FINITE ELEMENT HISTORICAL DEFORMATION ANALYSIS IN PIECEWISE LINEAR PLASTICITY BY MATHEMATICAL PROGRAMMING

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SUMMARY

When loads increase proportionally beyond the elastic limit in the presence of elastic-plastic piecewise-linear constitutive laws, the problem of finding the whole evolution of the plastic strain and displacements of structures was recently shown to be amenable to a parametric linear complementarity problem (PLCP) in which the parameter is represented by the load factor, the matrix is symmetric positive definite or at least semi-definite (for perfect plasticity) and the variables with a direct mechanical meaning are the plastic multipliers. With reference to plane trusses and frames with elastic-plastic linear work-hardening material behaviour numerical solutions were also fairly efficiently obtained using a recent mathematical programming algorithm (due to R. W. Cottle) which is able to provide the whole deformation history of the structure and, at the same time, to rule out local unloadings along the given proportional loading process by means of "a priori" checks carried out before each pivotal step of the procedure. Hence it becomes possible to use the holonomic (reversible, path-independent) constitutive laws in finite terms and to benefit by all the relevant numerical and computational advantages despite the non-holonomic nature of plastic behaviour. An extension to the evolution of plastic deformations under stepwise proportional loads, allowing for irreversibility from step to step, was also pointed out.

In the present paper the method of solution is re-examined in view to overcome an important drawback of the algorithm deriving from the size of PLCP fully populated matrix when structural problems with large number of variables are considered and, consequently, the updating, the storing or, generally, the handling of the current tableau may become prohibitive. The aim is achieved through the mechanical interpretation of some phases of the algorithm, which leads to the characterization, for each pivotal step, of an elastic structure corresponding to the actual updated tableau. This allows to avoid the evaluation and updating of the whole PLCP matrix and allows to translate all the mathematical programming algorithm into a finite termination procedure which provides the complete deformation history of the structure along the given loading path through the elastic analysis of structures which are suitable modifications of the original one. This is shown to lead to a dramatic reduction of the variables to be handled and to a substantial simplification from the numerical and computational standpoint. The extension of the procedure to the historical deformation analysis under sequence of individually proportional loadings is also pointed out.

Finally, the method is applied to several problems for some of which the solution is available by other iterative asymptotical or with finite termination solution procedures, so that comparisons become possible and are made.

1. Introduction

The historical deformation analysis problem, i.e. the determination of the whole evolution of the plastic strain and displacement fields for loads increasing beyond the elastic limit, has been recently subject to several studies. Different solution algorithms have been suggested instead of the trivial (but costly and laborious) application of the step by step solution process. In particular, Hodge |1| proposed a method based on a combined use of the static and kinematic minimum principles of the elastic-plastic rates by an application generalizing a three-bar truss example considered by Drucker in |2|; in |3| and |4|, on the basis of a finite element piecewiselinear plasticity approach, the same problem was shown to be amenable for beams and frames under a proportional (or a "stepwise proportional") loading process to a parametric linear complementarity problem (PLCP) and was solved with a recent mathematical programming algorithm (due to Cottle |5|).

The interest of the procedure proposed for improving the numerical efficiency of the last method in the presence of large size problems rests on the peculiar mathematical programming technique from which it is derived, which is able to foresee or to rule out the presence of local unloadings along the prescribed loading process and provides at the same time the complete deformation history of the structure. Under a proportional loading path this allows to use the holonomic (reversible, path independent) finite constitutive laws; the same is for a stepwise proportional loading path, when the non-holonomic nature of plastic behaviour is fully allowed for from step to step, even if not within each step, and the multistage technique |6| |7| is used. For these aspects the procedure proposed herein represent an improvement in terms of computational and numerical efficiency with respect the similar quadratic programming multistage approach |7| while with respect to well known iterative asymptotical solution procedure (initial stress, initial strain, etc.) it is characterized as a numerical technique with finite termination and known convergence property.

Notation: underlined symbols denote matrices and column vectors, $\underline{0}$ is a matrix or vector whose entries are zeros, a dot means time derivation, a ~ (tilde) means transposition. A vector inequality applies to each pair of corresponding components.

2. Historical analysis under a proportional loading process.

Denoting with $\underline{P}_o$ the n-vector of a basic load distribution, a proportional loading process is defined by the relations

$$\underline{P} = \mu(t)\underline{P}_o \quad , \quad \mu \geq 0 \quad , \quad \dot{\mu} = \frac{d\mu}{dt} \geq 0 \quad , \quad \mu(0) = 0 \tag{1}$$

where the load factor μ is a monotonously increasing function of the time t. The problem consist in determining the whole evolution of stresses and strains as functions of $\mu(t)$ when loads go beyond the elastic limit of the structure.

2.1 Element piecewise linear constitutive laws.

Reference is made to continua discretized in m finite elements with piecewise linearized yield surface and linearly work-hardening interacting planes. Let $\underline{Q}$ and $\underline{q}$ be c-vectors of element generalized stresses and strains corresponding to each other in the sense that the product $\tilde{\underline{Q}}\,\dot{\underline{q}}\,dt$ represents the first order work on the structural element considered. The vector $\underline{Q}$ is assumed as self-equilibrated and the component q_i of $\underline{q}$ unaffected by rigid motions. If r is the number of planes of the element yield surface in the space of superimposed stresses and strains of fig.1,the element piecewise linear non-holonomic laws may be described as follows(see fig.1):

$$\underline{q} = \underline{e} + \underline{p} \quad , \quad \underline{Q} = \underline{E}\,\underline{e} \quad , \quad \underline{\phi} = \tilde{\underline{N}}\,\underline{Q} - \underline{H}\,\underline{\lambda} - \underline{K} \quad , \quad \underline{\phi} \leq \underline{0} \qquad (2),(3),(4),(5)$$

$$\underline{p} = \underline{N}\,\underline{\lambda} \quad , \quad \dot{\underline{\lambda}} \geq \underline{0} \quad , \quad \tilde{\underline{\phi}}\,\dot{\underline{\lambda}} = 0 \quad , \quad \tilde{\dot{\underline{\phi}}}\,\dot{\underline{\lambda}} = 0 \qquad (6),(7),(8),(9)$$

where eq.(2) subdivides $\underline{q}$ in the elastic and plastic strains $\underline{e}$ and $\underline{p}$ respectively. Matrix $\underline{E}$ in eq.(3) is the element elastic stiffness matrix,assumed symmetric definite positive.Eq.(4) defines the r-vector $\underline{\phi}$ of the yield functions(or plastic potential) as linear both in the stress vector $\underline{Q}$ and in the vector $\underline{\lambda}$ of the plastic multipliers which measure the contribution to $\underline{p}$ of each yield plane during the previous history of plastic deformations. $\underline{N}$,$\underline{H}$,$\underline{K}$ are constant matrices and vector; $\underline{N}$ is the (r x c)-matrix whose columns are the $\underline{N}_j$ outward normal unit c-vectors of the r planes of the yield polyhedron,$\underline{H}$ is the work-hardening matrix which governs the translation of the yield planes due to plastic flow and $\underline{K}$ is the r-vector of the non negative distances of the planes from the origin. Eq.(5) and (4) define the yield surface.Eqs. (6) to (9) express the normality of flow rule :eqs.(6) and (7) require that the strain vector $\underline{p}$ be a linear combination,through the plastic multiplier vector $\underline{\lambda}$,of the outward normal unit vectors $\underline{N}_j$ (j=1,..r)and that the contribution to $\dot{\underline{p}}$ of each yield plane be directed as the outwârd normal to that plane. The fact that only the yield planes which contain the current stress point $\underline{Q}$ can contribute to $\underline{p}$ (or be activated) is expressed by the orthogonality condition (8),while eq.(9) states that the activation of a plane(i.e. $\dot{\lambda}_j > 0$) and the local unloading ($\dot{\phi}_j \leq 0$) on the same plane are mutually exclusive facts.

Eqs.(2) to (9) describe a broad class of piecewise linearized path dependent stress-strain laws with translating interacting planes once a suitable definition of the matrix $\underline{H}$ is given. Special cases are |8| : (i) perfect plasticity,with $\underline{H} = \underline{0}$, (ii) Koiter hardening rule of independently acting yield modes, when $\underline{H} = \mathrm{diag}|H_i|$, (iii) isotropic hardening,i.e. expansion with unaltered shape (see fig.1), when $\underline{H} = (K_r H_{ss}/K_s)$. In this latter case $\underline{H}$ is symmetric when all K_i are equal or $H_{ii} = h\,K_i^2$ h being a positive constant hardening rate common to all planes, (iv) kinematic hardening,implying a rigid translation of the elastic region with velocity proportional to plastic strain rates,when $\underline{H} = h\,\tilde{\underline{N}}\,\underline{N}$. In the following, matrix $\underline{H}$ is assumed to be symmetric.

The absence of local unloadings during a loading process(i.e. no stress points loosing contact with a plane j after this has been activated) will be referred to as progressive yielding or "regular progression". In the assumption of regular progression it may be observed that $\lambda_j > 0$ implies $\dot{\phi}_j = 0$ for any j and t. Then, $\tilde{\dot{\underline{\phi}}}\,\underline{\lambda} = 0$ for any t and eqs.(7) to (9) change into the eqs. :

$$\underline{\lambda} \geq \underline{0} \quad , \quad \tilde{\underline{\phi}}\,\underline{\lambda} = 0 \qquad (10),(11)$$

which, together with eqs.(2) to (6) represent a new set of constitutive laws (now in finite terms) and will be called piecewise linear holonomic laws corresponding to the non-holonomic ones given by eqs.(2) to (9). They ignore the irreversible nature of plastic deformations but in spite of this, they may be applied just as well when the absence of local unloading phenomena can be stated. Such statement may derive from a simplifying assumption or from a conjecture (frequently acceptable in the presence of proportionally increasing loads, particularly for hardening structures) or finally from a suitable algorithm able to detect " a priori" the absence of local unloading.

2.2 Formulation of the problem as a PLCP.

It can be easily verified that the set of equations (2) to (6) and (10)-(11) may represent in compact form the constitutive laws of all the m finite elements of the structure when suitable supervectors ($\underline{\tilde{\lambda}} = |\tilde{\lambda}_1 \ldots \tilde{\lambda}_m|$, $\underline{\tilde{\phi}} = |\tilde{\lambda}_1 \ldots \tilde{\phi}_m|$, $\underline{\tilde{Q}} = |\tilde{Q}_1 \ldots \tilde{Q}_m|$, etc.) and block diagonal matrices ($\underline{N} = \mathrm{diag}|N_i|$, $\underline{H} = \mathrm{diag}|H_i|$,etc i=1,..m) of the corresponding elementary vectors and matrices pertaining to the individual members are introduced. If $\underline{Q}_o^e$ denotes the stress vector of the linear elastic response of the structure to the basic loads $\underline{P}_o$, and $\underline{Z}$ is a semidefinite negative matrix which transforms an arbitrary set of plastic deformations $\underline{p}$ in the corresponding selfequilibrated linear elastic stress vector $\underline{Q}^s$,the actual stress vector $\underline{Q}$ for a given value of the load factor μ ,is given by:

$$\underline{Q} = \mu\, \underline{Q}_o^e + \underline{Q}^s \quad , \qquad \underline{Q}^s = \underline{Z}\, \underline{p} \qquad (12),(12')$$

and the history of the plastic deformations along the proportional loading process is governed by the relations :

$$\underline{\phi} = \mu\, \underline{\bar{N}}\, \underline{Q}_o^e - (\underline{H} - \underline{\bar{N}}\, \underline{Z}\, \underline{N})\, \underline{\lambda} - \underline{K} \;, \quad \underline{\phi} \leq \underline{0} \;, \quad \underline{\lambda} \geq \underline{0} \quad \underline{\bar{\phi}}\, \underline{\lambda} = 0 \qquad (13),(14),(15),(16)$$

When the load factor μ is regarded as a parameter increasing from 0,eqs.(13) to (16) represent a "parametric linear complementarity problem" (PLCP) in the unknown vector $\underline{\lambda}$ (μ). From this through eqs.(12),(6),(2) the stresses and strain evolution is easily derived.From the linear complementarity programming theory the existence and uniqueness of the solution $\underline{\lambda}$ (μ) for a given parameter μ depends from the definiteness in sign of the matrix $\underline{A}=\underline{H} - \underline{\bar{N}ZN}$. Uniqueness is assured for any load factor μ when $\underline{A}$ is a P-matrix(with all principal subdeterminants strictly positive); this occur in particular when $\underline{A}$ is symmetric positive definite (the symmetric positive definite matrices are a subset of P-matrices),which happens whenever $\underline{H}$ is so too.

Finally,when $\underline{H} = \underline{0}$, $\underline{A}$ becomes semidefinite positive and the solution vector $\underline{\lambda}$ (μ) may not exist or may not be unique. Lack of existence would correspond to a load factor greater than the ultimate load factor of the structure.

2.3 Remarks on Cottle's algorithm.

In fig.2 the main steps of the solution algorithm so far adopted(see |3| and |4|) are shown. Matrix $\underline{B}$ in the initial tableau may be any (n x n)-matrix having linearly independent lexicographically positive rows and $\underline{K}$ as its first column. This condition is satisfied in particular by the identity matrix where the first column(if K_1 is the first positive entry of $\underline{K}$) is $\underline{K}$ and the remaining entries of the first row are zeros. B_j is defined as the j-th row of the matrix. Each loop increases the value of the load factor μ and,through step 5,leads to the corresponding solution(which is given,for each variable λ_j entered into the basis,by $\lambda_j = K_j + \mu_{new} v_j$).The steps 2 and 4 are monotonicity checks before the principal pivoting (step 5). A positive answer to step 2 means that from the actual value μ_{new} on,the solution $\underline{\lambda}$ (μ) will be a monotone function of μ,i.e.,in mechanical terms,that progressive yielding without local unloading is ensured for incresing values of μ up to infinity. This justifies the adoption of the holonomic(in finite terms)piecewise linear constitutive laws (2)-(6), (10)-(11) instead of the nonholonomic ones (2)-(9).A positive answer to step 4 means that the value μ_{old} is the last load factor for which(from zero) monotonicity is ensured and reaching μ_{new} unloadings will appear. The solution obtained from the following step 5 would be incorrect because the non-holonomic constitutive laws (2)-(9) under step wise proportional loading process should be used.

As it may be easily seen,the main drawback of the algorithm derives,from a computational standpoint,from the need to compute and to store the current tableau whose size is twice the already large dimension (r x m,r x m) of A matrix. Besides,often the costly updating of the overal tableau may be a waste. In fact when the solution is monotone the rows and columns of A corresponding to yielded planes,as soon as activated , may be ignored,as the relevant variable remains in the basis in all following steps; on the other hand when the solution is not monotone from a particular μ on ,all the variables corresponding to yield planes not yet activated at that particular μ could have been ignored,as different constitutive laws will have to be used from that μ on.

However these remarks cannot really help in reducing the size od A as it is not possible to foresee in which of the preceding situations the problem in hand will fall. Secondly the number of activated planes at the solution might be close to the number of variables of the problem and no reducing procedure could be adopted.

This means that more drastic alternatives must be considered and the attention be directed towards procedure computationally less sensitive to the number of variables and,if possible,avoiding to compute and store the whole PLCP tableau.

3. A modified stiffnesses method for the historical analysis by PLCP.

The present section aims to find a phisical meaning for matrices and vectors appearing in the initial and the subsequent PLCP tableaux,in order to replace by suitable mechanical operations on the original structure the costly numerical calculations required to transform the tableau.

No attention is devoted to matrix B(except to its first column) as it may be seen that it is very seldom involved in the calculations. Indeed this may happen:

(i) at the initial tableau when some rates K_i/v_i are assumed equal to the minimum K_r/v_r;

(ii) because of the presence of a geometric structural symmetry;

(iii) during the calculations,when two or more yielding planes are activated at the same time.

However,the cases may be avoided by slightly modifying the components of K for case(i) and (ii),and reducing the analysis to 1/m part of the symmetric structure(m being the degree of symmetry). In this way the lexicographic rule of step 3 is simply reduced to find the index r such that:

$$- K_r/v_r = \min \{- K_i/v_i \mid v_i < 0 \} , \qquad (17)$$

which involves only the first column of B. Besides it is worth noting that even without the above modifications the matrix B is nearly always useless. It has been numerically ascertained that usually an arbitrary choice of the pivotal index r among those for which the rate K_i/v_i is minimum leads to the same solution obtained using the whole matrix B.

In fact,after the first choice of r,.no increase of the parameter μ("zero pivoting") was noticed for each of the subsequent choices untill all yielding planes relevant to all indices i were activated.

For the above reasons attention will be paid from now on only to matrix A of the PLCP tableau.

3.1 Mechanical interpretation of the PLCP initial tableau.

Let $\underline{\lambda}_j^u$ indicate the elementary unit vector with all components equal to zero except for $\lambda_j^{uj} = 1$. The j-th column (or row) of the symmetric matrix $\underline{A}$ may then be expressed through eqs.(6),(12') in the form:

$$(\underline{H} - \tilde{\underline{N}}\,\underline{Z}\,\underline{N})_j = (\underline{H} - \tilde{\underline{N}}\,\underline{Z}\,\underline{N})\,\underline{\lambda}_j^u = H_j - \tilde{\underline{N}}\,\underline{Q}^s(\underline{p}_i^u) \qquad \text{where} \qquad (18)$$

$$\underline{p}_i^u = \underline{N}\,\underline{\lambda}_j^u \qquad (19)$$

and $\underline{Q}^s$ represents the vector of the selfequilibrated linear elastic stresses in all the elements of the structure under the unit dislocation vector $\underline{p}_i^u$ given by (19), i being the element index relevant to the j-th component of $\underline{\lambda}$. In other words |10|, from eq.(18), a part from the known vector H_j, the j-th column of $\underline{A}$ may be evaluated through a linear elastic analysis of the structure subject to a unit dislocation. This fact may be of interest if the general phylosophy of generating in each pivotal step the column A_r of the current tableau required for deriving the subsequent one (r being the actual pivotal index) is adopted.

In particular eq.(18) may be applied to the initial tableau, j being the first pivotal index r_1.

In order to use again eq.(18) for the subsequent tableau it would be necessary to know to which new structure the tableau transformed by the first pivoting on the element (r_1, r_1) corresponds. In such case it would just be a question of evaluating the elastic response of the new structure to a unit dislocation vector $\underline{p}_i^u$ given by (19) (j being now the new pivoting index r_2 etc.). So the original structure and, subsequently, the modified ones, would be charged to "store" informations on the current $\underline{A}$ matrix.

These remarks justify the attempt made in the following sections in order to establish some relationship between the PLCP transformed tableau and some suitable modification of the original structure.

3.2 The modified influence matrix $\overline{Z}$.

Let the symmetric semidefinite negative influence matrix $\underline{Z}$ be transformed by a pivoting on a diagonal element r,r. The new $\underline{Z}^1$ matrix will be characterized by the following entries |11|:

$$Z^1_{ij} = Z_{ij} - Z_{ir} Z_{jr} / Z_{rr} \quad , \quad Z^1_{ir} = Z_{ir} / Z_{rr} \qquad \forall\, i,j \neq r \qquad (20),(21)$$

$$Z^1_{rj} = - Z_{rj} / Z_{rr} \quad , \quad Z^1_{rr} = 1 / Z_{rr} \qquad \forall\, j \neq r \qquad (22),(23)$$

Denoting with $\underline{T}_z^r$ the identity matrix with the r-th row substituted by the pivotal row of $\underline{Z}^1$, the relation between $\underline{Z}^1$ and $\underline{Z}$ becomes:

$$\underline{Z}^1 = \tilde{\underline{T}}_z^r\,\underline{Z}\,\underline{T}_z^r + (\tilde{\underline{T}}_z^r - \underline{T}_z^r) \qquad (24)$$

The above relations exclude the possibility for $\underline{Z}^1$ to be still an influence matrix, as the fundamental property of symmetry is lost by virtue of eqs.(21),(22). However, except for the row and column r, the remaining part of $\underline{Z}^1$ is still symmetric semidefinite positive, so that it could represent the influence matrix of some modification of the original structure.

To this end let us confine ourselves, for semplicity, to the analysis of truss structures characterized by a single generalized stress and strain component per element and by (m x m) influence matrices $\underline{Z}$, m being the number of elements of the structure. In this case it is easy to see that when the stiffness $(EA/1)_r$ of a generic element r of the original structure S is modified into a new value $(EA/l)^M_r$, the influence matrix $\bar{\underline{Z}}^1$ of the new structure S^1 has entries related to those of $\underline{Z}$ by the relations:

$$\bar{Z}^1_{ij} = Z_{ij} - Z_{ir}Z_{jr}/ Z_{rr}.(1 - \alpha) \quad , \qquad \bar{Z}^1_{ir} = \bar{Z}^1_{ri} = Z_{ri}.\ \alpha \qquad (25),(26)$$

$$\bar{Z}^1_{rr} = Z_{rr}.\ \alpha \qquad \text{where} \quad \alpha = |1 + Z_{rr}/(EA)_r - Z_{rr}/(EA)^M_r|^{-1} \qquad (27),(28)$$

or,in more compact form, $\underline{O}^r$ being an (m x m)-matrix having a single nonzero entry equal to $(1-\alpha)/Z_{rr}$ in(r,r) ,

$$\underline{\bar{Z}}^1 = (1 - \alpha)\ \tilde{\underline{T}}^r_z\ \underline{Z}\ \underline{T}^r_z + \alpha\ \underline{Z} - \underline{O}^r \qquad (29)$$

From (29), eq.(24) becomes :

$$\underline{Z}^1 = (\ 1 - \alpha\ )^{-1}(\underline{\bar{Z}}^1 - \alpha\ \underline{Z} + \underline{O}^r) + (\underline{T}^r_z - \tilde{\underline{T}}^r_z) \qquad (30)$$

which relates the influence matrix $\underline{\bar{Z}}^1$ of the modified structure S^1 to the transformed matrix $\underline{Z}^1$. No application is known of the above equation except for α = 0. In that case in fact eq.(30) is reduced to

$$\underline{Z}^1 = \underline{\bar{Z}}^1_o + (\ \underline{T}^r_z - \tilde{\underline{T}}^r_z) + \underline{O}^r = \underline{\bar{Z}}^1_o + \underline{R}^1 \qquad (31)$$

which states apart from the pivotal row and column,the equivalence of matrix $\underline{Z}^1$ to the matrix $\underline{\bar{Z}}^1_o$ relevant to the structure S^1 derived from S by elimination of the r-th element. Further pivotings lead to the following generalization of (31):

$$\underline{Z}^n = \underline{\bar{Z}}^n_o + \underline{R}^n \qquad (32)$$

where $\underline{\bar{Z}}^n_o$ refers to the structure S^n derived from S by elimination of the elements corresponding to all the n pivotal indices of transformation.

Eqs.(31) and (32) have already found applications in the step by step approach to limit analysis(where the generalized stress components relevant to activated plastic modes are subsequently eliminated until a structural mechanism is attained). Recently in | 9| this technique was applied also to the stability and collapse analysis of elastoplastic structures in the presence of second order geometric effects.

As to the possibility to apply the above approach,based on modifications of the original structure S ,directly to the historical analysis,serious obstacles met with,mostly deriving from the fact that the equivalence stated by eq.(32) is confined only to the rows and columns of $\underline{Z}^n$ relative to the remaining elements of S^n after n element eliminations,i.e. refers to an always smaller part of $\underline{Z}^n$ for increasing n. This prevents from using it in the case,fundamental for the historical analysis,of a new pivoting step on a previous pivotal index.

In order to leave this possibility,let the rule (24) be applied to matrix $(\underline{D} - \underline{Z})$, $\underline{D}$ being a given diagonal matrix. It is then possible,through eq.(29) and (30) to derive the following relation:

$$(\ \underline{D} - \underline{Z}\ )^1 = \underline{D} - \underline{U}^-_r\ \underline{\bar{Z}}^1\ \underline{U}_r + \underline{\bar{O}}^r \qquad (33)$$

where $\underline{U}^-_r$, $\underline{U}_r$ are the unit matrices with $1/D_{rr}$,$-1/D_{rr}$ as r-th diagonal elements,while $\underline{\bar{Z}}^1$ refers to a modified structure S^1 obtained from the original S by a reduction of r-th element stiffness according to the equation:

$$(\ EA\ )^M_r = D_{rr}(EA)_r/(\ D_{rr}+(EA)_r) \qquad \text{i.e for} \quad \alpha_1 = D_{rr}/(D_{rr}-Z_{rr}) \qquad (34),(34')$$

In the same way a further pivoting on the elemnt (s,s) of $(\underline{D} - \underline{Z})$ leads to:

$$(\underline{D} - \underline{Z})^2 = \underline{D} - \underline{U}_r^- \underline{U}_s^- \overline{\underline{Z}}^2 \underline{U}_s \underline{U}_r + \underline{O}^s + \underline{O}^r \qquad (35)$$

where matrix $\overline{\underline{Z}}^2$ now corresponds to a modified structure S^2 derived from S^1 by a reduction of the s-th element stiffness again according to eq.(34) (with index s instead of r),i.e. for $\alpha_2 = D_{ss}/(D_{ss} - \overline{Z}^1_{ss})$. In the case of n principal pivotings eq.(33) and(35) may be generalized as:

$$(\underline{D} - \underline{Z})^n = \underline{D} - (\prod_{r=1}^{n} \underline{U}_r^-) \overline{\underline{Z}}^n (\prod_{r=1}^{n} \underline{U}_r) + \sum_{r=1}^{n} \overline{\underline{O}}^r \qquad (36)$$

with a modification to be operated on the n element stiffnesses relevant to the pivoting indices r,∀ r=1,...n given by:

$$(EA)^M_r = D_{rr}(EA)^{r-1}_r/(D_{rr}+(EA)^{r-1}_r) \quad , \text{ i.e. } \quad \alpha_r = D_{rr}/(D_{rr} - \overline{\underline{Z}}^{r-1}_{rr}) \qquad (37),(37')$$

Eqs. (36) and (37) show the possibility to compute the transformed matrix $\underline{D} - \underline{Z}$ after n principal pivotings simply evaluating the $\overline{\underline{Z}}^n$ influence matrix relevant to the structure S^n. This last is derived from the original structure S by means of suitable modifications of the stiffnesses of the element associated to the n pivoting indices.

In contrast with the case of α=0 (eq.32) the present interpretation now extends to all the rows and columns of $\underline{Z}^n$ (including those corresponding to previous pivotal rows).Possible transformations on formerly used pivotal indices (an important feature for historical analysis) are here allowed for.

3.3 Mechanical interpretation of PLCP transformed tableau.

Let s be the index of the pivot entry for the transformation of matrix $\underline{A}=\underline{H} - \underline{\tilde{N}}\,\underline{Z}\,\underline{N}$ of the tableau,and r the index of the corresponding element of the original structure S. In the following the matrix $\underline{H}$ is assumed diagonal (as in Koiter hardening case). Using the transformation (32) it is easy to prove that:

$$\underline{A}^1 = (\underline{H} - \underline{\tilde{N}}\,\underline{Z}\,\underline{N})^1 = \underline{H} - \underline{U}_s^- \underline{\tilde{N}}\,\overline{\underline{Z}}^1 \underline{N}\,\underline{U}_s + \underline{O}^s_A \qquad (38)$$

where symbols have the same meaning as in sect.3.2, $\underline{O}^s_A$ being a matrix with a single non zero entry equal to $(1/H_{ss}-H_{ss})$ at (s,s). Then, after n principal pivoting transformations,the tableau will become:

$$\underline{A}^n = (\underline{H} - \underline{\tilde{N}}\,\underline{Z}\,\underline{N})^n = \underline{H} - (\prod_{s=1}^{n} \underline{U}_s^-) \underline{\tilde{N}}\,\overline{\underline{Z}}^n \underline{N} (\prod_{s=1}^{n} \underline{U}_s) + \sum_{s=1}^{n} \underline{O}^s_A \qquad (39)$$

where $\overline{\underline{Z}}^n$ now refers to a structure S^n derived from S having modified as follows the n element stiffnesses $(EA/1)_r$ relevant to the pivoting indices s(∀ r=1,...n):

$$(EA)^M_r = H_{ss}/(H_{ss}+(EA)^{r-1}_r) \,.EA^{r-1}_r \quad , \text{ i.e. } \quad \alpha_r = H_{ss}/(H_{ss} - \underline{\tilde{N}}\,\overline{\underline{Z}}^{r-1}_{rr}\underline{N}) \qquad (40),(40')$$

Eqs. (39) and (40) state once more the possibility to use the concept of "modified structure" in order to derive the current tableau. The novelty is now given by the presence of matrix $\underline{N}$ which could imply more than one subsequent modification for the stiffness of one element accordin to the number of plastic modes relevant to the element that could be activated during the loading history. From eq.(40) it appears that the modified element stiffness $(EA/1)^M_r$ is always smaller than the previous one (in the limit it has the same value for $H_{ss} \to \infty$) and has a minimum(equal to zero) for ideal plasticity case.

Finally,as already pointed out in sect.3.1,only one column of the current tableau $\underline{A}^n$ is required in order to obtain the subsequent one. The direct determination of this column instead of the overall $\underline{A}$ matrix represents an important tool drastically reducing the computational effort required at each step of the solution procedure.To this end,let both sides of eq.(39) be multiplied by the unit vector $\underline{\lambda}_j^u$,j being the requested column of $\underline{A}^n$. Through the the positions:

$$\underline{u}_j^n = \left(\prod_{s=1}^{n} \underline{U}_s \right) \underline{\lambda}_j^u \quad , \quad \underline{o}_j^n = \sum_{s=1}^{n} \underline{O}_A^s \, \underline{\lambda}_j^u \tag{41}$$

the column j of $\underline{A}^n$ will be given by:

$$\underline{A}_j^n = (\underline{H} - \underline{\tilde{N}} \, \underline{Z} \, \underline{N})_j^n = \underline{H}_j - \left(\prod_{s=1}^{n} \underline{U}_s^- \right) \underline{\tilde{N}} \, \overline{\underline{Q}}_n^s(\underline{\lambda}_j^u) + \underline{o}_j^n \tag{42}$$

where

$$\overline{\underline{Q}}_n^s(\underline{\lambda}_j^u) = \underline{\overline{Z}}^n \underline{N} \, \underline{u}_j^n \tag{43}$$

When $j \neq r \ \forall \ r=1,\ldots n$,$\overline{\underline{Q}}_n^s$ represents the selfequilibrated stresses in the structure S^n for a unit dislocation at the r_j element associated to index j and $\underline{o}_j^n = \underline{0}$. ; the same holds when j corresponds to one of the preceding pivotal column indices,except for the r-th elements (r=1,...n) of $\overline{\underline{Q}}_n^s$which must be multiplied by $1/H_{rr}$,and $\underline{o}_j^n = \underline{o}_j^j$.

3.4 Outline of the modified stiffnesses procedure.

Some important simplifications may be easily introduced in Cottle solution procedure of fig.2,on the basis of the conclusions of the preceding sections. They are mainly related to the possibility to avoid the storing of the current tableau and to reduce its calculations to one column only for each step. However this implies a strict compenetration of mathematical and statical aspects which lead,in their combination to a substantial new procedure. Its main points may be summarized in the following:

1. calculate the elastic response $\underline{Q}^e$ to basic loads $\underline{P}_o$.Take n=0;
2. if $-\underline{\tilde{N}}\,\underline{Q}_n^e$ has non negative components ,stop. The solution is monotone for any $\mu > \mu_{new}$ Otherwise go to 3;
3. find index j such that $-K_j^n/v_j^n = \min \{-K_i^n/v_i^n \mid v_i^n < 0\}$ and set $\mu_{new} = -K_j^n/v_j^n$;
4. if $\mu_{new} > \mu_{old}$ and vector $\underline{v}$ has a negative component at a row already used as pivotal row,stop; the solution is not monotone (for $\mu_{old} < \mu \leq \mu_{new}$).Otherwise go to 5.;
5. find the selfequilibrated stresses $\overline{\underline{Q}}_n^s$ of the structure S^n according to eq.(43) for a unit dislocation at the element r_j associated to index j. The pivotal column is given by $\underline{H} - \underline{\tilde{N}}\,\underline{Q}_o^s$ for n =0, and by eq.(42) for n>0 ;
6. transform vectors $\underline{K}^n$ and $\underline{v}^n$ by the pivotal rules on the basis of the known pivotal column.Then take n=n+1 and go to 2.

The application of this procedure to same example is illustrated in the last section.

4. Examples.

Figures 3 and 4 show the structural examples considered for applications of the proposed modified stiffnesses procedure and for comparisons with other solution procedures. The three bar truss of fig.3,already considered in |4|was solved for increasing number of hardening branches(up to 9) and for combinations of the parameters leading both to monotone or non monotone solutions. The smallness of the example was not yet able to point out particularly strong savings in computer time with respect Cottle's solution procedure; however the reduction of the computational effort and of the required storing was already evident(see the three tableaux of fig.3 required by Cottle's procedure where the boxes represent the columns evaluated by the proposed algorithm).

The interest of the example of fig.4 is two-fold. Firstly for increasing number of variables,it easily provides a better evidence of the computational relative efficiency between Cottle's algorithm and the proposed procedure. Secondly,as it was solved also in |13| with classical solution procedures,it allows to make a comparison of merit between these algorithms and the PLCP approach.

As to the first point,the proposed procedure and Cottle's algorithm were applied for increasing number of hardening branches. It turned out that from 34 unknowns onwards(i.e. about two hardening brances per element) durind the execution the CPU time for the second method became greater than for the first. It was expected to be so as the computer time for an elastic analysis of the structure under given dislocations is independent from the number of unknowns n,while the time required by a direct pivotal transformation of the tableau goes with twice the square of n.

The table of fig.4 compares the rate of convergence of PLCP algorithm(whatever it is, Cottle's or the modified stiffness one,as the rate of convergence is the same) and of initial stress and initial strain approachs using the improved iterative method(VIM). The comparison is made using as common term the "iteration" which is,or is amenable(in PLCP algorithm) to an elastic calculation of the structure. For each value of μ/μ_1 the table gives the total number of iterations required to reach the solution. Using the PLCP procedure over 1/2 of the structure(which is symmetric) the solution was reached on the assumption of one and two branches per element. In order to obtain better approximate solutions an increasing number of hardening branches in the constitutive law must be considered.This could reduce the rate of convergence.However in any case the total number of iterations required to complete a historical analysis may always be controled beforehand as the algorithm assure such number is less than that of unknowns of the problem.

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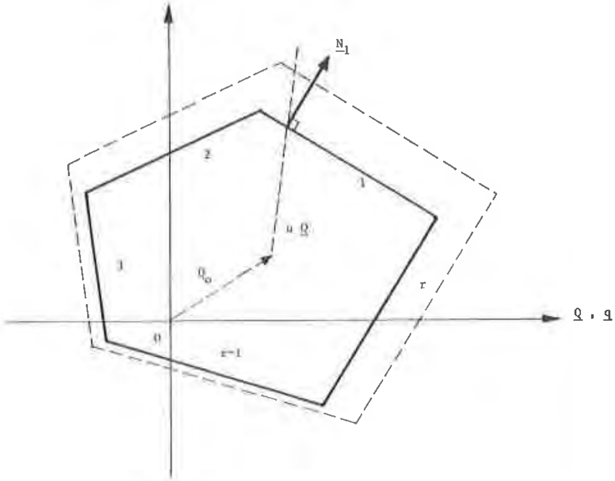


Fig.1 - Piecewise-linear yield surface,associated flow rule and isotropic hardening: representation in the space of generalized stress components of a proportional stress path with progressive yielding.

$$\text{P.L.C.P.}\begin{cases} -\underline{\phi} = (\underline{H} - \underline{N}\,\underline{Z}\,\underline{N})\underline{\lambda} + \mu\underline{v} + \underline{K} \\ \underline{\lambda} \geq \underline{0}\,,\ \bar{\underline{\phi}}\,\underline{\lambda} = 0 \end{cases}$$

$$\underline{v} = (-\bar{\underline{N}}\,\underline{Q}^{*}_{o})$$

$$\underline{A} = (\underline{H} - \bar{\underline{N}}\,\underline{Z}\,\underline{N})$$

	1	0	...	0	μ	λ_1		λ_n	
$-\phi_1$	K_1	0	0 ...	0	v_1	a_{11}	...	a_{1n}	
		1							matrix $\underline{A}$
			. .						
$-\phi_n$	K_n			1	v_n	a_{n1}		a_{nn}	

matrix $\underline{B}$

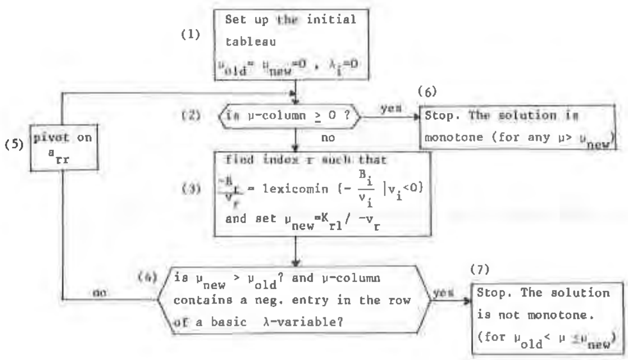
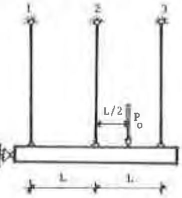
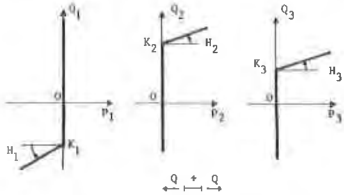


Fig.2 - Cottle's monotonicity-chequing algorithm for the historical analysis of hardening structures. General flow-chart and initial tableau.



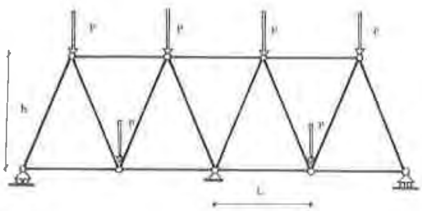
L = 150 cm
P = 50 Kg
EA_1=.3+07 Kg/cm^2
EA_2=.25+07 "
EA_3=.2+07 "
H_1=.2+07 "
H_2=.1+07 "
H_3=.1+07 "
K_1=.1+05 "
K_2=.12+05 "
K_3=.9+04 "



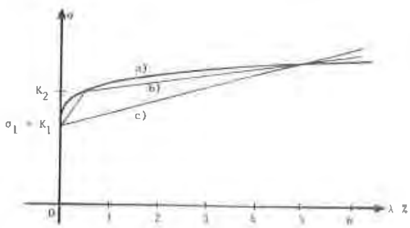
	.1000+05	.0000	.0000	-.9247+01	.2411+07	.8219+06	-.4110+06	1°
	.1200+05	.1000+01	.0000	-.4349+02	.8219+06	.2644+07	-.8219+06	
(μ_1=.27591+03)	.9000+04	.0000	.1000+01	-.7877-05	-.4110+06	-.8219+06	.1411+07	
	.6269+04	-.3109+00	.0000	.4275+01	.2155+07	.3109+00	-.1554+06	2°
	-.4539+02	-.3782-06	.0000	.1645-04	-.3109+00	.3782-06	.3109+00	
(μ_2=.94153+03)	.1273+05	.3109+00	.1000+01	-.1352+02	-.1554+06	-.3109+00	.1155+07	
	.7982+04	-.2691+00	-.1345+00	.2456+01	.2135+07	.2691+00	-.1345+00	3°
	-.7964-02	-.4619-06	.2691-06	.2009-04	-.2691+00	.4619-06	.2691-06	
	-.1102-01	-.2691-06	-.8655-06	.1170-04	.1345+00	.2691-06	.8655-06	

The solution is monotone for $\mu > \mu_2$

Fig.3 - Example of application of the modified stiffnesses method for the historical analysis by PLCP and comparisons with Cottle's solution procedure.



N. of nodal points = 9
N. of elements = 15
N. of unknowns (b) = 30
(c) = 15



a) Ramberg-Osgood

$\lambda = 1.1\,\sigma_1/(mE)\,[(\sigma/1.1\sigma_1)^m - (1/1.1)^m]$

m= 10 E= 801420 kg/cm^2 σ_1= 2425 Kg/cm^2

b) Two branch piecewise linearization

$K_1 = \sigma_1$ H_1= 219387 Kg/cm^2
K_2= 3500 Kg/cm^2 H_2= 20814 "

c) One branch piecewise linearization

$K_1 = \sigma_1$ H_1 = 39980 Kg/cm^2

μ/μ_1	Initial stress VIM	Initial strain VIM	PLCP (c)	PLCP (b)
1	-	-	1	1
1.16	4	3	2	2
2.64	47	28	3	5
2.87	computation not continued	computation not continued	4	6
3.67			5	7
4.14			6	10
4.37			7	11
10.74			8	15
15.32			-	16

Fig.4 - Rate of convergence of PLCP approach and comparisons with initial stress, initial strain methods using VIM.