdot \gamma_0 \mathfrak{H}(d\mathcal{O}) = 0$$

and Proposition 4.2 yields that $\Psi_n(\mathcal{O})$ converges in law to $(t_0 + t_1 c_1 + t_2 c_2) M_\infty^{(\alpha)}$ where

$$c_1 := \int_{\mathrm{SO}(d)} \int_{\{y: y\cdot\mathcal{O}\mathbf{e}_d > 1\}} \phi(dy) \mathfrak{H}(d\mathcal{O}),\ c_2 := \int_{\mathrm{SO}(d)} \int_{\{y: y\cdot\mathcal{O}\mathbf{e}_d < -1\}} \phi(dy) \mathfrak{H}(d\mathcal{O}).$$

Since $O\mathbf{e}_d$ is uniformly distributed on $\mathbb{S}^{d-1}$ whenever O has Haar distribution on $\mathrm{SO}(d)$ (see (3.3)), then

$$c_1 = \int_{\mathbb{S}^{d-1}} \int_{\{y: y\cdot s > 1\}} \phi(dy) \mathfrak{u}_{\mathbb{S}}(ds) = c = \int_{\mathbb{S}^{d-1}} \int_{\{y: y\cdot s < -1\}} \phi(dy) \mathfrak{u}_{\mathbb{S}}(ds) = c_2.$$

This yields that the vector

$$Z_n := (M_n^{(\alpha)}, \sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \phi(B_{\varpi_{j,n}}), \sum_{j=1}^{n+1} \beta_{j,n}^{\alpha} \phi(B_{-\varpi_{j,n}}), Q_{3,n})$$

converges in law to $(M_\infty^{(\alpha)}, c M_\infty^{(\alpha)}, c M_\infty^{(\alpha)}, 0)$. Since

$$(M_n^{(\alpha)}, Q_{1,n}(y), Q_{2,n}(y), Q_{3,n}) = Z_n + (0, R_{1,n}(y), R_{2,n}(y), 0)$$

using (5.2) and Lemma 3.31 Chapter VI of [37] one obtains the thesis. □

5.2 Proof of Theorem 2.3.

The proof is split into three steps. In the first step we introduce a Skorohod-type representation which is inspired to the one used in [31] as an essential ingredient to prove a central limit theorem for arrays of partially exchangeable random variables. This technique has been already employed in a fruitful way in the context of the asymptotic study of kinetic equations, see e.g. [6, 43, 28, 33]. In the second step we prove that the classical conditions for the convergence to a (one-dimensional) stable law hold almost surely in the Skorohod representation. In the third step we conclude the proof.

Step 1: Skorohod representation. For every $n \geq 1$ and for $j > n+1$, let us define $\beta_{j,n} = 0$ and $\varpi_{j,n} = \mathrm{e}_d$, while for $j \leq n+1$ they are defined as in the previous sections.

Let $\mathcal{B}_n$ denote the σ–field generated by the $\beta_{j,n}$'s and $\varpi_{j,n}$'s, i.e. $\mathcal{B}_n = \sigma(\beta_{j,n}, \varpi_{j,n}; j \geq 1)$. Let $\lambda_{j,n}$ denote the conditional law of $\beta_{j,n}\varpi_{j,n} \cdot X_j$ given $\mathcal{B}_n$ and λ_n the conditional law of $\sum_{j=1}^{n+1} \beta_{j,n}\varpi_{j,n} \cdot X_j$, given $\mathcal{B}_n$. Hence, $\lambda_{j,n}(-\infty, x] = F_0(x/\beta_{j,n}, \varpi_{j,n})$ and $\lambda_n = \lambda_{1,n} * \cdots * \lambda_{n+1,n}$. Let $Q_{3,n} = \sum_{j=1}^{n+1} \beta_{j,n}\varpi_{j,n} \cdot \gamma_0$ with γ_0 as in Theorem 2.3 if $\alpha = 1$ and with $\gamma_0 = \mathbf{0}$ otherwise. Let us consider

$$W_n = \Big(\lambda_n, (\lambda_{j,n})_{j\geq 1}, \beta_{(n)}, (\beta_{j,n})_{j\geq 1}, (\varpi_{j,n})_{j\geq 1}, M_n^{(\alpha)}, Q_{1,n}(\cdot), Q_{2,n}(\cdot), Q_{3,n}\Big)$$

as a random element from $(\Omega, \mathcal{F}, \mathbb{P})$ in $(S, \mathcal{B}(S))$, where $S := \mathcal{P}(\bar{\mathbb{R}})^\infty \times \bar{\mathbb{R}}^\infty \times (\mathbb{S}^{d-1})^\infty \times \mathbb{D}(\mathbb{R}_+, \mathbb{R}^4)$. Here $\bar{\mathbb{R}}$ denotes the extended real line, $\mathcal{P}(\bar{\mathbb{R}})$ the set of all probability measures on borel σ-field $\mathcal{B}(\bar{\mathbb{R}})$ with the topology of the complete convergence and $\mathcal{B}(S)$ denotes the borel $\sigma-$field on S.

The sequence $(W_n)_{n\geq 1}$ is tight since $\mathcal{P}(\bar{\mathbb{R}})^\infty$ and $(\mathbb{S}^{d-1})^\infty$ are compact, $\beta_{(n)} \to 0$ in probability by Lemma 4.1 and the sequence $(M_n^{(\alpha)}, Q_{1,n}(\cdot), Q_{2,n}(\cdot), Q_{3,n})_{n\geq 1}$ of random elements in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^4)$ converges in law to $(M_\infty^{(\alpha)}, cM_\infty^{(\alpha)}, cM_\infty^{(\alpha)}, 0)$ in view of Proposition 5.1. Hence, every subsequence of (n) includes a subsequence (n') such that

$$W_{n'} \xrightarrow{\mathcal{L}} W'_\infty.$$

Since S is Polish, from the Skorohod representation theorem (see, e.g., Theorem 4.30 [38]) one can determine a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and random elements on it taking value in S,

$$\hat{W}_\infty = \Big(\hat{\lambda}, (\hat{\lambda}_j)_{j\geq 1}, \hat{\beta}, (\hat{\beta}_j)_{j\geq 0}, (\hat{\varpi}_j)_{j\geq 1}, \hat{M}, \hat{Q}_1(\cdot), \hat{Q}_2(\cdot), \hat{Q}_3\Big)$$

$$\hat{W}_{n'} = \Big(\hat{\lambda}_{n'}, (\hat{\lambda}_{j,n'})_{j\geq 1}, \hat{\beta}_{(n')}, (\hat{\beta}_{j,n'})_{j\geq 1}, (\hat{\varpi}_{j,n'})_{j\geq 1}, \hat{M}_{n'}, \hat{Q}_{1,n'}(\cdot), \hat{Q}_{2,n'}(\cdot), \hat{Q}_{3,n'}\Big)$$

which have the same probability distribution of W'_∞ and $W_{n'}$, respectively and

$$\lim_{n'\to+\infty} \hat{W}_{n'}(\hat{\omega}) = \hat{W}_\infty(\hat{\omega})$$

for every $\hat{\omega} \in \hat{\Omega}$ in the metric of S. In view of the definition of W_n and since W_n and $\hat{W}_{n'}$

have the same probability distribution, the following statements hold, for each n', $\hat{\mathbb{P}}-$a.s.

$$
\begin{aligned}
&\hat{\lambda}_{n'} = \hat{\lambda}_{1,n'} * \cdots * \hat{\lambda}_{n'+1,n'}, \qquad \hat{\lambda}_{j,n'}(-\infty, x] = F_0(x/\hat{\beta}_{j,n'}, \hat{\varpi}_{j,n'}),\\
&\hat{\beta}_{(n')} = \max_{j=1,\dots,n'+1} \hat{\beta}_{j,n'}, \quad \hat{M}_{n'} = \sum_{j=1}^{n'+1} \hat{\beta}^{\alpha}_{j,n'},\\
&\hat{Q}_{1,n'}(y) = \frac{1}{y^\alpha} \sum_{j=1}^{n'+1} [1 - F_0((1/y\hat{\beta}_{j,n'})^-, \hat{\varpi}_{j,n'})] \text{ for every } y > 0,\\
&\hat{Q}_{2,n'}(y) = \frac{1}{y^\alpha} \sum_{j=1}^{n'+1} F_0(-1/y\hat{\beta}_{j,n'}, \hat{\beta}_{j,n'}) \text{ for every } y > 0,\\
&\hat{Q}_{3,n'} = \sum_{j=1}^{n'+1} \hat{\beta}^{\alpha}_{j,n'} \hat{\varpi}_{j,n'} \cdot \gamma_0.
\end{aligned}
\tag{5.3}
$$

Furthermore, since

$$
(\beta_{(n)}, M_n^{(\alpha)}, Q_{1,n}(\cdot), Q_{2,n}(\cdot), Q_{3,n}) \xrightarrow{\mathcal{L}} (0, M_\infty^{(\alpha)}, cM_\infty^{(\alpha)}, cM_\infty^{(\alpha)}, 0)
$$

then

$$
(\hat{\beta}, \hat{M}, \hat{Q}_1(\cdot), \hat{Q}_2(\cdot), \hat{Q}_3) = (0, \hat{M}, c\hat{M}, c\hat{M}, 0)
$$

$\hat{\mathbb{P}}-$a.s.. and the law of $\hat{M}$ is equal to the law of $M_\infty^{(\alpha)}$ and hence does not depend on the sequence (n').

Step 2: sufficient conditions for the convergence to a stable law. The next step is to prove that the following conditions hold $\hat{\mathbb{P}}-$a.s.:

i) $\sup_{1\le j\le n'} \hat{\lambda}_{j,n'}([-\epsilon,\epsilon]^c) \to 0$, for every $\epsilon > 0$, as $n' \to +\infty$ (u.a.n. condition);

ii) $\lim_{n'\to+\infty} x^\alpha \sum_{j=1}^{n'+1} \hat{\lambda}_{j,n'}((-\infty,-x]) = c\hat{M}$ and $\lim_{n'\to+\infty} x^\alpha \sum_{j=1}^{n'+1} \hat{\lambda}_{j,n'}((x,+\infty]) = c\hat{M}$ for every $x > 0$, with c as in (5.1);

iii) $\lim_{\epsilon\downarrow 0} \limsup_{n'\to+\infty} \sum_{j=1}^{n'+1} \int_{(-\epsilon,\epsilon)} x^2 \hat{\lambda}_{j,n'}(dx) = 0$;

iv) $\lim_{n'\to+\infty} E_{n'} = 0$ where

$$
E_{n'} := \Big\{ -\sum_{j=1}^{n'+1} \hat{\lambda}_{j,n'}((-\infty,-1]) + \sum_{j=1}^{n'+1} (1 - \hat{\lambda}_{j,n'}(-\infty,1]) + \sum_{j=1}^{n'+1} \int_{(-1,1]} x\hat{\lambda}_{j,n'}(dx) \Big\}.
$$

In view of the well-known criteria for the convergence to a (one-dimensional) stable law - see, e.g., Theorem 30 in Section 16.9 and in Proposition 11 in Section 17.2 of [32] - the previous conditions yield that $\hat{\mathbb{P}}-$a.s.

$$
\int e^{i\rho x} \hat{\lambda}_{n'}(dx) \to \int e^{i\rho x} \hat{\lambda}(dx) = e^{-c\hat{M}|\rho|^\alpha} \tag{5.4}
$$

and this will lead easily to the conclusion.

Let us first prove *i)*. Recall that from Lemma A.3 we know that

$$
\lim_{x\to+\infty} \{ \sup_{u\in\mathbb{S}^{d-1}} |x^\alpha(1 - F_0(x,u)) - \phi(B_u)| + \sup_{u\in\mathbb{S}^{d-1}} |x^\alpha F_0(-x,u) - \phi(B_{-u})| \} = 0 \tag{5.5}
$$

and hence, in particular,

$$\sup_{x>0}\{\sup_{u\in\mathbb{S}^{d-1}}|x^\alpha(1-F_0(x,u))-\phi(B_u)|+\sup_{u\in\mathbb{S}^{d-1}}|x^\alpha F_0(-x,u)-\phi(B_{-u})|\}<K. \tag{5.6}$$

Since for every $u\in\mathbb{S}^{d-1}$, one has $\phi(B_u)\le\phi\{y:|y|\ge 1\}<+\infty$, then (5.6) yields

$$\sup_{x>0}\{\sup_{u\in\mathbb{S}^{d-1}}|x^\alpha(1-F_0(x,u))|+\sup_{u\in\mathbb{S}^{d-1}}|x^\alpha F_0(-x,u)|\}<K'. \tag{5.7}$$

In view of (5.3) we have

$$\begin{aligned}\hat\lambda_{j,n'}([-\epsilon,\epsilon]^c)&\le 1-F_0(\epsilon/\hat\beta_{j,n'},\hat\varpi_{j,n'})+F_0(-\epsilon/\hat\beta_{j,n'},\hat\varpi_{j,n'})\\&\le\frac{\hat\beta^\alpha_{j,n'}}{\epsilon^\alpha}\sup_{u\in\mathbb{S}^{d-1}}\Big\{\Big[1-F_0(\epsilon/\hat\beta_{j,n'},u)+F_0(-\epsilon/\hat\beta_{j,n'},u)\Big]\frac{\epsilon^\alpha}{\hat\beta^\alpha_{j,n'}}\Big\}\\&\le K'\frac{\hat\beta^\alpha_{j,n'}}{\epsilon^\alpha}\le K'\frac{\hat\beta^\alpha_{(n')}}{\epsilon^\alpha}\end{aligned}$$

and the last term converges to zero for $n'\to+\infty$.

As for ii), if $x>0$

$$x^\alpha\sum_{j=1}^{n'+1}\hat\lambda_{j,n'}((x,+\infty))=\hat Q_{1,n'}\Big(\Big(\frac{1}{x}\Big)^-\Big)$$

and

$$x^\alpha\sum_{j=1}^{n'+1}\hat\lambda_{j,n'}((-\infty,-x])=\hat Q_{2,n'}\Big(\frac{1}{x}\Big).$$

Since $(\hat Q_{1,n'}(\cdot),\hat Q_{2,n'}(\cdot))$ converges for every $\hat\omega\in\Omega$ in the topology of $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^2)$ to the constant function $(c\hat M,c\hat M)$ then, by using Proposition 2.4 Chapter VI of [37], one gets for every $y>0$

$$\hat Q_{1,n'}(y^-)\to c\hat M\quad\text{and}\quad\hat Q_{2,n'}(y)\to c\hat M.$$

Hence ii) is proved.

In order to prove iii) note that integration by parts, gives

$$\int_{(-\epsilon,\epsilon)}x^2dG(x)\le 2\int_{-\epsilon}^0|x|G(x)dx+2\int_0^\epsilon x(1-G(x))dx.$$

Hence the last inequality and (5.7) yield

$$\begin{aligned}\sum_{j=1}^{n'+1}\int_{(-\epsilon,\epsilon)}x^2\hat\lambda_{j,n'}(dx)&=\sum_{j=1}^{n'+1}\hat\beta^2_{j,n'}\int_{(-\frac{\epsilon}{\hat\beta_{j,n'}},\frac{\epsilon}{\hat\beta_{j,n'}})}x^2F_0(dx,\hat\varpi_{j,n'})\\&\le 2\sum_{j=1}^{n'+1}\Big\{\int_{-\epsilon}^0|x|F_0\Big(\frac{x}{\hat\beta_{j,n'}},\hat\varpi_{j,n'}\Big)dx+\int_0^\epsilon x\big[1-F_0\Big(\frac{x}{\hat\beta_{j,n'}},\hat\varpi_{j,n'}\Big)\big]\,dx\Big\}\\&\le 2\sum_{j=1}^{n'+1}\hat\beta^\alpha_{j,n'}K'\int_0^\epsilon x^{1-\alpha}dx=\hat M_{n'}\frac{2K'\epsilon^{2-\alpha}}{2-\alpha}\end{aligned}$$

which gives the result since $\hat M_{n'}$ converges $\hat{\mathbb{P}}$-a.s. to $\hat M$.

Concerning iv), assume first that $\alpha<1$. Then, integration by parts gives

$$\begin{aligned}E_{n'}=&-\sum_{j=1}^{n'+1}\int_{(-1,0]}\hat\lambda_{j,n'}((-\infty,x])dx+\sum_{j=1}^{n'+1}\int_{(0,1]}(1-\hat\lambda_{j,n'}((-\infty,x]))dx\\&\le-\sum_{j=1}^{n'+1}\int_{(-1,0]}F_0\Big(\frac{x}{\hat\beta_{j,n'}},\hat\varpi_{j,n'}\Big)dx+\sum_{j=1}^{n'+1}\int_{(0,1]}(1-F_0\Big(\frac{x}{\hat\beta_{j,n'}},\hat\varpi_{j,n'}\Big)dx.\end{aligned}$$

We know that, if $x < 0$, $-|x|^\alpha \sum_{j=1}^{n'+1} F_0\Big(\frac{x}{\hat\beta_{j,n'}}, \hat\varpi_{j,n'}\Big) = \hat Q_{1,n'}(1/|x|) \to -c\hat M$ on $\hat\Omega$. Furthermore for every $x \in (0,1)$

$$\Big|\sum_{j=1}^{n'+1} F_0\Big(\frac{x}{\hat\beta_{j,n'}}, \hat\varpi_{j,n'}\Big)\Big| \le \frac{1}{|x|^\alpha} \sup_{y<0} \sup_{u\in\mathbb{S}^{d-1}} F_0(y,u)|y|^\alpha \hat M_{n'} \le \frac{1}{|x|^\alpha} K' \sup_{n'} \hat M_{n'}.$$

Analogously, for $x > 0$, $|x|^\alpha \sum_{j=1}^{n'+1} \Big(1 - F_0\Big(\frac{x}{\hat\beta_{j,n'}}, \hat\varpi_{j,n'}\Big)\Big) = \hat Q_{2,n'}(1/|x|) \to c\hat M$ on $\hat\Omega$. Finally, for every $x \in (-1,0)$

$$\begin{aligned}\Big|\sum_{j=1}^{n'+1} \Big(1 - F_0\Big(\frac{x}{\hat\beta_{j,n'}}, \hat\varpi_{j,n'}\Big)\Big)\Big| &\le \frac{1}{|x|^\alpha} \sup_{y<0} \sup_{u\in\mathbb{S}^{d-1}} (1 - F_0(y,u))|y|^\alpha \hat M_{n'} \\ &\le \frac{1}{|x|^\alpha} K' \sup_{n'} \hat M_{n'}.\end{aligned}$$

Hence, dominated convergence (for any $\hat\omega$) yields that

$$E_{n'} = -\sum_{j=1}^{n'+1} \int_{(-1,0]} \hat\lambda_{j,n'}((-\infty,x])dx + \sum_{j=1}^{n'+1} \int_{(0,1]} (1 - \hat\lambda_{j,n'}((-\infty,x]))dx$$

converges (as $n \to +\infty$) to

$$-\int_{(-1,0]} \frac{c\hat M}{|x|^\alpha} dx + \int_{(0,1]} \frac{c\hat M}{|x|^\alpha} dx = 0.$$

When $1 < \alpha < 2$, since $\int_{\mathbb{R}} yF_0(dy,u) = 0$ for every u in $\mathbb{S}^{d-1}$, we can write

$$E_{n'} = -\int_{(-\infty,-1]} (1+x) \sum_{j=1}^{n'+1} \hat\lambda_{j,n'}(dx) - \int_{(1,+\infty)} (x-1) \sum_{j=1}^{n'+1} \hat\lambda_{j,n'}(dx).$$

Integration by parts gives

$$\begin{aligned}&\int_{(-\infty,-1]} (1+x) \sum_{j=1}^{n'+1} \hat\lambda_{j,n'}(dx) = \lim_{T\to+\infty} \int_{(-T,-1]} (1+x) \sum_{j=1}^{n'+1} \hat\lambda_{j,n'}(dx) \\ &= \lim_{T\to+\infty} \Big[-\sum_{j=1}^{n'+1} \hat\lambda_{j,n'}((-\infty,1-T])(1-T) - \int_{(-T,-1]} \sum_{j=1}^{n'+1} \hat\lambda_{j,n'}((-\infty,x])dx.\Big]\end{aligned}$$

Now

$$\limsup_{T\to+\infty} \sum_{j=1}^{n'+1} \hat\lambda_{j,n'}((-\infty,1-T])(1-T) \le \limsup_{T\to+\infty} KM_n^{(\alpha)}(1-T)^{1-\alpha} = 0,$$

and hence

$$\int_{(-\infty,-1]} (1+x) \sum_{j=1}^{n'+1} \hat\lambda_{j,n'}(dx) = -\int_{(-\infty,-1]} \sum_{j=1}^{n'+1} \hat\lambda_{j,n'}((-\infty,x])dx.$$

In an analogous way one shows that

$$\int_{(1,+\infty)} (x-1) \sum_{j=1}^{n'+1} \hat\lambda_{j,n'}(dx) = \int_{(1,+\infty)} \sum_{j=1}^{n'+1} (1 - \hat\lambda_{j,n'}((-\infty,x])dx,$$

so that

$$E_{n'} = \int_{(-\infty,-1]} \sum_{j=1}^{n'+1} \hat{\lambda}_{j,n'}((-\infty,x])dx - \int_{(-\infty,-1]} \sum_{j=1}^{n'+1} (1-\hat{\lambda}_{j,n'}((-\infty,x]))dx.$$

Arguing as in the case $\alpha < 1$ one proves that

$$E_{n'} \to +\int_{(-\infty,-1]} \frac{c\hat{M}}{|x|^\alpha}dx - \int_{(1,+\infty)} \frac{c\hat{M}}{|x|^\alpha}dx = 0.$$

It remains to consider the case $\alpha = 1$. Note that by point $ii)$ with $x = 1$

$$\lim_{n'\to+\infty} E_{n'} = \lim_{n'\to+\infty} \sum_{j=1}^{n'+1} \int_{(-1,1]} x\hat{\lambda}_{j,n'}(dx)$$

if the limit exists and

$$\begin{aligned} \sum_{j=1}^{n'+1} \int_{(-1,1]} x\hat{\lambda}_{j,n'}(dx) &= \sum_{j=1}^{n'+1} \hat{\beta}_{j,n'} \Big[\int_{(-1/\hat{\beta}_{j,n'},1/\hat{\beta}_{j,n'}]} x dF_0(x, \hat{\varpi}_{j,n'}) - \gamma_0 \cdot \hat{\varpi}_{j,n'} \Big] \\ &+ \sum_{j=1}^{n'+1} \hat{\beta}_{j,n'} \gamma_0 \cdot \hat{\varpi}_{j,n'} \\ &=: E^*_{n'} + \sum_{j=1}^{n'+1} \hat{\beta}_{j,n'} \gamma_0 \cdot \hat{\varpi}_{j,n'} = E^*_{n'} + \hat{Q}_{3,n} \end{aligned}$$

and $\hat{Q}_{3,n} \to 0$ $\hat{\mathbb{P}}$-a.s.. Furthermore

$$|E^*_{n'}| \leq \hat{M}_{n'} \sup_{R\geq 1/\hat{\beta}_{(n')}} \sup_{u\in\mathbb{S}^{d-1}} |\int_{(-R,R]} x dF_0(x,u) - \gamma_0 \cdot u|.$$

Since $\beta_{(n')} \to 0$ and $\hat{M}_{n'} \to \hat{M}$ it follows from assumption (2.9) that $\lim_{n'\to+\infty} E_{n'} = 0$ in the case $\alpha = 1$ too. At this stage the proof of $iv)$ is completed.

Step 3: conclusion of the proof. By (5.4) and dominated convergence theorem one has

$$\begin{aligned} \mathbb{E}[e^{i\rho \sum_{k=1}^{n'+1} \beta_{k,n'}\varpi_{k,n'}\cdot X_k}] &= \hat{\mathbb{E}}[\int e^{i\rho x} \hat{\lambda}_{n'}(dx)] \\ &\to \hat{\mathbb{E}}[\int e^{i\rho x}\hat{\lambda}(dx)] = \hat{\mathbb{E}}[e^{-c\hat{M}|\rho|^\alpha}] = \mathbb{E}[e^{-cM_\infty^{(\alpha)}|\rho|^\alpha}] \end{aligned}$$

where $\hat{\mathbb{E}}$ denotes the expectation with respect to $\hat{\mathbb{P}}$ and the last equality is due to the fact that we proved that $M_\infty^{(\alpha)}$ and $\hat{M}$ have the same probability distribution. In particular we have stated that the limit does not depend on the subsequence (n') and hence the convergence is true for the entire sequence (n). Hence, using also Proposition 2.5, one has that for every $\mathbf{e} \in \mathbb{S}^{d-1}$ and any $\rho > 0$

$$\lim_{n\to\infty} \mathcal{U}_n(\rho\mathbf{e}) = \mathbb{E}[e^{-cM_\infty^{(\alpha)}|\rho|^\alpha}].$$

At this stage, the convergence of $\mathcal{U}(t)$ to $\hat{\mu}^c_\infty$ follows from (2.7).

In order to prove the last part of the theorem it is enough to check that since μ^c_∞ is a scale mixture of a spherically symmetric stable law, it belongs to NDA of the same stable law.

A Multivariate stable laws and their domain of attractions

A random vector Z taking values in $\mathbb{R}^d$ has a centered α-stable distribution, for α in $(0,2)$, if and only if its characteristic function is

$$\mathbb{E}[e^{i\xi\cdot Z}] = \exp\Big\{ -\int_{\mathbb{S}^{d-1}} |\xi\cdot s|^{\alpha}\eta(\xi,s)\Lambda(ds)\Big\} \qquad (\xi \in \mathbb{R}^d) \tag{A.1}$$

where Λ is a finite measure on $\mathbb{S}^{d-1}$ and

$$\eta(\xi,s) := \begin{cases} 1 - i\mathrm{sign}(\xi\cdot s)\tan(\frac{\pi\alpha}{2}) & \text{if } \alpha \neq 1 \\ 1 + i\frac{2}{\pi}\mathrm{sign}(\xi\cdot s)\log|\xi\cdot s| & \text{if } \alpha = 1. \end{cases}$$

See, e.g., Theorem 7.3.16 in [41].

A random vector Z has a centered α-stable spherically symmetric distribution if

$$\mathbb{E}[e^{i\xi\cdot Z}] = \exp\{-c|\xi|^{\alpha}\} \qquad (\xi \in \mathbb{R}^d) \tag{A.2}$$

for some $c > 0$. Clearly, in this case, $\Lambda(A) \propto |A|$.

As in the one-dimensional case, one says that:

A random vector X_0 (or equivalently its law μ_0) belongs to the normal domain of attraction (NDA, for short) of an α-stable law if for any sequence $(X_i)_{i\geq 1}$ of i.i.d. random vectors with the same law of X_0, there is a sequence of vectors $(b_n)_{n\geq 1}$ such that $n^{-1/\alpha}\sum_{i=1}^n X_i - b_n$ converges in law to an α-stable random vector.

Given any a finite measure Λ on $\mathbb{S}^{d-1}$ the so-called *Lévy measure* $\phi = \phi_\Lambda$ on $\mathbb{R}^d \setminus \{0\}$ is given in polar coordinates by

$$\phi(d\theta dr) = \Lambda(d\theta)\frac{\alpha k_\alpha}{r^{\alpha+1}}dr. \tag{A.3}$$

A stable law is said to be *full* if it is not supported on any $d-1$ dimensional subspace of $\mathbb{R}^d$. In this case, it is possible to characterize the NDA in terms of the tails of μ_0 in the following way:

X_0 belongs to the NDA of a stable law with Lévy measure $\phi = \phi_\Lambda$ if and only if for every $r > 0$ and every Borel set $B \subset \mathbb{S}^{d-1}$ such that $\Lambda(\partial B) = 0$

$$\lim_{t\to+\infty} t^{\alpha}\mathbb{P}\Big\{|X_0| > rt,\ \frac{X_0}{|X_0|} \in B\Big\} = \frac{k_\alpha}{r^\alpha}\Lambda(B), \tag{A.4}$$

with

$$k_\alpha = \frac{2\Gamma(\alpha)\sin(\alpha\pi/2)}{\pi}.$$

See Theorems 6.20 and 7.11 in [3].

We collect some results on the NDA of an α-stable law, which are used in Section 5.

Lemma A.1. *If a stable law is full, then the corresponding Lévy measure ϕ is full, that is ϕ is not supported on any $d-1$ dimensional subspace of $\mathbb{R}^d$.*

Proof. The thesis can be deduced combining Proposition 3.1.20 and Theorem 7.3.3 in [41]. □

Recall that, for every $x \in \mathbb{R}^d$,

$$B_x = \{y \in \mathbb{R}^k : x\cdot y > 1\}.$$

Lemma A.2. *Let ϕ be a full Lévy measure, then*

$$x \mapsto \phi(B_x)$$

is a continuous function on $\mathbb{R}^d \setminus \{0\}$.

Proof. The proof is essentially the same as the proof of Lemma 6.1.25 in [41] and it is left to the reader. □

Lemma A.3. *Let S be a compact subset of $\mathbb{R}^d \setminus \{0\}$. If X_0 belongs to the normal domain of attraction of a full α-stable law with Lévy measure ϕ, then*

$$\lim_{t\to+\infty} \sup_{u\in S} |t^\alpha \mathbb{P}\{X_0 \cdot u > t\} - \phi(B_u)| = 0 \tag{A.5}$$

and

$$\lim_{t\to-\infty} \sup_{u\in S} ||t|^\alpha \mathbb{P}\{X_0 \cdot u \le t\} - \phi(B_{-u})| = 0. \tag{A.6}$$

Moreover (A.5) *remains true if one replace $>$ with $\ge$.*

Proof. The proof of this result can be obtained with minor modifications from the proof of a similar result contained in Lemma 6.1.26 of [41]. The details are left to the reader. □

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Acknowledgments. The authors would like to thank E. Breuillard, E. Dolera and A. Ghigi for helpful comments. We would also like to thank the referee for his or her very careful reading of our manuscript and the manifold pieces of advice for improvement.