The Hamiltonian Generating Quantum Stochastic Evolutions in the Limit from Repeated to Continuous Interactions

Matteo Gregoratti

Department of Mathematics "F. Brioschi", Politecnico di Milano
Piazza Leonardo da Vinci 32, I-20133 Milano, Italy

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1. Introduction

Quantum Stochastic Calculus was founded in the '80 by Hudson and Parthasarathy as a noncommutative generalization of Itō calculus [18, 24]. Stochastic processes are generalized by adapted families of operators acting on $\mathcal{H}\otimes\Gamma$, the tensor product between a complex separable Hilbert space $\mathcal{H}$, the initial space, and the symmetric Fock space Γ over $L^2(\mathbb{R};\mathfrak{Z})$, $\mathfrak{Z}$ being another complex separable Hilbert space, the multiplicity space. One of the first achievements of the new calculus was the introduction of Quantum Stochastic Differential Equations (Hudson-Parthasarathy equation) defining Quantum Stochastic Evolutions V_t, $t \geq 0$, strongly continuous unitary adapted processes allowing to represent a uniformly continuous Quantum Dynamical Semigroup on $\mathcal{H}$ by the conditional expectation of a Quantum Markov Process on $\mathcal{H} \otimes \Gamma$, analogously to the representation of a Classical Markov Semigroup by the conditional expectation of a Classical Markov Process.

Immediately Frigerio and Maassen realized [13, 14, 20, 21] that a Quantum Stochastic Evolution V_t enjoys the cocycle property, previously introduced by Accardi [1, 2], and thus it is naturally associated to a strongly continuous unitary group U_t, $t \in \mathbb{R}$, providing V_t with a quantum mechanical interpretation:

it describes a Hamiltonian coupling between a quantum system $\mathcal{H}$ and a boson field Γ in interaction picture with respect to the left shift Θ_t on Γ, which models the field free evolution. In other words,

$$U_t = \begin{cases} \Theta_t V_t & \text{if} \quad t \geq 0, \\ V_{|t|}^* \Theta_t & \text{if} \quad t \leq 0, \end{cases}$$

is a strongly continuous unitary group on $\mathcal{H} \otimes \Gamma$ and so there exists a Hamiltonian K generating U_t, that is $U_t = e^{-iKt}$, the evolution in Schrödinger picture. Roughly speaking, U_t describes a boson field Γ continuously flowing on a system $\mathcal{H}$ and interacting in such a way that each boson of the field can have a unique singular instantaneous interaction with $\mathcal{H}$, exactly when the free evolution Θ_t brings it to hit $\mathcal{H}$, and then it will be brought away by Θ_t never hitting $\mathcal{H}$ again. Thus the boson field Γ plays the role of a quantum noise in the dynamics of $\mathcal{H}$. Applications of Quantum Stochastic Evolutions in Physics can be found in the theories of open quantum systems, continuous measurements, quantum filtering, quantum optics, electronic transport or thermalization.

The problem that a Quantum Stochastic Evolution satisfies two differential equations, an ordinary one for U_t coming from Stone's theorem, and a quantum stochastic one for V_t, was raised by Accardi [5]. The characterization of the Hamiltonian K generating a Quantum Stochastic Evolution started in [10 − 12] by Chebotarev and it was completed in [15 − 17] for the general case of a Hudson-Parthasarathy equation with bounded coefficients (the coefficients are operators on $\mathcal{H}$) and arbitrary multiplicity. It is a singular perturbation of the unbounded Hamiltonian E_0 generating Θ_t, with the interaction partially encoded as boundary conditions defining the domain $\mathcal{D}(K)$. The Hamiltonian K is important because it gives the total energy of the coupled system $\mathcal{H} \otimes \Gamma$, it gives the solution of the Hudson-Parthasarathy equation $V_t = \exp\{iE_0 t\} \exp\{-iKt\}$, and it summarizes all the model assumptions leading to a Quantum Stochastic Evolution. Indeed, the singular features of a Quantum Stochastic Evolution often represent some ideal situation which is reached by some suitable limit, such as flat-spectrum and broad-band approximation, weak coupling limit, singular coupling limit, low density limit, stochastic limit, or a continuous limit of repeated interactions. Actually, all of these limits are not on the same ground, as some are heuristic, while the stochastic limit and the continuous limit of repeated interactions are mathematically rigorous and they are more powerful techniques.

In this paper we are interested in the last limit. The idea of approximating Quantum Stochastic Evolutions in continuous time with quantum evolutions in discrete time goes back, with different approaches, to Meyer [22], Accardi and Bach [3, 4], Parthasarathy and Lindsay [23, 19], and it has been used in a number of variants by other authors. In particular, Attal and Pautrat [6, 25] showed how to obtain Quantum Noises and Quantum Stochastic Evolutions in continuous time from Quantum Stochastic Calculus in discrete time and evolutions defined by repeated interactions: they showed how to embed the discrete time model in the continuous time model and how to perform the limit in the strong operator topology.

This approach has recently gained the attention of other theoretical physicists and mathematicians [7−9, 26], who studied how to get, in the limit, also continuous measurements of the system $\mathcal{H}$.

Our interest, instead, is to get, in the limit, the Hamiltonian K. Of course, once the temporal step Δt of the discrete time model has gone to 0 and the cocycle V_t has been obtained, one implicitly also has the group U_t and, by differentiation, also the Hamiltonian K. Anyway, following a suggestion by Attal, our aim is to show that K can be obtained directly by a suitable unique limit when $\Delta t \to 0$. This is interesting in order to understand how the structure of the singular and unbounded Hamiltonian K emerges in the limit $\Delta t \to 0$. Moreover, it could even be an alternative tool to characterize the Hamiltonian K, maybe working also in the case of unbounded coefficients.

We consider the case of 1-dimensional multiplicity space $\mathfrak{Z} = \mathbb{C}$ and of 1-dimensional system space $\mathcal{H} = \mathbb{C}$. This last assumption is very strong. From a physical point of view, it reduces the role of the system $\mathcal{H}$ to that of a singular potential acting on the boson field Γ producing scattering, absorption and emission phenomena (e.g. a beam splitter acting on the electromagnetic field). From a mathematical point of view, it implies several simplifications: operators on $\mathcal{H} = \mathbb{C}$ are just commuting numbers, the Hudson-Parthasarathy equation admits an explicit solution, the exponential domain is invariant for the quantum stochastic evolution and its intersection with $\mathcal{D}(K)$ is not only dense but even a domain of essential self-adjointness for K. Thus we can study the right formulation of the problem and we can find the right limit giving K as $\Delta t \to 0$.

The paper is organized as follows. Section 2 summarizes notations and results for Quantum Stochastic Evolutions in continuous time, Sect. 3 summarizes notations and results for Quantum Stochastic Evolutions in discrete time, Sect. 4 deals with the limit from discrete to continuous time, first summarizing the results by Attal and Pautrat, then stating and proving our new results.

2. Continuous Quantum Stochastic Evolutions

Given a measurable set $I \subseteq \mathbb{R}$, let us consider the *symmetric Fock space* over $L^2(I)$

$$\Gamma[I] \;=\; \bigoplus_{n=0}^{\infty} L^2_{\mathrm{symm}}(I^n)\,,$$

the complex separable Hilbert space of sequences $\xi = (\xi_n)_{n=0}^{\infty}$ with totally symmetric components $\xi_n \in L^2_{\mathrm{symm}}(I^n)$, with

$$\|\xi\|^2 \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \|\xi_n\|^2_{L^2(\mathbb{R}^n)} \,.$$

As usual $L^2_{\mathrm{symm}}(\mathbb{R}^0) = \mathbb{C}$. If $L^2(I)$ is the Hilbert space associated to some bosonic particle, then $\Gamma[I]$ is the Hilbert space associated to a field of such bosons.

For every f in $L^2(I)$, let $\psi(f)$ be the corresponding *exponential vector* in $\Gamma[I]$,

$$\psi(f) \;=\; (1, f, f^{\otimes 2}, \ldots, f^{\otimes n}, \ldots)\,, \qquad \|\psi(f)\|^2 \;=\; \exp \|f\|^2.$$

Exponential vectors are linearly independent and their linear span is dense in $\Gamma[I]$. Even better: for every subspace $\mathfrak{s}$ of $L^2(I)$, the corresponding exponential domain $\mathcal{E}(\mathfrak{s})$ of $\Gamma[I]$,

$$\mathcal{E}(\mathfrak{s}) \;=\; \mathrm{span}\Big\{\psi(f) \,:\, f \in \mathfrak{s}\Big\}\,,$$

is dense in $\Gamma[I]$ if $\mathfrak{s}$ dense in $L^2(I)$. Thanks to the properties of the exponential vectors, we have the factorization property of the symmetric Fock space

$$\Gamma[I] \;=\; \Gamma[B] \otimes \Gamma[B^{\mathrm{c}}]\,, \qquad \forall\, B \subseteq I\,, \quad B^{\mathrm{c}} = I \setminus B$$

based on the identification $\psi(f) = \psi(f|_B) \otimes \psi(f|_{B^{\mathrm{c}}})$, and we have the natural immersion

$$\Gamma[B] \;=\; \Gamma[B] \otimes \psi(0|_{B^{\mathrm{c}}}) \;\subseteq\; \Gamma[I]\,, \qquad \forall\, B \subseteq I\,,$$

based on the identification $\psi(f|_B) = \psi(f I_B)$, where I_B denotes the indicator function of a (measurable) set B.

For every vector $g \in L^2(I)$ and every unitary operator $\mathtt{U}$ on $L^2(I)$, let $W(g, \mathtt{U})$ be the corresponding *Weyl operator*, the unitary operator on $\Gamma[I]$ defined by

$$W(g, \mathtt{U})\, \psi(f) \;=\; e^{-\frac{1}{2}\|g\|^2 - \langle g|\mathtt{U}f\rangle}\, \psi(\mathtt{U}f + g)\,, \qquad \forall\, f \in L^2(I)\,.$$

Then

$$W(g, \mathtt{U})\, W(f, \mathtt{V}) \;=\; e^{-i\mathrm{Im}\langle g|\mathtt{U}f\rangle}\, W(g + \mathtt{U}f, \mathtt{UV})\,.$$

The *second quantization* of a strongly continuous unitary group $\mathtt{U}_t$ on $L^2(I)$ is $W(0, \mathtt{U}_t)$, which is a strongly continuous unitary group on $\Gamma[I]$. It describes the evolution of a field of non-interacting bosons, each one with Hilbert space $L^2(I)$ and evolution $\mathtt{U}_t$.

For every vector $g \in L^2(I)$, let $A(g)$ and $A^\dagger(g)$ be the corresponding *annihilation* and *creation* operators defined by

$$A(g)\,\psi(f) \;=\; \langle g|f\rangle\psi(f)\,, \quad A^\dagger(g)\,\psi(f) \;=\; \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\psi(f+\varepsilon g)\Big|_{\varepsilon=0}\,, \quad \forall\, f \in L^2(I)$$

and, for every bounded operator $\mathtt{N}$ on $L^2(I)$, let $\Lambda(\mathtt{N})$ be the corresponding *conservation* operator defined by

$$\Lambda(\mathtt{N})\psi(f) \;=\; \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,\psi(e^{\varepsilon\mathtt{N}}f)\Big|_{\varepsilon=0}\,, \qquad \forall\, f \in L^2(I)\,.$$

The operators $A(g)$, $A^\dagger(g)$ and $\Lambda(\mathtt{N})$ are unbounded closed operators, respectively antilinear, linear and linear in the arguments g, g and $\mathtt{N}$. The operators $A(g)$ and $A^\dagger(g)$ are mutually adjoint, as are $\Lambda(\mathtt{N})$ and $\Lambda(\mathtt{N}^*)$.

The *differential second quantization* of a bounded Hamiltonian $\mathtt{H} = \mathtt{H}^*$ on $L^2(I)$ is $\Lambda(\mathtt{H})$, which is the unbounded Hamiltonian on $\Gamma[I]$ generating the second quantization of $e^{-i\mathtt{H}t}$, that is

$$e^{-i\Lambda(\mathtt{H})t} \;=\; W(0, e^{-i\mathtt{H}t})\,.$$

The *differential second quantization* of an unbounded Hamiltonian $\mathtt{H} = \mathtt{H}^*$ on $L^2(I)$ is just the Hamiltonian on $\Gamma[I]$ generating $W(0, e^{-i\mathtt{H}t})$, it is always denoted by $\Lambda(\mathtt{H})$, and we have

1. $\mathcal{D}(\Lambda(\mathtt{H})) \supseteq \mathcal{E}(\mathcal{D}(\mathtt{H}))\,,$
2. $\Lambda(\mathtt{H})\,\psi(f) = A^\dagger(\mathtt{H}f)\,\psi(f)\,, \quad \forall\, f \in \mathcal{D}(\mathtt{H}),$
3. $\Lambda(\mathtt{H})|_{\mathcal{E}(\mathcal{D}(\mathtt{H}))}$ is essentially self-adjoint.

In order to introduce Quantum Stochastic Evolutions, now we consider the symmetric Fock space $\Gamma[\mathbb{R}]$, the Hilbert space associated to a field of bosonic particles of Hilbert space $L^2(\mathbb{R})$. The bosonic degree of freedom is understood to be the conjugate momentum of the free field energy, so that the free evolution of the bosons will be modelled by a left shift.

The canonical *quantum noises* on $\Gamma[\mathbb{R}]$ are the adapted processes of operators

$$\begin{aligned} A(t) &= A(I_{(0,t)})\,, && t \geq 0\,, \\ A^\dagger(t) &= A^\dagger(I_{(0,t)})\,, && t \geq 0\,, \\ \Lambda(t) &= \Lambda(\pi_{(0,t)})\,, && t \geq 0\,, \end{aligned}$$

which act non-trivially only on the corresponding factor of the field space $\Gamma = \Gamma[(-\infty,0)] \otimes \Gamma[(0,t)] \otimes \Gamma[(t,+\infty)]$. For every measurable $B \subseteq \mathbb{R}$, the operator π_B is the multiplication operator by I_B.

We are interested in *Quantum Stochastic Evolutions* V_t defined by the *Hudson-Parthasarathy equation*, that is in the adapted processes of operators V_t on $\Gamma[\mathbb{R}]$ which are solutions of the Quantum Stochastic Differential Equation

$$\mathrm{d}V_t \;=\; \Big[(\sigma-1)\,\mathrm{d}\Lambda_t - \bar\rho\sigma\,\mathrm{d}A_t + \rho\,\mathrm{d}A_t^\dagger - \Big(i\eta + \frac{1}{2}|\rho|^2\Big)\,\mathrm{d}t\Big]V_t\,, \quad V_0 = \mathbf{1}\,, \tag{1}$$

where

$$\sigma \;=\; e^{-i\alpha}, \quad \alpha\in\mathbb{R}\,, \quad \rho\in\mathbb{C}\,, \quad \eta\in\mathbb{R}\,.$$

The properties of the coefficients guarantee that (1) admits a unique adapted solution V_t, which is a strongly continuous unitary cocycle. As we are considering the case of a 1-dimensional initial space, the solution admits an explicit representation by Weyl operators:

$$V_t \;=\; e^{-i\eta t}\,W(\rho I_{(0,t)}, \mathtt{Q}_t)\,, \quad \mathtt{Q}_t \;=\; e^{-i\alpha\,\pi_{(0,t)}} \;=\; 1 + (\sigma-1)\pi_{(0,t)}\,, \quad t\geq 0\,, \tag{2}$$

where $\mathtt{Q}_t$, $t\geq 0$, is a strongly continuous family of unitary operators on $L^2(\mathbb{R})$.

In order to introduce the group U_t on the field space $\Gamma[\mathbb{R}]$ associated to V_t, it is convenient to introduce first a group $\mathtt{P}_t$ on the one-boson space $L^2(\mathbb{R})$ associated to $\mathtt{Q}_t$.

Let θ_t be the left shift on $L^2(\mathbb{R})$,

$$\theta_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})\,, \qquad f(r) \mapsto (\theta_t\, f)(r) = f(r+t)\,, \qquad t\in\mathbb{R}\,,$$

which is a strongly continuous unitary group describing a quantum particle whose degree of freedom r is the conjugate momentum of the energy, travelling from right to left. This evolution is generated by the unbounded Hamiltonian ϵ_0,

$$\theta_t \;=\; e^{-it\epsilon_0}, \quad \mathcal{D}(\epsilon_0) \;=\; H^1(\mathbb{R}) \;=\; \Big\{f\in L^2(\mathbb{R}) \,:\, f'\in L^2(\mathbb{R})\Big\}, \quad \epsilon_0 f \;=\; if'\,,$$

where f' is the derivative of f in the sense of distributions on $\mathbb{R}$.

For every $\alpha\in\mathbb{R}$, let $\mathtt{P}_t$ be the strongly continuous unitary group on $L^2(\mathbb{R})$ defined by

$$\mathtt{P}_t \;=\; \theta_t\mathtt{Q}_t \;=\; \theta_t\, e^{-i\alpha\pi_{(0,t)}} \;=\; e^{-i\alpha\pi_{(-t,0)}}\theta_t\,, \quad t\geq 0\,, \tag{3}$$

and by complex conjugation for $t\leq 0$. This is the same evolution given by θ_t, perturbed by a phase change when the quantum particle's degree of freedom hits $r=0$. Its Hamiltonian $\mathtt{H}$ is a singular perturbation of ϵ_0. If we set $\mathbb{R}_* = \mathbb{R}\setminus\{0\}$, we have

$$\mathtt{P}_t \;=\; e^{-i\mathtt{H}t}, \quad \mathcal{D}(\mathtt{H}) \;=\; \Big\{f\in H^1(\mathbb{R}_*) : f(0^-) = e^{-i\alpha}f(0^+)\Big\}, \quad \mathtt{H}f \;=\; if'\,, \tag{4}$$

where f' is the derivative of f in the sense of distributions on $\mathbb{R}_*$.

Note that $\mathtt{H}$ is the limit in the strong resolvent sense, as $\beta \downarrow 0$, of the Hamiltonian $\epsilon_0 - \alpha \mathtt{V}_\beta$, where $\mathtt{V}_\beta$ is the (bounded) multiplication operator by $\mathtt{v}_\beta(r) = \dfrac{1}{\sqrt{2\pi\beta}} \exp\Big\{ -\dfrac{r^2}{2\beta}\Big\}$, which describes a potential acting on the particle. Since $\mathtt{v}_\beta(r) \to \delta(r)$ in the sense of distributions, heuristically we could write $\mathtt{H}v(r) = iv'(r) - \alpha\delta(r)v(r)$, where $\alpha\delta$ would be a "function" describing a singular potential located at $r = 0$. Actually, the Hamiltonian $\mathtt{H}$ does not comprehend a multiplication operator term, but the whole perturbation is encoded in the boundary condition defining the domain of the Hamiltonian.

Going back to the Fock space, let Θ_t be the left shift on $\Gamma[\mathbb{R}]$, that is the second quantization of θ_t,

$$\Theta_t : \Gamma[\mathbb{R}] \to \Gamma[\mathbb{R}], \qquad \Theta_t\, \psi(f) \;=\; \psi(\theta_t f)\,,$$

which is the strongly continuous unitary group generated by the unbounded Hamiltonian $\Lambda(\epsilon_0)$,

$$\Theta_t \;=\; e^{-itE_0}\,, \quad E_0 \;=\; \Lambda(\epsilon_0)\,.$$

Finally, let U_t be the strongly continuous unitary group on $\Gamma[\mathbb{R}]$ associated to the Hudson-Parthasarathy equation, defined by

$$U_t \;=\; \Theta_t\, V_t \;=\; e^{-i\eta t} W(\rho I_{(-t,0)}, \mathtt{P}_t)\,, \quad t \geq 0\,,$$

and by complex conjugation for $t \leq 0$. The group U_t models an evolution, in Schrödinger picture, where the field continuously flows from right to left on some singular potential localized at $r = 0$, so that each boson of the field can have a unique singular instantaneous interaction with the potential, exactly when the free evolution Θ_t brings it in $r = 0$. Thus, the cocycle V_t models the same evolution as U_t, but in interaction picture with respect to Θ_t, and each factor $\Gamma[(s,t)]$ of $\Gamma[\mathbb{R}]$ is associated to those bosons of the field which interact with the singular potential in the time interval (s,t).

The Hamiltonian K generating such an evolution U_t,

$$U_t \;=\; e^{-iKt},$$

is a singular perturbation of E_0. As we are considering the case of a 1-dimensional initial space, it is completely characterized by its behaviour on the exponential domain [15–17]:

1. $\mathcal{D}(K) \cap \mathcal{E}(L^2(\mathbb{R})) = \mathcal{E}(\mathfrak{C})$,

   where $\mathfrak{C} = \Big\{ f \in H^1(\mathbb{R}_*) : f(0^-) \;=\; \sigma f(0^+) + \rho \Big\}$,

2. $U_t\, \mathcal{E}(\mathfrak{C}) = \mathcal{E}(\mathfrak{C}), \quad \forall\, t \in \mathbb{R}$,

3. $K|_{\mathcal{E}(\mathfrak{C})}$ is essentially self-adjoint,

4. For every $f \in \mathfrak{C}$,

$$K\psi(f) \;=\; \Big[\eta + A^{\dagger}(if') - i\bar{\rho}\sigma\, f(0^{+}) - \frac{i}{2}|\rho|^{2}\Big]\psi(f)\,,$$

where f' is the derivative of f in the sense of distributions on $\mathbb{R}_{*}$.

Note that, when $\rho = 0$, we have $U_t = e^{-i\eta t}W(0, \mathtt{P}_t)$ and so we simply have $K = \eta + \Lambda(\mathtt{H})$ for every $\alpha \in \mathbb{R}$. Thus, up to the irrelevant constant η, the evolution U_t is just a second quantization, that is an evolution of non-interacting bosons, where each boson singularly interacts with the same potential which can change its phase. When $\rho \neq 0$, the evolution U_t is no longer a second quantization of a single boson evolution: the interaction with the potential includes emission and absorption phenomena which cannot be described in the one boson space $L^2(\mathbb{R})$, but only in the Fock space $\Gamma[\mathbb{R}]$.

3. Discrete Quantum Stochastic Evolutions

For every $n \in \mathbb{Z}$ let us consider a 2-dimensional complex Hilbert space $\widehat{\mathfrak{Z}}_n$ with basis $\{\omega_n, z_n\}$,

$$\widehat{\mathfrak{Z}}_n \;=\; \mathrm{span}\{\omega_n, z_n\}\,.$$

Then we introduce the *Toy Fock space*

$$T\Gamma \;=\; \bigotimes_{n\in\mathbb{Z}} \widehat{\mathfrak{Z}}_n \quad \text{w.r.t. the stabilizing sequence } \omega_n,$$

which is a complex separable Hilbert space with basis $\{Z_A\}_{A\in\mathcal{P}_0(\mathbb{Z})}$, where $\mathcal{P}_0(\mathbb{Z})$ is the collection of the finite subsets $A = \{n_1 < n_2 < \ldots < n_k\}$ of $\mathbb{Z}$, and where

$$Z_A \;=\; \Big(\bigotimes_{n\in A} z_n\Big) \otimes \Big(\bigotimes_{n\notin A} \omega_n\Big)\,,$$

so that

$$\Phi \in T\Gamma \implies \Phi \;=\; \sum_{A} \Phi_A\, Z_A\,, \qquad \|\Phi\|^2 \;=\; \sum_{A} |\Phi_A|^2.$$

For every f in $\ell^2(\mathbb{Z})$, let $\phi(f)$ be the corresponding *discrete exponential vector* in $T\Gamma$,

$$\phi(f) \;=\; \bigotimes_{n\in\mathbb{Z}}(\omega_n + f_n\, z_n)\,, \qquad (\phi(f))_A \;=\; \prod_{n\in A} f_n\,,$$

$$\|\phi(f)\|^2 \;=\; \prod_{n\in\mathbb{Z}}\,(1 + |f_n|^2) \;=\; \exp\Big\{\sum_{n\in\mathbb{Z}} \log\,(1 + |f_n|^2)\Big\}\,.$$

The linear span of discrete exponential vectors is dense in $T\Gamma$, but exponentials of distinct functions f are not necessarily linearly independent.

As usual, any operator acting on some factor $\widehat{\mathfrak{Z}}_n$ of $T\Gamma$ will be extended to the whole Toy Fock space by tensorizing with the identity.

The canonical *quantum noises* on $T\Gamma$ are the processes of bounded operators

$$b(n) \;=\; |\omega_n\rangle\langle z_n| \,, \qquad b^\dagger(n) \;=\; |z_n\rangle\langle \omega_n| \,, \qquad b^\dagger(n)\, b(n) \;=\; |z_n\rangle\langle z_n| \,,$$

which, actually, will correspond to the increments of the noises introduced in continuous time.

We are interested in *Quantum Stochastic Evolutions* in discrete time $v(n)$ defined by repeated interactions, that is in adapted unitary cocycles

$$v(n) \;=\; e^{-i\Delta t\, h(n)} \cdots e^{-i\Delta t\, h(1)},$$

defined by the Hamiltonians

$$h(n) \;=\; \eta_0 + \frac{1}{\sqrt{\Delta t}}(\lambda\, b^\dagger(n) + \bar{\lambda}\, b(n)) + \frac{\alpha}{\Delta t} b^\dagger(n)\, b(n) \,, \quad n \in \mathbb{N} \,,$$

where $\eta_0, \alpha \in \mathbb{R}$, $\lambda \in \mathbb{C}$. The parameter Δt is the temporal step of the discrete evolution and it will play a role only in the limit from discrete to continuous time. If $\omega_n = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $z_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have the matrix representation

$$e^{-i\Delta t\, h(n)} \;=\; e^{-i\Delta t\, \eta_0 - i\frac{\alpha}{2}} \times \begin{bmatrix} \cos\sqrt{\frac{\alpha^2}{4} + |\lambda|^2\Delta t} + i\frac{\alpha}{2}\, \frac{\sin\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}{\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}} & -i\sqrt{\Delta t}\, \frac{\sin\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}{\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}\, \bar{\lambda} \\ -i\sqrt{\Delta t}\, \frac{\sin\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}{\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}\, \lambda & \cos\sqrt{\frac{\alpha^2}{4} + |\lambda|^2\Delta t} - i\frac{\alpha}{2}\, \frac{\sin\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}{\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}} \end{bmatrix}.$$

Let $\widehat{\theta}$ be the left shift on $T\Gamma$,

$$\widehat{\theta} : T\Gamma \to T\Gamma \,, \qquad \widehat{\theta}\, \phi(f) \;=\; \bigotimes_{n\in\mathbb{Z}} (\omega_n + f_{n+1}\, z_n) \,, \qquad \widehat{\theta}\, Z_A \;=\; Z_{A-1} \,,$$

where $A - 1 = \{n_1 - 1 < n_2 - 1 < \ldots < n_k - 1\}$. Of course, $\widehat{\theta}$ is a unitary operator.

Finally, let u be the unitary operator

$$u \;=\; \widehat{\theta}\, v(1)$$

and let us consider the evolution given by u^n, $n \in \mathbb{Z}$, the corresponding unitary group on $T\Gamma$. Note that

$$u^n = \widehat{\theta}^n v(n) \;\forall\, n \in \mathbb{N} \,.$$

Similarly to the continuous time case, the group u^n models an evolution, in Schrödinger picture, where the quantum system $T\Gamma$ flows from right to left on some localized potential, and each factor $\widehat{\mathfrak{z}}_n$ describes the fraction of the system which interacts with the potential (only) during the n-th temporal step. The cocycle $v(n)$ models the same evolution as u^n, but in interaction picture with respect to the free evolution $\widehat{\theta}^n$.

4. From Discrete to Continuous Quantum Stochastic Evolutions

In order to recover the continuous time evolution from the repeated interactions model, we embed the Toy Fock space $T\Gamma$ in the symmetric Fock space $\Gamma[\mathbb{R}]$ and then we take the limit $\Delta t \downarrow 0$. For every given $\Delta t > 0$, we set $t_n = n\Delta t$, $n \in \mathbb{Z}$, and we get

$$\Gamma[\mathbb{R}] = \bigotimes_{n\in\mathbb{Z}} \Gamma[(t_{n-1}, t_n)] \quad \text{w.r.t. the stabilizing sequence } \Omega_n = \psi(0|_{(t_{n-1},t_n)}).$$

The Toy Fock space is embedded in the symmetric Fock space by the isometries

$$J_n : \widehat{\mathfrak{z}}_n \to \Gamma[(t_{n-1}, t_n)]\,, \qquad \omega_n \mapsto \Omega_n \;=\; \psi(0|_{(t_{n-1},t_n)})\,,$$

$$z_n \mapsto X_n \;=\; \frac{1|_{(t_{n-1},t_n)}}{\sqrt{\Delta t}}\,, \qquad J_{\Delta t} \;=\; \bigotimes_{n\in\mathbb{Z}} J_n : T\Gamma \to \Gamma[\mathbb{R}]$$

with ranges

$$\gamma_n \;=\; J_n(\widehat{\mathfrak{z}}_n) \;=\; \mathrm{span}\Big\{\Omega_n, X_n\Big\}$$

$$\gamma_{\Delta t} \;=\; J_{\Delta t}(T\Gamma) \;=\; \bigotimes_{n\in\mathbb{Z}} \gamma_n \quad \text{w.r.t. the stabilizing sequence } \Omega_n$$

and projections

$$P_n : \Gamma[(t_{n-1}, t_n)] \to \gamma_n\,, \qquad P_{\Delta t} \;=\; \bigotimes_{n\in\mathbb{Z}} P_n : \Gamma[\mathbb{R}] \to \gamma_{\Delta t}\,.$$

Then $J^*_{\Delta t} = J^{-1}_{\Delta t}\, P_{\Delta t} : \Gamma[\mathbb{R}] \to T\Gamma$. Let us note that J^*_{Deltat} maps exponential vectors to discrete exponential vectors:

$$J^*_{\Delta t}\psi(f) \;=\; \phi(\widehat{f}_{\Delta t}) \;=\; \bigotimes_{n\in\mathbb{Z}} (\omega_n + \widehat{f}_{\Delta t}(n)\, z_n)\,, \tag{5}$$

$$\widehat{f}_{\Delta t}(n) \;=\; \langle X_n | f|_{(t_{n-1},t_n)}\rangle \;=\; \frac{1}{\sqrt{\Delta t}} \int_{t_{n-1}}^{t_n} f(r)\,\mathrm{d}r\,.$$

In order to embed the noises, for every $n \in \mathbb{Z}$ let us introduce $E_1(n)$, the projection from $\Gamma[(t_{n-1}, t_n)]$ to its one-boson subspace $L^2((t_{n-1}, t_n))$, tensorized with the identity on the other factors of $\Gamma[\mathbb{R}]$, and then the operators

$$a(n) \;=\; A\Big(\frac{I_{(t_{n-1},t_n)}}{\sqrt{\Delta t}}\Big) E_1(n) : \Gamma[\mathbb{R}] \to \Gamma[\mathbb{R}]\,.$$

Then $J_{\Delta t} b(n) J^*_{\Delta t} \,:\, \Gamma[\mathbb{R}] \to \Gamma[\mathbb{R}]$, $J_{\Delta t} b(n) J^*_{\Delta t} = a(n)$. The evolutions in discrete time embedded in the symmetric Fock space are

$$J_{\Delta t} v(n) J^{-1}_{\Delta t} : \gamma_{\Delta t} \to \gamma_{\Delta t}\,, \qquad J_{\Delta t} \widehat{\theta}^n J^{-1}_{\Delta t} \;=\; \Big(J_{\Delta t} \widehat{\theta} J^{-1}_{\Delta t}\Big)^n : \gamma_{\Delta t} \to \gamma_{\Delta t}\,,$$

$$J_{\Delta t} u^n J^{-1}_{\Delta t} \;=\; \Big(J_{\Delta t} u J^{-1}_{\Delta t}\Big)^n : \gamma_{\Delta t} \to \gamma_{\Delta t}\,.$$

Then, taking the limit $\Delta t \downarrow 0$, we have [6]:

1. $P_{\Delta t} \to \mathbf{1}_{\Gamma[\mathbb{R}]}$ strongly,
2. $\displaystyle\sum_{n=1}^{[\frac{t}{\Delta t}]} a^\dagger(n)\, a(n) \to \Lambda_t$ strongly on $\Big\{\xi \in \Gamma[\mathbb{R}] : \displaystyle\sum_{n=0}^{\infty} n \|\xi_n\|^2_{L^2(\mathcal{P}_n)} < \infty\Big\}$,
3. $\sqrt{\Delta t} \displaystyle\sum_{n=1}^{[\frac{t}{\Delta t}]} a(n) \to A_t$ strongly on $\Big\{\xi \in \Gamma[\mathbb{R}] : \displaystyle\sum_{n=0}^{\infty} n \|\xi_n\|^2_{L^2(\mathcal{P}_n)} < \infty\Big\}$,
4. $\sqrt{\Delta t} \displaystyle\sum_{n=1}^{[\frac{t}{\Delta t}]} a^\dagger(n) \to A^\dagger_t$ strongly on $\Big\{\xi \in \Gamma[\mathbb{R}] : \displaystyle\sum_{n=0}^{\infty} n \|\xi_n\|^2_{L^2(\mathcal{P}_n)} < \infty\Big\}$,
5. $\Delta t \displaystyle\sum_{n=1}^{[\frac{t}{\Delta t}]} |\Omega_n\rangle\langle\Omega_n| \to t$ strongly on $\Big\{\xi \in \Gamma[\mathbb{R}] : \displaystyle\sum_{n=0}^{\infty} n \|\xi_n\|^2_{L^2(\mathcal{P}_n)} < \infty\Big\}$,
6. $J_{\Delta t}\, v([\frac{t}{\Delta t}])\, J^*_{\Delta t} = J_{\Delta t}\, v([\frac{t}{\Delta t}])\, J^{-1}_{\Delta t}\, P_{\Delta t} \to V_t$ strongly if

$$\eta \;=\; \eta_0 + |\lambda|^2\, \frac{\sin\alpha - \alpha}{\alpha^2}\,, \qquad \sigma \;=\; e^{-i\alpha}, \qquad \rho \;=\; \frac{\sigma - 1}{\alpha}\,\lambda\,.$$

To these limits we can add the following ones, regarding the evolutions in Schrödinger picture and their Hamiltonians.

THEOREM 1 *As $\Delta t \downarrow 0$, we have*

7. $J_{\Delta t}\, \widehat{\theta}^{[\frac{t}{\Delta t}]}\, J^*_{\Delta t} = \Big(J_{\Delta t}\, \widehat{\theta}\, J^{-1}_{\Delta t}\Big)^{[\frac{t}{\Delta t}]}\, P_{\Delta t} \to \Theta_t$ *strongly,*

8. $J_{\Delta t}\, u^{[\frac{t}{\Delta t}]}\, J^*_{\Delta t} = \big(J_{\Delta t}\, u\, J^{-1}_{\Delta t}\big)^{[\frac{t}{\Delta t}]}\; P_{\Delta t} \to U_t$ *strongly,*

9. $i\,\dfrac{J_{\Delta t}\,\widehat{\theta}\,J^{-1}_{\Delta t} - \mathbf{1}}{\Delta t}\, P_{\Delta t} \to E_0$ *strongly on* $\mathcal{D}(E_0)$*,*

10. $i\,\dfrac{J_{\Delta t}\,u\,J^{-1}_{\Delta t} - \mathbf{1}}{\Delta t}\, P_{\Delta t} \to K$ *strongly on* $\mathcal{E}(\mathfrak{C})$*.*

Let us remark that we recover the Hamiltonians E_0 and K by taking a unique limit which combine the limit from repeated to continuous interactions with the limit of the difference quotient of the evolution. This limit gives E_0 on its full domain and K at least on $\mathcal{E}(\mathfrak{C})$, which is anyway a domain of essential self-adjointness. It is not obvious that it should work, even if $P_{\Delta t} \to \mathbf{1}$ strongly, as $P_{\Delta t}$ projects outside the domains $\mathcal{D}(E_0)$ and $\mathcal{D}(K)$ for every $\Delta t > 0$. Indeed, if we consider the Hilbert space $L^2(\mathbb{R})$, the evolution $\mathtt{P}_t$ (3) with Hamiltonian $\mathtt{H}$ (4), the projections $\pi_{(-\Delta t,\Delta t)^{\mathrm{c}}}$, and we take the limit $\Delta t \downarrow 0$, then $\pi_{(-\Delta t,\Delta t)^{\mathrm{c}}} \to \mathbf{1}$ strongly, but

$$\frac{\mathtt{P}_t - \mathbf{1}}{\Delta t}\pi_{(-\Delta t,\Delta t)^{\mathrm{c}}}\, f$$

has divergent norm for every $f \in \mathcal{D}(\mathtt{H})$ with $f(0^+) \neq 0$.
Proof.
7. Since

$$J_{\Delta t}\,\widehat{\theta}\,J^*_{\Delta t} \;=\; P_{\Delta t}\,\Theta_{\Delta t} \tag{6}$$

we have

$$J_{\Delta t}\,\widehat{\theta}^{[\frac{t}{\Delta t}]}\,J^*_{\Delta t} \;=\; P_{\Delta t}\,\Theta_{\Delta t[\frac{t}{\Delta t}]} \;\longrightarrow\; \Theta_t \quad \text{strongly,}$$

as $P_{\Delta t} \to \mathbf{1}$ and $\Theta_{\Delta t[\frac{t}{\Delta t}]} \to \Theta_t$ strongly and they all have norms bounded by 1: taken $\xi \in \Gamma[\mathbb{R}]$,

$$\begin{aligned} \Big\|\Big(J_{\Delta t}\,\widehat{\theta}^{[\frac{t}{\Delta t}]}J^*_{\Delta t} - \Theta_t\Big)\xi\Big\| \;&\leq\; \Big\|P_{\Delta t}\Big(\Theta_{\Delta t[\frac{t}{\Delta t}]} - \Theta_t\Big)\xi\Big\| + \Big\|\Big(P_{\Delta t}\,\Theta_t - \Theta_t\Big)\xi\Big\| \\ &\leq\; \Big\|\Big(\Theta_{\Delta t[\frac{t}{\Delta t}]} - \Theta_t\Big)\xi\Big\| + \Big\|\Big(P_{\Delta t} - \mathbf{1}\Big)\Theta_t\,\xi\Big\| \;\longrightarrow\; 0\,. \end{aligned}$$

8. Similarly to the previous point,

$$J_{\Delta t}\,u^{[\frac{t}{\Delta t}]}\,J^*_{\Delta t} \;=\; \Big(J_{\Delta t}\,\widehat{\theta}^{[\frac{t}{\Delta t}]}\,J^*_{\Delta t}\Big)\Big(J_{\Delta t}\,v([\tfrac{t}{\Delta t}])\,J^*_{\Delta t}\Big) \;\longrightarrow\; U_t \quad \text{strongly.}$$

9. Taken $\xi \in \mathcal{D}(E_0)$, thanks to (6), we have

$$\begin{aligned} i\,\frac{J_{\Delta t}\,\widehat{\theta}\,J^{-1}_{\Delta t} - \mathbf{1}}{\Delta t}\,P_{\Delta t}\,\xi - E_0\,\xi \;&=\; i\,P_{\Delta t}\,\frac{\Theta_{\Delta t} - \mathbf{1}}{\Delta t}\,\xi - E_0\,\xi \\ &=\; iP_{\Delta t}\Big(\frac{\Theta_{\Delta t} - \mathbf{1}}{\Delta t} + iE_0\Big)\xi + \Big(P_{\Delta t} - \mathbf{1}\Big)E_0\xi \;\longrightarrow\; 0\,. \end{aligned}$$

10. For this limit we cannot repeat the argument used for E_0, as $J_{\Delta t} u J_{\Delta t}^* \neq P_{\Delta t} U_{\Delta t}$. Taken a vector $\xi \in \mathcal{D}(K)$, we have

$$
\begin{aligned}
& i \frac{J_{\Delta t} u J_{\Delta t}^{-1} - \mathbf{1}}{\Delta t} P_{\Delta t} \xi - K \xi \\
& \qquad = \Big(i P_{\Delta t} \frac{U_{\Delta t} - \mathbf{1}}{\Delta t} \xi - K \xi \Big) + i \frac{J_{\Delta t} u J_{\Delta t}^{-1} P_{\Delta t} - P_{\Delta t} U_{\Delta t}}{\Delta t} \xi ,
\end{aligned}
$$

where the first term goes to 0 as before. Let us show also that the second term goes to 0 when ξ belongs to $\mathcal{E}(\mathfrak{C}) \subseteq \mathcal{D}(K)$, that is if $\xi = \psi(f)$ with $f \in \mathfrak{C}$. First of all, let us note that $J_{\Delta t}^* \psi(f) = \phi(\widehat{f}_{\Delta t})$ by (5) where, as $\mathfrak{C} \subseteq H^1(\mathbb{R}_*)$, we have

$$
\widehat{f}_{\Delta t}(1) \;=\; \frac{1}{\sqrt{\Delta t}} \int_0^{\Delta t} f(r) \mathrm{d}r \;=\; f(0^+)\sqrt{\Delta t} + o(\sqrt{\Delta t}) , \quad \text{as } \Delta t \to 0.
$$

Moreover, we can compute both

$$
\begin{aligned}
J_{\Delta t} u J_{\Delta t}^{-1} P_{\Delta t} \psi(f) \;&=\; J_{\Delta t} \widehat{\theta} v(1) \phi(\widehat{f}_{\Delta t}) \;=\; J_{\Delta t} \widehat{\theta} v(1) \bigotimes_{n \in \mathbb{Z}} \Big(\omega_n + \widehat{f}_{\Delta t}(n) \, z_n \Big) \\
&= \exp\Big\{ - i\eta_0 \Delta t - i \frac{\alpha}{2} \Big\} \Big(\bigotimes_{n \neq 0} \Big(\Omega_n + \widehat{f}_{\Delta t}(n+1) \, X_n \Big) \Big) \\
& \quad \otimes \Bigg[\Bigg(\cos \sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t} + i \frac{\alpha}{2} \frac{\sin \sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t}}{\sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t}} \\
& \qquad\qquad - i \sqrt{\Delta t} \frac{\sin \sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t}}{\sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t}} \bar{\lambda} \, \widehat{f}_{\Delta t}(1) \Bigg) \Omega_0 \\
& \quad + \Bigg(- i \sqrt{\Delta t} \frac{\sin \sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t}}{\sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t}} \lambda + \cos \sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t} \, \widehat{f}_{\Delta t}(1) \\
& \qquad\qquad - i \frac{\alpha}{2} \frac{\sin \sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t}}{\sqrt{\frac{\alpha^2}{4} + |\lambda|^2 \Delta t}} \widehat{f}_{\Delta t}(1) \Bigg) X_0 \Bigg]
\end{aligned}
$$

and

$$
P_{\Delta t} U_{\Delta t} \psi(f) \;=\; P_{\Delta t} \exp\Big\{ - i\eta_0 \Delta t - \frac{1}{2} |\rho|^2 \Delta t - \bar{\rho} \sigma \int_0^{\Delta t} f(r) \mathrm{d}r \Big\}
$$

$$\times\,\psi\Big(\theta_{\Delta t}e^{-i\alpha\pi_{(0,\Delta t)}}f+\rho\,I_{(-\Delta t,0)}\Big)$$

$$=\ \exp\Big\{-i\eta_0\Delta t-\frac{1}{2}|\rho|^2\Delta t-\bar\rho\sigma\int_0^{\Delta t}f(r)\,\mathrm{d}r\Big\}$$

$$\Big(\bigotimes_{n\neq 0}\Big(\Omega_n+\widehat{f}_{\Delta t}(n+1)\,X_n\Big)\Big)\otimes\Big(\Omega_0+(\sigma\widehat{f}_{\Delta t}(1)+\rho\sqrt{\Delta t})X_0\Big)\,.$$

Therefore

$$\frac{J_{\Delta t}\,u\,J_{\Delta t}^{-1}\,P_{\Delta t}-P_{\Delta t}\,U_{\Delta t}}{\Delta t}\,\psi(f)\ =\ \frac{1}{\Delta t}\left(\bigotimes_{n\neq 0}\Big(\Omega_n+\widehat{f}_{\Delta t}(n+1)\,X_n\Big)\right)$$

$$\otimes\Bigg\{\Bigg[e^{-i\eta_0\Delta t-i\frac{\alpha}{2}}\Bigg(\cos\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}+i\frac{\alpha}{2}\,\frac{\sin\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}{\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}$$

$$-\,i\sqrt{\Delta t}\,\frac{\sin\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}{\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}\,\bar\lambda\,\widehat{f}_{\Delta t}(1)\Bigg)-e^{-i\eta_0\Delta t-\frac{1}{2}|\rho|^2\Delta t-\bar\rho\sigma\int_0^{\Delta t}f(r)\,\mathrm{d}r}\Bigg]\Omega_0$$

$$+\Bigg[e^{-i\eta_0\Delta t-i\frac{\alpha}{2}}\Bigg(-\,i\sqrt{\Delta t}\,\frac{\sin\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}{\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}\,\lambda+\cos\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}\,\widehat{f}_{\Delta t}(1)$$

$$-\,i\frac{\alpha}{2}\,\frac{\sin\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}{\sqrt{\frac{\alpha^2}{4}+|\lambda|^2\Delta t}}\,\widehat{f}_{\Delta t}(1)\Bigg)$$

$$-\,e^{-i\eta_0\Delta t-\frac{1}{2}|\rho|^2\Delta t-\bar\rho\sigma\int_0^{\Delta t}f(r)\,\mathrm{d}r}\Big(\sigma\,\widehat{f}_{\Delta t}(1)+\rho\sqrt{\Delta t}\Big)\Bigg]X_0\Bigg\}$$

$$=\ \Big(\bigotimes_{n\neq 0}\Big(\omega_n+\widehat{f}_{\Delta t}(n+1)\,z_n\Big)\Big)\otimes\frac{\Big(o(\Delta t)\,\omega_0+o(\Delta t)\,z_0\Big)}{\Delta t}\ \longrightarrow\ 0\,.$$

Thus

$$\lim_{\Delta t\downarrow 0}i\frac{J_{\Delta t}uJ_{\Delta t}^{-1}-\mathbf{1}}{\Delta t}\,P_{\Delta t}$$

is the right limit to find directly the Hamiltonian K in the limit from repeated to continuous interactions. Anyhow, the generalization of this result to the case of an arbitrary initial space $\mathcal{H}$ is not trivial, as one would lose the explicit solution (2) of the Hudson-Parthasarathy equation and the straightforward computation of the limit.

Then one could study under which conditions the existence of such a limit is an alternative characterization of K, giving its full domain or some domain of essential self-adjointness.

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