Minimax Control of Markov Jump Linear Systems*

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1 Introduction

This paper aims at providing new results on Markov Jump Linear Systems (MJLS) control design

The paper is organized as follows.

The notation used throughout is standard. For square matrices, $\mathrm{Tr}(\cdot)$ denotes the trace function. For real vectors or matrices, $(')$ refers to their transpose. For symmetric matrices, $(\bullet)$ denotes each of their symmetric blocks. The symbols $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. For any symmetric matrix, $X > 0$ $(X \geq 0)$ denotes a positive (semi)definite matrix. The expected value operator is $\mathcal{E}(\cdot)$ and $\mathbb{P}(\cdot)$ is the probability of the event $(\cdot)$. The unitary simplex composed by all nonnegative vectors $\mu \in \mathbb{R}^N$ such that $\sum_{j\in\mathbb{K}} \mu_j = 1$ is denoted by Ω. The i-th column of any identity matrix is denoted by e_i. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted as $\|x\|_2^2 = x'x$.

2 Problem Statement and Preliminaries

Consider the following Markov jump linear system, denominated σ-MJLS, with state space realization

$$\dot{x}(t) = A_{\theta\sigma}x(t) + E_\theta w(t) \tag{1}$$

$$z(t) = C_{\theta\sigma}x(t) \tag{2}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^r$ and $z \in \mathbb{R}^s$ are the state, the exogenous input and the controlled output, respectively. Denote the set of Markov modes as $\mathbb{K} = \{1, \cdots, N\}$ and $\{\theta(t) = \theta \in \mathbb{K}\}$, a time-varying function governed by a continuous-time Markov process with transition rate matrix $\{\lambda_{ij}\} = \Lambda \in \mathbb{R}^{N\times N}$ given by

$$\mathbb{P}(\theta(t+h) = j|\theta(t) = i) = \delta_{i-j} + \lambda_{ij}h + o(h) \tag{3}$$

where δ_{i-j} is the Kronecker delta function, that means $\delta_{i-j} = 1$ if $i = j \in \mathbb{K}$ and $\delta_{i-j} = 0$, otherwise, and $\lim_{h\to 0^+} o(h)/h = 0$. The elements of matrix

Λ are such that $\lambda_{ij} \geq 0$, $\forall i \neq j$, and $\sum_{j \in \mathbb{K}} \lambda_{ij} = 0$, $\forall i \in \mathbb{K}$, which implies that $\lambda_{ii} \leq 0$, $\forall i \in \mathbb{K}$. It is assumed that the system evolves from initial conditions $x(0) = 0$ and $\theta(0) = \theta_0$, with $\mathbb{P}(\theta_0 = i) = \pi_{i0} > 0$, $\forall i \in \mathbb{K}$.

The minimax control action is exerted by the deterministic function $\sigma(\cdot)$ whose domain can be either $\mathbb{K}$ or Ω. Clearly, the effectiveness obtained from the adoption of the second strategy is higher than the one produced by the first because, naturally, the set $\mathbb{K}$ can be interpreted as the vertices of Ω. Two classes well known in the literature are of interest. The first one, denominated *mode dependent*, is characterized by $\sigma = \sigma(x, \theta)$ which makes clear that the control function depends on both the state and the Markov mode. The second class is characterized by $\sigma = \sigma(x)$ and, for this reason, it is denominated *mode independent*. Of course, the design conditions under the assumption of mode independent are much more involved than the ones of mode dependent. For simplicity, it is assumed that the input matrices E_i for all $i \in \mathbb{K}$ do not depend on $\sigma(\cdot) \in \mathbb{K}$. As it can be verified, this assumption can be removed with no difficulty at expense of more complicated formulas.

Our objective is twofold. The first goal is to determine an upper bound $J(\sigma)$ such that

$$\mathcal{E}\left\{\sum_{\ell=1}^{r} \int_0^\infty z_\ell(t)' z_\ell(t) dt\right\} \leq J(\sigma) \tag{4}$$

where z_ℓ is the controlled output associated to the input $w(t) = e_\ell \delta(t)$. This means that, preserving stochastic stability, a guaranteed $\mathcal{H}_2$ cost is established. The second is to provide a solution to the optimal control problem

$$\inf_{\sigma} J(\sigma) \tag{5}$$

One of the main contribution of this paper is to propose a guaranteed cost $J(\sigma)$ such that the optimal control problem (5) is expressed by means of LMIs, exclusively. Moreover, the quality in terms of performance loss appears to be reduced as the numerical example presented in a forthcoming section indicates.

The literature of MJLS, see [4] and the references therein, is impressive. A large number of state feedback control design problems involving, for instance, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, have been proposed and solved mainly under the assumption that the Markov modes are available for feedback. Mainly in continuous time, the case of mode independent control has received much less attention due to the nonconvex nature of the design problem to deal with. In a forthcoming section, the possibility to imbed the classical $\mathcal{H}_2$ control problem for MJLS, in the present context will be discussed. Likewise, from now on, consider the stochastic Lyapunov function

$$V(x,\theta) = x' P_\theta x \tag{6}$$

defined at any time $t \geq 0$ for $x(t) = x$ and $\theta(t) = \theta$ with $\{P_i\}_{i\in\mathbb{K}}$ positive definite matrices properly determined. The infinitesimal generator $\mathcal{L}V(x,\theta)$ is defined as

$$\mathcal{L}V(x,\theta) = \lim_{h\to 0} \frac{1}{h}(\mathcal{E}(V(x(t+h),\theta(t+h))|(x,\theta)) - V(x,\theta)) \tag{7}$$

and can be calculated for each $\theta = i \in \mathbb{K}$. As a consequence, the inequality

$$\mathcal{L}V(x,i) < -z'z \tag{8}$$

whenever valid for all $0 \neq x \in \mathbb{R}^n$ and all $i \in \mathbb{K}$ implies that for a given deterministic control $\sigma(\cdot)$ the σ-MJLS (1)-(3) with zero input $w = 0$ and initial conditions $x(0) = x_0$ and $\theta(0) = \theta_0$, with $\mathbb{P}(\theta_0 = i) = \pi_{i0} > 0$, $\forall i \in \mathbb{K}$ is stochastically stable and Dynkin's formula yields the upper bound

$$\begin{aligned} \mathcal{E}\left\{\int_0^\infty z(t)'z(t)dt\right\} &< \mathcal{E}\left\{\int_0^\infty -\mathcal{L}V(x(t),\theta(t))dt\right\} \\ &< \mathcal{E}\{V(x_0,\theta_0)|(x_0,\theta_0)\} \\ &< \sum_{i\in\mathbb{K}} \pi_{i0} x_0' P_i x_0 \end{aligned} \tag{9}$$

A key property of MJLS still valid for σ-MJLS is due, exclusively, to the stochastic behavior of the Markov chain. It follows from the fact that the differential equation $\dot{\pi}(t) = \Gamma'\pi(t)$ with initial condition $\pi(0) = \pi_0$ produces the probability $\pi_i(t) = \mathbb{P}(\theta(t) = i)$ for each $i \in \mathbb{K}$ at any instant of time

$t \geq 0$. Consequently, $\mathcal{E}(V(x,\theta)|(x_0,\theta_0)) = \sum_{i\in\mathbb{K}} \pi(t)x'P_i x$. At this point it is important to mention that these algebraic manipulations remains true in our framework since the control action exerted by the deterministic function $\sigma(\cdot)$ does not modify the Markovian nature of the stochastic process defined by the Markov chain $\{\theta(t) = \theta \in \mathbb{K}\}$.

2.1 Aspects of Minimax Theory

The material of this section is mainly based on [11]. This is not the case of the next three lemmas that are specific to deal with the forthcoming control design problems. Let us consider the following minimax problem of interest

$$\max_{\pi\in\Omega}\min_{\mu\in\Omega} \pi' H \mu \tag{10}$$

where $H = \{h_{ij}\} \in \mathbb{R}^{N\times N}$ is a given matrix and remember that Ω is the unitary simplex. Since Ω is convex, the objective is linear for each $\pi \in \Omega$ or $\mu \in \Omega$ fixed then the conditions for the existence of a saddle point (π_*, μ_*) are fulfilled. Hence, the optimal equilibrium $f_* = \pi_*' H \mu_*$ satisfies

$$\pi' H \mu_* \leq f_* \leq \pi_*' H \mu \quad \forall \pi \in \Omega, \quad \forall \mu \in \Omega \tag{11}$$

In the sequel, some properties of this equilibrium solution which follows from the celebrated von Neumann's minimax theorem will be used. First, using duality, the optimal value f_* can be determined from the solution of one of the following linear programming problems

$$\max_{f,\pi\in\Omega} \left\{ f \ : \ \sum_{i\in\mathbb{K}} \pi_i h_{ij} \geq f \right\} = \min_{f,\mu\in\Omega} \left\{ f \ : \ \sum_{j\in\mathbb{K}} h_{ij}\mu_j \leq f \right\} \tag{12}$$

whose pair of primal-dual optimal variables is just (π_*, μ_*). On the other hand, the equality $f_* = \max_{\pi\in\Omega} \pi' H \mu_* = \min_{\mu\in\Omega} \pi_*' H \mu$ makes clear that the optimal value of the equilibrium is also attained at some vertex of Ω

because the former relations imply that

$$\max_{i \in \mathbb{K}} \left\{ \sum_{j \in \mathbb{K}} h_{ij} \mu_{j*} \right\} = \min_{j \in \mathbb{K}} \left\{ \sum_{i \in \mathbb{K}} \pi_{i*} h_{ij} \right\} \tag{13}$$

The next instrumental lemmas bring to light three specific properties that are central in the development of a mode independent control for the σ-MJLS. As it will be clear in the sequel, the optimal equilibrium f_* is associated to the time derivative of the stochastic Lyapunov function (6) which naturally calls the necessity to characterize the conditions assuring that $f_* < 0$.

Lemma 1 *The optimal equilibrium f_* is negative whenever $H + H' < 0$.*

Proof: It follows from the saddle point condition (11) which states that $f_* \leq \pi_*' H \mu \; \forall \mu \in \Omega$. However, taking into account that $\pi_* \in \Omega$ we obtain

$$\begin{aligned} f_* &\leq \pi_*' H \pi_* \\ &\leq \frac{1}{2} \pi_*' (H + H') \pi_* \\ &< 0 \end{aligned} \tag{14}$$

from which the claim follows. □

Clearly, this a well known sufficient condition to assure that $f_* < 0$ but in compensation it is well adapted to be handled by any LMI solver. The conservatism introduced by this condition will be verified by means of some numerical examples. The following simple multiplicative and additive robustness properties are relevant to our purposes and so their proofs are included for completeness.

Lemma 2 *Let a diagonal positive matrix $\Delta = \text{diag}\{\delta_1, \cdots, \delta_N\} > 0$ be given. Define the equilibrium solution of the minimax problem*

$$f_{\delta *} = \max_{\pi \in \Omega} \min_{\mu \in \Omega} \pi' \Delta H \mu \tag{15}$$

Then, $f_* < 0$ if and only if $f_{\delta *} < 0$.

Proof: Similarly of what we have been done to determine f_*, the optimal equilibrium $f_{\delta *}$ can be determined from

$$f_{\delta *} = \max_{\pi \in \Omega} \left\{ f \ : \ \sum_{i \in \mathbb{K}} \pi_i \delta_i h_{ij} \geq f \right\} \tag{16}$$

Assuming $\pi_{\delta *} \in \Omega$ solves (16) then there exists a scalar $\alpha_* > 0$ such that $\pi' = \pi'_{\delta *} D/\alpha_* \in \Omega$ is feasible to problem (12) and yields $f_* \geq f_{\delta *}/\alpha_*$. On the other hand, assuming that $\pi_* \in \Omega$ solves (12) then there exists a scalar $\beta_* > 0$ such that $\pi' = \pi'_* \Delta^{-1}/\beta_* \in \Omega$ is feasible to problem (16) and yields $f_{\delta *} \geq f_*/\beta_*$. Consequently

$$\left(\frac{1}{\beta_*} \right) f_* \leq f_{\delta *} \leq \alpha_* f_* \tag{17}$$

which implies the desired result. □

Lemma 3 *Let a matrix $\Delta = \{\delta_{ij}\} \in \mathbb{R}^{N \times N}$ with nonnegative elements. Define the equilibrium solution of the minimax problem*

$$f_{\delta *} = \max_{\pi \in \Omega} \min_{\mu \in \Omega} \pi' (H + \Delta) \mu \tag{18}$$

Then, $f_* \leq f_{\delta *}$.

Proof: Once again we characterize the equilibrium (18) be means of

$$f_{\delta *} = \max_{\pi \in \Omega} \left\{ f \ : \ \sum_{i \in \mathbb{K}} \pi_i (\delta_i + h_{ij}) \geq f \right\} \tag{19}$$

and the solution $\pi_* \in \Omega$ is plugged on it to provide

$$\begin{aligned} f_{\delta *} &\geq \min_{j \in \mathbb{K}} \sum_{i \in \mathbb{K}} \pi_{i*} (h_{ij} + \delta_{ij}) \\ &\geq f_* + \min_{j \in \mathbb{K}} \sum_{i \in \mathbb{K}} \pi_{i*} \delta_{ij} \end{aligned} \tag{20}$$

which is the claim since by assumption $\delta_{ij} \geq 0$. The proof is concluded.

As already commented these two properties are essential to obtain the theoretical results to be presented in the next section. In the context of mode independent control design for σ-MJLS, the only source of conservativeness has been introduced by the sufficient nature (and hence, in general, conservative) of Lemma 1 which, however, is amenable in the numeric viewpoint.

It is interesting to see what happens when the following problem is considered

$$\max_{i \in \mathbb{K}} \min_{j \in \mathbb{K}} h_{ij} \tag{21}$$

The advantage when compared to (10) is that the optimal solution is much simpler to calculate. However, the main difficulty to be faced is that, in general, $\max_{i \in \mathbb{K}} \min_{j \in \mathbb{K}} h_{ij} < \min_{j \in \mathbb{K}} \max_{i \in \mathbb{K}} h_{ij}$ implying that a saddle point does not exist and a similar relation as (11) does not hold. This occurs because, obviously, the set $\mathbb{K}$ is not convex. However, if we search for a solution of the form $j(i)$ then $\max_{i \in \mathbb{K}} \min_{j \in \mathbb{K}} h_{ij} \leq \max_{i \in \mathbb{K}} h_{ii}$ which means that the objective function is negative whenever $h_{ii} < 0$ for all $i \in \mathbb{K}$. This strategy is of mode dependent type being thus useless in the context of mode independent control design.

3 Minimax Control Design

This section provides the main contributions of this paper. To this end, let us consider the σ-MJLS with state space realization given in (1)-(2) and introduce the following matrices

$$\mathcal{R}_{ij} = A'_{ij} P_i + P_i A_{ij} + C'_{ij} C_{ij} + \sum_{k \in \mathbb{K}} \lambda_{ik} P_k \tag{22}$$

for all $i \in K$ and $j \in \mathbb{K}$ which allows, for any given $x \in \mathbb{R}^n$, the determination of all elements of the matrix $\{h_{ij}\} = H \in \mathbb{R}^{N \times N}$ from

$$h_{ij} = x' \mathcal{R}_{ij} x \tag{23}$$

Formally, at each instant of time $t \geq 0$ the state variable $x(t) = x$ is supposed to be available and matrix H is calculated whenever matrices P_i for all $i \in \mathbb{K}$ are known. It is used on the determination of the minimax strategy in both cases, namely, mode dependent and mode independent control.

3.1 Mode Dependent Control

Under the assumption that at each instant of time $t \geq 0$ the current vale of the state variable $x(t) = x \in \mathbb{R}^n$ and the Markov mode $\theta(t) = \theta \in \mathbb{K}$ are available, that is, both are measured online, let us consider the switching function

$$\sigma(x,\theta) = \arg\min_{j\in\mathbb{K}} h_{\theta j} \tag{24}$$

and determine matrices $P_i > 0$ for all $i \in \mathbb{K}$ to assure stochastic stability and guaranteed $\mathcal{H}_2$ performance.

Theorem 1 *If there exist symmetric matrices V_i and W_{ik} for all $k \neq i \in \mathbb{K} \times \mathbb{K}$ satisfying the LMIs*

$$\begin{bmatrix} A_{ii}V_i + V_iA'_{ii} + \sum_{k\neq i\in\mathbb{K}} \lambda_{ik}W_{ik} & V_iC'_{ii} \\ \bullet & -I \end{bmatrix} < 0 \tag{25}$$

$$\begin{bmatrix} W_{ik} + V_i & V_i \\ \bullet & V_k \end{bmatrix} > 0, \;\; k \neq i \in \mathbb{K} \tag{26}$$

then, with $P_i = V_i^{-1}$ for all $i \in \mathbb{K}$, the mode dependent switching function (24) assures global stochastic stability and the guaranteed $\mathcal{H}_2$ performance

$$\mathcal{E}\left\{\sum_{\ell=1}^{r}\int_0^\infty z_\ell(t)'z_\ell(t)dt\right\} < \sum_{i\in\mathbb{K}} \pi_{i0}\mathrm{Tr}(E'_iV_i^{-1}E_i) \tag{27}$$

Proof: Applying the Schur Complement to the second diagonal element of (25), multiplying the result by $\lambda_{ik} \geq 0$ for all $k \neq i$ and adding all terms it follows that

$$\begin{aligned}\sum_{k\neq i\in\mathbb{K}} \lambda_{ik} W_{ik} &> \sum_{k\neq i\in\mathbb{K}} \lambda_{ik}\left(V_i V_k^{-1} V_i - V_i\right)\\ &> \sum_{k\in\mathbb{K}} \lambda_{ik} P_i^{-1} P_k P_i^{-1} \end{aligned} \tag{28}$$

where the fact that $\lambda_{ii} = -\sum_{k\neq i\in\mathbb{K}} \lambda_{ik}$ and $P_i = V_i^{-1}$ for all $i \in \mathbb{K}$ has been used. Hence, multiplying (25) to the right and to the left by diag$\{V_i^{-1}, I\}$ after performing again the Schur Complement, the coupled Lyapunov inequalities $\mathcal{R}_{ii} < 0$ for $i \in \mathbb{K}$ are obatained.

At an arbitrary $t \geq 0$ set $x(t) = x \neq 0$ and $\theta(t) = \theta \in \mathbb{K}$. The mode dependent switching function (24) imposes

$$\begin{aligned}\mathcal{L}V(x,\theta) + z'z &= x'\mathcal{R}_{\theta\sigma} x\\ &= \min_{j\in\mathbb{K}} h_{\theta j}\\ &\leq h_{\theta\theta} < 0 \end{aligned} \tag{29}$$

where the last inequality follows immediately from (25) which also implies that $\mathcal{L}V(x,i) + z'z \leq h_{ii} = x'\mathcal{R}_{ii}x < 0$ for all $i \in \mathbb{K}$, that is, the system is stochastically stable. On the other hand, (9) holds and applying it to the initial conditions $x_0 = E_i e_\ell$, successively, for $\ell = 1, \cdots, r$ and adding terms we conclude that the guaranteed $\mathcal{H}_2$ performance (27) holds. The proof is concluded.

The design conditions for mode dependent switched control provided in Theorem 1 are expressed by LMIs of reduced dimensions when compared to those normally adopted in the literature, see [4]. Of course, the fact that the Markov mode $\theta(t) = \theta \in \mathbb{K}$ is supposed to be known, the conditions of Theorem 1 may produce $j(\theta) = \theta \in \mathbb{K}$ since $h_{ii} < 0$ for all $i \in \mathbb{K}$. In this case, the switching rule (24) imposes to the closed-loop σ-MJLS exactly the same performance as the one of the optimal linear state feedback $\mathcal{H}_2$ control of MJLS. This important feature will be addressed in the next section. The switching control (24) is alternatively given by

$$\sigma(x,\theta) = \arg\min_{\mu\in\Omega} \sum_{j\in\mathbb{K}} h_{\theta j}\mu_j \tag{30}$$

which means that there is no improvement if we adopt, instead (24) this linear parameter varying strategy $\mu(t) = \mu \in \Omega$. In other words, both control strategies produce the same result. The next section puts in evidence that this is not true for mode independent control.

3.2 Mode Independent Control

Mode independent control is a key issue in the context of σ-MJLS because it avoids the necessity to implement online measurement of the Markov mode $\theta(t) = \theta \in \mathbb{K}$. In this section, a new solution to the mode independent guaranteed $\mathcal{H}_2$ control design problem is proposed. It follows from the adoption of the minimax linear parameter varying control law

$$\sigma(x) = \mu_* = \arg\min_{\mu\in\Omega} \pi_*' H\mu \tag{31}$$

where $\pi_* \in \Omega$ is the optimal solution of the linear programming problem (12). This function becomes effective for stability and guaranteed performance once the Lyapunov matrices $P_i > 0$ for all $i \in \mathbb{K}$ are adequately determined as the next theorem indicates.

Theorem 2 *If there exist symmetric matrices $V_i > 0$, W_{ik} for all $k \neq i \in \mathbb{K}\times\mathbb{K}$ and a matrix $Q = \{q_{ij}\} \in \mathbb{R}^{N\times N}$ satisfying the LMIs (26),*

$$\begin{bmatrix} A_{ij}V_i + V_iA_{ij}' - q_{ij}I + \sum_{k\neq i\in\mathbb{K}} \lambda_{ik}W_{ik} & V_iC_{ij}' \\ & -I \end{bmatrix} < 0 \tag{32}$$

for all $i,j \in \mathbb{K}\times\mathbb{K}$ and $Q + Q' < 0$ then, with $P_i = V_i^{-1}$ for all $i \in \mathbb{K}$, the mode independent switching function (31) assures global stochastic stability *and the guaranteed $\mathcal{H}_2$ performance*

$$\mathcal{E}\left\{\sum_{\ell=1}^{r}\int_0^\infty z_\ell(t)' z_\ell(t)dt\right\} < \sum_{i\in\mathbb{K}} \pi_{i0}\mathrm{Tr}(E_i' V_i^{-1} E_i) \tag{33}$$

Proof: At an arbitrary $t \geq 0$ set $x(t) = x \neq 0$. For any $i \in \mathbb{K}$, the minimax linear parameter varying control (31) implies that

$$\begin{aligned}
\mathcal{L}V(x,i) + z'z &= x'\mathcal{R}_{i\sigma}x \\
&\leq \max_{\pi\in\Omega} x'\left(\sum_{i\in\mathbb{K}}\pi_i\mathcal{R}_{i\sigma}\right)x \\
&\leq \max_{\pi\in\Omega}\min_{\mu\in\Omega} x'\left(\sum_{i\in\mathbb{K}}\sum_{j\in\mathbb{K}}\pi_i\mu_j\mathcal{R}_{ij}\right)x \\
&\leq \max_{\pi\in\Omega}\min_{\mu\in\Omega} \pi' H\mu
\end{aligned} \tag{34}$$

Now, adopting the same algebraic manipulations as before in the proof of Theorem 1 to inequalities (26), performing the Schur Complement with respect to the second diagonal element of (32) and multiplying the result from both sides by $V_i^{-1} = P_i$ it follows that

$$\begin{aligned}
h_{ij} &= x'\mathcal{R}_{ij}x \\
&< q_{ij}\|P_i x\|_2^2
\end{aligned} \tag{35}$$

Defining the positive diagonal matrix $D = \mathrm{diag}\{\|P_1 x\|_2^2, \cdots \|P_N x\|_2^2\}$ and using Lemma 1, together with Lemma 2 and Lemma 3 it is seen that

$$\max_{\pi\in\Omega}\min_{\mu\in\Omega} \pi' H\mu < \max_{\pi\in\Omega}\min_{\mu\in\Omega} \pi' DQ\mu < 0 \tag{36}$$

because $Q + Q' < 0$. Hence, stochastic stability and (9) hold. Applying (9) to the initial conditions $x_0 = E_i e_\ell$, successively, for $\ell = 1, \cdots, r$ and adding terms the conclusion is that the guaranteed $\mathcal{H}_2$ performance (33) holds. The proof is concluded.

This result is somewhat surprising because all matrix variables including $Q = \{q_{ij}\} \in \mathbb{R}^{N\times N}$ are determined by means of LMIs. Hence, no kind of relaxation procedure to deal with temporarily fixed variables is necessary. In addition, the products of matrices and variables appears to be adequate to include state feedback gains in the set of matrix variables, keeping unchanged the convex nature of the problem, an aspect to be addressed in future works. Finally, it is important to stress that to be feasible for $Q + Q' < 0$, the diagonal elements must be strictly negative but the off diagonal ones are allowed to be positive, a fact that certainly contributes to increase the quality of the control policy (31) as far as the minimization of the guaranteed $calH_2$ cost is concerned.

3.3 MJLS Control Design

This section is devoted to apply the previous results to design mode dependent and mode independent state feedback control of MJLS. See [4] and the references therein for a rather complete study on optimal $\mathcal{H}_2$ control including historical and numerical aspects. The MJLS state space realization is given by

$$\dot{x}(t) = A_\theta x(t) + B_\theta u(t) + E_\theta w(t) \tag{37}$$

$$z(t) = C_\theta x(t) + D_\theta u(t) \tag{38}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^r$ and $z \in \mathbb{R}^s$ are the state, the control, the exogenous input and the controlled output, respectively. As usual, it is assumed that the system evolves from initial conditions $x(0) = 0$ and $\theta(0) = \theta_0$, with $\mathbb{P}(\theta_0 = i) = \pi_{i0} > 0$, $\forall i \in \mathbb{K}$ and $D_i' D_i > 0$ for all $i \in \mathbb{K}$. Under the assumption that the Markov modes are available for feedback, the optimal state feedback control is $u(t) = L_\theta x(t)$ with the state feedback gains given by

$$\min_{L_i, P_i > 0} \left\{ \sum_{i\in\mathbb{K}} \pi_{i0} \mathrm{Tr}(E_i' P_i E_i) \ : \ \mathcal{R}_{ii} < 0, \ i \in \mathbb{K} \right\} \tag{39}$$

where $\mathcal{R}_{ii}$ for all $i \in \mathbb{K}$ are given in (22) with $A_{ii} = A_i + B_i L_i$ and $C_{ii} = C_i + D_i L_i$. It is well known that the optimal solution of problem (39) lies on the border of the feasible set defined by the inequalities $\mathcal{R}_{ii} < 0$ and, consequently, it can be calculated from the stabilizing solution of the set of coupled Riccati equations $\mathcal{R}_{ii} = 0$ for $i \in \mathbb{K}$. This requires an iterative procedure whose description and convergence are discussed in [4]. A simpler alternative follows from the reformulation of problem (39) in terms of the new matrix variables $V_i = P_i^{-1} > 0,\ i \in \mathbb{K}$, yielding

$$\min_{L_i, V_i, W_{ij}} \left\{ \sum_{i \in \mathbb{K}} \pi_{i0} \mathrm{Tr}(E_i' V_i^{-1} E_i) \ : \ (25) - (26),\ i \in \mathbb{K} \right\} \tag{40}$$

which is immediately converted to a convex programming problem by introducing the one to one change of variables $L_i = Y_i V_i^{-1},\ i \in \mathbb{K}$. It is important to mention that when compared to the various numerical methods available in the literature to deal with the optimal $\mathcal{H}_2$ control for MJLS, problem (40) appears to have an advantage. Actually, LMIs of smaller dimensions certainly causes a reduction on the computational burden. Once the optimal gains are determined we are able to determine $\mathcal{R}_{ij}$ by setting $A_{ij} = A_i + B_i L_j$ and $C_{ii} = C_i + D_i L_j$ for all $i \in \mathbb{K}$ and $j \in \mathbb{K}$.

Hence, the switching mode dependent control provided by Theorem 1 is implemented as $u(t) = L_{\sigma(x,\theta)} x(t)$ where the switching control $\sigma(x, \theta) \in \mathbb{K}$ is given in (24). For mode independent control, notice that due to convexity, it can be verified that for each $i \in \mathbb{K}$ and for all $\mu \in \Omega$, the inequality

$$\sum_{j \in \mathbb{K}} R_{ij} \mu_j \geq (A_i + B_i L)' P_i + P_i (A_i + B_i L) + (C_i + D_i L)'(C_i + D_i L) \tag{41}$$

with $L = \sum_{j \in \mathbb{K}} L_j \mu_j$, holds. Consequently, the minimax state feedback control is implemented through $u(t) = \sum_{j \in K} \mu_{j*}(t) L_j x(t)$ where $\sigma(x) = \mu_* \in \Omega$ is given in (31).

A final remark concerns the concept of *consistency* of switched linear systems, see [6]. The switching control provided by Theorem 1 is consistent because the comparison of (40) and (25)-(27) shows that the switching control imposes to the closed-loop system a guaranteed $\mathcal{H}_2$ performance that can not be worse than the minimum cost provided by (40).

4 Illustrative Example

To illustrate the theoretical result obtained so far, the following academical example has been borrowed from [2]. The matrices of the state space realization (37)-(38) are

$$A_1 = \begin{bmatrix} 0.14 & 0.80 \\ 4.00 & -1.01 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1.44 \\ 2.52 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1.74 & 0.20 \\ 0.00 & -2.51 \end{bmatrix}, \; B_2 = \begin{bmatrix} 2.44 \\ 5.52 \end{bmatrix}$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; D_1 = D_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; E_1 = E_2 = \begin{bmatrix} 1.70 \\ -1.20 \end{bmatrix}$$

The Markov chain is characterized by the transition rate matrix

$$\Lambda = \begin{bmatrix} -1.09 & 1.09 \\ 2.87 & -2.87 \end{bmatrix}$$

and the initial probability $\pi_0 = [1\ 0]'$. First of all, a near optimal solution to the convex problem (40) with minimum cost $J_{opt} \approx 2.4260$ and the associated state feedback gains

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} -1.6077 & -0.7937 \\ -2.0740 & -0.5569 \end{bmatrix}$$

has been calculated. Then, mode dependent and mode independent control have been designed from the direct application of Theorem 1 and Theorem 2, respectively. Each case has being validated by Monte Carlo simulation of 500 runs in the time interval $t \in [0, 4]$ seconds using the method of [8] to describe adequate and efficiently the Markov jump process.

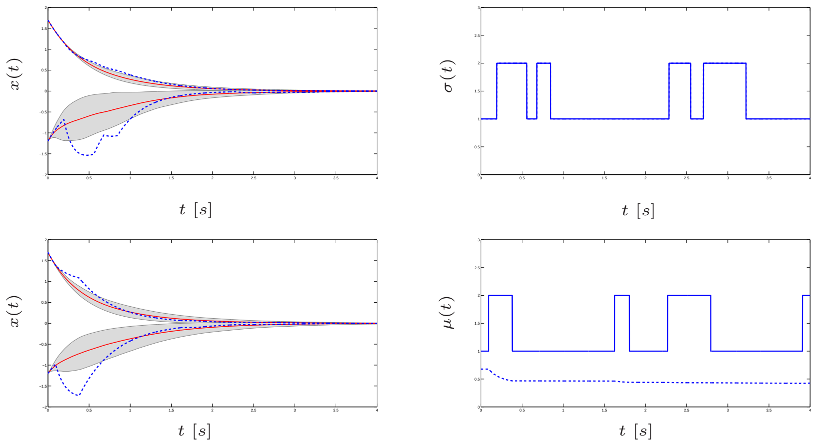

Figure 1: Time simulation

	Optimal	Mode Dependent	Mode Independent
Guaranteed	2.4260	2.4245	2.6554
Simulated		2.4287	2.4559

Table 1: Theoretical and simulated $\mathcal{H}_2$ performance

The costs indicated in Table 1 deserve some remarks. The guaranteed cost and the respective value obtained by simulation agree, in the cases of mode dependent and mode independent control. As expected, the guaranteed cost of the mode dependent control is approximately J_{opt} which allows the conclusion that, in this particular exemple, the obvious mode dependent control $\sigma(x, \theta) = \theta \in \mathbb{K}$ produces almost the same performance.

On the other hand, the case of mode independent control, puts in evidence the quality of our design. Again, the guaranteed and simulated $\mathcal{H}_2$ performances are very close to J_{opt} which indicates that the proposed control is actually effective even thought the Markov mode is unknown.

Figure 1 shows two time simulations corresponding to mode independent (top row) and mode independent (bottom row) control. In both cases, the graphics on the left show the closed-loop system mean trajectories (in red) of each state inside a region (in gray) defined by one standard deviation. A single trajectory is also shown. On the top right the graphic shows the time evolution of the corresponding Markov mode and the switching control (24) which coincides in this cases. On the bottom right it is presented the time evolution of the Markov mode and the minimax control (31) which evolves continuously on time from $\mu_1(0) = 0.6784$ to $\mu_1(4) = 0.4268$. Mainly in the mode independent case, the minimax control appears to be very effective.

5 Conclusion

References

[1] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

[2] H. Liu, E. Boukas, F. Sun, and D. W. C. Ho, Controller design for Markov jumpinj systems subject to actuator saturation, Automatica, vol. 42, pp. 459–465, 2006.

[3] P. Colaneri, J. C. Geromel, and A. Locatelli, *Control Theory and Design: An R2 / Roo Viewpoint*, Academic Press, London, 1997.

[4] O. L. V. Costa, M. D. Fragoso, and M. G. Todorov, *Continuous-time Markov Jump Linear Systems*, Probability and Its Applications, Springer-Verlag, Berlim, 2013.

[5] J. C. Geromel, and P. Colaneri, Stability and stabilization of continuous-time switched linear systems, SIAM Journal on Control and Optimization, vol. 45, pp. 1915–1930, 2006.

[6] J. C. Geromel, G. S. Deaecto, and J. Daafouz, Suboptimal switching control consistency analysis for switched linear systems, IEEE Transactions on Automatic Control, vol. 58, pp. 1857–1861, 2013.

[7] A. P. C. Gonçalves, A. R. Fioravanti, and J. C. Geromel, Markov jump linear systems and filtering through network transmitted measurements, Signal Processing, vol. 90, pp. 2842 – 2850, Oct. 2010.

[8] A. Leon-Garcia, *Probability, Statistics, and Ramdom Processes for Electrical Engineering*, Pearson Education Inc., NJ, 2008.

[9] D. Liberzon, *Switching in Systems and Control*, Birkhauser, 2003.

[10] D. Liberzon, and A. S. Morse, Basic problems in stability and design of switched systems, IEEE Control Systems Magazine, vol. 19, pp. 59–70, 1999.

[11] R. T. Rockafellar, *Convex Analysis*, Princeton Univesity Press, 1997.