REGULARITY OF HIGHER ORDER IN TWO-PHASE FREE BOUNDARY PROBLEMS

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Abstract. We develop further our strategy in [DFS] to show that flat or Lipschitz free boundaries of two-phase problems with forcing terms are locally $C^{2,\gamma}$.

1. Introduction

Let $f_\pm \in C^{0,\gamma}(B_1)$, where B_1 denotes the unit ball in $\mathbb{R}^n$, $n \geq 2$, centered at 0, and consider the two-phase problem

$$
\begin{cases}
\Delta u = f_+ & \text{in } B_1^+(u), \\
\Delta u = f_- & \text{in } B_1^-(u), \\
u_\nu^+ = G(u_\nu^-) & \text{on } F(u) := \partial B_1^+(u) \cap B_1.
\end{cases} \tag{1.1}
$$

Here

$$B_1^+(u) := \{x \in B_1 : u(x) > 0\}, \quad B_1^-(u) := \{x \in B_1 : u(x) \leq 0\}^\circ,$$

while u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $B_1^+(u)$ and $B_1^-(u)$ respectively. The function $G : [0,\infty) \to \mathbb{R}^+$ satisfies the usual ellipticity assumption:

$$G \quad \text{is strictly increasing,} \quad G(0) > 0, \text{ and } G(b) \to \infty \text{ as } b \to \infty. \tag{1.2}$$

For simplicity, we assume that $G \in C^2([0,\infty))$ and say $G(0) = 1$.

Typical examples of inhomogeneous two-phase problems are the Prandtl-Bachelor model in fluid-dynamics (see e.g. [B1, EM]), or the eigenvalue problem in magneto-hydrodynamics considered in [FL]. Other examples come from limits of singular perturbation problems with forcing term as in [LW], where the authors analyze solutions to (1.1), arising in the study of flame propagation with nonlocal effects.

Our main result gives C^{2,γ^*} regularity of flat free boundaries. Precisely, we prove the following theorem, where we call universal any constant depending on $n, \gamma, L := Lip(u), \|f_\pm\|_{C^{0,\gamma}(B_1)}$, and $\|G\|_{C^2([0,L+1])}$.

Theorem 1.1. *Let u be a (Lipschitz) viscosity solution to* (1.1) *in B_1. There exists a universal constant $\bar\eta > 0$ such that, if*

$$\{x_n \leq -\eta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \eta\}, \quad \text{for } 0 \leq \eta \leq \bar\eta, \tag{1.3}$$

then $F(u)$ is C^{2,γ^*} in $B_{1/2}$ for a small γ^* universal, with the C^{2,γ^*} norm bounded by a universal constant.

In view of Theorem 1.3 in [DFS], if (1.3) holds then the free boundary $F(u)$ is locally $C^{1,\bar{\gamma}}$. Note that the same conclusion holds if $F(u)$ is a graph of a Lipschitz function (see Theorem 1.4 in [DFS]). Therefore, throughout the paper we will assume that $F(u)$ is $C^{1,\bar{\gamma}}$ and hence u is a classical solution, i.e. the free boundary condition is satisfied in a pointwise sense. To fix ideas, let us say that $0 < \gamma \leq \bar{\gamma}$.

Our result extends without much effort to more general linear uniformly elliptic equations with $C^{0,\gamma}$ coefficients and to more general free boundary jump conditions $u_\nu^+ = G(u_\nu^-, \nu, x)$, where G is C^2 with respect to all its arguments. For those operators, also considering the existence paper [DFS2], the theory of viscosity solutions to inhomogeneous free boundary problems has reached a considerable level of completeness. Perhaps, the only relevant open question remains the analysis of singular ("nonflat") points.

For fully nonlinear operators, we proved in [DFS3] that for a fairly general class of problems (with right-hand side), Lipschitz viscosity solutions with Lipschitz or flat (in the sense of (1.3)) free boundaries are indeed classical. The questions of Lipschitz continuity of solutions and higher regularity of the free boundary remain open problems.

In order to explain the significance of our main theorem, we describe here the state of the art about the higher regularity theory for two-phase free boundary problems. In the seminal paper [KNS], the authors use a zero order hodograph transformation and a suitable reflection map, to locally reduce a two-phase problem to an elliptic and coercive system of nonlinear equations (see Appendix B). The existing literature on the regularity of solutions to nonlinear systems developed in [ADN, M] can be applied as long as the solution u is $C^{2,\alpha}$ (for some $\alpha > 0$) up to the free boundary (from either side). Hence, the following corollary of Theorem 1.1 holds.

Corollary 1.2. *Let k be a nonnegative integer. Assume that $f_\pm \in C^{k,\gamma}(B_1)$ and G is C^{2+k}. Then $F(u) \cap B_{1/2}$ is C^{k+2,γ^*}. If $f_\pm$ are C^∞ or real analytic in B_1, then $F(u) \cap B_{1/2}$ is C^∞ or real analytic, respectively.*

As noted in the recent work [KL], in the case when the governing equation in (1.1) is in divergence form the initial assumption to obtain the Corollary above is that $u \in C^{1,\alpha}$. It is not evident that the general case of linear uniformly elliptic equations with $C^{0,\gamma}$ coefficients can also be treated in a similar manner. On the other hand, the case when the leading operator is say a convex (or concave) fully nonlinear operator definitely requires the solution to have Hölder second derivatives (from both sides).

Our purpose is to develop a general strategy that would apply to a larger class of problems, possibly to include also the case of fully nonlinear operators.

Other related higher regularity results can be found in [E, K].

The overall strategy for the proof of Theorem 1.1 is based, as in Theorem 1.3 of [DFS], on a compactness argument leading to a limiting linearized problem in which the information for an improvement of flatness is stored. However, reaching the $C^{2,\gamma}$ regularity requires a much more involved process because of the possible degeneracy of the negative part. Indeed this causes a delicate interplay between the two phases, as we shall try to explain in the next section. Ultimately the main source of difficulties is due to the presence of a forcing term of general sign in the negative phase. Indeed, if $f_- \geq 0$, Hopf maximum principle would imply nondegeneracy (also) on the negative side, making the two-phases of comparable size and considerably simplifying the final iteration procedure. It is worth noticing, that however even in this easier scenario (and in particular in the homogeneous case), if one wants to attain uniform estimates with universal constants, then one must employ the more involved methods developed here for the degenerate case.

The paper is organized as follows. In Section 2 we outline the strategy of the proof and in Section 3 through 7 we implement it. In Appendix A, we provide a refinement of the classical pointwise $C^{1,\alpha}$ estimates for elliptic equations. This is a technical tool used in the paper. In Appendix B we sketch the main steps for the reduction of our problem (1.1) to a system of nonlinear equations as in [KNS].

2. Outline and strategy

In this section, we outline the main strategy in the proof of Theorem 1.1, trying to emphasize the key points, also in comparison to the *flatness implies $C^{1,\gamma}$ case* in [DFS]. The first thing to do is to reinforce the notion of flatness, tailoring it for the attainment of $C^{2,\gamma}$ regularity. This can be done by introducing a suitable class of functions that we call *two-phase* and *one-phase* polynomials. In principle second order polynomials should be enough but it turns out that we need a small third order perturbation.

Given $\omega \in \mathbb{R}^n$, with $|\omega| = 1$, and let S_ω be an orthonormal basis containing ω. Let $M \in S^{n\times n}$ satisfy

$$M\omega = 0$$

and define

$$P_{M,\omega}(x) = x\cdot\omega - \frac{1}{2}x^T M x.$$

Set,

$$V^{\alpha,\beta}_{M,\omega,a,b}(x) = \alpha(1 + a\cdot x)P^+_{M,\omega}(x) - \beta(1 + b\cdot x)P^-_{M,\omega}(x), \quad \alpha > 0, \beta \geq 0, a, b \in \mathbb{R}^n$$

where the superscripts $\pm$ denote as usual the positive/negative part of a function. These are our two-phase polynomials, one-phase if $\beta = 0$. In the particular case when $M = 0, a = b = 0, \omega = e_n$ we obtain the two-plane function:

$$U_\beta(x) = \alpha x_n^+ - \beta x_n^-.$$

The unit vector ω establishes the "direction of flatness".

We shall need to work with a subclass, strictly related to problem (1.1), at least at the origin. We denote by $\mathcal{V}_{f_\pm,G}$ the class of functions of the form $V^{\alpha,\beta}_{M,\omega,a,b}$ for which

$$2\alpha a\cdot\omega - \alpha trM = f_+(0)$$

$$2\beta b\cdot\omega - \beta trM = f_-(0) \quad \text{if } \beta \neq 0,$$

$$\alpha = G(\beta), \quad \text{if } \beta \neq 0,$$

and

$$\alpha a\cdot\omega^\perp = \beta G'(\beta)\, b\cdot\omega^\perp, \quad \forall\omega^\perp \in S_\omega.$$

The role of the last condition will be clear in the sequel (e.g. Proposition 3.3).

When $\beta = 0$, then there is no dependence on b and $a\cdot\omega^\perp = 0$. Thus, we drop the dependence on β, b, G and f_- in our notation above and we indicate the dependence on $a_\omega := a\cdot\omega$.

We introduce the following definitions.

Definition 2.1. Let $V = V^{\alpha,\beta}_{M,\omega,a,b}$. We say that u is (V, ϵ, δ) flat in B_1 if

$$V(x - \epsilon\omega) \leq u(x) \leq V(x + \epsilon\omega) \quad \text{in } B_1$$

and

$$|a|, |b'|, \|M\| \leq \delta\epsilon^{1/2}, \quad |b_n| \leq \delta^2, \quad |b_n|\|M\| \leq \delta^2\epsilon.$$

Given $V = V^{\alpha,\beta}_{M,\omega,a,b}$, set

$$V_r(x) = \frac{V(rx)}{r}$$

and notice that

$$V_r = V^{\alpha,\beta}_{rM,\omega,ra,rb}.$$

Definition 2.2. Let $V = V^{\alpha,\beta}_{M,\omega,a,b}$. We say that u is (V, ϵ, δ) flat in B_r if the rescaling

$$u_r(x) := \frac{u(rx)}{r}$$

is $(V_r, \frac{\epsilon}{r}, \delta)$ flat in B_1.

Notice that if u is (V, ϵ, δ) flat in B_r then

$$V(x - \epsilon\omega) \leq u(x) \leq V(x + \epsilon\omega) \quad \text{in } B_r.$$

The parameter ϵ measures the level of polynomial approximation and δ is a flatness parameter (also controlling the $C^{0,\gamma}$ norms of f_+ and f_-).

Thus roughly our purpose is to show that u is $(V_k, \lambda_k^{2+\gamma^*}, \delta)$ flat in B_{λ_k} for $\lambda_k = \eta^k$ and all $k \geq 0$, for some δ, η small and a sequence of V_k converging to a final profile V_0. This would give uniform pointwise C^{2,γ^*} regularity both for the solution and the free boundary in $B_{1/2}$.

The starting point is to show (Section 3, Lemma 3.2) that the flatness condition (1.3) allows us to normalize our solution so that a rescaling $u_{\bar{r}}$ of u falls into one of the following cases, with suitable $\bar{\lambda}, \bar{\delta}$. This kind of dichotomy parallels in a sense what happens in [DFS].

Case a). $u_{\bar{r}}$ is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat for some $V = V^{\alpha,\beta}_{0,e_n,a,b} \in \mathcal{V}_{f_\pm,G}$. Moreover, $\beta\bar{\delta}$ controls the $C^{0,\gamma}$ seminorms of f_-.This case corresponds to a nondegenerate configuration, in which the two phases have comparable size and $u_{\bar{r}}$ is trapped between two translations of a genuine two-phase polynomial.

Case b). $u^+_{\bar{r}}$ is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat for some $V = V^1_{0,e_n,a_n} \in \mathcal{V}_{f_+}$, and $u^-_{\bar{r}}$ is close to a purely quadratic profile cx_n^2. This case corresponds to a degenerate configuration, where the negative phase has either zero slope or a small one (but not negligible) with respect to $u^+_{\bar{r}}$, and $u^+_{\bar{r}}$ is trapped between two translations of a one-phase polynomial. Note that this situation cannot occur if $f_- \geq 0$, unless u^- is identically zero.

Next we examine how the initial flatness corresponding to cases a) and b) above improves successively at a smaller scale. In Section 4, we construct the following "subroutines", to be implemented in the course of the final iteration towards C^{2,γ^*} regularity.

Two-phase flatness improvement (Proposition 4.4): if u is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat for some $V = V^{\alpha,\beta}_{M,\omega,a,b} \in \mathcal{V}_{f_\pm,G}$ in B_λ, the $C^{0,\gamma}$ seminorms of f_+ and f_- are controlled by $\bar{\delta}$ and $\beta\bar{\delta}$, respectively, then, in $B_{\lambda\eta}$, u enjoys a $C^{2,\gamma}$ flatness improvement, i.e. u is $(\bar{V}, (\eta\lambda)^{2+\gamma}, \bar{\delta})$ flat for some $\bar{V} \in \mathcal{V}_{f_\pm,G}$, properly close to V.

One-phase flatness improvement (Proposition 4.3): if u^+ is $(V, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat for some $V = V^{\alpha}_{M,\omega,a_\omega} \in \mathcal{V}_{f_+}$ in B_λ, the $C^{0,\gamma}$ seminorm of f_+ is controlled by $\bar{\delta}$ and u^+_ν is close to α on $F(u)$, then u^+ enjoys a $C^{2,\gamma}$ flatness improvement, with $\bar{V} \in \mathcal{V}_{f_+}$, properly close to V.

The achievement of the improvements above relies on a Harnack inequality, and a higher order refinement of Theorems 4.1 and 4.4 in [DFS]. This gives the necessary compactness to pass to the limit in a sequence of suitable renormalizations of u and obtain a limiting transmission problem (Neumann problem in the one phase-case). From the regularity of the solution of this problem we get the information to improve the two-phase or one-phase approximation for u or u^+ respectively, and hence their flatness.

Now we start iterating. As we have seen, according to Lemma 3.2, after a suitable rescaling, we face a first dichotomy "degenerate versus nondegenerate".

In the latter case the two-phase subroutine in Proposition 4.4 can be applied indefinitely to reach pointwise C^{2,γ^*} regularity for some universal γ^*.

When u falls into the degenerate case a new kind of dichotomy appears. This is the deepest part of the paper. First of all, to run the subroutine in Proposition 4.3 one needs to make sure that the closeness of u^- to a purely quadratic profile makes u^+ to be a (viscosity) solution of a one-phase free boundary problem with u^+_ν close to an appropriate α on $F(u)$. This is the content of Lemma 6.2, in Section 6. At this point two alternatives occur at a smaller scale (Proposition 6.3).

- D1 : either u^- is closer to a purely quadratic profile at a proper $C^{2,\gamma}$ rate and u^+ enjoys a $C^{2,\gamma}$ flatness improvement;
- D2 : or u^- is closer (at a $C^{2,\gamma}$ rate) to a one-phase polynomial profile with a small non-zero slope but u^+ only enjoys an "intermediate" C^2 flatness improvement.

If D1 occurs indefinitely we are done. If not, we prove that (Proposition 7.1) the intermediate improvement in D2 is kept for a while, at smaller and smaller scale. The final and crucial step is to prove (Proposition 7.2) that, at a given universally small enough scale, the $C^{2,\gamma}$ one-phase approximation of u^-, together with the intermediate C^2 flatness improvement of u^+, is good enough to recover a full C^{2,γ^*} two-phase improvement of u with a universal $\gamma^* < \gamma$.

As we have mentioned at the end of Section 1, we emphasize that it is the interplay between the parallel improvements on both sides of the free boundary that makes possible to obtain the full two-phase improvement, at the price of a little decrease of the Hölder exponent. This kind of situation has no counterpart in the *flatness implies $C^{1,\gamma}$ case* of [DFS].

From this point on we can go back to subroutine of Proposition 4.4 and finally reach pointwise C^{2,γ^*} regularity.

In the next section we start implementing the above strategy. In the course of a proof, universal constants possibly changing from line to line will be denoted by c, C. Dependence on other parameters, will be explicitly noted.

3. Initial configurations

As we mentioned in Section 2, we start by showing that the flatness condition (1.3) allows us to normalize our solution so that a rescaling $u_{\bar{r}}$ of u satisfies a suitable (V,ε,δ) flatness. We first recall the following result proved in [DFS]. Set

$$u_r(x) := \frac{u(rx)}{r}, \quad f_{\pm r}(x) = rf_{\pm}(rx), \quad x \in B_1.$$

Lemma 3.1. *Let u be a (Lipschitz) solution to* (1.1) *in B_1 with $Lip(u) \leq L$ and $\|f_\pm\|_{L^\infty} \leq L$. For any $\epsilon > 0$, $\epsilon < \epsilon_0 = \epsilon_0(n,L)$, there exist $\bar\eta$ depending on ϵ, n and L, $\bar\eta \leq \epsilon^4$, such that if u satisfies* (1.3) *for some $\eta \leq \bar\eta$ then*

$$\|u_r - U_\beta\|_{L^\infty(B_1)} \leq \epsilon, \quad \text{for some } 0 \leq \beta \leq L, \tag{3.1}$$

$$\|f_{\pm r}\|_\infty \leq \epsilon, \quad |f_{\pm r}(x) - f_{\pm r}(0)| \leq \epsilon|x|^\gamma \tag{3.2}$$

$$\{x_n \leq -\epsilon\} \subset \{u_r^+ = 0\} \subset \{x_n \leq \epsilon\}, \tag{3.3}$$

and $r = \epsilon^3$.

Let u be as in Lemma 3.1 and for a given ϵ, let $\bar\eta(\epsilon)$ and $r(\epsilon)$ be the corresponding parameters provided by the lemma.

In the next Lemma, we denote by $\bar\delta, \bar\lambda$ the universal constants which will be chosen later in Proposition 4.4 and Proposition 6.3 (say for $\beta_1 = L+1$).

Lemma 3.2. *There exists $\bar\epsilon$ universal such that if u satisfies* (1.3) *with $\bar\eta = \bar\eta(\bar\epsilon)$ then either of these flatness conditions holds with $\bar r = \bar r(\bar\epsilon)$.*

(i) *Degenerate case:*
$u_{\bar r}^+$ is $(V, \bar\lambda^{2+\gamma}, \bar\delta)$ flat in B_1, for $V = V^1_{0,e_n,a_n} \in \mathcal{V}_{f_+}$,

$$|u_{\bar r}^- + \frac{1}{2} f_{-\bar r}(0) x_n^2| \leq \bar\delta^{1/2}\bar\lambda^{2+\gamma} \quad \text{in } B_1^-(u_{\bar r})$$

and

$$\|f_{-\bar r}\|_\infty \leq \bar\delta, \quad |f_{\pm\bar r}(x) - f_{\pm \bar r}(0)| \leq \bar\delta|x|^\gamma$$

(ii) *Non-degenerate case:*
$u_{\bar r}$ is $(V, \bar\lambda^{2+\gamma}, \bar\delta)$ in B_1, with $V = V^{\alpha,\beta}_{0,e_n,a,b} \in \mathcal{V}_{f_{\pm\bar r},G}$,

$$a' = b' = 0, \quad \beta \geq \frac{1}{2}\bar\delta^{1/2}\bar\lambda^{2+\gamma},$$

and

$$|f_{+\bar r}(x) - f_{+\bar r}(0)| \leq \bar\delta|x|^\gamma \quad |f_{-\bar r}(x) - f_{-\bar r}(0)| \leq \beta\bar\delta|x|^\gamma.$$

Proof. Call $\epsilon^* = \bar\lambda^{2+\gamma}$ and $\tilde\delta = \frac{1}{2}\bar\delta^{1/2}\epsilon^*$. Let $\bar\epsilon << \tilde\delta < \epsilon^*$ and $\bar\epsilon < \bar\delta$, to be made precise later. For such $\bar\epsilon$ the conclusion of Lemma 3.1 above gives that in B_1

$$\|u_{\bar r} - U_\beta\|_\infty \leq \bar\epsilon, \quad \|f_{\pm\bar r}\|_\infty \leq \bar\epsilon, \quad |f_{\pm\bar r}(x) - f_{\pm\bar r}(0)| \leq \bar\epsilon|x|^\gamma \tag{3.4}$$

and

$$\{x_n \leq -\bar\epsilon\} \subset \{u_{\bar r}^+ = 0\} \subset \{x_n \leq \bar\epsilon\}, \tag{3.5}$$

for $\bar r = \bar r(\bar\epsilon)$ and some $0 \leq \beta \leq L$.

We distinguish two cases. For notational simplicity we drop the subindex $\bar r$.

Degenerate Case. $\beta < \tilde\delta$. In this case we wish to show that

$$V(x_n - \epsilon^*) \leq u^+(x) \leq V(x_n + \epsilon^*) \tag{3.6}$$

with

$$V(x_n) := (1 + a_n x_n)x_n^+, \quad a_n = \frac{f_+(0)}{2}.$$

We prove the upper bound, as the lower bound can be obtained similarly. From the first two equations in (3.4) we get (for $\bar\epsilon << \epsilon^*$, $\alpha = G(\beta)$)

$$u^+ \leq \alpha x_n^+ + \bar\epsilon \leq \alpha V(x_n) + c\bar\epsilon \leq \alpha V(x_n + 2\epsilon^*),$$

hence

$$u^+ \leq V(x_n + 2\epsilon^*) + 2\tilde\delta \leq V(x_n + \epsilon^*)$$

as long as $\tilde\delta \leq \frac{1}{4}\epsilon^*$ (which is clearly satisfied because $\bar\delta$ is very small).

The bound for u^- follows immediately by noticing that in view of (3.4) and (3.5),

$$|u^- + \frac{1}{2}f_-(0)x_n^2| \leq \bar\epsilon + \alpha\bar\epsilon + \tilde\delta \quad \text{in } B_1^-(u).$$

Non-degenerate Case. $\beta \geq \tilde\delta$. In this case we want to show that

$$V(x_n - \epsilon^*) \leq u(x) \leq V(x_n + \epsilon^*) \quad \text{in } B_1$$

with

$$V(x_n) = \alpha(1+a_n x_n)x_n^+ - \beta(1+b_n x_n)x_n^-, \quad 2\alpha a_n = f_+(0), \quad 2\beta b_n = f_-(0) \quad \alpha = G(\beta).$$

Let us prove the upper bound. In view of (3.4) we get,

$$u \leq V(x_n) + 2\bar\epsilon \leq v(x_n + \epsilon^*)$$

where in the last inequality we have used that $V' \geq \frac{1}{2}\tilde\delta$ and $\bar\epsilon << \tilde\delta < \epsilon^*$. The bound on the modulus of continuity of f_- also follows because $\beta \geq \tilde\delta >> \bar\epsilon$. □

We conclude this section by providing sufficient conditions for a two-phase/one-phase polynomial V to be a strict subsolution (resp. supersolution). We will work simultaneously with the two-phase problem (1.1) and with the one-phase problem

$$\begin{cases} \Delta v = f_+ & \text{in } B_1^+(v), \\ |v_\nu^+ - \alpha| \leq \delta^{1/2}\epsilon & \text{on } F(v), \end{cases} \tag{3.7}$$

with δ and ϵ sufficiently small constants. The free boundary will always be $C^{1,\gamma}$.

The two-phase results are needed to deal with the non-degenerate case, that is the case when our flat solution u to (1.1) is trapped between two translates of a function $V \in \mathcal{V}_{f_\pm,G}$ with a positive slope β (not too small). The one-phase results will be of use when we will deal with the degenerate case, that is when the flatness of the free boundary only guarantees closeness of the positive part u^+ to a quadratic profile, i.e. $\beta = 0$.

Precisely we prove the following Proposition. The corresponding statement for V to be a strict supersolution can also be obtained. Here, $0 \leq \beta \leq \beta_1, 1 = G(0) \leq \alpha \leq \alpha_1 = G(\beta_1)$. Dependence of the constants on β_1 is not noted (as it will be fixed universal.)

Proposition 3.3. *Assume that in B_1*

$$|f_+(x) - f_+(0)| \leq \delta\epsilon, \quad \text{and} \quad |f_-(x) - f_-(0)| \leq \beta\delta\epsilon \quad \text{if } \beta \neq 0. \tag{3.8}$$

Given $V = V^{\alpha,\beta}_{M,\omega,a,b}$ with

$$\|M\|, |a|, |b'| \leq \delta\epsilon^{1/2}, \tag{3.9}$$

$$|b_n| \leq \bar C\delta^2, \quad |b_n|\|M\| \leq \bar C\delta^2\epsilon, \tag{3.10}$$

V is a strict subsolution to (1.1) for $\beta \neq 0$ or to (3.7) for $\beta = 0$, if

$$2\alpha a \cdot \omega - \alpha tr M \geq f_+(0) + 2\delta\epsilon \tag{3.11}$$

$$2\beta b \cdot \omega - \beta tr M \geq f_-(0) + 2\beta\delta\epsilon \quad \text{if } \beta \neq 0 \tag{3.12}$$

and

$$\alpha + \alpha t a \cdot \omega^\perp \geq G(\beta) + \beta G'(\beta) t b \cdot \omega^\perp + \delta^{1/2}\epsilon, \quad \forall t \in [0,1] \tag{3.13}$$

as long as δ is small (depending on $\bar{C}$).

Proof. Say $\omega = e_n$. Since $|a|, |b| < 1$, we have that

$$V(x) = \alpha(x_n - \frac{1}{2}x^T M x)(1 + a \cdot x), \quad \text{in } B_1^+(V).$$

Thus,

$$\Delta V = -\alpha tr M(1 + a \cdot x) + 2\alpha(e_n - \frac{1}{2}Mx) \cdot a \quad \text{in } B_1^+(V).$$

Using assumptions (3.8)-(3.9)-(3.10)-(3.11) we get

$$\Delta V \geq f_+(0) + \delta\epsilon - C\delta^2\epsilon \geq f_+(x) + \delta\epsilon - C\delta^2\epsilon > f_+(x)$$

if δ is chosen universally small. The computation in the negative phase follows similarly.

To check the free boundary condition we must verify that, say for $\beta > 0$,

$$|\nabla V^+| - G(|\nabla V^-|) > 0 \quad \text{on } F(V). \tag{3.14}$$

We compute that on $F(V)$, since $M \cdot e_n = 0$,

$$|\nabla V^+| = \alpha|e_n - Mx|(1 + a \cdot x) \geq \alpha(1 + a \cdot x).$$

Similarly, using assumption (3.9)

$$|\nabla V^-| = \beta|e_n - Mx|(1 + b \cdot x) \leq \beta(1 + b \cdot x) + C\delta^2\epsilon.$$

Thus,

$$G(|\nabla V^-|) \leq G(\beta) + \beta G'(\beta) b \cdot x + C\delta^2\epsilon$$

and (3.14) is satisfied in view of (3.9)-(3.10)-(3.13), as long as δ is small enough. In fact, (3.13) gives that

$$\alpha(1 + a \cdot x) \geq G(\beta) + \beta G'(\beta) b \cdot x + \alpha a_n x_n - \beta G'(\beta) b_n x_n + \delta^{1/2}\epsilon.$$

Using that on $F(V)$ the size of x_n is bounded by $\|M\|$ and from assumptions (3.9)-(3.10), we conclude that

$$\alpha(1 + a \cdot x) \geq G(\beta) + \beta G'(\beta) b \cdot x - C\delta^2\epsilon + \delta^{1/2}\epsilon,$$

from which the desired claim follows.

A similar computation holds for $\beta = 0$. □

Remark 3.4. We can consider the larger class of functions

$$P := x \cdot \omega + \xi' \cdot x' - \frac{1}{2}x^T M x,$$

and the corresponding V's (for $A \in \mathbb{R}$)

$$V = \alpha(1 + a \cdot (x + Ae_n))P^+ - \beta(1 + b \cdot (x + Ae_n))P^-.$$

The proposition above remains valid if $|\xi'|, |A| \leq \bar{C}\delta\epsilon^{1/2}$, $|b_n||A| \leq \bar{C}\delta^2\epsilon$.

4. The Improvement of flatness

In this section we prove our main improvement of flatness theorem. Let u solve

$$\Delta u^+ = f_+ \quad \text{in } B_1^+(u), \quad \Delta u^- = f_- \quad \text{in } B_1^-(u), \quad \text{if } \beta > 0$$

or

$$\Delta u^+ = f_+ \quad \text{in } B_1^+(u), \quad \text{if } \beta = 0$$

with $0 \in F(u) \in C^{1,\gamma}$.

Denote by $V = V^{\alpha,\beta}_{M,e_n,a,b} \in \mathcal{V}_{f_\pm,G}$ with $0 \le \beta \le \beta_1, 1 = G(0) \le \alpha \le \alpha_1 := G(\beta_1)$. In what follows, given V, for any function v defined in B_1, we will use the notation:

$$\tilde{v}^\epsilon(x) = \begin{cases} \dfrac{v(x) - \alpha(1 + a\cdot x)P_{M,e_n}}{\alpha\epsilon}, & x \in B_1^+(u) \cup F(u), \\[2ex] \dfrac{v(x) - \beta(1 + b\cdot x)P_{M,e_n}}{\beta\epsilon}, & x \in B_1^-(u), \quad \beta > 0, \\ 0, \quad x \in B_1^-(u), \quad \beta = 0. \end{cases} \tag{4.1}$$

Proposition 4.1 (Improvement of Flatness). *There exist $\bar\eta, \bar\delta, \bar\epsilon$ universal, such that*

(i) *Two-phase case: $\beta > 0$, if*

$$u \text{ is } (V, \epsilon, \bar\delta) \text{ flat in } B_1, \quad 0 < \epsilon \le \bar\epsilon \tag{4.2}$$

$$|f_+(x) - f_+(0)| \le \bar\delta\epsilon, \quad |f_-(x) - f_-(0)| \le \beta\bar\delta\epsilon, \tag{4.3}$$

and

$$u_\nu^+ = G(u_\nu^-) \quad \text{on } F(u) \cap B_{2/3} \tag{4.4}$$

then

$$u \text{ is } (\bar V, \bar\eta^{2+\gamma}\epsilon, \bar\delta) \text{ flat in } B_{\bar\eta}$$

with $\bar V = V^{\bar\alpha,\bar\beta}_{\bar M,\bar\nu,\bar a,\bar b} \in \mathcal{V}_{f_\pm,G}$ and $|\beta - \bar\beta| \le C\epsilon$, for C universal.

(ii) *One-phase case: $\beta = 0$, if*

$$u^+ \text{ is } (V, \epsilon, \bar\delta) \text{ flat in } B_1, \quad 0 < \epsilon \le \bar\epsilon, \tag{4.5}$$

$$|f_+(x) - f_+(0)| \le \bar\delta\epsilon, \tag{4.6}$$

and

$$|u_\nu^+ - \alpha| \le \bar\delta^{1/2}\epsilon \quad \text{on } F(u) \cap B_{2/3} \tag{4.7}$$

then

$$u^+ \text{ is } (\bar V, \bar\eta^{2+\gamma}\epsilon, \bar\delta) \text{ flat in } B_{\bar\eta} \tag{4.8}$$

with $\bar V = V^{\alpha}_{\bar M,\bar\nu,\bar a_{\bar\nu}} \in \mathcal{V}_{f_+}$.

Proof. Let $\bar\eta$ be given (to be specified later).

Step 1. By contradiction assume that there exist $\epsilon_k, \delta_k \to 0$ and $u_k, V_k, f_{\pm k}, G_k$ as above, with $\|G_k\| \le L, G_k(0) = 1$, for which the assumptions above hold but the conclusion does not. Now, let us define the corresponding $\tilde{u}_k^{\epsilon_k}$ as in (4.1). For notational simplicity we call $w_k := \tilde{u}_k^{\epsilon_k}$

Then (4.2)-(4.5) give,

$$-2 \le w_k(x) \le 2 \quad \text{for } x \in B_1. \tag{4.9}$$

Up to a subsequence, G_k converges, locally uniformly in C^1, to some C^1 function G_0, while $\beta_k \to \tilde{\beta}$ so that $\alpha_k \to \tilde{\alpha} = G_0(\tilde{\beta})$. Moreover, by Harnack inequality (see Lemma 5.1 in the next section) the graphs of w_k converge in the Hausdorff distance to a Hölder continuous w.

Step 2 – Limiting Solution. We now show that, say in the case $\beta_k > 0$ for all k's, w solves the following linearized problem

$$\begin{cases} \Delta w = 0 & \text{in } B_{1/2} \cap \{x_n \neq 0\}, \\ \\ \tilde{\alpha} w_n^+ - \tilde{\beta} G_0'(\tilde{\beta}) w_n^- = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{4.10}$$

One can argue similarly in the case $\beta_k = 0$ for all k's, with w satisfying:

$$\begin{cases} \Delta w = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \\ w_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{4.11}$$

It is easy to check that, from our assumptions,

$$|\Delta w_k| \leq C\delta_k \quad \text{in } B_1^+(u_k) \cup B_1^-(u_k),$$

hence one easily deduces that w is harmonic in $B_{1/2} \cap \{x_n \neq 0\}$.

Next, we prove that w satisfies the boundary condition in (4.10) in the viscosity sense.

Let ϕ be a function of the form

$$\phi(x) = A + px_n^+ - qx_n^- + \xi' \cdot x' + \frac{1}{2} x^T N x$$

with

$$N \in S^{n\times n}, trN = 0, \quad A \in \mathbb{R},$$

and

$$\tilde{\alpha} p - \tilde{\beta} G_0'(\tilde{\beta}) q > 0. \tag{4.12}$$

Then we must show that ϕ cannot touch w strictly by below at a point $x_0 = (x_0', 0) \in B_{1/2}$ (the analogous statement by above follows with a similar argument.)

Suppose that such a ϕ exists and let x_0 be the touching point. Without loss of generality, we can assume that ϕ is globally below w (this observation will be used in the Remark 4.2.)

We construct now a subsolution to the free boundary problem satisfied by the u_k's. We will use these computations also in the proof of Lemma 5.1 in the next section.

Call

$$W_k(x) = \tilde{\alpha}_k(1 + \tilde{c}_k \cdot (x + \epsilon_k A e_n))Q^+ - \tilde{\beta}_k(1 + \tilde{d}_k \cdot (x + \epsilon_k A e_n))Q^-,$$

where

$$Q = P_{\tilde{M}_k, e_n} + \epsilon_k \xi' \cdot x' + A\epsilon_k, \quad \tilde{M}_k = M_k - \epsilon_k N,$$
$$\tilde{\alpha}_k = \alpha_k(1 + \epsilon_k p), \quad \tilde{\beta}_k = \beta_k(1 + \epsilon_k q)$$
$$\tilde{c}_k = a_k + \epsilon_k c, \quad \tilde{d}_k = b_k + \epsilon_k d$$

and c, d to be specified later.

From Proposition 3.3 and Remark 3.4, W_k is a strict subsolution for k large as long as

$$2\tilde{\alpha}_k\tilde{c}_k\cdot e_n-\tilde{\alpha}_k trM_k\geq f_{+k}(0)+2\delta_k\epsilon_k$$
$$2\tilde{\beta}_k\tilde{d}_k\cdot e_n-\tilde{\beta}_k trM_k\geq f_{-k}(0)+2\tilde{\beta}_k\delta_k\epsilon_k$$

and

$$\tilde{\alpha}_k+\tilde{\alpha}_k\tilde{c}_k\cdot x'\geq G_k(\tilde{\beta}_k)+\tilde{\beta}_k G_k'(\tilde{\beta}_k)\tilde{d}_k\cdot x'+\delta_k^{1/2}\epsilon_k.$$

The first two equations are satisfied if we choose

$$c_n=C(p)\delta_k,\quad d_n=C(q)\delta_k.$$

Indeed, say for the first one, we compute,

$$\begin{aligned}2\tilde{\alpha}_k\tilde{c}_k\cdot e_n-\tilde{\alpha}_k trM_k&=f_{+k}(0)+2\alpha_k\epsilon_k c_n+2\alpha_k\epsilon_k pa_k\cdot e_n+2\alpha_k\epsilon_k^2 pc_n-\epsilon_k\alpha_k ptrM_k\\&\geq f_{+k}(0)+2\alpha_0\epsilon_k c_n+O(p\epsilon_k^{3/2}\delta_k),\end{aligned}$$

and the conclusion follows with an appropriate choice of C.

We will also choose $c'=d'=0$. Then, the third equation is satisfied for k large in view of (4.12).

In fact,

$$\begin{aligned}G_k(\tilde{\beta}_k)+\tilde{\beta}_k G_k'(\tilde{\beta}_k)\tilde{d}_k\cdot x'&\leq G_k(\beta_k)+\beta_k G_k'(\beta_k)q\epsilon_k+\beta_k G_k'(\beta_k)b_k\cdot x'+\epsilon_k O(|q|(\epsilon_k+\delta_k))\\&=\alpha_k+\alpha_k a_k\cdot x'+\beta_k G_k'(\beta_k)q\epsilon_k+\epsilon_k O(|q|(\epsilon_k+\delta_k)).\end{aligned}$$

Thus we need,

$$\alpha_k p-\beta_k G_k'(\beta_k)q\geq C(p,q)\delta_k+\delta_k^{1/2}$$

which is satisfied for k large in view of (4.12).

Define now, as in (4.1), $W_k^*:=\tilde{W}_k^{\epsilon_k}$. We observe that W_k^* converges uniformly to ϕ on $B_{1/2}$. Indeed, one can easily compute that in $B_1^+(W_k)$

$$W_k(x)-\alpha_k(1+a_k\cdot x)P_{M_k,e_n}=\alpha_k\epsilon_k(px_n+\frac{1}{2}x^TNx+\xi'\cdot x'+A)+\alpha_k\epsilon_k O(\delta_k)$$

and similarly, in $B_1^-(W_k)$

$$W_k(x)-\beta_k(1+b_k\cdot x)P_{M_k,e_n}=\beta_k\epsilon_k(qx_n+\frac{1}{2}x^TNx+\xi'\cdot x'+A)+\beta_k\epsilon_k O(\delta_k).$$

Since w_k converges uniformly to w, W_k^* converges uniformly to ϕ and ϕ touches w strictly by below at x_0 we can conclude that there exist a sequence of constants $t_k\to 0$ and of points $x_k\to x_0$ such that the function

$$\psi_k(x)=W_k(x+\epsilon_k t_k e_n)$$

touches u_k by below at x_k. We thus get a contradiction if we prove ψ_k is a strict subsolution to the free boundary problem satisfied by the u_k's. This follows from the fact that W_k is a strict subsolution and the translation in the $e_n$direction only perturbs A into $A+t_k$.

Step 3. Since w_k converges uniformly to w and $w(0)=0$ we get that

$$|w_k-\psi(x)|\leq\frac{1}{8}\bar{\eta}^{2+\gamma},\quad\text{in }B_{\bar{\eta}}$$

with

$$\psi(x)=px_n^+-qx_n^--\xi'\cdot x'-\frac{1}{2}x^TNx+\hat{a}\cdot xx_n^+-\hat{b}\cdot xx_n^-$$

and

$$\tilde{\alpha}p-\tilde{\beta}G_0'(\tilde{\beta})q=0,\quad trN=2\hat{a}_n=2\hat{b}_n,\quad Ne_n=0$$
$$\tilde{\alpha}\hat{a}'\cdot x'=\tilde{\beta}G_0'(\tilde{\beta})\hat{b}'\cdot x'.$$

Thus,

$$|u_k-\alpha_k(1+a_k\cdot x)P_{M_k,e_n}-\alpha_k\epsilon_k\psi(x)|\leq\frac{1}{8}\alpha_k\epsilon_k\bar\eta^{2+\gamma},\quad\text{in } B_{\bar\eta}\cap(B_1^+(u_k)\cup F(u_k)):=B_F^+$$

and

$$|u_k-\beta_k(1+b_k\cdot x)P_{M_k,e_n}-\beta_k\epsilon_k\psi(x)|\leq\frac{1}{8}\beta_k\epsilon_k\bar\eta^{2+\gamma},\quad\text{in } B^-_{\bar\eta}(u_k).$$

Hence, since in the region B_F^+ we have $x_n\geq-2\epsilon$, we conclude that

$$|u_k-\alpha_k(1+a_k\cdot x)P_{M_k,e_n}-\alpha_k\epsilon_k(px_n-\xi'\cdot x'-\frac{1}{2}x^TNx+\hat a\cdot xx_n)|\leq\frac{1}{4}\alpha_k\epsilon_k\bar\eta^{2+\gamma},\quad\text{in } B_F^+$$

and similarly

$$|u_k-\beta_k(1+b_k\cdot x)P_{M_k,e_n}-\beta_k\epsilon_k(qx_n-\xi'\cdot x'-\frac{1}{2}x^TNx+\hat b\cdot xx_n)|\leq\frac{1}{4}\beta_k\epsilon_k\bar\eta^{2+\gamma},\quad\text{in } B^-_{\bar\eta}(u_k).$$

Set

$$\alpha_k^*=\alpha_k(1-\epsilon_kp),\quad\beta_k^*=\beta_k(1-\epsilon_kq),$$
$$N_k^*=M_k+\epsilon_kN,\quad\nu_k^*=e_n+\epsilon_k\xi,\quad\xi_n=0,$$

and

$$a_k^*=a_k+\epsilon_k\hat a,$$
$$b_k^*=b_k+\epsilon_k\hat b.$$

Then,

$$|u_k-\alpha_k^*(1+a_k^*\cdot x)P_{N_k^*,\nu_k^*}|\leq\frac{1}{2}\alpha_k^*\epsilon_k\bar\eta^{2+\gamma},\quad\text{in } B_F^+$$

and

$$|u_k-\beta_k^*(1+b_k^*\cdot x)P_{N_k^*,\nu_k^*}|\leq\frac{1}{2}\beta_k^*\epsilon_k\bar\eta^{2+\gamma},\quad\text{in } B^-_{\bar\eta}(u_k).$$

Now choose,

$$\bar\nu_k=\frac{\nu_k^*}{|\nu_k^*|}=e_n+\epsilon_k\xi+\epsilon_k^2\tau,\quad|\tau|\leq C.$$

For notational simplicity, we drop the dependence on k from ν_k^* and $\bar\nu_k$.

We write

$$N_k^*=\bar N_k+L_k,\quad\bar N_k\cdot\bar\nu=0.$$

Then, since $M_ke_n=Ne_n=0$, we get

$$\|L_k\|=O(\epsilon_k^{3/2}).$$

Moreover set,

$$\bar\beta_k=\beta_k^*\quad\bar\alpha_k=G_k(\bar\beta_k)=\alpha_k^*+O(\epsilon_k^2)$$

where we have used that $\tilde\alpha p-\tilde\beta G_0'(\tilde\beta)q=0$.

Thus,

$$|u_k-\bar\alpha_k(1+a_k^*\cdot x)P_{\bar N_k,\bar\nu}|\leq\frac{2}{3}\bar\alpha_k\epsilon_k\bar\eta^{2+\gamma},\quad\text{in } B_F^+$$

and

$$|u_k-\bar\beta_k(1+b_k^*\cdot x)P_{\bar N_k,\bar\nu}|\leq\frac{2}{3}\bar\beta_k\epsilon_k\bar\eta^{2+\gamma},\quad\text{in } B^-_{\bar\eta}(u_k).$$

Let $S_{\bar\nu}:=\{\bar\nu_i\}_{i=1,\dots n}$ be an orthonormal system containing $\bar\nu$. Say, $\bar\nu=\bar\nu_n$. Recall that

$$\bar\nu_n=e_n+\epsilon\omega,\quad|\omega|\leq C,$$

hence

$$\bar\nu_i=e_i+\epsilon v,\quad|v|\leq C,\quad i\neq n$$

Define,

$$\bar{a}_k = a_k^* + \sum_{i=1}^{n} z_i \bar{\nu}_i$$

$$\bar{b}_k = b_k^* + \zeta \bar{\nu}_n$$

where z_i, ζ are chosen so that

$$2\bar{\alpha}_k \bar{a}_k \cdot \bar{\nu}_n - \bar{\alpha}_k tr \bar{N}_k = f_{+k}(0)$$

$$2\bar{\beta}_k \bar{b}_k \cdot \bar{\nu}_n - \bar{\beta}_k tr \bar{N}_k = f_{-k}(0), \quad \text{if } \beta_k > 0$$

and

$$\bar{\alpha}_k \bar{a}_k \cdot \bar{\nu}_i = \bar{\beta}_k G'_k(\bar{\beta}_k) \bar{b}_k \cdot \bar{\nu}_i, \quad i \neq n.$$

Unravelling these identities using all of the definitions above and the compatibility conditions for ψ we can estimate that

$$|z_i|, |\zeta| = O(\epsilon_k^{3/2}).$$

Therefore we conclude that

$$|u_k - \bar{\alpha}_k(1 + \bar{a}_k \cdot x) P_{\bar{N}_k, \bar{\nu}}| \leq \frac{4}{5} \bar{\alpha}_k \epsilon_k \bar{\eta}^{2+\gamma}, \quad \text{in } B_F^+$$

and

$$|u_k - \bar{\beta}_k(1 + \bar{b}_k \cdot x) P_{\bar{N}_k, \bar{\nu}}| \leq \frac{4}{5} \bar{\beta}_k \epsilon_k \bar{\eta}^{2+\gamma}, \quad \text{in } B_{\bar{\eta}}^-(u_k).$$

Set, $\bar{V}_k = V^{\bar{\alpha}_k, \bar{\beta}_k}_{\bar{N}_k, \bar{\nu}, \bar{a}_k, \bar{b}_k}$ we conclude that

$$\bar{V}(x - \epsilon_k \bar{\nu}) \leq u_k(x) \leq \bar{V}(x + \epsilon_k \bar{\nu}) \quad \text{in } B_{\bar{\eta}}$$

and we reach a contradiction as long as

$$\bar{\eta}|\bar{a}_k|, \bar{\eta}|\bar{b}_k \cdot \bar{\nu}^\perp|, \bar{\eta}\|\bar{M}_k\| \leq \delta_k (\epsilon_k \bar{\eta}^{1+\gamma})^{1/2}$$

and

$$\bar{\eta}|\bar{b}_k \cdot \bar{\nu}| \leq \delta_k^2, \quad \bar{\eta}^2 |\bar{b}_k \cdot \bar{\nu}| \|\bar{M}_k\| \leq \delta_k^2 (\epsilon_k \bar{\eta}^{1+\gamma}).$$

This follows (for $\bar{\eta}$ possibly smaller) from the initial bounds on $|a_k|, |b_k|, |M_k|$ and the fact that

$$|a_k - \bar{a}_k|, |b_k - \bar{b}_k|, \|M_k - \bar{M}_k\|, |e_n - \bar{\nu}| \leq C\epsilon_k.$$

□

Remark 4.2. We observe that it is enough for the free boundary condition to be satisfied in the following viscosity sense. At all points where u is touched globally by the positive side in $B_{2/3}$ by a test function with free boundary $\Gamma := \{x_n = \mathcal{P}\}$ with $\mathcal{P}$ a quadratic polynomial with coefficients of size 1, then

$$u_\nu^+ \leq G(u_\nu^-) \quad \text{if } \beta > 0, \quad u_\nu^+ - \alpha \leq \bar{\delta}^{1/2}\epsilon \quad \text{if } \beta = 0$$

and similarly the lower bound is satisfied at all points where u is touched by above on the free boundary by a surface Γ as before.

This can be easily seen from the proof (see the conclusion of Step 2 in the proof above.)

As a consequence we obtain the following two propositions. Let u be as at the beginning of this section and $0 \leq \beta \leq \beta_1, 1 \leq \alpha \leq \alpha_1$.

Proposition 4.3 (One-phase $C^{2,\gamma}$ improvement of flatness)**.** *There exist $\bar\eta, \bar\delta, \bar\lambda$ such that if for $\beta = 0$*

$$u^+ \text{ is } (V, \lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_\lambda, \lambda \leq \bar\lambda \tag{4.13}$$

with $V = V^{\alpha}_{M,e_n,a_n} \in \mathcal{V}_{f_+}$,

$$|f_+(x) - f_+(0)| \leq \bar\delta |x|^\gamma \tag{4.14}$$

and

$$|u^+_\nu - \alpha| \leq \bar\delta^{1/2} \lambda^{1+\gamma} \quad \text{on } F(u) \cap B_{2/3\lambda}, \tag{4.15}$$

in the viscosity sense, then

$$u^+ \text{ is } (\bar V, (\bar\eta\lambda)^{2+\gamma}, \bar\delta) \text{ in } B_{\bar\eta\lambda} \tag{4.16}$$

with $\bar V = V^{\alpha}_{\bar M, \bar\nu, \bar a_{\bar\nu}} \in \mathcal{V}_{f_+}$.

Proposition 4.4 (Two-phase $C^{2,\gamma}$ improvement of flatness)**.** *There exist $\bar\eta, \bar\delta, \bar\lambda$ universal, such that if for $\beta > 0$*

$$u \text{ is } (V, \lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_\lambda, \lambda \leq \bar\lambda \tag{4.17}$$

with $V = V^{\alpha,\beta}_{M,e_n,a,b} \in \mathcal{V}_{f_\pm,G}$,

$$|f_+(x) - f_+(0)| \leq \bar\delta |x|^\gamma, \quad |f_-(x) - f_-(0)| \leq \beta\bar\delta |x|^\gamma \tag{4.18}$$

and

$$u^+_\nu = G(u^-_\nu) \quad \text{on } F(u) \cap B_{2/3\lambda}$$

then

$$u \text{ is } (\bar V, (\bar\eta\lambda)^{2+\gamma}, \bar\delta) \text{ in } B_{\bar\eta\lambda} \tag{4.19}$$

with $\bar V = V^{\bar\alpha,\bar\beta}_{\bar M,\bar\nu,\bar a,\bar b} \in \mathcal{V}_{f_\pm,G}$ and $|\beta - \bar\beta| \leq C\lambda^{1+\gamma}$ for C universal.

5. Harnack inequality

In this section we prove a Harnack type inequality which is the key ingredient in the compactness argument used to prove our improvement of flatness proposition.

Let $V = V^{\alpha,\beta}_{M,e_n,a,b} \in \mathcal{V}_{f_\pm,G}$ with $0 \leq \beta \leq \beta_1, 1 \leq \alpha \leq \alpha_1$. Let u solve

$$\Delta u^+ = f_+ \quad \text{in } B_1^+(u), \quad \Delta u^- = f_- \quad \text{in } B_1^-(u), \quad \text{if } \beta > 0$$

or

$$\Delta u^+ = f_+ \quad \text{in } B_1^+(u), \quad \text{if } \beta = 0$$

with $0 \in F(u) \in C^{1,\gamma}$.

We need the following key lemma. The free boundary condition in the lemma is assumed to hold in the viscosity sense of Remark 4.2.

Lemma 5.1. *There exist $\bar\epsilon, \bar\delta$ universal such that*

(i) *Two-phase case: $\beta > 0$, if*

$$u \text{ is } (V, \epsilon, \bar\delta) \text{ flat in } B_1, \quad 0 < \epsilon \leq \bar\epsilon, \tag{5.1}$$

$$|f_+(x) - f_+(0)| \leq \bar\delta\epsilon, \quad |f_-(x) - f_-(0)| \leq \beta\bar\delta\epsilon, \tag{5.2}$$

and

$$u^+_\nu = G(u^-_\nu) \quad \text{on } F(u) \cap B_{2/3} \tag{5.3}$$

then either

$$u(x) \leq V(x + (1-\eta)\epsilon e_n) \quad \text{in } B_\eta$$

or

$$u(x) \geq V(x - (1-\eta)\epsilon e_n) \quad \text{in } B_\eta$$

for a small universal $\eta \in (0,1)$.

(ii) *One-phase case:* $\beta = 0$, *if*

$$u^+ \text{ is } (V, \epsilon, \bar{\delta}) \text{ flat in } B_1, \quad 0 < \epsilon \leq \bar{\epsilon}, \tag{5.4}$$

$$|f_+(x) - f_+(0)| \leq \bar{\delta}\epsilon, \tag{5.5}$$

and

$$|u_\nu^+ - \alpha| \leq \bar{\delta}^{1/2}\epsilon \quad \text{on } F(u) \cap B_{2/3} \tag{5.6}$$

then either

$$u^+(x) \leq V(x + (1-\eta)\epsilon e_n) \quad \text{in } B_\eta$$

or

$$u^+(x) \geq V(x - (1-\eta)\epsilon e_n) \quad \text{in } B_\eta$$

for a small universal $\eta \in (0,1)$.

Proof. To fix ideas let $\beta > 0$. The case $\beta = 0$ follows in the same way.

Let $\bar{x} = \frac{1}{5}e_n$ and assume that

$$u(\bar{x}) \geq V(\bar{x}) > 0. \tag{5.7}$$

We prove that the second statement holds.

Define:

$$\phi_t(x) := t + px_n^+ - 2nx_n^- + \frac{1}{2}x^T N x \tag{5.8}$$

where $N \in \mathcal{S}^{n\times n}$

$$N_{ii} = -2, \quad \text{if } i,j = 1, \dots n-1$$
$$N_{1,j} = 0 \quad \text{if } j = 1, \dots n-1$$
$$N_{nn} = 4n.$$

and

$$\alpha p = 1 + 2n\beta G'(\beta). \tag{5.9}$$

We show that there is a constant $r_0 \leq 1/16$ (r_0 universal) such that

(5.10)

$$\phi_{1/8} < -1/16 \quad \text{on } -1/2 \leq x_n \leq r_0, |x'| = 1/2 \text{ and on } x_n = -1/2, |x'| \leq 1/2.$$

Indeed, on the first region above, when $x_n > 0$ we have:

$$\phi_{1/8}(x) = 1/8 + 2nx_n^2 - 1/4 + px_n \leq -1/8 + (2n+p)r_0 < -1/16$$

as long as

$$(2n+p)r_0 \leq 1/16.$$

When $-1/2 \leq x_n \leq 0$ and $|x'| = 1/2$

$$\phi_{1/8}(x) = 1/8 + 2nx_n(x_n+1) - 1/4 < -1/16.$$

Finally, on $x_n = -1/2$ and $|x'| \leq 1/2$ we have

$$\phi_{1/8}(x) = 1/8 - |x'|^2 - n/2 < -1/16.$$

Finally notice that (η universal),

$$\phi_{1/8} \geq \frac{1}{16} \quad \text{in } B_\eta. \tag{5.11}$$

Call

$$D'' := \{|x'| \leq 2/3, r_0/2 \leq x_n \leq 2/5\}$$

and

$$D' := \{|x'| \leq 1/2, r_0/3 \leq x_n \leq 1/5\}.$$

Finally,

$$D := \{|x'| \leq 1/2, -1/2 \leq x_n \leq r_0\}.$$

Notice that (for ϵ small)

$$D'' \subset B_1^+(V^\epsilon) \subset B_1^+(u). \tag{5.12}$$

We have that $v(x) := u(x) - V(x - \epsilon e_n) \geq 0$ and because of (5.2) and the sizes of the coefficients of V, we have $\Delta v \geq -\bar{\delta}\epsilon - \bar{c}\alpha\bar{\delta}^2\epsilon$ in D''. Thus we can apply Harnack inequality to obtain

$$v \geq cv(\bar{x}) - C(\bar{\delta}\epsilon + \bar{c}\alpha\bar{\delta}^2\epsilon) \quad \text{in } D'.$$

From (5.7) we conclude that (for $\bar{\delta}$ small enough)

$$u(x) - V(x - \epsilon e_n) \geq c\alpha\epsilon - C\bar{\delta}\epsilon - \bar{c}\alpha\bar{\delta}^2\epsilon \geq \alpha c_0 \epsilon \quad \text{in } D'. \tag{5.13}$$

Now set,

$$\tilde{\alpha} = \alpha(1 + \epsilon c_0 p), \quad \tilde{\beta} = \beta(1 + \epsilon 2nc_0), \quad \tilde{M} = M - \epsilon c_0 N, \quad \tilde{c} = a + \epsilon c, \quad \tilde{d} = b + \epsilon d$$

with

$$c_i = d_i = 0 \quad \forall i = 1, \dots n-1, \quad c_n = d_n = O(\bar{\delta}),$$

and call

$$W_t := \tilde{\alpha}(1+\tilde{c}\cdot(x-\epsilon e_n+t\epsilon e_n))(P_{\tilde{M},e_n}-\epsilon+t\epsilon)^+ - \tilde{\beta}(1+\tilde{d}\cdot(x-\epsilon e_n+t\epsilon e_n))(P_{\tilde{M},e_n}-\epsilon+t\epsilon)^-.$$

Now, the same computation as in Step 2 of Proposition 4.1 guarantees that $\tilde{W}_t^\epsilon$ (defined as in (4.1)) converges uniformly to $-1 + c_0\phi_t$ as $\epsilon \to 0$. Clearly, $\tilde{V}^\epsilon(x - \epsilon e_n) = -1 + O(\bar{\delta}^{1/2}\epsilon)$. Since for $\bar{t} << 0$ we have $c_0\phi_{\bar{t}} \leq -2$ in D, we conclude that

$$W_{\bar{t}} \leq V(x - \epsilon e_n) \leq u \quad \text{on } D.$$

Let $\bar{s}$ be the largest t such that

$$W_t \leq u \quad \text{on } D.$$

We want to show that $\bar{s} \geq 1/8$. Then, in view of the uniform convergence of $\tilde{W}_{\bar{s}}^\epsilon$ to $c_0\phi_{\bar{s}}$ and the bound (5.11) we get

$$u(x) \geq W_{\bar{s}} \geq V(x - (1 - \bar{c})\epsilon e_n) \quad \text{in } B_\eta.$$

Assume that $\bar{s} < 1/8$. Then the first touching point $\tilde{x}$ of u and $W_{\bar{s}}$ in D, occurs on $x_n = r_0$. Indeed, as shown in Step 2 of Proposition 4.1, $W_{\bar{s}}$ is a strict subsolution to (1.1) which lies below u in D. Thus the touching point can only occur on ∂D.

By (5.10) and the uniform convergence of the $\tilde{W}_{\bar{s}}^\epsilon$ we have $W_{\bar{s}} < V^\epsilon \leq u$ on $\partial D \setminus \{x_n = r_0.\}$ Thus, again by the uniform convergence of the $\tilde{W}_{\bar{s}}^\epsilon$ and the fact that $\phi_{1/8} \leq 1/2$ on $x_n = r_0$, we get

$$u(\tilde{x}) = W_{\bar{s}}^\epsilon(\tilde{x}) < V^\epsilon(\tilde{x}) + \alpha c_0 \epsilon$$

which contradicts (5.13).

∎

Corollary 5.2. *Let u be as in Lemma* 5.1*. Then the modulus of continuity of $\tilde{u}^\epsilon$ is Hölder outside $[0,\sigma(\epsilon)]$ with $\sigma(\epsilon)\to 0$ as $\epsilon\to 0$.*

6. The dichotomy

Throughout this section, u is a (Lipschitz) solution to (1.1).

We introduce the class of functions $\mathcal{Q}_{f_-}$ defined as

$$Q_{p,q,\omega,M} = (x\cdot\omega - \frac{1}{2}x^T Mx)(p+q\cdot x) - \frac{1}{2}(f_-(0)+ptrM)(x\cdot\omega)^2,$$

with $p\in\mathbb{R}, q\in\mathbb{R}^n, M\in S^{n\times n}$, such that

$$q\cdot\omega = 0,\quad M\omega=0,\quad \|M\|\le 1.$$

In the degenerate case, we use these functions to approximate u^- in a $C^{2,\gamma}$ fashion at a smaller and smaller scale. The goal is to reach a scale ρ where u^- is trapped between two translations of Q of size $\rho^{2+\gamma^*}$. This would guarantee that the full u is $(V,\rho^{2+\gamma^*},\bar\delta)$ flat at that scale, which allows us to apply the improvement of flatness result of the non-degenerate case.

Initially u^+ is $(V,\bar\lambda^{2+\gamma},\bar\delta)$ flat while u^- is $C^{2,\gamma}$ close to the configuration $Q_{0,0,e_n,0}$. This closeness improves at a $C^{2,\gamma}$ rate until (possibly) the slope p of the approximating polynomial $Q=Q_{p,q,\omega,M}$ is no longer zero, say at scale λ. However, to obtain the desired flatness of u, we need to reach a scale $\rho=\lambda r$ for $r << \lambda^{1/\gamma}$. It is necessary to exploit also the information that the flatness of u^+ is in fact improving at a C^2 rate for a little while, hence allowing us to continue the iteration on the negative side and obtain that u^- is $C^{2,\gamma}$ close to a configuration Q at an even smaller scale. As already pointed out, in the case of the $C^{1,\gamma}$ estimates of [DFS] this issue was not present, as in the degenerate case it was sufficient to reach a scale $\rho=\lambda r$ with $r=\lambda^{\gamma/2}$. This makes the $C^{2,\gamma}$ proof much more sophisticated and technically involved.

We are ready to start developing the tools to apply the strategy described above.

The next lemma relates the closeness of u^- to a function in the class $\mathcal{Q}_{f_-}$ with a one-phase free boundary condition for u^+_ν of the type in Proposition 4.1-(ii) (one-phase improvement of flatness). The condition is satisfied in the viscosity sense of Remark 4.2. The proof relies on a variant of the pointwise $C^{1,\gamma}$ estimate, which we describe in Appendix A. We also, need the following easy remark.

Remark 6.1. We remark that

$$|Q_{p,q,\omega,M} - Q_{p,q,\omega,0}| \le C|p|r^2 + |q|r^3 \quad \text{in } B_r;$$

$$|Q_{p,q,e_n,0} - Q_{p,\tilde q,\omega,0}| \le (|p|r + (2|q| + |f_-(0)|)r^2)|e_n-\omega| \quad \text{in } B_r;$$

where $\tilde q$ is a rotation of q by the angle between e_n and ω.

Let $V=V^\alpha_{M,e_n,a_n}$ and $Q=Q_{p,q,e_n,M}$ with $\alpha=G(|p|)$.

Lemma 6.2. *Let u^+ be $(V,r^2\lambda^{2+\gamma},\bar\delta)$ flat in $B_{r\lambda}$,*

$$|f_-(x)-f_-(0)|\le\bar\delta|x|^\gamma,\quad \|f_-\|_\infty\le\bar\delta$$

and

$$|u^- - Q|\le\bar\delta^{1/2}(r\lambda)^{2+\gamma},\quad \text{in } B^-_{r\lambda}(u),$$

for $r\le 1$, with $p\le 0$, $|p|\sim(\bar\delta^{1/2}\lambda^{1+\gamma})$, $|q|=O(\bar\delta^{1/2}\lambda^\gamma)$, and

$$\bar\delta^{1/2}r^\gamma \ge 2\lambda^{1+\gamma}. \tag{6.1}$$

Then,

$$|u_\nu^+ - \alpha| \leq \bar{C}\bar{\delta}^{1/2} r\lambda^{1+\gamma} \quad \text{on } F(u) \cap B_{r\lambda} \tag{6.2}$$

in the viscosity sense, with $\bar{C}$ universal.

Proof. Let $\mathcal{P}(x)$ be a quadratic polynomial and let $x_n = \mathcal{P}(x)$ touch $F(u)$ by the negative side at $x_0 \in F(u) \cap B_{r\lambda/2}$, with the coefficients of $\mathcal{P}$ of size 1. Let ν be the normal to $F(u)$ at x_0 (pointing toward the positive phase). Then, we want to show that

$$u_\nu^+(x_0) \geq \alpha - \bar{\delta}^{1/2} r\lambda^{1+\gamma} \tag{6.3}$$

Denote $\nu^- = -\nu$. Since (for some t),

$$u_\nu^+(x_0) = G(u_\nu^-(x_0)) = G(|p|) + G'(t)(\nabla u^-(x_0)\cdot \nu^- - |p|),$$

it suffices to show that

$$\nabla u^-(x_0)\cdot \nu \leq p + C\bar{\delta}^{1/2} r\lambda^{1+\gamma}. \tag{6.4}$$

Let

$$u_{r\lambda}(x) := u(r\lambda x), \quad Q_{r\lambda}(x) := Q(r\lambda x), \quad \mathcal{P}_{r\lambda}(x) := \mathcal{P}(r\lambda x),$$

and set

$$v(x) := \bar{\delta}^{-1/2}(r\lambda)^{-(2+\gamma)}(u_{r\lambda}^- - Q_{r\lambda})(x), \quad x \in B_1, \tag{6.5}$$

Then,

$$|v| \leq 1, \quad |\Delta v| \leq 2\bar{\delta}^{1/2} \quad \text{in } B_1^-(u_{r\lambda}). \tag{6.6}$$

Moreover, since r satisfies (6.1), using the estimates for $|p|, |q|$ and the flatness of the free boundary, we obtain that

$$|v| \leq \bar{\delta}^{1/2} \quad \text{on } F(u_{r\lambda}).$$

We claim that

$$v_\nu(y_0) \leq C r^{-\gamma}, \quad y_0 = \frac{x_0}{r\lambda}.$$

To prove the claim, we wish to apply Theorem 8.3 in the Appendix to $-v$. Indeed, assume $F(u_{r\lambda})$ is the graph of a function g in the ν^- direction. Then, since $x_n = \frac{1}{r\lambda}\mathcal{P}_{r\lambda}$ touches $F(u_{r\lambda})$ at y_0 by the negative side we conclude that

$$g \leq C(r\lambda)|x-y_0|^2.$$

Thus, it suffices to prove that at the point y_0, for any $\nu^\perp$ perpendicular to ν^- we have

$$|\nabla \bar{Q}_{r\lambda}\cdot \nu^\perp| = O(r^{-\gamma}), \quad -O(r^{-\gamma}) \leq \nabla \bar{Q}_{r\lambda}\cdot \nu^- \leq O(\tfrac{r^{-\gamma}}{r\lambda}) \tag{6.7}$$

$$|D^2\bar{Q}_{r\lambda}(\nu,\nu^\perp)| = O(r^{-\gamma}), \quad |D^2\bar{Q}_{r\lambda}(\nu^-,\nu^-)| = O(\tfrac{r^{-\gamma}}{r\lambda}), \tag{6.8}$$

where we denoted

$$\bar{Q}_{r\lambda} = \bar{\delta}^{-1/2}(r\lambda)^{-(2+\gamma)}Q_{r\lambda}.$$

Indeed, it is easy to check that in $B_\rho \cap \{|x_n| < \rho^2\}$ we have:

$$\nabla Q = pe_n + O(|q|\rho + |p|\|M\|\rho + |f_-(0)|\rho^2 + |p||trM|\rho^2 + |q|\|M\|\rho^2)$$

and

$$D^2 Q = -pM - 2(\frac{1}{2}f_-(0) + ptrM)e_n\otimes e_n + O(|q| + \|M\||q|\rho).$$

In particular, for $\rho = r\lambda$, using the bounds for $|p|, |q|, |f_-(0)|$ we conclude that

$$\nabla Q = pe_n + O(\bar\delta^{1/2} r\lambda^{1+\gamma})$$

and

$$D^2 Q = -(f_-(0) + 2ptrM) e_n \otimes e_n + O(\bar\delta^{1/2}\lambda^\gamma).$$

Thus, we easily obtain the second estimate in (6.7)-(6.8) (recall that $p \leq 0$ and $e_n \cdot \nu^- \leq 0$). The first one follows by using that at y_0 we have $|\nu - e_n| \leq \lambda r$.

Hence the claim holds and rescaling back we get

$$(\nabla u^-(x_0) - \nabla Q(x_0)) \cdot \nu \leq C\bar\delta^{1/2} r\lambda^{1+\gamma}. \tag{6.9}$$

Moreover, at such point $|\nu - e_n| \leq \lambda r$ hence we have

$$\begin{aligned} |\nabla Q(x_0) \cdot \nu - p| &\leq |\nabla Q(x_0) \cdot \nu - \nabla Q(x_0) \cdot e_n| + |Q_n(x_0) - p| \\ &\leq \|\nabla Q\|_\infty |\nu - e_n| + O(\bar\delta^{1/2} r\lambda^{1+\gamma}) \leq Cr\bar\delta^{1/2}\lambda^{1+\gamma} \end{aligned}$$

and we reach the desired conclusion. □

In the next proposition we show that if we are in a degenerate setting, that is u^- is very close to the configuration $Q_{0,0,e_n,0}$, then either this is preserved at a smaller (universal) scale or u^- becomes close to a configuration $Q_{p,q,\mathbf{e},M}$ with a non-zero slope p. In either case the positive part u^+ also improves. Without loss of generality, we still denote the universal constants below as in previous propositions.

Proposition 6.3. *There exist universal constants $\bar\lambda, \bar\delta, \bar\eta$ such that if*

$$u^+ \text{ is } (V, \lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_\lambda, \lambda \leq \bar\lambda \tag{6.10}$$

with $V = V^1_{M,e_n,a_n} \in \mathcal{V}_{f_+}$,

$$|f_\pm(x) - f_\pm(0)| \leq \bar\delta |x|^\gamma, \quad \|f_-\|_\infty \leq \bar\delta \tag{6.11}$$

and

$$|u^- - Q_{0,0,e_n,0}| \leq \bar\delta^{1/2}\lambda^{2+\gamma}, \quad \text{in } B^-_\lambda(u) \tag{6.12}$$

then either one of the following holds:

(i) *there exists $\bar V = V^1_{\bar M, \bar{\mathbf{e}}, \bar a_{\bar{\mathbf{e}}}} \in \mathcal{V}_{f_+}$, such that*

$$u^+ \text{ is } (\bar V, (\bar\eta\lambda)^{2+\gamma}, \bar\delta) \text{ flat in } B_{\bar\eta\lambda}, \tag{6.13}$$

and

$$|u^- - Q_{0,0,\bar{\mathbf{e}},0}| \leq \bar\delta^{1/2}(\bar\eta\lambda)^{2+\gamma}, \quad \text{in } B^-_{\bar\eta\lambda}(u); \tag{6.14}$$

(ii) *there exists $V^* = V^{\alpha^*}_{M^*,\mathbf{e}^*,a^*_{\mathbf{e}^*}} \in \mathcal{V}_{f_+}$, such that*

$$u^+ \text{ is } (V^*, \bar\eta^2\lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_{\bar\eta\lambda},$$

and

$$|u^- - Q_{p^*,q^*,\mathbf{e}^*,M^*}| \leq \bar\delta^{1/2}(\bar\eta\lambda)^{2+\gamma}, \quad \text{in } B^-_{\bar\eta\lambda}(u),$$

for $\alpha^* = G(|p^*|)$ and $p^* < 0, |p^*| \sim (\bar\delta^{1/2}\lambda^{1+\gamma}), |q^*| = O(\bar\delta^{1/2}\lambda^\gamma)$.

Proof. All the universal constants will be specified throughout the proof. In particular, for $\bar{\eta}$ fixed, $\bar{\lambda} << \bar{\delta} << \bar{\eta}$ and they are small enough so that Proposition 4.3 can be applied.

In view of Lemma 6.2 and Proposition 4.3 (see also Remark 6.9), then (6.13) holds.

Now, for $x \in B_1$, set

$$\tilde{u}(x) = \frac{1}{\lambda}u(\lambda x), \quad \tilde{f}_-(x) = \lambda f_-(\lambda x)$$

$$v(x) = \bar{\delta}^{-1/2}\lambda^{-(1+\gamma)}(\tilde{u}^-(x) + \frac{1}{2}\tilde{f}_-(0)x_n^2).$$

Since $\tilde{u}^- \geq 0$ and $|\tilde{f}_-(0)| \leq \lambda\bar{\delta}$

$$v \geq -\lambda^{-\gamma}\frac{\bar{\delta}^{1/2}}{2}x_n^2. \tag{6.15}$$

If we prove that

$$|v| \leq \frac{1}{2}\bar{\eta}^{2+\gamma}, \quad \text{in } B^-_{\bar{\eta}}(\tilde{u}), \tag{6.16}$$

then in view of Remark 6.1, we can conclude that (6.14) holds as well, by choosing $\bar{\delta}$ small enough (depending on $\bar{\eta}$.)

From assumptions (6.10)-(6.11)-(6.12) we get (λ small)

$$F(\tilde{u}) \subset \{-\lambda \leq x_n \leq \lambda\} = S_\lambda,$$

$$|v| \leq 1 \quad \text{in } B^-_1(\tilde{u}), \quad |\Delta v| \leq \bar{\delta}^{1/2} \quad \text{in } B^-_1(\tilde{u}), \quad |v| \leq \delta^{1/2}\lambda \quad \text{on } F(\tilde{u}).$$

Using a barrier, we can prove that

$$|v| \leq C\lambda \quad \text{in } S_\lambda \cap B^-_{2/3}(\tilde{u}). \tag{6.17}$$

Indeed, let ξ satisfy

$$\Delta\xi = -\delta^{1/2} \quad \text{in } B_1 \cap \{x_n < \lambda\}$$

with

$$\xi = 0 \quad \text{on } x_n = \lambda, \quad \xi = 1 \quad \text{on } \partial B_1 \cap \{x_n < \lambda\}.$$

Then, by the maximum principle,

$$\xi + \delta^{1/2}\lambda \geq v \quad \text{in } B^-_1(\tilde{u}) \tag{6.18}$$

and

$$\xi \leq C|x_n - \lambda| \quad \text{in } B_{2/3} \cap \{x_n < \lambda\},$$

from which the desired upper bound follows. The lower bound is proved similarly.

Thus, for λ small (depending on $\bar{\eta}$), (6.16) holds in S_λ.

We now analyze what happens in $B_{\bar{\eta}} \cap \{x_n < -\lambda\}$. Denote by w the solution to

$$\Delta w = 0 \quad \text{in } B^- := B_{2/3} \cap \{x_n < 0\}$$

with boundary data

$$w = v \quad \text{on } \partial B^- \cap \{x_n < 0\}, \quad w = 0 \quad \text{on } \{x_n = 0\}.$$

Notice that,

$$|w| \leq C\lambda \quad \text{on } x_n = -\lambda.$$

This can be obtained using the barrier ξ above and (6.18) (and the corresponding upper bound for v). Thus, by the maximum principle

$$|w - v| \leq C(\lambda + \bar{\delta}^{1/2}) \quad \text{in } B_{1/2} \cap \{x_n \leq -\lambda\}$$

In particular, if

$$|w| \leq \frac{1}{4}\bar{\eta}^{2+\gamma}, \quad \text{in } B_{\bar{\eta}} \cap \{x_n \leq -\lambda\}, \tag{6.19}$$

then (6.16) holds as long as λ and $\bar{\delta}$ are small enough (depending on $\bar{\eta}$.)

We now determine under which conditions (6.19) is valid. First, expanding w near 0, we have

$$w(x) = x_n(p + q \cdot x + O(|x|^2)), \quad q \cdot e_n = 0, \quad |p|, |q| \leq C. \tag{6.20}$$

Thus, there exists $\bar{\eta}$ universal, such that if

$$|p| \leq \bar{\eta}^2, \quad |q| \leq \bar{\eta} \tag{6.21}$$

then (6.19) holds.

We now prove that:

$$p \geq -\bar{\eta}^4 \Rightarrow (6.19) \text{ holds } \Rightarrow (i)$$

$$p < -\bar{\eta}^4 \Rightarrow (ii).$$

First, we observe that

$$|v - w| \leq C(\lambda + \delta^{1/2}|x_n|), \quad \text{in } B_{1/2} \cap \{x_n \leq -\lambda\}. \tag{6.22}$$

Indeed, let ψ, ϕ solve

$$\Delta\psi = 0, \Delta\phi = -1 \quad \text{in } B_{2/3} \cap \{x_n < -\lambda\}$$

with

$$\psi = \phi = 0 \quad \text{on } \partial B_{2/3} \cap \{x_n < -\lambda\}$$

$$\psi = C\lambda, \phi = 0 \quad \text{on } B_{2/3} \cap \{x_n = -\lambda\}.$$

Then,

$$v - w \leq \psi + \delta^{1/2}\phi \quad \text{in } B_{2/3} \cap \{x_n \leq -\lambda\}$$

and the desired upper bound follows. The lower bounds is obtained similarly.

We now distinguish three cases.

Case 1. $|p| \leq \bar{\eta}^4$.

In this case we show that $|q| \leq \bar{\eta}$ hence (6.21) is satisfied.

Indeed, assume $|q| > \bar{\eta}$, say to fix ideas $|q_1| > \frac{1}{\sqrt{n}}\bar{\eta}$. Let $\bar{x} = ((sign q_1)\bar{\eta}^2, 0, \ldots, -\lambda^\beta)$ with $\beta = (1+\gamma)/2$. Then, using (6.15)-(6.22) we get

$$-\frac{1}{2}\lambda\bar{\delta}^{1/2} \leq v(\bar{x}) \leq w(\bar{x}) + C\lambda + C\bar{\delta}^{1/2}\lambda^\beta$$

from which, using (6.20), we deduce that (for $\bar{\lambda}, \bar{\delta} << \bar{\eta}$)

$$C\lambda \geq \lambda^\beta(p + |q_1|\bar{\eta}^2 - \bar{C}\bar{\eta}^4 - C\bar{\delta}^{1/2}) \geq C\bar{\delta}^{1/2}\lambda^\beta$$

and we reach a contradiction if $\bar{\lambda}^{1-\gamma} << \bar{\delta}$.

Case 2. $p > \bar{\eta}^4$.

In this case we can argue similarly as in Case 1, by choosing $\bar{x} = (0, -\lambda^\beta)$.

Case 3. $p < -\bar{\eta}^4$.

In this case, we first notice that in view of (6.17)-(6.22) and the linear growth of w (extended to zero in $x_n > 0$), we have

$$|v - w| \leq C(\lambda + \bar{\delta}^{1/2}) \quad \text{in } B^-_{\bar{\eta}}(\tilde{u}).$$

Moreover, in view of (6.20) we also have that ($B^-_{\bar\eta}(\tilde u) \subset \{x_n < \lambda\}$)

$$|w - x_n(p + q\cdot x)| \leq C\bar\eta^3 \quad \text{in } B^-_{\bar\eta}(\tilde u).$$

(for λ small.)

Combining these two inequalities and using the formula for v and rescaling back, we obtain

$$|u^- - Q_{\bar p,\bar q,e_n,0}| \leq \frac{1}{2}\bar\delta^{1/2}(\bar\eta\lambda)^{2+\gamma} \quad \text{in } B^-_{\bar\eta\lambda}(u)$$

with

$$\bar p = \bar\delta^{1/2}\lambda^{1+\gamma}p, \quad \bar q = \bar\delta^{1/2}\lambda^{\gamma}q.$$

Thus, in view of Remark 6.1, if $\bar{\mathbf{e}}, \bar M$ are given by (6.13) (which we have already observed to hold), then

$$|u^- - Q_{\bar p,q^*,\bar{\mathbf{e}},\bar M}| \leq \bar\delta^{1/2}(\bar\eta\lambda)^{2+\gamma} \quad \text{in } B^-_{\bar\eta\lambda}(u), \quad |\bar q - q^*| \leq C\lambda^{1+\gamma}.$$

Finally, setting $\alpha^* = G(|\bar p|)$, and letting $V^* = V^{\alpha^*}_{\bar M,\bar{\mathbf{e}},a^*_{\bar{\mathbf{e}}}}$ we obtain the desired conclusion in (ii) (again for $\bar\delta << \bar\eta$). Here a^* is chosen so that $V^* \in \mathcal{V}_{f_+}$, i.e.

$$2\alpha^* a^*\cdot\bar{\mathbf{e}} - \alpha^* tr\bar M = f_+(0).$$

Thus, the claim follows from the fact that

$$|\alpha^* - 1| = O(\delta^{1/2}\lambda^{1+\gamma}), \quad |a^* - \bar a| = O(|\alpha^* - 1||2\bar a_{\bar{\mathbf{e}}} - tr\bar M|),$$

with

$$r|\bar a|, r\|M\| \leq \bar\delta r^{\frac{1+\gamma}{2}}, \quad r = \bar\eta\lambda,$$

and one can easily check that

$$|V^1_{\bar M,\bar{\mathbf{e}},\bar a_{\bar{\mathbf{e}}}} - V^{\alpha^*}_{\bar M,\bar{\mathbf{e}},a^*_{\bar{\mathbf{e}}}}| \leq C(|\alpha^* - 1|r + |a^* - \bar a|r^2) \leq \frac{1}{2}\bar\eta^2\lambda^{2+\gamma} \quad \text{in } B_r.$$

□

7. The iteration.

In the next proposition we show that if we are in an "intermediate degenerate setting", that is if at a small enough scale $u^+$is flat and u^- is close to a configuration Q with a small non-zero slope, then the flatness of u^+ improves for a little while at a C^2 rate while the estimate on u^- improves at a $C^{2,\gamma}$ rate.

Let $V = V^\alpha_{M,e_n,a_n} \in \mathcal{V}_{f_+}$ and $Q = Q_{p,q,e_n,M}$ with $\alpha = G(|p|)$.

Proposition 7.1. *There exist universal constants $\bar\lambda, \bar\delta, \bar\eta$, such that if*

$$u^+ \text{ is } (V, r^2\lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_{r\lambda}, \lambda \leq \bar\lambda \tag{7.1}$$

for some $r \leq \bar\eta$ with $\bar\delta^{1/2}r^\gamma \geq 2\lambda^{1+\gamma}$, and

$$|f_\pm(x) - f_\pm(0)| \leq \bar\delta|x|^\gamma, \quad \|f_-\|_\infty \leq \bar\delta \tag{7.2}$$

$$|u^- - Q| \leq \bar\delta^{1/2}(r\lambda)^{2+\gamma}, \quad \text{in } B^-_{r\lambda}(u), \tag{7.3}$$

with $p < 0, |p| \sim \bar\delta^{1/2}\lambda^{1+\gamma}, |q| = O(\bar\delta^{1/2}\lambda^\gamma)$, then

$$u^+ \text{ is } (\bar V, (\bar\eta r)^2\lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_{r\bar\eta\lambda}, \tag{7.4}$$

and

$$|u^- - \bar Q| \leq \bar\delta^{1/2}(r\bar\eta\lambda)^{2+\gamma}, \quad \text{in } B^-_{\lambda r\bar\eta}(u) \tag{7.5}$$

with $\bar V = V^{\bar\alpha}_{\bar M,\bar{\mathbf{e}},\bar a_{\bar{\mathbf{e}}}} \in \mathcal{V}_{f_+}$, $\bar Q = Q_{\bar p,\bar q,\bar{\mathbf{e}},\bar M}$, $\bar\alpha = G(|\bar p|)$, $\bar p < 0, |\bar p| \sim \bar\delta^{1/2}\lambda^{1+\gamma}, |\bar q| = O(\bar\delta^{1/2}\lambda^{\gamma})$.

Proof. Call,

$$\tilde u(x) = \frac{1}{r\lambda}u(r\lambda x), \quad \tilde V(x) = \frac{1}{r\lambda}V(r\lambda x), \quad \tilde Q(x) = \frac{1}{r\lambda}Q(r\lambda x) \quad x \in B_1.$$

Step 1. As usual, universal constants are small enough so that previous results can be applied. They will be made smaller in the proof, with $\bar\lambda << \bar\delta << \bar\eta$. Let,

$$\epsilon = r\lambda^{1+\gamma}.$$

In view of Lemma 6.2, $\tilde u/\alpha$ satisfies the assumptions of Proposition 4.1 (see also Remark 4.2) hence

$$\tilde u^+ \text{ is } (\tilde V^*, \bar\eta^{2+\gamma}\epsilon, \bar\delta) \text{ flat in } B_{\bar\eta} \tag{7.6}$$

with $\tilde V^* = V^{\alpha}_{M^*,\bar{\mathbf{e}},a^*_n}$. Let $\bar M = M^*/(r\lambda), \bar a = a^*/(r\lambda)$.

Step 2. Let

$$v(x) = \bar\delta^{-1/2}(r\lambda)^{-(1+\gamma)}(\tilde u^-(x) - \tilde Q(x)).$$

We argue similarly as in Proposition 6.3.

From assumptions (7.1)-(7.2)-(7.3) and the estimates for the sizes of p, q, r we get

$$F(\tilde u) \subset \{-r\lambda \le x_n \le r\lambda\} = S_{r\lambda},$$
$$|v| \le 1 \quad \text{in } B_1^-(\tilde u), \quad |\Delta v| \le \bar 2\delta^{1/2} \quad \text{in } B_1^-(\tilde u), \quad |v| \le \bar\delta^{1/2} \quad \text{on } F(\tilde u).$$

Using a barrier, we can prove that

$$|v| \le C\lambda + \bar\delta^{1/2} \quad \text{in } S_{r\lambda} \cap B^-_{2/3}(\tilde u). \tag{7.7}$$

Indeed, let ξ satisfy

$$\Delta\xi = -\delta^{1/2} \quad \text{in } B_1 \cap \{x_n < r\lambda\}$$

with

$$\xi = 0 \quad \text{on } x_n = r\lambda, \quad \xi = 1 \quad \text{on } \partial B_1 \cap \{x_n < r\lambda\}.$$

Then, by the maximum principle,

$$\xi + \bar\delta^{1/2} \ge v \quad \text{in } B_1^-(\tilde u) \tag{7.8}$$

and

$$\xi \le C|x_n - r\lambda| \quad \text{in } B_{2/3} \cap \{x_n < r\lambda\},$$

from which the desired upper bound follows. The lower bound is proved similarly.

We now analyze the region where $x_n \le -r\lambda$.

Denote by w the solution to

$$\Delta w = 0 \quad \text{in } B^+ := B_{2/3} \cap \{x_n < 0\}$$

with boundary data

$$w = v \quad \text{on } \partial B^+ \cap \{x_n < 0\}, \quad w = 0 \quad \text{on } \{x_n = 0\}.$$

Using the barrier ξ above and (7.8) we get that

$$|w| \le C\lambda + \bar\delta^{1/2} \quad \text{on } B_{2/3} \cap \{x_n = -r\lambda\}.$$

Hence by the maximum principle, $(\lambda << \bar\delta)$

$$|v - w| \le C\bar\delta^{1/2} \quad \text{on } B_{2/3} \cap \{x_n < -r\lambda\}. \tag{7.9}$$

In fact, in view of (7.7) and the linear growth of w we get (say $w = 0$ on $x_n > 0$)

$$|v - w| \leq C\bar{\delta}^{1/2} \quad \text{on } B^-_{1/2}(\tilde{u}). \tag{7.10}$$

We now expand w near 0, i.e.

$$w(x) = x_n(p^* + q^* \cdot x + O(|x|^2)), \quad q^* \cdot e_n = 0, \quad |p^*|, |q^*| \leq C. \tag{7.11}$$

Hence, for $\bar{\eta}$ small universal, and $\lambda << \bar{\eta}$, ($B^-_{\bar{\eta}}(\tilde{u}) \subset \{x_n < r\lambda\}$)

$$|w - x_n(p^* + q^* \cdot x)| \leq \frac{1}{2}\bar{\eta}^{2+\gamma} \quad \text{in } B^-_{\bar{\eta}}(\tilde{u}). \tag{7.12}$$

In this case, using the formula for v and rescaling back, we obtain from (7.10)-(7.12) that

$$|u^- - Q_{\bar{p},\bar{q},e_n,M}| \leq \frac{4}{5}\bar{\delta}^{1/2}(\bar{\eta}r\lambda)^{2+\gamma} \quad \text{in } B^-_{\bar{\eta}r\lambda}(u),$$

with

$$\bar{p} = \bar{\delta}^{1/2}(r\lambda)^{1+\gamma}p^* + p, \quad \bar{q} = \bar{\delta}^{1/2}(r\lambda)^{\gamma}q^* + q.$$

Using that $r \leq \bar{\eta}$, it easily follow that for $\bar{\eta}$ small universal,

$$\bar{p} < 0, \quad |\bar{p}| \sim \bar{\delta}^{1/2}\lambda^{1+\gamma}$$

and clearly

$$|\bar{q}| = O(\bar{\delta}^{1/2}\lambda^{\gamma}).$$

If we replace M with $\bar{M}$, then for $\bar{\delta} << \bar{\eta}$, the error E has size

$$E = O(\|M - \bar{M}\||x|^2(|\bar{p}| + |\bar{q}||x|)) \leq \frac{1}{10}\bar{\delta}^{1/2}(\bar{\eta}r\lambda)^{2+\gamma}$$

where in the last inequality we used that $\lambda^{1+\gamma} \leq \frac{1}{2}\bar{\delta}^{1/2}r^{\gamma}$. Similarly, if we now replace e_n with $\bar{\mathbf{e}}$ and $\bar{q}$ with the corresponding $\tilde{q}$ such that $\tilde{q} \cdot \bar{\mathbf{e}} = 0$ we get an error E of size

$$E = O(|x||e_n - \bar{\mathbf{e}}|(|\bar{p}| + |\bar{q}||x|) + (|f_-(0)| + |\bar{p}|\|\bar{M}\|)|x|^2|e_n - \bar{\mathbf{e}}|^2 + |\bar{q} - \tilde{q}|(|x| + \|M\||x^2|))$$

and again

$$E \leq \frac{1}{10}\bar{\delta}^{1/2}(\bar{\eta}r\lambda)^{2+\gamma}$$

using that $\lambda^{1+\gamma} \leq \frac{1}{2}\bar{\delta}^{1/2}r^{\gamma}$. Thus,

$$|u^- - Q_{\bar{p},\tilde{q},\bar{\mathbf{e}},\bar{M}}| \leq \bar{\delta}^{1/2}(\bar{\eta}r\lambda)^{2+\gamma}$$

and $|\tilde{q}| = O(\bar{\delta}^{1/2}\lambda^{\gamma})$. Finally, let $\bar{\alpha} = G(|\bar{p}|)$. Then,

$$|\alpha - \bar{\alpha}| = O(|p - \bar{p}|) = O(\bar{\delta}^{1/2}(r\lambda)^{1+\gamma}).$$

Thus, dropping the dependence on the subscripts $\bar{a}, \bar{\mathbf{e}}, \bar{M}$

$$|V^{\alpha} - V^{\bar{\alpha}}| \leq C|\alpha - \bar{\alpha}|\bar{\eta}r\lambda \leq C\bar{\delta}^{1/2}(r\lambda)^{2+\gamma}\bar{\eta}$$

and for $\bar{\eta}$ small universal and $\bar{\delta} << \bar{\eta}$

$$V^{\bar{\alpha}}(x + \bar{\eta}^{2+\gamma}r^2\lambda^{2+\gamma}\bar{\mathbf{e}}) + C\bar{\delta}^{1/2}(r\lambda)^{2+\gamma}\bar{\eta} \leq V^{\bar{\alpha}}(x + \frac{1}{2}\bar{\eta}^2r^2\lambda^{2+\gamma}\bar{\mathbf{e}}).$$

Hence, scaling back, we conclude from (7.6) that (7.4) holds for $\bar{V} = V^{\bar{\alpha}}_{\bar{M},\bar{\mathbf{e}},\bar{a}_{\bar{\mathbf{e}}}}$ (arguing similarly for the lower bound.) As at the end of the proof of Proposition 6.3 we can now modify $\bar{a}$ slightly to guarantee that $\bar{V} \in \mathcal{V}_{f_+}$. □

Finally, in the proposition below, we show that after reaching a small enough scale, the approximation of u^- with a configuration Q is good enough to recover the full C^{2,γ^*} flatness of u (in a non-degenerate setting.)

Proposition 7.2. *There exist $\bar\lambda, \bar\delta, \gamma^*$ universal such that if*

$$u^+ \text{ is } (V, r^2\lambda^{2+\gamma}, \bar\delta) \text{ flat in } B_{r\lambda}, \lambda \leq \bar\lambda \tag{7.13}$$

with $V = V^{\alpha}_{M,e_n,a_n}$, for r with $\bar\delta^{1/2} r^\gamma \in [2\bar\eta^\gamma \lambda^{1+\gamma}, 2\lambda^{1+\gamma})$ and

$$|u^- - Q_{p,q,e_n,M}| \leq \bar\delta^{1/2}(r\lambda)^{2+\gamma}, \quad \text{in } B^-_{r\lambda}(u), \tag{7.14}$$

for $\alpha = G(|p|)$ and $p < 0, |p| \sim \bar\delta^{1/2}\lambda^{1+\gamma}, |q| = O(\bar\delta^{1/2}\lambda^\gamma)$, then

$$u \text{ is } (\bar V, (r\lambda)^{2+\gamma^*}, \bar\delta) \text{ flat in } B_{r\lambda} \tag{7.15}$$

with $\bar V = V^{\alpha,\beta}_{M,e_n,a,b} \in \mathcal{V}_{f_\pm,G}, \beta = |p|$.

Proof. Let $\bar\lambda, \bar\delta$ be the constants in Proposition 7.1 , with $\bar\lambda << \bar\delta$ to be made possibly smaller. Let γ^* be given, to be specified later.

Call

$$W_\beta := \beta(1 + b\cdot x)(x_n - \frac{1}{2}x^T Mx)$$

with

$$\beta = |p|, \quad b' = \frac{1}{|p|}q', \quad 2\beta b_n = \beta trM + f_-(0).$$

Then it follows from (7.14) that

$$|u - W_\beta| \leq C\bar\delta^{1/2}(r\lambda)^{2+\gamma}, \quad \text{in } B^-_{r\lambda}(u). \tag{7.16}$$

Hence,

$$W_\beta(x - (r\lambda)^{2+\gamma^*}e_n) \leq u \leq W_\beta(x + (r\lambda)^{2+\gamma^*}e_n) \quad \text{in } B^-_{r\lambda}(u), \tag{7.17}$$

as long as

$$r \leq C\lambda^{\frac{1+\gamma^*}{\gamma-\gamma^*}}. \tag{7.18}$$

Moreover, call

$$\epsilon := (r\lambda)^{1+\gamma^*},$$

then

$$(r\lambda)|b'| \leq \bar\delta^2 \quad (r\lambda)^2|b_n|\|M\| \leq \bar\delta^2\epsilon \tag{7.19}$$

as long as

$$r \leq C\lambda^{\frac{\gamma+\gamma^*}{1-\gamma^*}}. \tag{7.20}$$

Now, let

$$W_\alpha := \alpha(1 + a\cdot x)(x_n - \frac{1}{2}x^T Mx)^+$$

with

$$\alpha a' := \beta G'(\beta)b'.$$

Then, $\alpha a' = O(|q'|)$ and in view of (7.13) we get

$$W_\alpha(x - Cr^2\lambda^{2+\gamma}e_n) \leq u^+(x) \leq W_\alpha(x + Cr^2\lambda^{2+\gamma}e_n) \quad \text{in } B_{r\lambda} \tag{7.21}$$

and conclude that

$$W_\alpha(x - (r\lambda)^{2+\gamma^*}e_n) \leq u^+(x) \leq W_\alpha(x + (r\lambda)^{2+\gamma^*}e_n) \quad \text{in } B_{r\lambda} \tag{7.22}$$

as long as

$$r \geq C\lambda^{\frac{\gamma}{\gamma^*}-1}. \tag{7.23}$$

Notice that all bounds on r are satisfied as long as

$$\gamma^* < \frac{\gamma^2}{1+2\gamma} \tag{7.24}$$

and $\bar{\lambda}$ is small enough. Now the conclusion follows combining (7.17) and (7.22). $\square$

We conclude this section by exhibiting the proof of our main Theorem 1.1.

Proof of Theorem 1.1. According to Lemma 3.2, after rescaling, we can assume that u satisfies either the assumptions of Proposition 6.3 or Proposition 4.4 with $\lambda = \bar{\lambda}$ (say $0 \in F(u)$). In the latter case, we can apply Proposition 4.4 indefinitely. If u falls in the degenerate case of Proposition 6.3, then either we can iterate the conclusion (i) indefinitely or we denote by $\lambda^* = \bar{\eta}^k\bar{\lambda}$ the first value for which (ii) holds i.e, without loss of generality,

$$u^+ \text{ is } (V, \bar{\eta}^2\lambda^{*2+\gamma}, \bar{\delta}) \text{ flat in } B_{\bar{\eta}\lambda^*},$$

for some $V = V^{\alpha,\beta}_{M,e_n,a,b} \in \mathcal{V}_{f_\pm,G}$

$$|u^- - Q_{p,q,e_n,M}| \leq \bar{\delta}^{1/2}(\bar{\eta}\lambda^*)^{2+\gamma}, \quad \text{in } B^-_{\bar{\eta}\lambda^*}(u),$$

for $\alpha = G(|p|)$ and $p < 0, |p| \sim (\bar{\delta}^{1/2}\lambda^{*1+\gamma}), |q| = O(\bar{\delta}^{1/2}\lambda^{*\gamma})$.

We now follow under the assumptions of Proposition 7.1 or possibly Proposition 7.2, with $r = \bar{\eta}$, $\lambda = \lambda^*$. We apply the conclusion of Proposition 7.1 till the first $\bar{r} = \bar{\eta}^{m_0}$ (possibly $m_0 = 1$) for which $\bar{\delta}^{1/2}\bar{r}^\gamma \in [2\bar{\eta}^\gamma\lambda^{*1+\gamma}, 2\lambda^{*1+\gamma})$. Then we conclude by Proposition 7.2 that

$$u \text{ is } (\bar{V}, (\bar{r}\lambda)^{2+\gamma^*}, \bar{\delta}) \text{ flat in } B_{\bar{r}\lambda} \tag{7.25}$$

with $\bar{V} = V^{\alpha,\beta}_{M,e_n,a,b} \in \mathcal{V}_{f_\pm,G}, \beta = |p|$ and we can apply indefinitely Proposition 4.4. To guarantee the C^{2,γ^*} improvement, we have to check that as r decreases from $\bar{\eta}$ to $\bar{r}$ we have

$$\lambda^{*2+\gamma}r^2 \leq (\lambda^* r)^{2+\gamma^*}.$$

Thus we need,

$$\bar{r} \geq (\lambda^*)^{\frac{\gamma-\gamma^*}{\gamma^*}},$$

which follows from the fact that (see (7.24)),

$$\gamma^* < \frac{\gamma^2}{1+2\gamma}.$$

8. Appendix A

In this short section we recall standard pointwise $C^{1,\alpha}$ estimates for solutions to elliptic equations in $C^{1,\alpha}$ domains (see for example [MW] for further details.). We also presents a few variants which are needed in the previous section.

Let

$$\Omega := \{x_n > g(x')\} \cap B_1 \quad \Gamma := \{x_n = g(x')\} \cap B_1, \quad g(0) = 0, \quad \nabla_{x'} g(0) = 0$$

with $g \in C^{1,\alpha}$ and say

$$|g| \leq |x|^{1+\alpha}. \tag{8.1}$$

Let u be a bounded solution to

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma, \end{cases} \tag{8.2}$$

for $f \in L^\infty(\Omega)$, $\varphi \in C^{1,\alpha}(\Gamma)$.

Theorem 8.1. *Assume that*

$$|\varphi(x) - l(x')| \le |x|^{1+\alpha} \tag{8.3}$$

with l a linear function. If

$$\|l\|_\infty, \|u\|_\infty, \|f\|_\infty \le 1 \quad \text{in } \Omega, \tag{8.4}$$

then u is $C^{1,\alpha}$ at 0 and

$$|\nabla u(0)| \le C \tag{8.5}$$

with $C = C(n, \alpha)$.

It follows also that

$$\partial_i u(0) = \partial_i l \quad i \ne n.$$

Remark 8.2. It easy to see that if (8.3) is replaced by

$$\varphi(x) - l(x') \le |x|^{1+\alpha} \quad (\text{rsp. } \ge |x|^{1+\alpha}) \tag{8.6}$$

then the following conclusion holds: $\partial_n u(0)$ exists and

$$\partial_n u(0) \le C.$$

Indeed we can apply Theorem 8.1 to the function v which solves problem (8.2) with φ replaced by $l(x') + |x|^{1+\alpha}$. By the maximum principle $u \le v$, and the conclusion follows.

We need the following refinement of this remark. Let φ be defined in B_1.

Theorem 8.3. *Assume that*

$$-|x|^{1+\alpha} \le g \le \sigma |x|^{1+\alpha} \tag{8.7}$$

for some small $\sigma > 0$. If $\|u\|_\infty, \|f\|_\infty \le 1$, and $\varphi \in C^2$ satisfies

$$|\varphi(0)|, |\partial_i \varphi(0)| \le 1 \quad i \ne n \tag{8.8}$$

$$-1 \le \partial_n \varphi(0) \le \frac{1}{\sigma} \tag{8.9}$$

$$|\partial_{ij} \varphi| \le 1 \quad \text{in } B_1, \quad (i,j) \ne (n,n) \tag{8.10}$$

and

$$|\partial_{nn} \varphi| \le \frac{1}{\|g\|_\infty} \tag{8.11}$$

then $\partial_n u(0)$ exists and

$$\partial_n u(0) \le C.$$

To obtain this estimate it suffices to apply the Remark 8.2 with

$$l(x') = \varphi(0) + \sum_{i\neq n} \partial_i \varphi(0) x_i.$$

Indeed

$$\varphi(x) - l(x') \leq \partial_n \varphi(0) x_n + C|x|^2 + \frac{1}{\|g\|_\infty} x_n^2 \leq C|x|^{1+\alpha}.$$

A similar statement holds when g satisfies the inequality

$$g \geq -\sigma |x|^{1+\alpha}. \tag{8.12}$$

9. Appendix B

For the reader convenience we recall the technique of [KNS] to transform a general (possibly nonlinear) two-phase free boundary problem into an elliptic system with coercive boundary conditions.

Let u be a classical solution to a two-phase free boundary problem governed by a second order elliptic equation, say in B_1 with $0 \in F(u)$. For σ small, the partial hodograph map

$$y' = x', \quad y_n = u^+(x)$$

is $1-1$ from $\overline{B_1^+}(u) \cap B_\sigma(0)$ onto a neighborhood of the origin $U \subset \{y_n \geq 0\}$, and flattens $F(u)$ into a set $\Sigma \subset \{y_n = 0\}$. The inverse mapping is the partial Legendre transformation

$$x' = y', \quad x_n = \psi(y),$$

where ψ satisfies $y_n = u^+(y', \psi(y))$, $y \in U$. The free boundary is now the graph of $x_n = \psi(y', 0)$.

Concerning the negative part, let C be a constant larger than $\partial_{y_n}\psi$ in U. Introduce the reflection map

$$x' = y', \quad x_n = \psi(y) - Cy_n,$$

which is $1-1$ from a neighborhood of the origin $U_1 \subseteq U$ onto $\overline{B_1^-}(u) \cap B_\sigma(0)$ (choosing σ smaller, if necessary). Define in U_1

$$\phi(y) = u^-(y', \psi(y) - Cy_n).$$

It is easily seen that the x derivatives of $u^\pm$ are expressed in terms of the y derivatives of ψ and ϕ. Since u is a solution to a two-phase problem, it follows that ψ and ϕ solve in U_1 a nonlinear system of the type

$$\begin{cases} \mathcal{F}_1(D^2\psi, \nabla\psi, \psi, y) = 0 \\ \mathcal{F}_2(D^2\phi, D^2\psi, \nabla\phi, \nabla\psi, \psi, y) = 0 \end{cases} \tag{9.1}$$

Moreover the free boundary conditions

$$u^+ = u^- \quad \text{and} \quad |\nabla u^+| = G(|\nabla u^-|), \quad \text{on } F(u)$$

become (for an appropriate $\tilde{G}$)

$$\begin{cases} \phi(y') = 0 \quad \text{on } y_n = 0, \\ \\ \partial_{y_n}\psi = \tilde{G}\left(\partial_{y_n}\phi, \nabla_{y'}\psi\right) \quad \text{on } y_n = 0. \end{cases} \tag{9.2}$$

Since $|\nabla u^{+}| > 0$ on $F(u)$ and G is strictly increasing, the system (9.1) is elliptic with coercive boundary conditions (9.2), under the natural choice of weights (see [KNS], p. 94, 95).

In the particular case when the equation governing the problem is in divergence form, then (9.1) will also be in divergence form. On the other hand, if the system (9.1)-(9.2) has no special structure, then higher regularity follows by classical results on elliptic-coercive systems in [ADN, M], as long as u is in $C^{2,\alpha}(\overline{B_1^+(u)}) \cap C^{2,\alpha}(\overline{B_1^-(u)})$.

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