ON THE RELATIONSHIP BETWEEN A QUANTUM MARKOV SEMIGROUP AND ITS REPRESENTATION VIA LINEAR STOCHASTIC SCHRÖDINGER EQUATIONS

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Dedicated to Prof. Kalyan B. Sinha on occasion of his 70th birthday.

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1. Introduction

A quantum Markov semigroup (QMS) $\mathcal{T}$ is a weakly*-continuous semigroup $(\mathcal{T}_t)_{t\geq 0}$ of completely positive, identity preserving, normal maps on a von Neumann algebra. In this paper, we will only be concerned with QMS on a matrix algebra which are norm-continuous.

These QMS semigroups were introduced in the seventies (as quantum dynamical semigroups) to model the irreversible evolution of an open quantum system and are now an important tool to investigate quantum systems and quantum stochastic processes (see [4, 6, 7, 17, 19, 21, 24] and the references therein). The representation of QMS via solutions of classical stochastic differential equations, already noticed by Skorohod [33], is also well-known and plays a key role in quantum trajectory theory (see, e.g. [4] section 3.2.3, [2, 3, 5, 7, 22, 23] and the references therein). These equations, called stochastic Schrödinger equations (SSE) (see [16, 25, 26, 27, 28, 30] for recent results), turn out to be very useful to study open quantum systems through the interplay between classical and quantum stochastic analysis.

The aim of this note is to study the relationship between irreducibility of a QMS and diffusion processes solving the associated SSE driven by independent Brownian motions. Our motivation is to establish a bridge between classical and quantum stochastics, however, these results may turn out to be useful in the study of open quantum systems and their numerical simulations via SSEs since irreducibility enables one to apply powerful results from ergodic theory. Moreover, we want to find the range of solutions to SSE because it can be thought of as the set of reachable (random) states in a continuous measurement.

We first give a new characterisation (Theorem 6) irreducible QMS by a multiple commutator condition looking like the celebrated Lie algebra rank condition (LARC) and Hörmander condition.

Then we prove our main result (Theorem 8); a QMS is irreducible if and only if the associated diffusion processes via SSEs are total in the Hilbert space of the system.

Moreover, we study the relationship with other properties such as accessibility, the Lie algebra rank condition, and irreducibility. We prove that irreducibility of a QMS is, in general, a weaker property than irreducibility of diffusions solving the associated SSEs. It is also weaker of the LARC and Hörmander condition, although equivalent for some important classes of semigroups. We thus find a quantum version of these classical conditions.

The paper is organized as follows. In Section 2 we present a short account of the main results on irreducible QMSs and describe the support projection at time t of a state evolving under the action of a QMS together with the characterisation based on the multiple commutator condition (Theorem 6). In Section 3 we introduce SSE, driven by independent Brownian motions, and prove (Theorem 8) that a QMS is irreducible if and only if associated diffusion processes are total in the Hilbert space of the system. Then we turn our attention to the range of diffusion processes showing by simple argument and examples (Example 1) that we cannot expect these diffusion processes to be irreducible. In Section 4 we discuss the Stroock and Varadhan support theorem and the LARC condition showing (Theorem 11 and counter example 3) that our multiple commutator condition, equivalent to irreducibility of a QMS, is indeed weaker that the LARC condition. Finally, in Section 5, we show that both the LARC condition and the multiple commutator condition hold for generic QMSs.

2. Irreducible QMS

Let $\mathsf{h} = \mathbb{C}^d$ and let $\mathcal{T}$ be the QMS on the algebra $M_d(\mathbb{C})$ of $d \times d$ matrices generated by

$$\begin{aligned} \mathcal{L}(x) &= \mathrm{i}[H, x] + \frac{1}{2}\sum_{\ell=1}^{m}(-L_\ell^* L_\ell x + 2L_\ell^* x L_\ell - x L_\ell^* L_\ell) \\ &= G^* x + \sum_{\ell=1}^{m} L_\ell^* x L_\ell + xG \end{aligned} \tag{1}$$

where $L_1, \ldots, L_m, H \in M_d(\mathbb{C})$ with H self-adjoint and

$$G = -\frac{1}{2}\sum_{\ell=1}^{m} L_\ell^* L_\ell - \mathrm{i}H.$$

The representation (1) of the generator $\mathcal{L}$ is called a Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) representation of the generator. It is well-known to exist but it is not unique (see [29] Theorem 30.16 p. 271). In particular, one can always change a GKSL representation by translating the operators L_ℓ by multiples of the identity operator or increasing m and adding operators L_j which are multiples of the identity operator.

A representation with the smallest number of operators L_ℓ, i.e. the minimum m, is called *minimal*. In a minimal GKSL representation of $\mathcal{L}$, matrices $\mathbb{1}, L_1, \ldots, L_m$ are linearly independent ([29] Theorem 30.16 p. 271).

Definition 1 — For each non-zero $\xi \in \mathsf{h}$ we denote by $\mathcal{S}(\xi)$ the linear span of all vectors of the form

$$\xi,\ \delta_G^{n_1}(L_{\ell_1})\delta_G^{n_2}(L_{\ell_2})\cdots\delta_G^{n_k}(L_{\ell_k})\xi, \tag{2}$$

where $k \geq 1$, $n_1, \ldots, n_k \geq 0$ and $1 \leq \ell_1, \ldots, \ell_k \leq m$ and δ_G^n is defined recursively by $\delta_G^0(A) = A$, $\delta_G^{n+1}(A) = [G, \delta_G^n(A)]$.

The following results are proved in [15] Theorem 6 and 7 (see also their extensions in [20]).

***Theorem* 2** — *Let $(\mathcal{T}_t)_{t\geq 0}$ be a norm continuous QMS on $\mathcal{B}(\mathsf{h})$ with generator $\mathcal{L}$ as in (1) and let $P_t = \mathrm{e}^{tG}$. For all unit vector $\xi \in \mathsf{h}$ and all $t \geq 0$, the support projection of the state $\mathcal{T}_{*t}(|\xi\rangle\langle\xi|)$ is the closed linear span of $P_t\xi$ and vectors*

$$P_{s_1} L_{\ell_1} P_{s_2-s_1} L_{\ell_2} P_{s_3-s_2} \ldots P_{s_n-s_{n-1}} L_{\ell_n} P_{t-s_n}\xi \tag{3}$$

for all $n \geq 1$, $0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq t$ and $\ell_1, \ldots, \ell_n \geq 1$.

A simple argument based on the analyticity of the semigroup $(P_t)_{t\geq 0}$ (see [15] Theorem 7) leads to the following simpler characterisation of the support of $\mathcal{T}_{*t}(|\xi\rangle\langle\xi|)$ as the linear manifold $P_t\,\mathcal{S}(\xi)$.

***Theorem* 3** — *Let $(\mathcal{T}_t)_{t\geq 0}$ be a norm continuous QMS on $\mathcal{B}(\mathsf{h})$ with generator $\mathcal{L}$ as in (1) and let $P_t = \mathrm{e}^{tG}$. For all unit vector $\xi \in \mathsf{h}$ and all $t > 0$, the support projection of the state $\mathcal{T}_{*t}(|\xi\rangle\langle\xi|)$ is the linear manifold $P_t\,\mathcal{S}(\xi)$ where $\mathcal{S}(\xi)$ is the linear span of vectors (2).*

Definition 4 — A QMS $\mathcal{T}$ is *irreducible* if there exists no non-trivial subharmonic projection p ($\mathcal{T}_t(p) \geq p$ for all $t \geq 0$).

In an equivalent way, a QMS $\mathcal{T}$ is irreducible there exists no non-trivial common invariant subspace for the operators G and L_ℓ ([12] Theorem III.1).

Let $\mathcal{F}(\mathcal{T})$ be the vector space of fixed points of $\mathcal{T}$

$$\mathcal{F}(\mathcal{T}) = \{\, x \mid \mathcal{T}_t(x) = x,\, \forall t \geq 0 \,\}.$$

It is well-known that, if $\mathcal{T}$ has a faithful invariant state, then $\mathcal{F}(\mathcal{T})$ is a sub-*-algebra of $M_d(\mathbb{C})$.

Let $\mathcal{N}(\mathcal{T})$ be the decoherence free algebra of $\mathcal{T}$

$$\mathcal{N}(\mathcal{T}) = \{\, x \mid \mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x),\, \mathcal{T}_t(xx^*) = \mathcal{T}_t(x)\mathcal{T}_t(x^*)\ \forall t \geq 0 \,\}.$$

We refer to [14] for properties of $\mathcal{N}(\mathcal{T})$. Both $\mathcal{F}(\mathcal{T})$ and $\mathcal{N}(\mathcal{T})$ contain scalar multiples of the identity matrix $\mathbb{1}$; we say that they are trivial if they do not contain other matrices, i.e. they coincide with $\mathbb{C}\mathbb{1}$.

We recall the following result on irreducible QMSs.

***Theorem* 5** — *An irreducible QMS $\mathcal{T}$ on $M_d(\mathbb{C})$ admits a unique faithful invariant state. Its fixed point set $\mathcal{F}(\mathcal{T})$ and decoherence free subalgebra $\mathcal{N}(\mathcal{T})$ are trivial.*

PROOF : By finite-dimensionality, the QMS $\mathcal{T}$ admits an invariant state ρ and its support projection is subharmonic (see e.g. [12] Theorem II.1) and non-zero. Thus it must coincide with $\mathbb{1}$ because $\mathcal{T}$ is irreducible and so ρ is faithful.

As a well-known consequence, $\mathcal{F}(\mathcal{T})$ is a *-subalgebra of $M_d(\mathbb{C})$ because, for any $x \in \mathcal{F}(\mathcal{T})$, by complete positivity, we have $\mathcal{T}_t(x^*x) \geq \mathcal{T}_t(x^*)\mathcal{T}_t(x) = x^*x$ and, by the invariance of ρ, we have

$$\mathrm{tr}\left(\rho\left(\mathcal{T}_t(x^*x) - \mathcal{T}_t(x^*)\mathcal{T}_t(x)\right)\right) = \mathrm{tr}\left(\rho\left(\mathcal{T}_t(x^*x) - x^*x\right)\right) = 0$$

Thus $x \in \mathcal{N}(\mathcal{T})$. Since the algebra $\mathcal{N}(\mathcal{T})$ is trivial by [10] Proposition 14, because $\mathcal{T}$ is irreducible, also $\mathcal{F}(\mathcal{T})$ is trivial.

We finally show that ρ is the unique invariant state of $\mathcal{T}$. Indeed, if it is not, then the dimension of the kernel of $\mathcal{L}_*$ is at least 2 and so, since $\ker(\mathcal{L}_*)$ is the orthogonal space of the range of $\mathcal{L}$, it follows that the dimension of $R(\mathcal{L})$ is not bigger than $d^2 - 2$. This implies that the dimension of the kernel of $\mathcal{L}$ is at least 2 contradicting the triviality of $\mathcal{F}(\mathcal{T})$.

The following new characterisation of irreducible QMS can be regarded as the starting point of our analysis.

***Theorem* 6** — *The following are equivalent:*

(1) *the QMS $\mathcal{T}$ is irreducible,*

(2) $\mathcal{S}(\xi) = \mathsf{h}$ *for all non-zero* $\xi \in \mathsf{h}$,

PROOF : (2) $\Rightarrow$ (1). If the QMS $\mathcal{T}$ is not irreducible there exists a nontrivial subharmonic projection p. The subspace determined by p is invariant under G and all the L_ℓ by Theorem III.1 of [12]. Therefore for all non-zero ξ in the range of p, $\mathcal{S}(\xi)$ is contained in the range of p.

(1) $\Rightarrow$ (2). If the QMS $\mathcal{T}$ is irreducible it admits a unique faithful invariant state ρ by Theorem 5. Moreover, since $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$ (indeed both are trivial), by result due to Frigerio and Verri (see Theorem 3.3 of [18]) for any unit vector ξ in $\mathbb{C}^d$

$$\lim_{t\to\infty} \mathcal{T}_{*t}(|\xi\rangle\langle\xi|) = \rho.$$

By finite dimensionality, it follows that the state $\mathcal{T}_{*t}(|\xi\rangle\langle\xi|)$ is faithful for all t bigger than some $t_0 < +\infty$. Hence $P_t S(\xi) = \mathbb{C}^d$ for all $t > t_0$ by Theorem 3, and so $S(\xi) = \mathbb{C}^d$ by the invertibility of P_t. □

The previous result yields an algebraic condition that implies some qualitative property of a QMS (see [13, 18, 34] for related algebraic conditions implying other properties).

3. Stochastic Schrödinger Equations

A linear SSE for the QMS generated by (1) is the stochastic differential equation

$$dX_t(\xi) = GX_t(\xi)dt + \sum_{\ell=1}^{m} L_\ell X_t(\xi)dW_t^\ell, \qquad X_0(\xi) = \xi \tag{4}$$

where $\xi \in \mathbb{C}^d$ and $W^1, \ldots, W^m$ are independent real-valued independent Wiener processes on a filtered complete probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$.

It is well-known that (see e.g. [4] Theorem 2.11 p.29)

$$\langle \eta, \mathcal{T}_t(a)\xi\rangle = \mathbb{E}\left[\langle X_t(\eta), aX_t(\xi)\rangle\right], \qquad \mathcal{T}_{*t}\left(|\xi\rangle\langle\eta|\right) = \mathbb{E}\left[|X_t(\xi)\rangle\langle X_t(\eta)|\right].$$

Moreover, since the operators G, L_ℓ are bounded and $\mathcal{L}(\mathbb{1}) = 0$, by [4] Theorem 2.11, we have also

$$\mathbb{E}\left[\|X_t(\xi)\|^2\right] = \|\xi\|^2. \tag{5}$$

The initial condition ξ will be always assumed to be non-zero. In this way we associate with a generator $\mathcal{L}$ a diffusion process on $\mathbb{C}^d - \{0\}$. In order to investigate the relationship between irreducibility the QMS generated by $\mathcal{L}$ and the diffusion process (4) we start with the following result.

Proposition 7 — The random variable $X_t(\xi)$ admits the chaos expansion

$$\begin{aligned} X_t(\xi) &= P_t\,\xi \\ &+ \sum_{n\geq 1, \ell_1,\ldots,\ell_n\geq 1} \int_0^t dW_{s_1}^{\ell_1} \cdot\cdot \int_0^{s_{n-1}} dW_{s_n}^{\ell_n} P_{t-s_1}L_{\ell_1}P_{s_1-s_2}\cdots P_{s_{n-1}-s_n}L_{\ell_n}P_{s_n}\xi. \end{aligned} \tag{6}$$

Proof : Recall that $P_t = \mathrm{e}^{tG}$, i.e., $(P_t)_{t\geq 0}$ is the contraction semigroup generated by G. For all $t > 0$ and $s \in]0, t[$ we have

$$dP_{t-s}X_s(\xi) = \sum_{\ell\geq 1} P_{t-s}L_\ell X_s(\xi)dW_s^\ell$$

so that, integrating on $[0, t]$,

$$X_t(\xi) = P_t\,\xi + \sum_{\ell\geq 1}\int_0^t P_{t-s}L_\ell X_s(\xi)dW_s^\ell.$$

Iterating this formula n times we can write $X_t(\xi)$ as the sum of $P_t\xi$ plus

$$\sum_{k=1}^{n}\sum_{\ell_1,\dots,\ell_k\geq 1}\int_0^t dW^{\ell_1}_{s_1}\cdots\int_0^{s_{k-1}}dW^{\ell_k}_{s_k}P_{t-s_1}L_{\ell_1}\cdots P_{s_{k-1}-s_k}L_{\ell_k}P_{s_k}\xi$$

and a remainder $R_n(\xi)$ given by

$$\sum_{\ell_1,\dots,\ell_{n+1}\geq 1}\int_0^t dW^{\ell_1}_{s_1}\cdots\int_0^{s_n}dW^{\ell_n}_{s_{n+1}}P_{t-s_1}L_{\ell_1}\cdots P_{s_n-s_{n+1}}L_{\ell_{n+1}}X_{s_{n+1}}(\xi).$$

Therefore, putting

$$c:=\max_{1\leq\ell\leq m}\|L_\ell\|,$$

from we can write $\mathbb{E}\left[\,\|R_n(\xi)\|^2\,\right]$ as

$$\begin{aligned}
&\sum_{\ell_1,\dots,\ell_{n+1}\geq 1}\int_0^t ds_1\cdots\int_0^{s_n}ds_{n+1}\,\mathbb{E}\left[\,\left\|P_{t-s_1}L_{\ell_1}\cdots L_{\ell_{n+1}}X_{s_{n+1}}(\xi)\right\|^2\,\right]\\
&\leq c^{n+1}m^{n+1}\sum_{\ell_1,\dots,\ell_{n+1}\geq 1}\int_0^t ds_1\cdots\int_0^{s_n}ds_{n+1}\\
&=\frac{(c\,m)^{n+1}\,t^{n+1}}{(n+1)!}.
\end{aligned}$$

The conclusion follows letting n go to infinity.

The solution $X_t(\xi)$ to (4), for $t>0$ and $\xi\in\mathbb{C}^d$ fixed, defines a family of random vectors on Ω. We recall that the essential range of a $\mathbb{C}^d$-valued random variable Y is the set of all $u\in\mathbb{C}^d$ such that $\mathbb{P}\{Y\in\mathcal{U}\}>0$ for each neighbourhood $\mathcal{U}$ of u.

***Theorem* 8** — *Let $X_t(\xi)_{t\geq 0}$ be the unique solution to (4). The following are equivalent:*

1. $S(\psi)=\mathbb{C}^d$ *for all* $\psi\in\mathbb{C}^d-\{0\}$,
2. *for all $t>0$ and all $\xi\in\mathbb{C}^d-\{0\}$ the essential range of $X_t(\xi)$ is total in $\mathbb{C}^d$.*

PROOF : By the chaos expansion of Proposition 7, for all $v\in\mathbb{C}^d$ we have

$$\begin{aligned}
&\mathbb{E}\left[\,|\langle v,X_t(\xi)\rangle|^2\,\right]=\|P_t\,\xi\|^2\\
&+\sum_{n\geq 1,\ell_1,\dots,\ell_n\geq 1}\int_0^t ds_1\dots\int_0^{s_{n-1}}ds_n\,|\langle v,P_{t-s_1}L_{\ell_1}P_{s_1-s_2}\cdots L_{\ell_n}P_{s_n}\xi\rangle|^2\,.
\end{aligned}$$

It follows that $\mathbb{P}\{\langle v,X_t(\xi)\rangle\neq 0\}=0$ if and only if v is orthogonal to all vectors $P_t\xi$, $P_{t-s_1}L_{\ell_1}P_{s_1-s_2}\cdots L_{\ell_n}P_{s_n}\xi$ with $1\leq\ell_1,\dots,\ell_n\leq m$, $0\leq s_n\ldots\leq s_1\leq t$.

Now, if 2 holds, $\mathbb{P}\{\langle v, X_t(\xi)\rangle \neq 0\} > 0$ for all $v \in \mathbb{C}^d - \{0\}$ and v is not orthogonal to all vectors $P_t\xi$, $P_{t-s_1}L_{\ell_1}P_{s_1-s_2}\cdots L_{\ell_n}P_{s_n}\xi$ with $1 \leq \ell_1, \ldots, \ell_n \leq m$, $0 \leq s_n \ldots \leq s_1 \leq t$. Thus no non-zero vector is orthogonal to $P_t S(\xi)$ by Theorem 3. Since P_t is invertible, and $S(\xi)$ is a subspace of $\mathbb{C}^d$, it turns out that $S(\xi) = \mathbb{C}^d$.

Conversely, if 1 holds, then, for all non-zero $v \in \mathbb{C}^d$, the expectation of the random variable $|\langle v, X_t(\xi)\rangle|^2$ is strictly positive by Theorems 2 and 3 so that $X_t(\xi)$ is not orthogonal to v on an event of strictly positive probability.

We can now proceed to study the relationship between irreducibility of a QMS and diffusion process $(X_t)_{t\geq 0}$ solving the associated SSE first recalling the usual definition.

Definition 9 — The diffusion process $(X_t)_{t\geq 0}$ is called *irreducible* on $\mathbb{C}^d - \{0\}$ if, for all $X_0 = \xi \in \mathbb{C}^d - \{0\}$ and all open set $O \subseteq \mathbb{C}^d - \{0\}$, there exists $t > 0$ such that

$$\mathbb{P}\{X_t(\xi) \in O\} > 0.$$

Clearly, even if the QMS associated with G, L_ℓ is irreducible, the diffusion process $(X_t(\xi))_{t\geq 0}$ in $\mathbb{C}^d - \{0\}$ may not be. This is the case, for instance, when the non-zero vector ξ has *real* components and matrices G, L_ℓ have *real* entries.

This is not just a matter of phase and length of vectors $X_t(\xi)$ because, if $d > 2$, the diffusion process takes values in a manifold of *real* dimension d which is strictly smaller than the real dimension $2(d-1)$ of the complex projective space $\mathbb{CP}^{d-1}$ obtained on taking the quotient with respect to a complex scalar.

Moreover, these situations indicate that irreducibility of the QMS associated with G, L_ℓ is much weaker than irreducibility of

1. the diffusion process solving (4) in $\mathbb{C}^d - \{0\}$,
2. the diffusion process on the unit sphere of $\mathbb{C}^d$ obtained by normalization of vectors $X_t(\xi)$,
3. the diffusion process on the complex projective space $\mathbb{CP}^{d-1}$ associated with the solution of (4).

We finish this section by showing another example illustrating that irreducibility of (4) is stronger than irreducibility QMS.

Example 1 : Suppose that the operators L_ℓ are anti-selfadjoint, namely $L_\ell^* = -L_\ell$ for $\ell = 1, \ldots, m$. Then, by the Ito formula,

$$\begin{aligned} d\left\langle X_t(\xi), X_t(\xi)\right\rangle &= \Big\langle X_t(\xi), (G^* + \sum_\ell L_\ell^* L_\ell + G) X_t(\xi)\Big\rangle dt \\ &+ \sum_\ell \left\langle X_t(\xi), (L_\ell^* + L_\ell) X_t(\xi)\right\rangle dW_t^\ell = 0. \end{aligned}$$

It follows that $\|X_t(\xi)\| = \|\xi\|$ for all $t > 0$, namely the range of the diffusion process is contained in the unit sphere of $\mathbb{C}^d$.

It is not hard, however, to produce an irreducible QMS with anti-selfadjoint operators L_ℓ. We may consider, for instance $d = 2, m = 1$, $L = \mathrm{i}\sigma_2$, $H = \sigma_3$ where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this case we have

$$\delta_G(L) = \delta_{-\mathrm{i}H}(L) = [-\mathrm{i}\sigma_3, \sigma_2] = 2\sigma_1, \quad \delta_G(L)L = 2\mathrm{i}\sigma_3$$

so that $S(\xi) = \mathbb{C}^2$ for all $\xi \in \mathbb{C}^2 - \{0\}$.

Further examples will be discussed in the next section.

4. The Range of Solutions to Linear SSE

Useful tools from control theory are available to study the range of solutions to SSE (see the survey [31] Sect. 6). For equation (4) let us replace the Wiener processes W^ℓ by piecewise polygonal approximations

$$W_t^{\ell,n} = W_{k/n}^{\ell,n} + (nt - k)\left(W_{(k+1)/n}^{\ell,n} - W_{k/n}^{\ell,n}\right), \qquad \frac{k}{n} \le t \le \frac{k+1}{n}.$$

A celebrated result by Stroock and Varadhan [35] shows that, for any $d \times d$ matrix $\widetilde{G}$, the solutions of

$$dX_t^{(n)}(\xi) = \widetilde{G} X_t^{(n)}(\xi) dt + \sum_{\ell=1}^m L_\ell X_t^{(n)}(\xi) dW_t^{\ell,n}, \qquad X_0^{(n)}(\xi) = \xi \tag{7}$$

converge almost surely to $X_t(\xi)$ uniformly in t on any compact interval to the solution of

$$dX_t(\xi) = \widetilde{G} X_t(\xi) dt + \sum_{\ell=1}^m L_\ell X_t(\xi) \circ dW_t^\ell, \qquad X_0(\xi) = \xi \tag{8}$$

where $\circ$ denotes the Stratonovich integral, namely, in terms of the Ito integral, to the solution of

$$dX_t(\xi) = \left(\widetilde{G} + \frac{1}{2}\sum_{\ell=1}^m L_\ell^2\right) X_t(\xi)dt + \sum_{\ell=1}^m L_\ell X_t(\xi) dW_t^\ell, \quad X_0(\xi) = \xi.$$

Thus, choosing

$$\widetilde{G} = G - \frac{1}{2}\sum_{\ell=1}^m L_\ell^2 \tag{9}$$

solutions of (7) converge to solutions of (4).

Equation (8) has the form

$$dx_t = \widetilde{G}x_t dt + \sum_{\ell=1}^m L_\ell x_t u_\ell(t)dt, \qquad x_0 = \xi \tag{10}$$

where u_ℓ are piecewise constant functions. This is an ordinary (non-autonomous) differential equation, functions u_ℓ are controls and (10) is a bilinear control system.

Unfortunately there are no general necessary and sufficient condition for deciding when a bilinear system is controllable.

A well-known necessary condition for controllability ([32] Theorem 2.3) is the *Lie algebra rank condition*.

Definition 10 — The Lie algebra rank condition (LARC) holds if the linear manifold generated by vectors

$$\widetilde{G}\xi,\ L_\ell\xi, [\widetilde{G}, L_\ell]\xi,\ [L_{\ell_1}, L_{\ell_2}]\xi,\ [\widetilde{G}, [\widetilde{G}, L_\ell]]\xi,\ [\widetilde{G}, [L_{\ell_1}, L_{\ell_2}]]\xi, \dots \tag{11}$$

is $\mathbb{C}^d$ for all non zero ξ.

The LARC implies that the control system (10) is *accessible* namely the set of points $(x_s)_{0\le s\le t}$ reachable from ξ with some choice of piecewise constant controls u_ℓ contains a non-empty open set in $\mathbb{C}^d$ for all $t > 0$.

It is worth noticing here that the linear manifold spanned by vectors (11) may depend on the particular choice of the operators G and L_ℓ in the GKSL representation of the generator $\mathcal{L}$ as shows the next example. For this reason, from now on, we consider only *minimal* GKSL representations of $\mathcal{L}$.

Example 2 : Let $\mathsf{h} = \mathbb{C}^2$, let H be a self-adjoint matrix which is not a multiple of the identity $\mathbb{1}$, $L_\ell = 0$ for all $\ell \ge 1$ and let $\mathcal{T}$ the QMS on $\mathbb{C}^2$ defined by $\mathcal{T}_t(x) = \mathrm{e}^{\mathrm{i}tH} x \mathrm{e}^{-\mathrm{i}tH}$.

The LARC does not hold because the dimension of the linear manifold (11) is at most 1 for all non zero $\xi \in \mathbb{C}^2$. However, if we consider a GKSL representation of the generator $\mathcal{L}(x) = \mathrm{i}[H, x]$ with operators $L_\ell = 0$ for all $\ell > 1$,

$$L_1 = z\mathbb{1}, \qquad G = -\frac{|z|^2}{2}\mathbb{1} - \mathrm{i}H$$

for some non zero complex number z, the linear manifold (11) contains the vectors ξ and $H\xi$. It follows that the LARC condition holds for all vectors ξ which are not eigenvectors of H.

Clearly $\mathcal{T}$ is not irreducible because any eigenprojection of H is an harmonic projection for $\mathcal{T}$.

***Theorem* 11** — *If the LARC holds for some minimal GKSL, then the QMS generated by $\mathcal{L}$ is irreducible.*

PROOF : Indeed, if the QMS generated by $\mathcal{L}$ is not irreducible, for any GKSL representation of $\mathcal{L}$ by means of operators G, L_ℓ, by Theorem III.1 of [12], there exists a non-trivial subspace V invariant for G and all L_ℓ. This subspace is also invariant for the operators $\widetilde{G}$ and L_ℓ because of (9). It follows that, for all $\xi \in \mathsf{V}$, the linear manifold generated by vectors (11) is contained in V.

It is worth noticing here that vectors (11) may *not* be contained in $S(\xi)$. Indeed, there is no reason why $\widetilde{G}\xi$ should be contained in $S(\xi)$.

The converse of Theorem 11 is not true, indeed, there exist irreducible QMSs with a given GKSL representation of their generator which do not satisfy the LARC condition as shows the following example.

Example 3 : Let $\mathcal{T}$ be the QMS on $\mathcal{B}(\mathbb{C}^3)$ generated by

$$\mathcal{L}(x) = G^*x + L^*xL + xG$$

where L, H are the 3×3 matrices

$$L = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{bmatrix} \tag{12}$$

and

$$G = -\frac{1}{2}L^*L - \mathrm{i}H = \begin{bmatrix} -1/2 & 0 & -1 \\ 0 & -1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Since L is anti-self-adjoint its invariant subspaces are generated by eigenvectors $(1, \mathrm{i}, 0)$, $(-1, \mathrm{i}, 0), (0, 0, 1)$ of $-\mathrm{i}L$. One immediately checks that no one-dimensional or two-dimensional subspace generated by these vectors, which is obviously L^2 invariant, is H invariant, therefore it is not G invariant and the QMS $\mathcal{T}$ is irreducible.

Clearly

$$\widetilde{G} = G - \frac{1}{2}L^2 = -\mathrm{i}H = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and $[\widetilde{G}, L] = [G, L]$. Straightforward computations yield

$$[\widetilde{G}, L] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

so that, defining

$$X_1 = [\widetilde{G}, L] \qquad X_2 = -\widetilde{G}, \qquad X_3 = -L$$

we find a basis of the Lie algebra of the rotation group $SO(3)$, which satisfies the commutation relations

$$[X_1, X_2] = X_3, \qquad [X_2, X_3] = X_1, \qquad [X_3, X_1] = X_2.$$

Consequently iterated commutators of X_1, X_2, X_3 do not give other operators. Let $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ be the canonical orthonormal basis of $\mathbb{C}^3$. It is now immediate to check that

$$X_1 e_2 = e_3, \quad X_1 e_3 = -e_2, \quad X_2 e_1 = -e_3$$
$$X_2 e_3 = e_1, \quad X_3 e_1 = e_2, \quad X_3 e_2 = -e_1$$

and $X_k e_k = 0$ for $k = 1, 2, 3$. It follows that the linear manifold $S(e_k)$ is two-dimensional for all $k = 1, 2, 3$ and the (11) condition does not hold.

5. Generic QMSs

In this section we show that irreducibility of generic QMSs is equivalent to the LARC condition with respect to the natural GKSL representation of $\mathcal{L}$.

Generic QMSs arise in the stochastic limit of a open discrete quantum system with generic Hamiltonian, interacting with Gaussian fields through a dipole type interaction (see [1, 8, 9]). Here, as in the previous sections, the system space is finite-dimensional $\mathsf{h} = \mathbb{C}^{\,d}$ with orthonormal basis $(e_k)_{k\le j\le d}$. The operators L_ℓ, in this case labeled by a double index (ℓ,k) with $\ell \neq k$, are

$$L_{\ell k} = \gamma_{\ell k}^{1/2} \left| e_k \right\rangle \left\langle e_\ell \right|$$

where are $\gamma_{\ell k} \geq 0$ positive constants and the Hamiltonian H is a self-adjoint operator diagonal in the given basis whose explicit form is not needed here. The generator $\mathcal{L}$ is

$$\mathcal{L}(x) = \mathrm{i}[H,x] + \frac{1}{2}\sum_{\ell\neq k}\left(-L_{\ell k}^* L_{\ell k} x + 2L_{\ell k}^* x L_{\ell k} - x L_{\ell k}^* L_{\ell k}\right). \tag{13}$$

The converse of Theorem 11 holds for generic QMS.

***Theorem* 12** — *A generic QMS is irreducible if and only if the LARC condition holds.*

Proof : The restriction of $\mathcal{L}$ to the algebra of diagonal matrices coincides with the generator of a time continuous classical Markov chain with states $1,\ldots,d$ and jump rates $\gamma_{\ell k}$ (see [1, 8]). It is easy to see as in [11] that the QMS generated by (13) is irreducible if and only if the classical Markov chain is irreducible, i.e. for all pair of states ℓ, m with $\ell \neq k$, there exist $n \geq 1$ and states $j_1,\ldots,j_n$ such that

$$\gamma_{\ell j_1}\gamma_{j_1 j_2}\cdots\gamma_{j_n k} > 0. \tag{14}$$

There is no loss of generality in assuming that each j_i is not equal to any $j_1,\ldots,j_{i-1}$, together with $j_i \neq \ell$. Indeed, if $j_i = j_{i'}$, we can delete all states $j_{i'},\ldots,j_{i-1}$ in the sequence of transitions $\ell \to j_1 \to \ldots \to j_n \to k$. Consequently, we find the commutation relation

$$[L_{j_i\, j_{i+1}}, L_{j_{i-1}\, j_i}] = L_{j_i\, j_{i+1}} L_{j_{i-1}\, j_i} = \left(\gamma_{j_{i-1} j_i}\gamma_{j_i j_{i+1}}\right)^{1/2} \left| e_{j_{i+1}} \right\rangle \left\langle e_{j_{i-1}} \right|$$

and compute the iterated commutator

$$[L_{j_n k}, [L_{j_{n-1} j_n}, \ldots]] = \left(\gamma_{\ell j_1}\gamma_{j_1 j_2}\cdots\gamma_{j_n k}\right)^{1/2} \left| e_k \right\rangle \left\langle e_\ell \right|.$$

For all non-zero $\xi = \sum_{1\le j\le d} \xi_j e_j \in \mathbb{C}^d$, choose an ℓ such that $\xi_\ell \neq 0$ and note that

$$[L_{j_n k}, [L_{j_{n-1} j_n}, \ldots]]\xi = \xi_\ell \left(\gamma_{\ell j_1}\gamma_{j_1 j_2}\cdots\gamma_{j_n k}\right)^{1/2} e_k.$$

It follows that, if the QMS is irreducible, i.e. (14) holds, then also the LARC condition holds. This, together with Theorem 11, implies that irreducibility and the LARC condition are equivalent for generic QMSs.

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