Entropy Production and Detailed Balance for a Class of Quantum Markov Semigroups

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1. Introduction

Steady states of an open quantum system are considered equilibrium or nonequilibrium states according to whether or not they satisfy a quantum detailed balance condition (see [3, 4, 16, 17, 18, 19, 23, 26, 29] and the references therein). Concepts of entropy production have been proposed in several papers ([5, 6, 14, 15, 20, 24, 29] is a short list far from being complete) as an index of deviation from detailed balance (see [22] also for classical Markov processes).

In [14, 15] we introduced a definition of entropy production rate for faithful normal invariant states of quantum Markov semigroups, inspired by the one brought into play for classical Markov processes, by considering the derivative of relative entropy of the one-step forward and backward two-point states at time $t = 0$. Moreover, we proved an explicit formula for the entropy production of a quantum Markov semigroup in terms of the completely positive part of the generator (Theorem 3 here). This formula shows that nonzero entropy production is closely related with the violation of quantum detailed balance conditions and singles out states with finite entropy production as a rich class of simple nonequilibrium invariant states.

In this paper we compute the entropy production for a class of quantum Markov semigroups, a faithful invariant state ρ, arising in the weak coupling limit of a system coupled with reservoirs, whose generators $\mathcal{L}$ are sums of other generators $\mathcal{L}_\omega$ associated with positive Bohr frequencies ω of the system (see [2, 10, 11]).

Our main result is the explicit formula (15) for the entropy production rate in terms of second order moments of Kraus operators in the GKSL (Gorini-Kossakowski-Sudarshan-Lindblad) representation of the generator. This formula shows that the entropy production of a semigroup in this class is the *sum* of non-negative entropy productions of all semigroups generated by each $\mathcal{L}_\omega$. As a consequence (Theorem 6) the semigroup generated by $\mathcal{L}$ satisfies the quantum detailed balance condition if and only if so does each semigroup generated by $\mathcal{L}_\omega$.

The plan of the paper is as follows. In Sect. 2 we introduce the class of quantum Markov semigroups we are dealing with. In Sect. 3 we recall various notions of quantum detailed balance. Our new formula for the entropy production is proved in Sect. 4 and, finally, in Sect. 5 we essentially show that equilibrium states for the semigroup generated by $\mathcal{L}$ are equilibrium state for all semigroups generated by each $\mathcal{L}_\omega$.

2. QMS of Stochastic Limit Type

We will be concerned with the class of quantum Markov semigroups (QMS) we describe below under some restrictive assumptions in order to avoid domain problems and similar technicalities. This class arises in the weak coupling limit as well as in the stochastic limit of a Hamiltonian system S interacting with a reservoir (see [2, 10, 11] and the references therein).

Let h be a fixed d-dimensional ($d < \infty$) Hilbert space and let H_S be a self-adjoint operator on h with spectral decomposition

$$H_S = \sum_n \varepsilon_n P_{\varepsilon_n} ,$$

where $\varepsilon_n \neq \varepsilon_m$ for $m \neq n$ and P_{ε_n} is the orthogonal projection onto the nullspace of $H_S - \varepsilon_n \mathbb{1}_{\mathsf{h}}$ (here $\mathbb{1}_h$ denotes the identity operator on h). We denote by $\mathcal{B}(\mathsf{h})$ the algebra of all bounded operators on h. We call *Bohr frequencies* the differences $\varepsilon_n - \varepsilon_m$ with $\varepsilon_n > \varepsilon_m$.

Choose an operator V on h and define

$$V_\omega = \sum_{\varepsilon_n - \varepsilon_m = \omega} P_{\varepsilon_m} V P_{\varepsilon_n} . \qquad (1)$$

Moreover, let H_ω be a self-adjoint operator on h commuting with H_S. For all Bohr frequency ω let $\mathcal{L}_\omega$ be the GKSL generator of a QMS on $\mathcal{B}(\mathsf{h})$

$$\begin{aligned}\mathcal{L}_\omega(x) &= \mathrm{i}[H_\omega, x] - \frac{\gamma_\omega^-}{2}\left(V_\omega^* V_\omega x - 2V_\omega^* x V_\omega + x V_\omega^* V_\omega\right) \\ &\quad - \frac{\gamma_\omega^+}{2}\left(V_\omega V_\omega^* x - 2V_\omega x V_\omega^* + x V_\omega V_\omega^*\right), \end{aligned} \tag{2}$$

where $\gamma_\omega^-, \gamma_\omega^+ > 0$. QMSs in our class are generated by the linear map $\mathcal{L}$

$$\mathcal{L} = \sum_\omega \mathcal{L}_\omega \,. \tag{3}$$

Note that, defining

$$G_\omega = -\frac{1}{2}\left(\gamma_\omega^- V_\omega^* V_\omega + \gamma_\omega^+ V_\omega V_\omega^*\right) - \mathrm{i}H_\omega \tag{4}$$

we can write the generator $\mathcal{L}_\omega$ simply as

$$\mathcal{L}_\omega(x) = G_\omega^* x + \gamma_\omega^- V_\omega^* x V_\omega + \gamma_\omega^+ V_\omega x V_\omega^* + x G_\omega \,.$$

Since the Hilbert space h is finite-dimensional, the QMS generated by $\mathcal{L}_\omega$ admits an invariant state ρ. Moreover, it is well-known (see e.g. [2]) that there exists an invariant state whose density matrix ρ commutes with the system Hamiltonian H so that it can be written as

$$\rho = \sum_{1\le j\le d} \rho_j |e_j\rangle\langle e_j| \,,$$

where $\rho_j \geq 0$, the above sum is finite, $\sum_{1\le j\le d}\rho_j = 1$, $(e_j)_{1\le j\le d}$ is an orthonormal basis of h and each e_j belongs to an eigenspace P_n of H_S. We shall also assume that ρ is faithful (if not we can reduce the semigroup by its recurrent projection [13]).

The generators of these QMSs turn out to admit a special GKSL representantion ([28] Theorem 30.16)

$$\mathcal{L}(x) = \mathrm{i}[H, x] - \frac{1}{2}\sum_{\ell=1}^{2b}\left(L_\ell^* L_\ell x - 2L_\ell^* x L_\ell + x L_\ell^* L_\ell\right), \tag{5}$$

where b is the number of Bohr frequencies, such that $\mathrm{tr}(\rho L_\ell) = 0$ for all $1 \le \ell \le b$ and operators $(L_\ell)_{\ell\ge1}$ are linearly independent in $\mathcal{B}(\mathsf{h})$. Indeed, it suffices to associate with each Bohr frequency ω a pair of operators

$$L_{2\ell} = (\gamma_\omega^-)^{1/2} V_\omega \,, \qquad L_{2\ell-1} = (\gamma_\omega^+)^{1/2} V_\omega^*, \tag{6}$$

where the indexes run over a finite set, and define $H = \sum_\omega H_\omega$.

3. Quantum Detailed Balance and Entropy Production

A number of conditions called *quantum detailed balance* (QDB) have been proposed in the literature to distinguish, among invariant states, those enjoying reversibility properties.

The first one, to the best of our knowledge, appeared in the work of Agarwal [3] in 1973 (see also Majewski [25]) and involves a reversing operation $\Theta : \mathcal{B}(\mathsf{h}) \to \mathcal{B}(\mathsf{h})$, namely a linear $*$-map ($\Theta(x^*) = \Theta(x)^*$ for all $x \in \mathcal{B}(\mathsf{h})$), that is also an antihomomorphism ($\Theta(xy) = \Theta(y)\Theta(x)$) and satisfies $\Theta^2 = I$, where I denotes the identity map on $\mathcal{B}(\mathsf{h})$. A QMS $\mathcal{T}$ satisfies the Agarwal-Majewski QDB condition with respect to a faihtful normal invariant state ρ if $\mathrm{tr}\,(\rho x \mathcal{T}_t(y)) = \mathrm{tr}\,(\rho\,\Theta(y)\mathcal{T}_t(\Theta(x)))$, for all $x, y \in \mathcal{B}(h)$. If the state ρ is invariant under the reversing operation, i.e. $\mathrm{tr}\,(\rho\Theta(x)) = \mathrm{tr}\,(\rho x)$ for all $x \in \mathcal{B}(\mathsf{h})$, as we shall assume throughout the paper, this condition can be written in the equivalent form $\mathrm{tr}\,(\rho x \mathcal{T}_t(y)) = \mathrm{tr}\,(\rho\,((\Theta \circ \mathcal{T}_t \circ \Theta)(x))\,y)$ for all $x, y \in \mathcal{B}(h)$. Therefore the Agarwal-Majewski QDB condition means that maps $\mathcal{T}_t$ admit dual maps coinciding with $\Theta \circ \mathcal{T}_t \circ \Theta$ for all $t \geq 0$; in particular dual maps must be positive since Θ is obviously positivity preserving. The map Θ often appears in the physical literature as a parity map; a self-adjoint x is an even (resp. odd) observable if $\Theta(x) = x$ (resp. $\Theta(x) = -x$).

In our framework, since H_S is the energy of the system, which is a typical even observable, a reasonable map Θ is the transpose $\Theta(a) = a^{\mathsf{T}}$ with respect to an orthonormal basis $(e_j)_{1\leq j\leq d}$ of h diagonalizing H_S as in [12]. Interested readers can consult [17, 18, 25] in more general situations.

The best known QDB notion, however, is due to Alicki [4] and Kossakowski, Frigerio, Gorini, Verri [23]. According to these authors, a QMS with generator $\mathcal{L}$ as in (5), with invariant state ρ whose density commutes with H, satisfies the quantum detailed balance condition if $\mathrm{tr}\,(\rho\, x\mathcal{L}(y)) = \mathrm{tr}(\rho\,\widetilde{\mathcal{L}}(x)y)$ where $\widetilde{\mathcal{L}} = \mathcal{L} - 2\mathrm{i}[H, \cdot\,]$. As a consequence, the QMS $\widetilde{\mathcal{T}}$ on $\mathcal{B}(\mathsf{h})$ generated by $\widetilde{\mathcal{L}}$ satisfies $\mathrm{tr}\,(\rho\, x\mathcal{T}_t(y)) = \mathrm{tr}(\rho\,\widetilde{\mathcal{T}}_t(x)y)$ for all $t \geq 0$.

Both the above QDB conditions depend in a crucial way from the bilinear form $(x, y) \to \mathrm{tr}\,(\rho xy)$. In particular, if they hold true, all positive maps $\mathcal{T}_t$ admit *positive* dual maps; as a consequence, all the maps $\mathcal{T}_t$ must commute with the modular group $(\sigma_t^\rho)_{t\in\mathbb{R}}$, given by $\sigma_t^\rho(x) = \rho^{\mathrm{i}t} x \rho^{-\mathrm{i}t}$, associated with the state ρ (see [23, Prop. 2.1], [26, Prop. 5] and also [9]) as well as the generator $\mathcal{L}$. This algebraic restriction is unnecessary if we consider the bilinear form $(x, y) \to \omega\left(\sigma_{\mathrm{i}/2}(x)y\right)$ for defining dual QMSs.

QDB conditions arising when we consider this bilinear form are called *standard* (see e.g. [11, 18]); we could not find them in the literature, but it seems that they belong to the folklore of the subject. In particular, they were considered by R. Alicki and A. Majewski (private communication).

DEFINITION 1 Let $\mathcal{T}$ be a QMS with a dual $\mathcal{T}'$ defined by

$$\omega\left(\sigma_{\mathrm{i}/2}(x)\mathcal{T}_t(y)\right) = \omega\left(\sigma_{\mathrm{i}/2}\left(\mathcal{T}_t'(x)\right)y\right) \quad \text{for all } x, y \in \mathcal{B}(\mathsf{h}), \quad t \geq 0\,.$$

The semigroup $\mathcal{T}$ satisfies:

1. the standard quantum detailed balance condition with respect to the reversing operation Θ (SQBD-Θ) if $\mathcal{T}_t' = \Theta \circ \mathcal{T}_t \circ \Theta$ for all $t \geq 0$,

2. the standard quantum detailed balance condition (SQDB) if the difference of generators $\mathcal{L} - \mathcal{L}'$ of $\mathcal{T}$ and $\mathcal{T}'$ is a derivation.

It is worth noticing here that the above *standard* QDB conditions coincide with the Agarwal-Majewski and Alicki-Gorini-Kossakowski-Frigerio-Verri respectively when the QMS $\mathcal{T}$ commutes with the modular group $(\sigma_t)_{t\in\mathbb{R}}$ associated with ω (see [16, 18]).

In the framework of this paper all states are normal and will be identified with their densities. In particular, $\omega(x) = \mathrm{tr}\,(\rho\, x)$, $\sigma_t(x) = \rho^{\mathrm{i}t} x \rho^{-\mathrm{i}t}$ and $\omega\left(\sigma_{\mathrm{i}/2}(x)y\right) = \mathrm{tr}\left(\rho^{1/2} x \rho^{1/2} y\right)$.

In [18, Theorems 5, 8 and Remark 4] we proved the following characterisations of QMS satisfying a standard QDB condition we recall here in the present framework.

THEOREM 1 *A QMS $\mathcal{T}$ satisfies the SQDB if and only if for any special GKSL representation of the generator $\mathcal{L}$ by means of operators G, L_ℓ there exists a unitary $(u_{m\ell})_{1\leq m,\ell\leq 2b}$ on a Hilbert space k (called the multiplicity space) which is also symmetric (i.e. $u_{\ell m} = u_{m\ell}$ for all m, ℓ) such that, for all $\ell \geq 1$,*

$$\rho^{1/2} L_\ell^* = \sum_{1\leq m\leq 2b} u_{\ell m} L_m \rho^{1/2}\,. \tag{7}$$

THEOREM 2 *A QMS $\mathcal{T}$ satisfies the SQBD-Θ condition if and only if for any special GKSL representation of $\mathcal{L}$ by means of operators G, L_ℓ, there exists a self-adjoint unitary $(u_{\ell m})_{1\leq m,\ell\leq 2b}$ such that:*

1. $\rho^{1/2} G^{\mathsf{T}} = G \rho^{1/2}$,

2. $\rho^{1/2} L_\ell^{\mathsf{T}} = \sum_{1\leq m\leq 2b} u_{\ell m} L_m \rho^{1/2}$ *for all* $1 \leq \ell \leq 2b$.

The SQBD-Θ condition is more restrictive than the SQDB condition because it involves also the identity $\rho^{1/2} G^{\mathsf{T}} = G \rho^{1/2}$ (see [15, Example 7.3]). However, this does not happen if $G^{\mathsf{T}} = G$ and ρ commutes with G. This is a reasonable physical assumption satisfied by many QMSs as, for instance, those of stochastic limit type we are considering in this paper. Conditions obtained including the reversing map Θ seem more suitable for studying quantum detailed balance [18, 21].

4. A Formula for Entropy Production

We begin this section by recalling our notion of entropy production [15]. Since it provides an index describing deviation from detailed balance, it was introduced in [14, 15] through the forward and backward two-point states on $\mathcal{B}(\mathsf{h}) \otimes \mathcal{B}(\mathsf{h})$

$$\begin{aligned} \overrightarrow{\Omega}_t (x \otimes y) &= \operatorname{tr}\left(\rho^{1/2} x^{\intercal} \rho^{1/2} \mathcal{T}_t(y)\right) \\ \overleftarrow{\Omega}_t (x \otimes y) &= \operatorname{tr}\left(\rho^{1/2} \mathcal{T}_t(x^{\intercal})^{\intercal} \rho^{1/2} y\right), \end{aligned}$$

which clearly coincide if and only if $\mathcal{T}$ satisfies the SQDB-Θ condition, and their relative entropy $S(\overrightarrow{\Omega}_t, \overleftarrow{\Omega}_t)$ as

$$\operatorname{ep}(\mathcal{T}, \rho) = \limsup_{t \to 0^+} \frac{S(\overrightarrow{\Omega}_t, \overleftarrow{\Omega}_t)}{t}. \tag{8}$$

Moreover, in [15, Theorem 5], we proved an explicit formula based on the Kraus operators L_ℓ in a GKSL decomposition of the generator $\mathcal{L}$. Let $\overrightarrow{\Phi}_*$ and $\overleftarrow{\Phi}_*$ be the linear maps on trace class operators on $\mathsf{h} \otimes \mathsf{h}$

$$\overrightarrow{\Phi}_*(X) = \sum_\omega \left(\gamma_\omega^- (\mathbb{1} \otimes V_\omega) X (\mathbb{1} \otimes V_\omega^*) + \gamma_\omega^+ (\mathbb{1} \otimes V_\omega^*) X (\mathbb{1} \otimes V_\omega)\right), \tag{9}$$

$$\overleftarrow{\Phi}_*(X) = \sum_\omega \left(\gamma_\omega^- (V_\omega \otimes \mathbb{1}) X (V_\omega^* \otimes \mathbb{1}) + \gamma_\omega^+ (V_\omega^* \otimes \mathbb{1}) X (V_\omega \otimes \mathbb{1})\right). \tag{10}$$

Let θ be the antilinear conjugation in a basis $(e_j)_{1 \le j \le d}$ diagonalizing ρ and let D be the entangled state on $\mathcal{B}(\mathsf{h}) \otimes \mathcal{B}(\mathsf{h})$ introduced in [15] as

$$D = |r\rangle\langle r|, \qquad r = \sum_j \rho_j^{1/2} \theta e_j \otimes e_j. \tag{11}$$

It is not hard to check as in [15] that D is the density of the state $\overrightarrow{\Omega}_0 = \overleftarrow{\Omega}_0$.

THEOREM 3 *Let $\mathcal{T}$ be QMS on $\mathcal{B}(\mathsf{h})$ and ρ a faithful invariant state. Assume:*

1. $\rho^{1/2} G^{\intercal} = G \rho^{1/2}$,
2. *the linear spans of* $\left\{L_\ell \rho^{1/2} \mid \ell \ge 1\right\}$ *and* $\left\{\rho^{1/2} L_\ell^{\intercal} \mid \ell \ge 1\right\}$ *coincide.*

Then the ranges of $\overrightarrow{\Phi}_*(D)$ and $\overleftarrow{\Phi}_*(D)$ coincide and the entropy production is

$$\begin{aligned} \operatorname{ep}(\mathcal{T}, \rho) &= \frac{1}{2} \operatorname{tr}\left(\left(\overrightarrow{\Phi}_*(D) - \overleftarrow{\Phi}_*(D)\right)\left(\log\left(\overrightarrow{\Phi}_*(D)\right) - \log\left(\overleftarrow{\Phi}_*(D)\right)\right)\right) \\ &= \operatorname{tr}\left(\overrightarrow{\Phi}_*(D)\left(\log\left(\overrightarrow{\Phi}_*(D)\right) - \log\left(\overleftarrow{\Phi}_*(D)\right)\right)\right). \end{aligned} \tag{12}$$

In order to compute explicitly the entropy production for QMS in the class described in Sect. 2 we begin by establishing a preliminary Lemma. We denote by $\langle \cdot\,, \cdot \rangle$ the scalar product in $\mathsf{h}\otimes\mathsf{h}$.

LEMMA 1 *Let X and Y be bounded operators on h, then*

$$\langle (Y\otimes \mathbb{1})r, (\mathbb{1}\otimes X)r\rangle = \mathrm{tr}\left((\rho^{1/2}\theta Y^*\theta)^* X\rho^{1/2}\right) \tag{13}$$

$$\langle (\mathbb{1}\otimes Y)r, (\mathbb{1}\otimes X)r\rangle = \mathrm{tr}\,(\rho\, Y^* X)\,. \tag{14}$$

Proof. Both formulas follow from straightforward computations

$$\begin{aligned}
\langle (Y\otimes \mathbb{1})r, (\mathbb{1}\otimes X)r\rangle &= \sum_{j,k} (\rho_j\rho_k)^{1/2}\langle Y\theta e_j, \theta e_k\rangle\langle e_j, Xe_k\rangle \\
&= \sum_{j,k} \langle e_k, \theta Y\theta\rho^{1/2} e_j\rangle\langle e_j, X\rho^{1/2} e_k\rangle \\
&= \sum_{j,k} \langle \rho^{1/2}\theta Y^*\theta e_k, e_j\rangle\langle e_j, X\rho^{1/2} e_k\rangle \\
&= \sum_{k} \langle \rho^{1/2}\theta Y^*\theta e_k, X\rho^{1/2} e_k\rangle \\
&= \mathrm{tr}\left((\rho^{1/2}\theta Y^*\theta)^* X\rho^{1/2}\right),
\end{aligned}$$

$$\begin{aligned}
\langle (\mathbb{1}\otimes Y)r, (\mathbb{1}\otimes X)r\rangle &= \sum_{j,k} (\rho_j\rho_k)^{1/2}\langle \theta e_j, \theta e_k\rangle\langle Ye_j, Xe_k\rangle \\
&= \sum_{j} \langle Y\rho^{1/2} e_j, X\rho^{1/2} e_j\rangle \\
&= \mathrm{tr}\,(\rho\, Y^* X)\,.
\end{aligned}$$

Replacing the operators X, Y in Lemma 1 by operators V_ω and taking into account that $\theta V_\omega^*\theta = V_\omega^*$ if V_ω is a real matrix, we have the following

COROLLARY 1 *If the operators V_ω, defined by (1), are represented by real matrices we have*

$$\begin{aligned}
\langle (V_{\omega'}\otimes \mathbb{1})r, (\mathbb{1}\otimes V_\omega)r\rangle &= \delta_{\omega,\omega'}\mathrm{tr}\left(\rho^{1/2}V_\omega\rho^{1/2}V_\omega\right), \\
\langle (\mathbb{1}\otimes V_{\omega'})r, (\mathbb{1}\otimes V_\omega)r\rangle &= \delta_{\omega,\omega'}\mathrm{tr}\,(\rho V_\omega^* V_\omega)\,,
\end{aligned}$$

where $\delta_{\omega,\omega'}$ is the Kronecker delta.

We are now in a position to prove our entropy production formula

THEOREM 4 *Assume that V_ω and H_ω are real matrices for all Bohr frequency ω, then the entropy production is*

$$\begin{aligned}\mathrm{ep}(\mathcal{T},\rho) &= \sum_\omega \Bigg[\gamma_\omega^- \,\mathrm{tr}\left(\rho V_\omega^* V_\omega \log\Big(\frac{\gamma_\omega^-\,[\mathrm{tr}\,(\rho V_\omega^* V_\omega)]^2}{\gamma_\omega^+\,[\mathrm{tr}\,(\rho^{1/2} V_\omega^* \rho^{1/2} V_\omega)]^2}\Big)\right) \\ &\quad + \gamma_\omega^+ \mathrm{tr}\left(\rho V_\omega V_\omega^* \log\Big(\frac{\gamma_\omega^+\,[\mathrm{tr}\,(\rho V_\omega V_\omega^*)]^2}{\gamma_\omega^-\,[\mathrm{tr}\,(\rho^{1/2} V_\omega^* \rho^{1/2} V_\omega)]^2}\Big)\right)\Bigg]. \quad (15)\end{aligned}$$

Proof. Replacing X by D in (9) and (10) and denoting $\overrightarrow{V}_\omega = \mathbb{1}\otimes V_\omega$, $\overleftarrow{V}_\omega = V_\omega \otimes \mathbb{1}$ one obtains

$$\overrightarrow{\Phi}_*(D) = \sum_\omega \Big(\gamma_\omega^- |\overrightarrow{V}_\omega r\rangle\langle \overrightarrow{V}_\omega r| + \gamma_\omega^+ |\overrightarrow{V}^*_\omega r\rangle\langle \overrightarrow{V}^*_\omega r|\Big) \quad (16)$$

$$\overleftarrow{\Phi}_*(D) = \sum_\omega \Big(\gamma_\omega^- |\overleftarrow{V}_\omega r\rangle\langle \overleftarrow{V}_\omega r| + \gamma_\omega^+ |\overleftarrow{V}^*_\omega r\rangle\langle \overleftarrow{V}^*_\omega r|\Big). \quad (17)$$

By Corollary 1, each vector $\overrightarrow{V}_\omega r$ is orthogonal to any vector $\overrightarrow{V}^*_\omega r$ and each $\overrightarrow{V}_\omega r$ (respectively $\overrightarrow{V}^*_\omega r$) is orthogonal to $\overrightarrow{V}_{\omega'} r$ (resp. $\overrightarrow{V}^*_{\omega'} r$) with $\omega' \neq \omega$. Therefore, normalizing vectors $\overrightarrow{V}_\omega r, \overrightarrow{V}^*_\omega r$ yield an orthonormal basis of $\mathsf{h}\otimes\mathsf{h}$ (by adding appropriate vectors if necessary, on these vectors $\overrightarrow{\Phi}_*(D)$ and $\overleftarrow{\Phi}_*(D)$ are null, so that the computations remain unchanged). In this basis $\overrightarrow{\Phi}_*(D)$ turns out to be a diagonal matrix with 2×2 blocks associated with each Bohr frequency ω given by

$$\begin{bmatrix} \gamma_\omega^- \mathrm{tr}\,(\rho V_\omega^* V_\omega) & 0 \\ 0 & \gamma_\omega^+ \mathrm{tr}\,(\rho V_\omega V_\omega^*) \end{bmatrix}.$$

In order to write $\overleftarrow{\Phi}_*(D)$, compute first $\overleftarrow{V}_\omega r$:

$$\begin{aligned}\overleftarrow{V}_\omega r &= \frac{\mathrm{tr}\left(\rho^{1/2} V_\omega^* \rho^{1/2} V_\omega^*\right)}{\mathrm{tr}\,(\rho V_\omega^* V_\omega)} \overrightarrow{V}_\omega r + \frac{\mathrm{tr}\left(\rho^{1/2} V_\omega \rho^{1/2} V_\omega^*\right)}{\mathrm{tr}\,(\rho V_\omega V_\omega^*)} \overrightarrow{V}^*_\omega r \\ &= \frac{\mathrm{tr}\left(\rho^{1/2} V_\omega \rho^{1/2} V_\omega^*\right)}{\mathrm{tr}\,(\rho V_\omega V_\omega^*)} \overrightarrow{V}^*_\omega r,\end{aligned}$$

since the first term is 0. In the same way we have

$$\begin{aligned}\overleftarrow{V}^*_\omega r &= \frac{\mathrm{tr}\left(\rho^{1/2} V_\omega^* \rho^{1/2} V_\omega\right)}{\mathrm{tr}\,(\rho V_\omega^* V_\omega)} \overrightarrow{V}_\omega r + \frac{\mathrm{tr}\left(\rho^{1/2} V_\omega \rho^{1/2} V_\omega\right)}{\mathrm{tr}\,(\rho V_\omega V_\omega^*)} \overrightarrow{V}^*_\omega r \\ &= \frac{\mathrm{tr}\left(\rho^{1/2} V_\omega^* \rho^{1/2} V_\omega\right)}{\mathrm{tr}\,(\rho V_\omega^* V_\omega)} \overrightarrow{V}_\omega r.\end{aligned}$$

Thus, by the cyclic property of the trace, we have

$$\overleftarrow{V}_\omega r \;=\; \frac{\mathrm{tr}\left(\rho^{1/2}V_\omega^*\rho^{1/2}V_\omega\right)}{\mathrm{tr}\,(\rho V_\omega V_\omega^*)^{1/2}}\frac{\overrightarrow{V}^*_\omega r}{\|\overrightarrow{V}^*_\omega r\|}\,, \qquad \overleftarrow{V}^*_\omega r \;=\; \frac{\mathrm{tr}\left(\rho^{1/2}V_\omega^*\rho^{1/2}V_\omega\right)}{\mathrm{tr}\,(\rho V_\omega^* V_\omega)^{1/2}}\frac{\overrightarrow{V}_\omega r}{\|\overrightarrow{V}_\omega r\|}\,.$$

It follows that, in the above orthonormal basis of $\mathsf{h}\times\mathsf{h}$, obtained normalizing vectors $\overrightarrow{V}_\omega r$, $\overrightarrow{V}^*_\omega r$, $\overleftarrow{\Phi}_*(D)$ becomes

$$\begin{aligned}\overleftarrow{\Phi}_*(D) \;=\; \sum_\omega \Bigg[&\gamma_\omega^+\frac{[\mathrm{tr}\left(\rho^{1/2}V_\omega^*\rho^{1/2}V_\omega\right)]^2}{\mathrm{tr}\,(\rho V_\omega^* V_\omega)}\frac{|\overrightarrow{V}_\omega r\rangle\langle\overrightarrow{V}_\omega r|}{\|\overrightarrow{V}_\omega r\|^2}\\ &+\gamma_\omega^-\frac{[\mathrm{tr}\left(\rho^{1/2}V_\omega^*\rho^{1/2}V_\omega\right)]^2}{\mathrm{tr}\,(\rho V_\omega V_\omega^*)}\frac{|\overrightarrow{V}^*_\omega r\rangle\langle\overrightarrow{V}^*_\omega r|}{\|\overrightarrow{V}^*_\omega r\|^2}\Bigg]\end{aligned}$$

and it turns out to be a matrix with 2×2 diagonal blocks associated with each Bohr frequency ω given by

$$\begin{bmatrix} \gamma_\omega^+\dfrac{[\mathrm{tr}\left(\rho^{1/2}V_\omega^*\rho^{1/2}V_\omega\right)]^2}{\mathrm{tr}\,(\rho V_\omega^* V_\omega)} & 0 \\ 0 & \gamma_\omega^-\dfrac{[\mathrm{tr}\left(\rho^{1/2}V_\omega^*\rho^{1/2}V_\omega\right)]^2}{\mathrm{tr}\,(\rho V_\omega V_\omega^*)} \end{bmatrix}$$

and our entropy production formula follows immediately.

5. Global and Local Equilibrium

In this section we show that the entropy production (15) vanishes if and only if all the semigroups generated by each $\mathcal{L}_\omega$ satisfy the SQDB-Θ condition.

It is useful to introduce some notation that allows us to focus more clearly contributions of each QMS generated by $\mathcal{L}_\omega$ to the entropy production:

$$\nu_\omega^- \;=\; \mathrm{tr}\,(\rho V_\omega^* V_\omega)\;, \qquad \nu_\omega^+ \;=\; \mathrm{tr}\,(\rho V_\omega V_\omega^*)\;, \qquad \mu_\omega \;=\; \mathrm{tr}\left(\rho^{1/2}V_\omega^*\rho^{1/2}V_\omega\right)\,.$$

In this notation the entropy production is written as

$$\mathrm{ep}(\mathcal{T},\rho) \;=\; \sum_\omega \left[\gamma_\omega^-\nu_\omega^-\log\left(\frac{\gamma_\omega^-\nu_\omega^{-\,2}}{\gamma_\omega^+\mu_\omega^2}\right) + \gamma_\omega^+\nu_\omega^+\log\left(\frac{\gamma_\omega^+\nu_\omega^{+\,2}}{\gamma_\omega^-\mu_\omega^2}\right)\right]. \qquad (18)$$

Note that, by the Schwarz inequality,

$$\mu_\omega^2 \;\le\; \nu_\omega^+\nu_\omega^-\,. \qquad (19)$$

Moreover,

$$\log\left(\frac{\gamma_\omega^{\mp}\nu_\omega^{\mp\,2}}{\gamma_\omega^{\pm}\mu_\omega^2}\right) = \log\left(\frac{\gamma_\omega^{\mp}\nu_\omega^{\mp}}{\gamma_\omega^{\pm}\nu_\omega^{\pm}}\right) + \log\left(\frac{\nu_\omega^{+}\nu_\omega^{-}}{\mu_\omega^2}\right)$$

so that we can rewrite the entropy production as

$$\begin{aligned} \mathrm{ep}(\mathcal{T},\rho) &= \sum_\omega \left(\gamma_\omega^{-}\nu_\omega^{-} - \gamma_\omega^{+}\nu_\omega^{+}\right)\log\left(\frac{\gamma_\omega^{-}\nu_\omega^{-}}{\gamma_\omega^{+}\nu_\omega^{+}}\right) \\ &\quad + \left(\gamma_\omega^{-}\nu_\omega^{-} + \gamma_\omega^{+}\nu_\omega^{+}\right)\log\left(\frac{\nu_\omega^{+}\nu_\omega^{-}}{\mu_\omega^2}\right). \end{aligned}$$

COROLLARY 2 *The entropy production is zero if and only if* $\gamma_\omega^{-}\nu_\omega^{-} = \gamma_\omega^{+}\nu_\omega^{+}$ *and* $\nu_\omega^{-}\nu_\omega^{+} = \mu_\omega^2$.

Proof. It suffices to note that $\log(\nu_\omega^{+}\nu_\omega^{-}/\mu_\omega^2)$ is non-negative by (19) and $(t-s)\log(t/s)$ is non-negative for all reals t, s with $t \neq s$.

The following result shows that QMSs of stochastic limit type have zero entropy production if and only if the standard quantum detailed balance condition with reversing map Θ (SQDB-Θ condition) holds. This is not true for an arbitrary QMS as shows in [15, Example 7.3].

THEOREM 5 *Assume that* V_ω *and* H_ω *are real matrices for all* ω, *so that the semigroup commutes with the reversing map* Θ. *Then the following are equivalent:*

1. *the entropy production is zero,*
2. $\rho^{1/2}L^*_{2\ell-1} = L_{2\ell}\,\rho^{1/2}$ *for all* $\ell = 1,\ldots,2b$,
3. $(\gamma_\omega^{+})^{1/2}\rho^{1/2}V_\omega = (\gamma_\omega^{-})^{1/2}V_\omega\rho^{1/2}$ *for all* ω,
4. *the SQDB-*Θ *condition holds.*

Proof. $2 \Leftrightarrow 3$. Clear from the definition (6) of $L_{2\ell}$ and $L_{2\ell+1}$. Indeed, $(\gamma_\omega^{+})^{1/2}\rho^{1/2}V_\omega = \rho^{1/2}L^*_{2\ell-1}$ and $(\gamma_\omega^{-})^{1/2}V_\omega\rho^{1/2} = L_{2\ell}\rho^{1/2}$ for all $\ell = 1,\ldots,b$.

$1 \Rightarrow 3$. By Corollary 2 we have $\nu_\omega^{-}\nu_\omega^{+} = \mu_\omega^2$ and so the Schwarz inequality (19) turns out to be an equality. It follows that the operators $V_\omega\rho^{1/2}$ and $V_\omega\rho^{1/2}$, thought of as vectors in the Hilbert space of Hilbert-Schmidt operators on h are parallel, i.e. $\rho^{1/2}V_\omega = c_\omega V_\omega\rho^{1/2}$ for some constant c_ω. Computing the scalar product with $V_\omega\rho^{1/2}$ we immediately find $\mu_\omega = c_\omega\nu_\omega^{-}$, i.e., since $\mu_\omega^2 = \nu_\omega^{-}\nu_\omega^{+}$, $c_\omega = (\nu_\omega^{+}/\nu_\omega^{-})^{1/2}$ so that

$$\rho^{1/2}V_\omega = \left(\frac{\nu_\omega^{+}}{\nu_\omega^{-}}\right)^{1/2} V_\omega\rho^{1/2}$$

and 3 follows from $\gamma_\omega^-\nu_\omega^- = \gamma_\omega^+\nu_\omega^+$.

$3 \Rightarrow 4$. The SQDB-Θ condition is characterized by [18, Theorem 8], namely Theorem 2 in this paper. Now, the identity $\rho^{1/2}\theta G^*\theta = G\rho^{1/2}$ holds because we have assumed that V_ω and H_ω are real matrices. Moreover, since $L_\ell = \theta L_\ell \theta$ and $L_\ell^* = \theta L_\ell^* \theta$ for all $\ell \geq 1$, condition 2 of Theorem 2 holds choosing as unitary self-adjoint the operator u flipping even and odd indexes ℓ, i.e. $u_{kj} = 1$ if either $k = 2\ell$ and $j = 2\ell - 1$ or $k = 2\ell - 1$ and $j = 2\ell$ and $u_{kj} = 0$ otherwise.

$4 \Rightarrow 1$. For all vector $v = \sum_{\alpha,\beta} v_{\alpha\beta}\, \theta e_\alpha \otimes e_\beta$ we have

$$\left\langle v, \overrightarrow{\Phi}_*(D)v \right\rangle = \sum_{\ell,j,k,\beta,\beta'} \overline{v}_{j\beta'} v_{k\beta} \left\langle e_{\beta'}, L_\ell \rho^{1/2} e_j \right\rangle \left\langle L_\ell \rho^{1/2} e_k, e_\beta \right\rangle \tag{20}$$

and also, by the properties of the antiunitary θ

$$\begin{aligned}\left\langle v, \overleftarrow{\Phi}_*(D)v \right\rangle &= \sum_{\ell,j,k,\alpha,\alpha'} \overline{v}_{\alpha' j} v_{\alpha k} \left\langle \theta e_{\alpha'}, L_\ell \rho^{1/2} \theta e_j \right\rangle \left\langle L_\ell \rho^{1/2} \theta e_k, \theta e_\alpha \right\rangle \\ &= \sum_{\ell,j,k,\alpha,\alpha'} \overline{v}_{\alpha' j} v_{\alpha k} \left\langle \theta L_\ell \theta \rho^{1/2} e_j, e_{\alpha'} \right\rangle \left\langle e_\alpha, \theta L_\ell \theta \rho^{1/2} e_k \right\rangle \\ &= \sum_{\ell,j,k,\alpha,\alpha'} \overline{v}_{\alpha' j} v_{\alpha k} \left\langle e_j, \rho^{1/2} \theta L_\ell^* \theta e_{\alpha'} \right\rangle \left\langle \theta L_\ell^* \theta \rho^{1/2} e_\alpha, e_k \right\rangle.\end{aligned}$$

Now, the SQDB-Θ condition holds, then $\rho^{1/2}\theta L_\ell^* \theta = \sum_m u_{\ell m} L_m \rho^{1/2}$ for a unitary self-adjoint $(u_{\ell m})_{1 \leq \ell, m \leq 2b}$ so that, $\sum_\ell \overline{u}_{\ell m'} u_{\ell m} = \delta_{m' m}$ and

$$\begin{aligned}\left\langle v, \overleftarrow{\Phi}_*(D)v \right\rangle &= \sum_{\ell,j,k,\alpha,\alpha',m,m'} \overline{v}_{\alpha' j} v_{\alpha k} \overline{u}_{\ell m'} u_{\ell m} \left\langle e_j, L_{m'} \rho^{1/2} e_{\alpha'} \right\rangle \left\langle L_m \rho^{1/2} e_\alpha, e_k \right\rangle \\ &= \sum_{j,k,\alpha,\alpha',m} \overline{v}_{\alpha' j} v_{\alpha k} \left\langle e_j, L_m \rho^{1/2} e_{\alpha'} \right\rangle \left\langle L_m \rho^{1/2} e_\alpha, e_k \right\rangle.\end{aligned}$$

Changing indexes and comparing with (20), by the arbitrarity of v, we find $\overrightarrow{\Phi}_*(D) = \overleftarrow{\Phi}_*(D)$ and the entropy production, given by (15), is zero. □

Remark. It is worth noticing here that conditions of Theorem 5 are also equivalent to the QDB-Θ condition and so in our class of QMSs of stochastic limit type the SQDB-Θ and QDB-Θ. Indeed, since the modular group is given by $\sigma_t(x) = \rho^{\mathrm{i}t} x \rho^{-\mathrm{i}t}$, the identity $(\gamma_\omega^+)^{1/2} \rho^{1/2} V_\omega = (\gamma_\omega^-)^{1/2} V_\omega \rho^{1/2}$ reads $\sigma_{-\mathrm{i}/2}(V_\omega) = (\gamma_\omega^- / \gamma_\omega^+)^{1/2} V_\omega$. Taking the adjoint of $(\gamma_\omega^+)^{1/2} \rho^{1/2} V_\omega = (\gamma_\omega^-)^{1/2} V_\omega \rho^{1/2}$ we find also, in the same way, $\sigma_{-\mathrm{i}/2}(V_\omega^*) = (\gamma_\omega^+ / \gamma_\omega^-)^{1/2} V_\omega^*$. It follows that

$$\begin{aligned}\sigma_{-\mathrm{i}}(V_\omega) &= \sigma_{-\mathrm{i}/2}\left(\sigma_{-\mathrm{i}/2}(V_\omega)\right) = \left(\gamma_\omega^- / \gamma_\omega^+\right)^{1/2} \sigma_{-\mathrm{i}/2}(V_\omega) = \left(\gamma_\omega^- / \gamma_\omega^+\right) V_\omega \\ \sigma_{-\mathrm{i}}(V_\omega^*) &= \sigma_{-\mathrm{i}/2}\left(\sigma_{-\mathrm{i}/2}(V_\omega^*)\right) = \left(\gamma_\omega^+ / \gamma_\omega^-\right)^{1/2} \sigma_{-\mathrm{i}/2}(V_\omega^*) = \left(\gamma_\omega^+ / \gamma_\omega^-\right) V_\omega^*\end{aligned}$$

and

$$\begin{array}{rclcrcl} \sigma_{-\mathrm{i}}(L_{2\ell}) & = & \left(\gamma_\omega^-/\gamma_\omega^+\right) L_{2\ell}\,, & \qquad & \sigma_{-\mathrm{i}}(L_{2\ell+1}) & = & \left(\gamma_\omega^+/\gamma_\omega^-\right) L_{2\ell-1} \\ \sigma_{-\mathrm{i}}(L_{2\ell}^*) & = & \left(\gamma_\omega^+/\gamma_\omega^-\right) L_{2\ell}^*\,, & \qquad & \sigma_{-\mathrm{i}}(L_{2\ell+1}^*) & = & \left(\gamma_\omega^-/\gamma_\omega^+\right) L_{2\ell-1}^*\,. \end{array}$$

Straightforward computations show that $\sigma_{-\mathrm{i}}(H_\omega) = H_\omega$. It follows then from [16, Theorem 9], that the QDB-Θ condition holds.

Theorem 5 and the above remark lead us to the following result essentially showing that ρ is an equilibrium state for the QMS generated by $\mathcal{L}$ if and only if it is an equilibrium state for the QMSs generated by *each* $\mathcal{L}_\omega$.

THEOREM 6 *Let $\mathcal{L}$ be the generator of a QMS as in Sect. 2, let ρ be a faithful invariant state. Assume that V_ω is a real matrix for all ω and H_ω is a linear combination of $V_\omega^* V_\omega$ and $V_\omega V_\omega^*$. Then the following are equivalent:*

1. *the QMS generated by $\mathcal{L}$ satisfies the SQDB-Θ condition,*
2. *for all ω, the QMSs generated by each $\mathcal{L}_\omega$ admits ρ as invariant state and satisfies the SQDB-Θ condition.*

Proof. Clearly $2 \Rightarrow 1$.

Conversely, if the QMS generated by $\mathcal{L}$ satisfies the SQDB-Θ condition, then by Theorem 5 and the above Remark we have

$$\rho V_\omega^* V_\omega \rho^{-1} \;=\; \rho V_\omega^* \rho^{-1}\, \rho V_\omega \rho^{-1} \;=\; \frac{\gamma_\omega^+}{\gamma_\omega^-}\, V_\omega^*\, \frac{\gamma_\omega^-}{\gamma_\omega^+}\, V_\omega \;=\; V_\omega^* V_\omega$$

and so ρ commutes with $V_\omega^* V_\omega$. In the same way, we can check that it commutes with $V_\omega V_\omega^*$. As a consequence, by the commutation rules found in the above Remark $\rho V_\omega^* = (\gamma_\omega^+/\gamma_\omega^-) V_\omega^* \rho$ and $\rho V_\omega = (\gamma_\omega^-/\gamma_\omega^+) V_\omega \rho$ and we have

$$\begin{aligned} & G_\omega \rho + \gamma_\omega^- V_\omega \rho V_\omega^* + \gamma_\omega^+ V_\omega^* \rho V_\omega + \rho G_\omega^* \\ & \qquad = \; (G_\omega + G_\omega^*)\rho + \gamma_\omega^+ V_\omega V_\omega^* \rho + \gamma_\omega^- V_\omega^* V_\omega \rho \\ & \qquad = \; \left(G_\omega + G_\omega^* + \gamma_\omega^+ V_\omega V_\omega^* + \gamma_\omega^- V_\omega^* V_\omega\right) \rho \;=\; 0\,. \end{aligned}$$

Thus ρ is an invariant state for the QMS generated by $\mathcal{L}_\omega$. This semigroup also satisfies the SQDB-Θ condition because, from $(\gamma_\omega^+)^{1/2}\, \rho^{1/2} V_\omega = (\gamma_\omega^-)^{1/2}\, V_\omega \rho^{1/2}$, condition 2 of Theorem 2 follows immediately. □

Remark. If we drop the assumptions on matrices V_ω and H_ω similar result holds considering the quantum detailed balance condition without reversing operation Θ. In this case, however, the forward $\overrightarrow{\Omega}_t$ and backward $\overleftarrow{\Omega}_t$ states used to define the entropy production, defined in the same way without transpositions, must be thought of as states on the tensor product of the *opposite* algebra $\mathcal{B}(\mathsf{h})^{\mathrm{o}}$ with $\mathcal{B}(\mathsf{h})$ (see [15, Remark 2]).

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