Asymptotic stability of thermoelastic systems of Bresse type

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1. Introduction

Given a real interval $\mathfrak{I} = [0, L]$, we consider the thermoelastic Bresse system with Gurtin–Pipkin thermal dissipation

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$$
\begin{cases}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0, \\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0, \\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0, \\
\rho_3\theta_t - k_1 \int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s + \gamma\psi_{tx} = 0,
\end{cases}
\tag{1.1}
$$

in the unknowns variables

$$
\varphi, \psi, w, \theta : (x,t) \in \mathfrak{I} \times [0,\infty) \mapsto \mathbb{R}.
$$

Here, ρ_1, ρ_2, ρ_3 as well as $b, l, \gamma, k, k_0, k_1$ are strictly positive fixed constants, while g is a bounded convex summable function on $[0,\infty)$ of total mass

$$
\int_0^\infty g(s)\,\mathrm{d}s = 1
$$

having the explicit form

$$
g(s) = \int_s^\infty \mu(r)\,\mathrm{d}r,
$$

where $\mu : \mathbb{R}^+ = (0,\infty) \to [0,\infty)$, the so-called memory kernel, is a nonincreasing absolutely continuous function such that

$$
\mu(0) = \lim_{s\to 0} \mu(s) \in (0,\infty).
$$

In particular, μ is summable on $\mathbb{R}^+$ with

$$
\int_0^\infty \mu(s)\,\mathrm{d}s = g(0),
$$

and the requirement that g has total mass 1 translates into

$$
\int_0^\infty s\mu(s)\,\mathrm{d}s = 1.
$$

Moreover, the kernel μ is supposed to satisfy the additional assumption

$$
\mu'(s) + \nu\mu(s) \le 0 \tag{1.2}
$$

for some $\nu > 0$ and almost every $s \in \mathbb{R}^+$. The system is complemented with the Dirichlet bound-ary conditions for φ and θ

$$\varphi(0,t) = \varphi(L, t) = \theta(0,t) = \theta(L, t) = 0,$$

and the Neumann ones for ψ and w

$$\psi_x(0,t) = \psi_x(L, t) = w_x(0,t) = w_x(L, t) = 0.$$

From the physical viewpoint, system (1.1) describes the vibrations of a linear planar and shear-able thermoelastic beam of Bresse type [2,19,20]. Accordingly, the functions φ, w, ψ denote the vertical, longitudinal and shear angle displacements, respectively, while θ stands for the relative temperature, that is, the temperature variation from an equilibrium reference value.

In recent years, the asymptotic properties of Bresse systems without thermal effects, i.e. when the fourth equation of system (1.1) is omitted, have been widely investigated (see e.g. [1,4,12,28, 29] and the references therein). In particular, it has been pointed out that the exponential stability of the associated solution semigroups is highly influenced by the structural parameters of the problem. Roughly speaking, we may summarize the current results as follows: in the presence of frictional dissipation acting for instance on the shear angle displacement, the related contraction semigroup is exponentially stable[1] if and only if the wave speeds of the first three hyperbolic equations are equal, namely,

$$\frac{k}{\rho_1} = \frac{b}{\rho_2} \quad \text{and} \quad k = k_0.$$

Actually, as shown in [13,21], the same conclusions can be drawn for the thermoelastic Bresse system with Fourier-type thermal dissipation, that is, when the fourth equation of system (1.1) is replaced by the classical parabolic heat equation

$$\rho_3\theta_t - k_1\theta_{xx} + \gamma\psi_{tx} = 0.$$

Nevertheless, such an equation (which predicts instantaneous propagation of thermal signals) has encountered some criticisms in the scientific community, mainly due to the increasing evidence, also supported by physical experiments, that thermal motion is indeed a wave-type mechanism. Therefore, through the years, several alternative “nonclassical” heat conduction theories have been proposed. Among the others, we mention the so-called Maxwell–Cattaneo approach [3] and the heat flux history models of Coleman and Gurtin [5] and Gurtin and Pipkin [17].

Currently, several results on the stability properties of simplified versions of the thermoelas-tic system (1.1) are available in the literature, for instance regarding Timoshenko-type systems where the longitudinal motion is neglected [11,14,22,27,30]. On the contrary, the picture con-cerning the full model accounting for longitudinal movements is essentially poorer. In particular, to the best of our knowledge, no results have been obtained for nonclassical variants of the Bresse

[1] A linear semigroup $\Sigma(t)$ acting on a Banach space $\mathcal{X}$ is said to be exponentially stable if there exist $\omega > 0$ and $K \geq 1$ such that

$$\|\Sigma(t)z\|_{\mathcal{X}} \leq K\mathrm{e}^{-\omega t}\|z\|_{\mathcal{X}}, \quad \forall z \in \mathcal{X}.$$

system studied in [13]. Motivated by these considerations, in the present paper we analyze the stability properties of the thermoelastic Bresse system (1.1) with Gurtin–Pipkin thermal dissipation. In contrast to the classical situation, we show that the hyperbolic character of the heat conduction law reflects on the stability conditions. More precisely, exploiting the history frame-work of Dafermos [10], we show that system (1.1) generates a contraction semigroup $S(t) = \mathrm{e}^{t\mathbb{A}}$ acting on a suitable Hilbert space $\mathcal{H}$ accounting for the presence of the memory. Then, introduc-ing the new stability number

$$\chi_g = \left(\frac{\rho_1}{\rho_3 k} - \frac{1}{g(0)k_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{1}{g(0)k_1}\frac{\rho_1 \gamma^2}{\rho_3 bk}$$

we can state our main theorem as follows.

Theorem 1.1. *The semigroup $S(t) = \mathrm{e}^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}$ generated by the thermoelastic Bresse–Gurtin–Pipkin system (1.1) is exponentially stable if and only if*

$$\chi_g = 0 \text{ and } k = k_0.$$

Going beyond a mere study of the Gurtin–Pipkin case, Theorem 1.1 actually provides a com-plete stability characterization of Bresse systems with Fourier, Maxwell–Cattaneo and Coleman–Gurtin thermal dissipation (see Section 2). It also subsumes the aforementioned achievements in the asymptotic properties of Timoshenko systems with classical and nonclassical heat conduction (see Section 3).

In order to discuss the mathematical difficulties encountered in the analysis, let us briefly recall two widely used techniques for the investigation of exponential stability in linear semi-groups. The first one is the classical strategy based on energy-type estimates which, although successfully employed in the study of Timoshenko-type systems (see e.g. [11,22]), seems not applicable to the complete model (1.1). Another possibility is to take advantage of linear tech-niques, for instance exploiting the following famous result due to Prüss [26] (but see also [9,15] for the statement used here).

Lemma 1.2. *Let A be the infinitesimal generator of a contraction semigroup $\Sigma(t)$ acting on a complex Hilbert space $\mathcal{X}$. Then, the following are equivalent:*

(i) *$\Sigma(t)$ is exponentially stable.*

(ii) *There exists $\varepsilon > 0$ such that*

$$\inf_{\lambda\in\mathbb{R}} \|\mathrm{i}\lambda z - Az\|_{\mathcal{X}} \geq \varepsilon \|z\|_{\mathcal{X}}, \quad \forall z \in \mathfrak{D}(A).$$

(iii) *The imaginary axis $\mathrm{i}\mathbb{R}$ is contained in the resolvent set $\rho(A)$ of the operator A and*

$$\sup_{\lambda\in\mathbb{R}} \left\| (\mathrm{i}\lambda - A)^{-1} \right\|_{L(\mathcal{X})} < \infty.$$

In light of Lemma 1.2 above, once one has proved that $\mathrm{i}\mathbb{R} \subset \rho(A)$ to reach the desired expo-nential stability it is sufficient to show that the (real-variable and continuous) function

$$\lambda \mapsto \left\| (\mathrm{i}\lambda - A)^{-1} \right\|_{L(\mathcal{X})}$$

is bounded outside a compact set. In the typical situation, arising for example in the models analyzed in [13,22,27], when the infinitesimal generator A has compact inverse, the inclusion $\mathrm{i}\mathbb{R} \subset \rho(A)$ can be obtained simply demonstrating that no eigenvalues of A lie on the imaginary axis, due to a well-known result of Kato [18, Theorem 6.29]. Still, for the semigroup $S(t) = \mathrm{e}^{t\mathbb{A}}$ generated by system (1.1), this strategy cannot be applied, since the inverse $\mathbb{A}^{-1}$ is not compact due to the memory component (see [24] for a counterexample). Therefore, in principle, the spec-trum of the operator $\mathbb{A}$ can be a complicated object, not simply made by (isolated) eigenvalues. We will overcome these difficulties by applying a suitable contradiction argument, together with some specific estimates in Sobolev spaces with negative exponent, in order to achieve exponential stability without making use of any *a priori* information on the spectrum of $\mathbb{A}$.

Plan of the paper. In the next Sections 2–3 we discuss the aforementioned implications of Theorem 1.1 on thermoelastic systems of Bresse and Timoshenko type. In the subsequent Sections 4–5 we introduce the functional setting of the problem and we establish the existence of the solution semigroup. The final Sections 6–7 are devoted to the proof of the main result.

2. Implications of Theorem 1.1 on Bresse systems

2.1. Bresse systems with Fourier heat conduction

According to the article [13], the classical thermoelastic Bresse–Fourier system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0, \\ \rho_3\theta_t - k_1\theta_{xx} + \gamma\psi_{tx} = 0, \end{cases} \tag{2.1}$$

is exponentially stable if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b} \quad \text{and} \quad k = k_0. \tag{2.2}$$

Actually, system (2.1) can be (formally) obtained from (1.1) in the singular limit when the memory kernel g collapses into the Dirac mass at zero δ_0. More precisely, introducing the family of rescaled kernels

$$g_\varepsilon(s) = \frac{1}{\varepsilon}g\left(\frac{s}{\varepsilon}\right), \quad \varepsilon > 0,$$

we have the convergence $g_\varepsilon \to \delta_0$ in the distributional sense, and thus system (1.1) with g replaced by g_ε reduces to (2.1) for $\varepsilon \to 0$. On the other hand, since

$$\chi_{g_\varepsilon} = \left(\frac{\rho_1}{\rho_3 k} - \frac{\varepsilon}{g(0)k_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\varepsilon}{g(0)k_1}\frac{\rho_1\gamma^2}{\rho_3 bk},$$

exploiting Theorem 1.1 and letting $\varepsilon \to 0$ we recover (2.2). This formal argument can be made rigorous within the proper functional setting, in the same spirit of [7].

2.2. Bresse systems with Maxwell–Cattaneo heat conduction

The thermoelastic Bresse–Maxwell–Cattaneo system reads as follows

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \tau k_1 q_t + q + k_1\theta_x = 0. \end{cases} \tag{2.3}$$

Here, the additional variable

$$q : (x, t) \in \mathfrak{I} \times [0, \infty) \mapsto \mathbb{R}$$

represents the so-called heat flux vector, satisfying the Maxwell–Cattaneo thermal law [3]

$$\tau k_1 q_t + q + k_1\theta_x = 0, \quad \tau > 0. \tag{2.4}$$

Arguing in a standard way, one can prove that system (2.3) generates a contraction semigroup of solutions $V(t)$. To the best of our knowledge, no results are currently available in the literature concerning the stability properties of such a semigroup. Nevertheless, the Bresse–Maxwell–Cattaneo model above can be obtained as a particular instance of the Bresse–Gurtin–Pipkin system (1.1), when

$$g(s) = g_\tau(s) = \frac{1}{\tau k_1}\,\mathrm{e}^{-\frac{s}{\tau k_1}}.$$

Indeed, setting

$$q(x,t) = -k_1\int_0^\infty g_\tau(s)\theta_x(x, t-s)\,\mathrm{d}s$$

and integrating by parts, it is immediate to see that (2.4) holds. In conclusion, substituting the explicit value $g_\tau(0) = 1/\tau k_1$ in the condition of Theorem 1.1, we obtain

Theorem 2.1. *The semigroup $V(t)$ generated by Bresse–Maxwell–Cattaneo system* (2.3) *is ex-ponentially stable if and only if*

$$\left(\frac{\rho_1}{\rho_3 k} - \tau\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \tau\frac{\rho_1\gamma^2}{\rho_3 bk} = 0 \quad \textit{and} \quad k = k_0.$$

Again, this formal argument can be made rigorous. Indeed, denoting by $U(t)$ the solution semigroup generated by (1.1) for the particular choice $g(s) = g_\tau(s)$, arguing as in [11, Sec-tion 8] it is possible to show that $U(t)$ is exponentially stable if and only if $V(t)$ is exponentially stable. Observe also that, in the limit situation when $\tau = 0$, system (2.3) reduces to (2.1) and the condition of Theorem 2.1 collapses into (2.2). In other words, the Fourier case is fully recovered in the limit $\tau \to 0$.

2.3. Bresse systems with Coleman–Gurtin heat conduction

Finally, we discuss the implications of Theorem 1.1 on the so-called Bresse–Coleman–Gurtin system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0, \\ \rho_3\theta_t - (1-\varpi)k_1\theta_{xx} - \varpi k_1 \displaystyle\int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s + \gamma\psi_{tx} = 0, \end{cases} \tag{2.5}$$

where $\varpi \in (0, 1)$ is a fixed constant. The limit cases $\varpi = 0$ and $\varpi = 1$ correspond to the parabolic Fourier model (2.1) and the hyperbolic Gurtin–Pipkin model (1.1), respectively. Once more, system (2.5) can be shown to generate a contraction semigroup of solutions $T(t)$ for which no stability results have been obtained so far. Still, as a corollary of Theorem 1.1, we obtain the following necessary and sufficient condition.

Theorem 2.2. *The semigroup $T(t)$ generated by the Bresse–Coleman–Gurtin system* (2.5) *is exponentially stable if and only if* (2.2) *is satisfied.*

Theorem 2.2 can be deduced from Theorem 1.1 considering the kernel

$$g_\varepsilon(s) = \frac{1-\varpi}{\varepsilon} g\left(\frac{s}{\varepsilon}\right) + \varpi g(s), \quad \varepsilon > 0.$$

Indeed, since

$$g_\varepsilon \to (1-\varpi)\delta_0 + \varpi g$$

in the distributional sense as $\varepsilon \to 0$, we obtain the convergence

$$\int_0^\infty g_\varepsilon(s)\theta_{xx}(t-s)\,\mathrm{d}s \to (1-\varpi)\theta_{xx} + \varpi \int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s.$$

On the other hand, in light of Theorem 1.1, exponential stability occurs if and only if

$$\left(\frac{\rho_1}{\rho_3 k} - \frac{\varepsilon}{(1-\varpi+\varpi\varepsilon)g(0)k_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\varepsilon}{(1-\varpi+\varpi\varepsilon)g(0)k_1}\frac{\rho_1\gamma^2}{\rho_3 bk} = 0$$

and

$$k = k_0.$$

We end up with (2.2) in the limit $\varepsilon \to 0$. It is worth noting that here the picture is the same as in the Fourier case due to the partial parabolic character of the heat conduction law. A similar feature was observed in [27] in the context of thermoelastic systems of Timoshenko type.

3. Implications of Theorem 1.1 on Timoshenko systems

As shown in [11], the Timoshenko–Gurtin–Pipkin system

$$
\begin{cases}
\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \\
\rho_3\theta_t - k_1 \int\limits_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s + \gamma\psi_{tx} = 0,
\end{cases}
\tag{3.1}
$$

is exponentially stable if and only if

$$
\left(\frac{\rho_1}{\rho_3 k} - \frac{1}{g(0)k_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{1}{g(0)k_1}\frac{\rho_1\gamma^2}{\rho_3 bk} = 0.
\tag{3.2}
$$

This result subsumes and generalizes the ones obtained in the papers [14,22,27], concerning Timoshenko systems with Fourier and Maxwell–Cattaneo heat conduction. In turn, system (3.1) can be seen as a particular instance of (1.1), in the limit case when $l = 0$. Indeed, in this situation, the Bresse–Gurtin–Pipkin system formally reduces to

$$
\begin{cases}
\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \\
\rho_3\theta_t - k_1 \int\limits_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s + \gamma\psi_{tx} = 0, \\
\rho_1 w_{tt} - k_0 w_{xx} = 0,
\end{cases}
\tag{3.3}
$$

where the last wave equation is decoupled from the others, and hence can be neglected. As a consequence, the condition $k = k_0$ provided by Theorem 1.1 is no longer relevant; the other one, namely $\chi_g = 0$, is exactly (3.2). This argument can be made rigorous taking $l = 0$ and revisiting the proof of Theorem 1.1. The details are left to the reader.

4. Functional setting and notations

Along the paper, $\langle\cdot\,,\cdot\rangle$ and $\|\cdot\|$ will denote the standard inner product and norm on the Hilbert space $L^2(\mathfrak{I})$. We also consider the Hilbert subspace of zero-mean functions

$$
L^2_*(\mathfrak{I}) = \left\{ f \in L^2(\mathfrak{I}) : \int\limits_0^L f(x)\,\mathrm{d}x = 0 \right\},
$$

along with the Hilbert spaces

$$
H^1_0(\mathfrak{I}) \quad \text{and} \quad H^1_*(\mathfrak{I}) = H^1(\mathfrak{I}) \cap L^2_*(\mathfrak{I}),
$$

both endowed with the gradient norm (due to the Poincaré inequality). Moreover, we introduce the so-called memory space

$$\mathcal{M} = L^2_\mu\big(\mathbb{R}^+; H^1_0(\mathfrak{I})\big)$$

of square summable H^1_0-valued functions on $\mathbb{R}^+$ with respect to the measure $\mu(s)\mathrm{d}s$, equipped with the weighted inner product

$$\langle \eta, \xi\rangle_{\mathcal{M}} = \int_0^\infty \mu(s)\big\langle \eta_x(s), \xi_x(s)\big\rangle \,\mathrm{d}s,$$

along with the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$

$$T\eta = -\eta' \quad \text{with domain } \mathfrak{D}(T) = \Big\{\eta \in \mathcal{M} : \eta' \in \mathcal{M},\ \lim_{s\to 0}\big\|\eta_x(s)\big\| = 0\Big\},$$

the *prime* standing for weak derivative. Finally, we define the phase space

$$\mathcal{H} = H^1_0(\mathfrak{I}) \times L^2(\mathfrak{I}) \times H^1_*(\mathfrak{I}) \times L^2_*(\mathfrak{I}) \times H^1_*(\mathfrak{I}) \times L^2_*(\mathfrak{I}) \times L^2(\mathfrak{I}) \times \mathcal{M}$$

endowed with the (equivalent) product norm

$$\begin{aligned}\big\|(\varphi, \Phi, \psi, \Psi, w, W, \theta, \eta)\big\|^2_{\mathcal{H}} &= \rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2 + \rho_1\|W\|^2 + b\|\psi_x\|^2 \\ &\quad + k\|\varphi_x + \psi + lw\|^2 + k_0\|w_x - l\varphi\|^2 + \rho_3\|\theta\|^2 + k_1\|\eta\|^2_{\mathcal{M}}.\end{aligned}$$

5. The contraction semigroup

We recast (1.1) in the history space framework devised by Dafermos [10]. To this end, intro-ducing the auxiliary variable

$$\eta = \eta^t(x, s) : (x, t, s) \in \mathfrak{I} \times [0, \infty) \times \mathbb{R}^+ \mapsto \mathbb{R}$$

formally defined as

$$\eta^t(x, s) = \int_0^s \theta(x, t - r)\,\mathrm{d}r,$$

we rewrite system (1.1) in the form

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0, \tag{5.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0, \tag{5.2}$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0, \tag{5.3}$$

$$\rho_3\theta_t - k_1 \int_0^\infty \mu(s)\eta_{xx}(s)\,\mathrm{d}s + \gamma\psi_{tx} = 0, \tag{5.4}$$

$$\eta_t = T\eta + \theta. \tag{5.5}$$

Next, introducing the state vector $z(t) = (\varphi(t),\ \Phi(t),\ \psi(t),\ \Psi(t), w(t),\ W(t),\ \theta(t),\ \eta^t)$, we view (5.1)–(5.5) as the abstract ODE in $\mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \mathbb{A}z(t), \tag{5.6}$$

where the (linear) operator $\mathbb{A}$ is defined as

$$\mathbb{A}\begin{pmatrix} \varphi \\ \Phi \\ \psi \\ \Psi \\ w \\ W \\ \theta \\ \eta \end{pmatrix} = \begin{pmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi + lw)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) \\ \Psi \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + lw) - \frac{\gamma}{\rho_2}\theta_x \\ W \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + \psi + lw) \\ \frac{k_1}{\rho_3}\int_0^\infty \mu(s)\eta_{xx}(s)\,\mathrm{d}s - \frac{\gamma}{\rho_3}\Psi_x \\ T\eta + \theta \end{pmatrix}$$

with domain

$$\mathfrak{D}(\mathbb{A}) = \left\{ (\varphi, \Phi, \psi, \Psi, w, W, \theta, \eta) \in \mathcal{H} \;\middle|\; \begin{array}{c} \varphi \in H^2(\mathfrak{I}) \\ \Phi \in H_0^1(\mathfrak{I}) \\ \psi_x \in H_0^1(\mathfrak{I}) \\ \Psi \in H_*^1(\mathfrak{I}) \\ w_x \in H_0^1(\mathfrak{I}) \\ W \in H_*^1(\mathfrak{I}) \\ \theta \in H_0^1(\mathfrak{I}) \\ \eta \in \mathfrak{D}(T) \\ \int_0^\infty \mu(s)\eta(s)\,\mathrm{d}s \in H^2(\mathfrak{I}) \end{array} \right\}.$$

Theorem 5.1. *The operator $\mathbb{A}$ is the infinitesimal generator of a contraction semigroup*

$$S(t) = \mathrm{e}^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}.$$

Theorem 5.1 above can be proved by means of the classical Lumer–Phillips theorem [25]. Indeed, for every $\eta \in \mathfrak{D}(T)$, the nonnegative functional

$$\Gamma[\eta] = -\int_0^\infty \mu'(s)\left\|\eta_x(s)\right\|^2 \mathrm{d}s$$

satisfies the identity

$$2\langle T\eta, \eta\rangle_{\mathcal{M}} = -\Gamma[\eta]$$

(see e.g. [16]). Therefore, for every fixed $z \in \mathfrak{D}(\mathbb{A})$, the equality

$$\langle \mathbb{A}z, z\rangle_{\mathcal{H}} = k_1 \langle T\eta, \eta\rangle_{\mathcal{M}} = -\frac{k_1}{2}\Gamma[\eta] \leq 0, \tag{5.7}$$

implies that $\mathbb{A}$ is dissipative. Moreover, arguing in a standard way (see e.g. [8]), one can show that the operator

$$1 - \mathbb{A} : \mathfrak{D}(\mathbb{A}) \subset \mathcal{H} \to \mathcal{H}$$

is onto, and the conclusion of Theorem 5.1 follows. As a consequence, for every initial datum

$$z_0 = (\varphi_0, \Phi_0, \psi_0, \Psi_0, w_0, W_0, \theta_0, \eta_0) \in \mathcal{H},$$

the unique solution at time $t > 0$ to (5.6) reads

$$z(t) = S(t)z_0 = \big(\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), w(t), w_t(t), \theta(t), \eta^t\big).$$

Remark 5.2. The choice of the spaces of zero-mean functions for the variables ψ and w and their derivatives is consistent. Indeed, setting

$$\Theta(t) = \int_0^L \psi(x,t)\,\mathrm{d}x \quad \text{and} \quad \Sigma(t) = \int_0^L w(x,t)\,\mathrm{d}x$$

and integrating (5.2) and (5.3) on $\mathfrak{I}$, we obtain the differential system

$$\begin{cases} \rho_2 \ddot{\Theta}(t) + k\Theta(t) + kl\Sigma(t) = 0, \\ \rho_1 \ddot{\Sigma}(t) + kl^2\Sigma(t) + kl\Theta(t) = 0. \end{cases}$$

Thus, if $\Theta(0) = \dot{\Theta}(0) = \Sigma(0) = \dot{\Sigma}(0) = 0$, it follows that $\Theta(t) \equiv \Sigma(t) \equiv 0$.

6. Proof of Theorem 1.1 (sufficiency)

Along the section, $C \geq 0$ will stand for a *generic* constant depending only on the structural quantities of the problem, while ε_n will indicate a *generic* complex sequence $\varepsilon_n \to 0$. Besides, we will tacitly use several times the Hölder, Young and Poincaré inequalities, as well as the inequality

$$\int_0^\infty \mu(s)\big\|\eta_x(s)\big\|\,\mathrm{d}s \leq \sqrt{g(0)}\,\|\eta\|_{\mathcal{M}},$$

and the control

$$\nu\|\eta\|^2_{\mathcal{M}} \leq \Gamma[\eta], \quad \forall \eta \in \mathfrak{D}(T), \tag{6.1}$$

ensured by (1.2). It is also understood that $\mathbb{A}$ and $S(t)$ denote the complexifications of the oper-ator $\mathbb{A}$ and the semigroup $S(t)$, respectively.

Suppose by contradiction that $S(t)$ is not exponentially stable. Then, in light of Lemma 1.2, there exist two sequences $\lambda_n \in \mathbb{R}$ and

$$z_n = (\varphi_n, \Phi_n, \psi_n, \Psi_n, w_n, W_n, \theta_n, \eta_n) \in \mathfrak{D}(\mathbb{A}) \text{ with } \|z_n\|_{\mathcal{H}} = 1$$

such that

$$\mathrm{i}\lambda_n z_n - \mathbb{A}z_n \to 0 \quad \text{in } \mathcal{H}. \tag{6.2}$$

Writing (6.2) componentwise, we obtain the system

$$\mathrm{i}\lambda_n \varphi_n - \Phi_n \to 0 \quad \text{in } H_0^1(\mathfrak{I}), \tag{6.3}$$

$$\mathrm{i}\lambda_n \rho_1 \Phi_n - k(\varphi_{nx} + \psi_n + l w_n)_x - k_0 l(w_{nx} - l\varphi_n) \to 0 \quad \text{in } L^2(\mathfrak{I}), \tag{6.4}$$

$$\mathrm{i}\lambda_n \psi_n - \Psi_n \to 0 \quad \text{in } H_\star^1(\mathfrak{I}), \tag{6.5}$$

$$\mathrm{i}\lambda_n \rho_2 \Psi_n - b\psi_{nxx} + k(\varphi_{nx} + \psi_n + l w_n) + \gamma\theta_{nx} \to 0 \quad \text{in } L_\star^2(\mathfrak{I}), \tag{6.6}$$

$$\mathrm{i}\lambda_n w_n - W_n \to 0 \quad \text{in } H_\star^1(\mathfrak{I}), \tag{6.7}$$

$$\mathrm{i}\lambda_n \rho_1 W_n - k_0(w_{nx} - l\varphi_n)_x + kl(\varphi_{nx} + \psi_n + l w_n) \to 0 \quad \text{in } L_\star^2(\mathfrak{I}), \tag{6.8}$$

$$\mathrm{i}\lambda_n \rho_3 \theta_n - k_1 \int_0^\infty \mu(s)\eta_{nxx}(s)\,\mathrm{d}s + \gamma\Psi_{nx} \to 0 \quad \text{in } L^2(\mathfrak{I}), \tag{6.9}$$

$$\mathrm{i}\lambda_n \eta_n - T\eta_n - \theta_n \to 0 \quad \text{in } \mathcal{M}. \tag{6.10}$$

Our aim is to reach a contradiction by showing that, up to a subsequence, every single component of z_n goes to zero in its norm. To this end, the boundedness in $\mathcal{H}$ of the sequence z_n will be used without explicit mention. Assuming $\lambda_n \not\to 0$[2] we infer that, up to a subsequence,

$$\inf_{n\in\mathbb{N}} |\lambda_n| > 0. \tag{6.11}$$

Next, taking the inner product in $\mathcal{H}$ of (6.2) with z_n and making use of (5.7), we have

$$k_1 \Gamma[\eta_n] = -2\mathfrak{Re}\langle \mathbb{A}z_n, z_n\rangle_{\mathcal{H}} = 2\mathfrak{Re}\langle \mathrm{i}\lambda_n z_n - \mathbb{A}z_n, z_n\rangle_{\mathcal{H}} \to 0 \text{ which,} \tag{6.12}$$

together with (6.1), yields the convergence

$$\lim_{n\to\infty} \|\eta_n\|_{\mathcal{M}} = 0. \tag{6.13}$$

The remaining part of the proof will be carried out in a number of lemmas.

[2] When $\lambda_n \to 0$ the argument is much simpler (see Remark 6.7).

Lemma 6.1. *Up to a subsequence,*

$$\lim_{n\to\infty} \|\theta_n\| = 0.$$

Proof. First, by the triangle inequality and (6.9),

$$\begin{aligned}\rho_3|\lambda_n|\|\theta_n\|_{H^{-1}(\mathfrak{I})} &\le \left\| \mathrm{i}\lambda_n\rho_3\theta_n - k_1\int_0^\infty \mu(s)\eta_{nxx}(s)\,\mathrm{d}s + \gamma\Psi_{nx}\right\|_{H^{-1}(\mathfrak{I})} \\ &\quad + \left\| k_1\int_0^\infty \mu(s)\eta_{nxx}(s)\,\mathrm{d}s - \gamma\Psi_{nx}\right\|_{H^{-1}(\mathfrak{I})} \\ &\le \varepsilon_n + C\|\eta_n\|_{\mathcal{M}} + C\|\Psi_n\|\end{aligned}$$

and hence we have the estimate

$$\sup_{n\in\mathbb{N}} |\lambda_n|\|\theta_n\|_{H^{-1}(\mathfrak{I})} < \infty.$$

Then, introducing $\hat{\theta}_n$ such that

$$\begin{cases} \hat{\theta}_{nxx} = -\theta_n, \\ \hat{\theta}_n(0) = \hat{\theta}_n(L) = 0, \end{cases}$$

we infer from (6.13) and the uniform bound above that

$$\left|\mathrm{i}\lambda_n\langle\eta_n, \hat{\theta}_n\rangle_{\mathcal{M}}\right| \le C|\lambda_n|\|\theta_n\|_{H^{-1}(\mathfrak{I})}\int_0^\infty \mu(s)\left\|\eta_{nx}(s)\right\|\mathrm{d}s \le C\|\eta_n\|_{\mathcal{M}} \to 0.$$

On the other hand, rewriting (6.10) as

$$\mathrm{i}\lambda_n\eta_n - T\eta_n - \theta_n = \zeta_n$$

with $\zeta_n \to 0$ in $\mathcal{M}$, we find the explicit expression

$$\eta_n(s) = \frac{1}{\mathrm{i}\lambda_n}\big(1 - \mathrm{e}^{-\mathrm{i}\lambda_n s}\big)\theta_n + \int_0^s \mathrm{e}^{-\mathrm{i}\lambda_n(s-r)}\zeta_n(r)\,\mathrm{d}r.$$

Therefore, setting

$$a_n = \int_0^\infty \mu(s)\big(1 - \mathrm{e}^{-\mathrm{i}\lambda_n s}\big)\,\mathrm{d}s,$$

$$b_n = \mathrm{i}\lambda_n \int_0^\infty \mu(s) \int_0^s \mathrm{e}^{-\mathrm{i}\lambda_n(s-r)} \langle \zeta_{nx}(r), \hat{\theta}_{nx} \rangle \,\mathrm{d}r\,\mathrm{d}s,$$

we obtain the convergence

$$a_n \|\theta_n\|^2 + b_n = \mathrm{i}\lambda_n \langle \eta_n, \hat{\theta}_n \rangle_{\mathcal{M}} \to 0. \tag{6.14}$$

Exploiting now the control

$$\mu(r+s) \le \mathrm{e}^{-\nu r}\mu(s), \quad \forall r \ge 0 \text{ and } s > 0$$

provided by (1.2), we get

$$|b_n| \le |\lambda_n| \|\theta_n\|_{H^{-1}(\mathfrak{I})} \int_0^\infty \sqrt{\mu(s)} \int_0^s \mathrm{e}^{-\frac{\nu}{2}(s-r)} \sqrt{\mu(r)} \|\zeta_{nx}(r)\| \,\mathrm{d}r\,\mathrm{d}s \le C\|\zeta_n\|_{\mathcal{M}},$$

and thus $b_n \to 0$. Moreover, let $\lambda_\star \in [-\infty, \infty] \setminus \{0\}$ such that $\lambda_n \to \lambda_\star$ up to a subsequence. If $\lambda_\star \in \{-\infty, \infty\}$, the Riemann–Lebesgue lemma yields

$$a_n \to \int_0^\infty \mu(s)\,\mathrm{d}s > 0,$$

whereas, if $\lambda_\star \in \mathbb{R} \setminus \{0\}$,

$$\mathfrak{Re}\, a_n \to \int_0^\infty \mu(s)(1 - \cos\lambda_\star s)\,\mathrm{d}s > 0.$$

In both cases, up to a subsequence, the real part of a_n is away from zero for large n, and the desired convergence $\|\theta_n\| \to 0$ follows from (6.14).

Lemma 6.2. *Up to a subsequence,*

$$\lim_{n\to\infty} \|\psi_n\| = \lim_{n\to\infty} \|\Psi_n\| = 0.$$

Proof. Setting

$$\hat{\Psi}_n(x) = \int_0^x \Psi_n(y)\,\mathrm{d}y \in H_0^1(\mathfrak{I})$$

and integrating (6.6) on $[0, x]$, one can easily see that

$$\sup_{n\in\mathbb{N}} |\lambda_n| \|\hat{\Psi}_n\| < \infty. \tag{6.15}$$

Next, taking the inner product in $L^2(\mathfrak{I})$ of (6.9) with $\hat{\Psi}_n$ we get

$$\gamma\|\Psi_n\|^2 - k_1 \int_0^\infty \mu(s)\big\langle \eta_{nx}(s), \Psi_n\big\rangle \mathrm{d}s - \mathrm{i}\lambda_n \rho_3 \langle \theta_n, \hat{\Psi}_n \rangle \to 0.$$

In light of (6.13), it is immediate to see that

$$\int_0^\infty \mu(s)\big\langle \eta_{nx}(s), \Psi_n\big\rangle \mathrm{d}s \to 0.$$

Moreover, thanks to (6.15) and Lemma 6.1,

$$\big|-\mathrm{i}\lambda_n \rho_3 \langle \theta_n, \hat{\Psi}_n\rangle\big| \le \rho_3 |\lambda_n| \|\hat{\Psi}_n\| \|\theta_n\| \to 0$$

forcing

$$\|\Psi_n\| \to 0.$$

Finally, an application of (6.5) and (6.11) completes the proof.

Lemma 6.3. *Up to a subsequence,*

$$\lim_{n\to\infty} \|\psi_{nx}\| = 0.$$

Proof. Multiplying in $L^2(\mathfrak{I})$ relation (6.6) by ψ_n and making use of (6.5) we have

$$b\|\psi_{nx}\|^2 - \rho_2\|\Psi_n\|^2 + k\langle \varphi_{nx} + \psi_n + lw_n, \psi_n\rangle - \gamma\,\langle \theta_n, \psi_{nx}\rangle \to 0.$$

The claimed convergence is then a direct consequence of Lemmas 6.1 and 6.2.

The equality $\chi_g = 0$, not used so far, will play a crucial role in the next lemma.

Lemma 6.4. *Up to a subsequence,*

$$\lim_{n\to\infty} \|\varphi_{nx} + \psi_n + lw_n\| = 0.$$

Proof. We divide the proof into three steps.

Step 1. Multiplying in $L^2(\mathfrak{I})$ relation (6.6) by $\varphi_{nx} + \psi_n + lw_n$ we infer that

$$k\|\varphi_{nx} + \psi_n + lw_n\|^2 + \gamma\,\langle \theta_{nx}, \varphi_{nx} + \psi_n + lw_n\rangle + P_n + Q_n = \varepsilon_n \tag{6.16}$$

where

$$P_n = b\big\langle \psi_{nx}, (\varphi_{nx} + \psi_n + l w_n)_x \big\rangle,$$

$$Q_n = \mathrm{i}\lambda_n \rho_2 \langle \Psi_n, \varphi_{nx} + \psi_n + l w_n \rangle.$$

Next, exploiting (6.4), (6.5) and (6.8),

$$\begin{aligned} P_n &= \frac{bk_0 l}{k}\big\langle \psi_n, (w_{nx} - l\varphi_n)_x \big\rangle - \frac{\mathrm{i}\lambda_n \rho_1 b}{k}\langle \psi_{nx}, \Phi_n \rangle + \varepsilon_n \\ &= bl^2 \langle \psi_n, \varphi_{nx} + \psi_n + l w_n \rangle - \frac{b\rho_1 l}{k}\langle \Psi_n, W_n \rangle + \frac{b\rho_1}{k}\langle \Psi_n, \Phi_{nx} \rangle + \varepsilon_n. \end{aligned}$$

In light of Lemma 6.2,

$$bl^2 \langle \psi_n, \varphi_{nx} + \psi_n + l w_n \rangle - \frac{b\rho_1 l}{k}\langle \Psi_n, W_n \rangle \to 0$$

and thus

$$P_n = \frac{b\rho_1}{k}\langle \Psi_n, \Phi_{nx} \rangle + \varepsilon_n. \tag{6.17}$$

Concerning Q_n, invoking (6.3), (6.5) and (6.7) we can write

$$Q_n = -\rho_2 \langle \Psi_n, \Phi_{nx} + \Psi_n + l W_n \rangle + \varepsilon_n$$

which, applying Lemma 6.2 once more, reduces to

$$Q_n = -\rho_2 \langle \Psi_n, \Phi_{nx} \rangle + \varepsilon_n.$$

Plugging (6.17) and the identity above into (6.16) we end up

$$k\|\varphi_{nx} + \psi_n + l w_n\|^2 + \gamma \langle \theta_{nx}, \varphi_{nx} + \psi_n + l w_n \rangle = \left(\rho_2 - \frac{b\rho_1}{k}\right)\langle \Psi_n, \Phi_{nx} \rangle + \varepsilon_n. \tag{6.18}$$

Step 2. Our next goal is to prove the equality

$$g(0)k_1 \langle \theta_{nx}, \varphi_{nx} + \psi_n + l w_n \rangle = \mathrm{i}\lambda_n \rho_3 \langle \theta_n, \Phi_n \rangle - \gamma \langle \Psi_n, \Phi_{nx} \rangle + \varepsilon_n. \tag{6.19}$$

To this end, setting

$$\hat{m}_n(x) = \int_0^x \big(\varphi_{nx}(y) + \psi_n(y) + l w_n(y)\big)\, \mathrm{d}y \in H_0^1(\mathfrak{I})$$

and taking the inner product in $\mathcal{M}$ of (6.10) with $\hat{m}_n$, we have

$$g(0)\langle \theta_{nx}, \varphi_{nx} + \psi_n + l w_n \rangle = K_n + H_n + \varepsilon_n, \tag{6.20}$$

where

$$K_n = \int_0^\infty \mu(s)\big\langle \mathrm{i}\lambda_n \eta_{nx}(s), \varphi_{nx} + \psi_n + l w_n \big\rangle \, \mathrm{d}s,$$

$$H_n = \int_0^\infty \mu(s)\big\langle T\eta_n(s), (\varphi_{nx} + \psi_n + l w_n)_x \big\rangle \, \mathrm{d}s.$$

An exploitation of (6.3), (6.5) and (6.7), together with (6.13), entails

$$K_n = -\int_0^\infty \mu(s)\big\langle \eta_{nx}(s), \Phi_{nx} + \Psi_n + l W_n \big\rangle \, \mathrm{d}s + \varepsilon_n = \int_0^\infty \mu(s)\big\langle \eta_{nxx}(s), \Phi_n \big\rangle \, \mathrm{d}s + \varepsilon_n.$$

Moreover, recalling (6.9),

$$\int_0^\infty \mu(s)\big\langle \eta_{nxx}(s), \Phi_n \big\rangle \, \mathrm{d}s = \frac{\mathrm{i}\lambda_n \rho_3}{k_1} \langle \theta_n, \Phi_n \rangle - \frac{\gamma}{k_1} \langle \Psi_n, \Phi_{nx} \rangle + \varepsilon_n,$$

and hence (6.20) turns into

$$g(0)\langle \theta_{nx}, \varphi_{nx} + \psi_n + l w_n \rangle = \frac{\mathrm{i}\lambda_n \rho_3}{k_1} \langle \theta_n, \Phi_n \rangle - \frac{\gamma}{k_1} \langle \Psi_n, \Phi_{nx} \rangle + H_n + \varepsilon_n.$$

In order to reach (6.19), we are left to show that $H_n \to 0$. To this aim, integrating by parts in s (as shown in [16] the boundary terms vanish)

$$\begin{aligned} |H_n| = \left| -\int_0^\infty \mu'(s)\big\langle \eta_{nx}(s), \varphi_{nx} + \psi_n + l w_n \big\rangle \, \mathrm{d}s \right| &\le C \int_0^\infty -\mu'(s) \big\| \eta_{nx}(s) \big\| \, \mathrm{d}s \\ &\le C\sqrt{\Gamma[\eta_n]}, \end{aligned}$$

and an application of (6.12) completes the argument.

Step 3. First observe that, in light of (6.4) and Lemma 6.1,

$$\mathrm{i}\lambda_n \rho_1 \langle \theta_n, \Phi_n \rangle = k\langle \theta_{nx}, \varphi_{nx} + \psi_n + l w_n \rangle + \varepsilon_n. \tag{6.21}$$

Then, calling[3]

$$\sigma_g = k - \frac{g(0) k_1 \rho_1}{\rho_3},$$

[3] It is readily seen that $\chi_g = 0 \Rightarrow \sigma_g \neq 0$.

we multiply (6.19) by $\frac{\rho_1\gamma}{\sigma_g\rho_3}$ and (6.21) by $\frac{\gamma}{\sigma_g}$. Summing up with (6.18), we finally obtain the identity

$$k\|\varphi_{nx}+\psi_n+lw_n\|^2=\chi_g\frac{g(0)k_1bk}{\sigma_g}\langle\Psi_n,\Phi_{nx}\rangle+\varepsilon_n,$$

and since $\chi_g=0$ by assumption the conclusion follows. □

Lemma 6.5. *Up to a subsequence,*

$$\lim_{n\to\infty}\|\varphi_n\|_{H^{-1}(\mathfrak{I})}=0.$$

Proof. Setting

$$\hat{W}_n(x)=\int_0^x W_n(y)\,\mathrm{d}y\in H_0^1(\mathfrak{I})$$

and making use of Lemma 6.4, we deduce from relations (6.4) and (6.8) that

$$\begin{aligned}
&\mathrm{i}\lambda_n\rho_1\Phi_n-k_0l(w_{nx}-l\varphi_n)\to 0\quad\text{in } H^{-1}(\mathfrak{I}),\\
&\mathrm{i}\lambda_n\rho_1 l\hat{W}_n-k_0l(w_{nx}-l\varphi_n)\to 0\quad\text{in } H^{-1}(\mathfrak{I}).
\end{aligned}$$

Taking the difference

$$\mathrm{i}\lambda_n\rho_1(\Phi_n-l\hat{W}_n)\to 0\quad\text{in } H^{-1}(\mathfrak{I})$$

and then, exploiting (6.3) and (6.7), together with (6.11),

$$\varphi_n-l\hat{w}_n\to 0\quad\text{in } H^{-1}(\mathfrak{I}),$$

where

$$\hat{w}_n(x)=\int_0^x w_n(y)\,\mathrm{d}y\in H_0^1(\mathfrak{I}).$$

On the other hand, in light of Lemmas 6.2 and 6.4,

$$\varphi_n+l\hat{w}_n\to 0\quad\text{in } H^{-1}(\mathfrak{I}).$$

Summing up the two relations above we are finished. □

Lemma 6.6. *Up to a subsequence*

$$\lim_{n\to\infty}\|\Phi_n\|=0.$$

Proof. Multiplying in $L^2(\mathfrak{I})$ relation (6.4) by φ_n and invoking (6.3) we obtain

$$-\rho_1\|\Phi_n\|^2 + k\langle\varphi_{nx} + \psi_n + lw_n, \varphi_{nx}\rangle + k_0 l\langle w_n, \varphi_{nx}\rangle + k_0 l^2\|\varphi_n\|^2 \to 0$$

which, by means of Lemma 6.4, improves to

$$-\rho_1\|\Phi_n\|^2 + k_0 l\langle w_n, \varphi_{nx}\rangle + k_0 l^2\|\varphi_n\|^2 \to 0. \tag{6.22}$$

Besides, by interpolation and Lemma 6.5,

$$\|\varphi_n\|^2 \le C\|\varphi_n\|_{H^{-1}(\mathfrak{I})} \to 0.$$

Moreover, since

$$k_0 l\langle w_n, \varphi_{nx}\rangle = k_0 l\langle w_n, \varphi_{nx} + \psi_n + lw_n\rangle - k_0 l\langle w_n, \psi_n\rangle - k_0 l^2\|w_n\|^2,$$

appealing to Lemma 6.2 and invoking once more Lemma 6.4, we conclude from (6.22) that

$$\rho_1\|\Phi_n\|^2 + k_0 l^2\|w_n\|^2 \to 0.$$

The lemma is proven.

In light of (6.13) and Lemmas 6.1–6.6, the desired contradiction is attained once we prove the convergence (up to a subsequence)

$$\lim_{n\to\infty}\|w_{nx} - l\varphi_n\| = \lim_{n\to\infty}\|W_n\| = 0, \tag{6.23}$$

provided that $k = k_0$. To this aim, multiplying in $L^2(\mathfrak{I})$ relation (6.4) by $w_{nx} - l\varphi_n$ and exploiting (6.3) and (6.7), together with Lemma 6.6, we are led to

$$k_0 l\|w_{nx} - l\varphi_n\|^2 - k\big\langle\varphi_{nx} + \psi_n + lw_n, (w_{nx} - l\varphi_n)_x\big\rangle + \rho_1\langle\Phi_n, W_{nx}\rangle \to 0. \tag{6.24}$$

Next, with the aid of (6.8) and Lemma 6.4, we rewrite the second term as

$$-k\big\langle\varphi_{nx} + \psi_n + lw_n, (w_{nx} - l\varphi_n)_x\big\rangle = \frac{\mathrm{i}\lambda_n\rho_1 k}{k_0}\langle\varphi_{nx} + \psi_n + lw_n, W_n\rangle + \varepsilon_n$$

which, by means of (6.3), (6.5), (6.7) and Lemma 6.2, yields

$$-k\big\langle\varphi_{nx} + \psi_n + lw_n, (w_{nx} - l\varphi_n)_x\big\rangle = -\frac{\rho_1 k}{k_0}\langle\Phi_n, W_{nx}\rangle + \frac{\rho_1 kl}{k_0}\|W_n\|^2 + \varepsilon_n.$$

Plugging the equality above into (6.24) and recalling that $k = k_0$ we arrive at (6.23).

Remark 6.7. When $\lambda_n \to 0$, in addition to (6.12) and (6.13) we deduce from (6.2) that

$$\lim_{n\to\infty}\|\Phi_{nx}\| = \lim_{n\to\infty}\|\Psi_{nx}\| = \lim_{n\to\infty}\|W_{nx}\| = 0.$$

Moreover, relations (6.4), (6.6), (6.8) and (6.10) turn into

$$k(\varphi_{nx} + \psi_n + l w_n)_x + k_0 l(w_{nx} - l\varphi_n) \to 0 \quad \text{in } L^2(\mathfrak{I}), \tag{6.25}$$

$$b\psi_{nxx} - k(\varphi_{nx} + \psi_n + l w_n) - \gamma\theta_{nx} \to 0 \quad \text{in } L^2_\star(\mathfrak{I}), \tag{6.26}$$

$$k_0(w_{nx} - l\varphi_n)_x - kl(\varphi_{nx} + \psi_n + l w_n) \to 0 \quad \text{in } L^2_\star(\mathfrak{I}), \tag{6.27}$$

$$T\eta_n + \theta_n \to 0 \quad \text{in } \mathcal{M}. \tag{6.28}$$

Introducing again $\hat{\theta}_n$ such that

$$\begin{cases} \hat{\theta}_{nxx} = -\theta_n, \\ \hat{\theta}_n(0) = \hat{\theta}_n(L) = 0, \end{cases}$$

the inner product in $\mathcal{M}$ of (6.28) with $\hat{\theta}_n$ gives

$$g(0)\|\theta_n\|^2 + \langle T\eta_n, \hat{\theta}_n\rangle_{\mathcal{M}} \to 0.$$

Integrating by parts as in the proof of Lemma 6.4, one can see that the second term above goes to zero, and hence

$$\lim_{n\to\infty} \|\theta_n\| = 0.$$

At this point, multiplying in $L^2(\mathfrak{I})$ relation (6.25) by φ_n, (6.26) by ψ_n, (6.27) by w_n and summing up we conclude that

$$\lim_{n\to\infty} \|\varphi_{nx} + \psi_n + l w_n\| = \lim_{n\to\infty} \|\psi_{nx}\| = \lim_{n\to\infty} \|w_{nx} - l\varphi_n\| = 0.$$

The sought contradiction follows.

Remark 6.8. Actually, hypothesis (1.2) can be relaxed: the proof carried out in this section holds even if μ satisfies for some $C \geq 1$ and $\nu > 0$ the weaker condition

$$\mu(r+s) \leq C\mathrm{e}^{-\nu r}\,\mu(s),$$

for every $r \geq 0$ and $s > 0$. Observe that the latter inequality boils down to (1.2) when $C = 1$. However, in this case, the argument becomes much more involved, as one cannot deduce the convergence (6.13) directly from (6.12). As a consequence, an additional reasoning is needed, in the same spirit of [6,23].

7. Proof of Theorem 1.1 (necessity)

Without loss of generality, we will assume along the section that $L = \pi$. The strategy consists in showing that the necessary and sufficient condition (ii) of Lemma 1.2 fails to hold. To this end, for every $n \in \mathbb{N}$, we consider the vector

$$\zeta_n = (0, \alpha\ \sin nx, 0, 0, 0, \beta\ \cos nx, 0, 0) \in \mathcal{H}$$

for some constants $\alpha, \beta \in \mathbb{R}$ to be fixed later in such a way that

$$\|\zeta_n\|_{\mathcal{H}}^2 = \frac{\pi}{2\rho_1}, \tag{7.1}$$

and we study the resolvent equation

$$\mathrm{i}\lambda_n z_n - \mathbb{A} z_n = \zeta_n$$

for some real $\lambda_n \to \infty$ to be suitably chosen in a second moment, in the unknown variable

$$z_n = (\varphi_n, \Phi_n, \psi_n, \Psi_n, w_n, W_n, \theta_n, \eta_n) \in \mathfrak{D}(\mathbb{A}).$$

Due to (7.1), our conclusion is reached if we show that z_n is not bounded in $\mathcal{H}$, up to a subse-quence.

Reformulating the resolvent equation above componentwise and performing straightforward calculations, we obtain the system

$$\begin{cases} \rho_1 \lambda_n^2 \varphi_n + k(\varphi_{nx} + \psi_n + l w_n)_x + k_0 l (w_{nx} - l\varphi_n) = -\alpha\rho_1 \sin nx, \\ \rho_2 \lambda_n^2 \psi_n + b\psi_{nxx} - k(\varphi_{nx} + \psi_n + l w_n) - \gamma\theta_{nx} = 0, \\ \rho_1 \lambda_n^2 w_n + k_0 (w_{nx} - l\varphi_n)_x - kl(\varphi_{nx} + \psi_n + l w_n) = -\beta\rho_1 \cos nx, \\ \mathrm{i}\lambda_n \rho_3 \theta_n - k_1 \displaystyle\int_0^\infty \mu(s)\eta_{nxx}(s)\,\mathrm{d}s + \mathrm{i}\lambda_n \gamma \psi_{nx} = 0, \\ \mathrm{i}\lambda_n \eta_n - T\eta_n - \theta_n = 0. \end{cases}$$

Next, looking for solutions (compatible with the boundary conditions) of the form

$$\begin{aligned} \varphi_n &= A_n \sin nx, \\ \psi_n &= B_n \cos nx, \\ w_n &= C_n \cos nx, \\ \theta_n &= D_n \sin nx, \\ \eta_n &= \phi_n(s) \sin nx, \end{aligned}$$

for some $A_n, B_n, C_n, D_n \in \mathbb{C}$ and some complex function $\phi_n \in L^2_\mu(\mathbb{R}^+)$ with $\phi_n(0) = 0$, we draw the set of equations

$$\big(\rho_1 \lambda_n^2 - kn^2 - k_0 l^2\big) A_n - knB_n - ln(k + k_0) C_n = -\alpha\rho_1, \tag{7.2}$$

$$knA_n - \big(\rho_2 \lambda_n^2 - bn^2 - k\big) B_n + klC_n + \gamma n D_n = 0, \tag{7.3}$$

$$ln(k + k_0) A_n + klB_n - \big(\rho_1 \lambda_n^2 - k_0 n^2 - kl^2\big) C_n = \beta\rho_1, \tag{7.4}$$

$$\mathrm{i}\lambda_n \rho_3 D_n + k_1 n^2 \int_0^\infty \mu(s)\phi_n(s)\,\mathrm{d}s - \mathrm{i}\lambda_n \gamma n B_n = 0, \tag{7.5}$$

$$\mathrm{i}\lambda_n\phi_n + \phi_n' - D_n = 0. \tag{7.6}$$

An integration of (7.6) yields

$$\phi_n(s) = \frac{D_n}{\mathrm{i}\lambda_n}\big(1 - \mathrm{e}^{-\mathrm{i}\lambda_n s}\big)$$

which, substituted into (7.5), entails

$$D_n = \frac{\lambda_n^2 \gamma n B_n}{\lambda_n^2\rho_3 - k_1 n^2 g(0) + k_1 n^2 F(\lambda_n)}$$

where

$$F(\lambda_n) = \int_0^\infty \mu(s)\mathrm{e}^{-\mathrm{i}\lambda_n s}\,\mathrm{d}s$$

denotes the Fourier transform[4] of μ. Plugging the equality above into (7.3), we arrive at

$$\begin{cases} p_1(n)A_n - knB_n - ln(k+k_0)C_n = -\alpha\rho_1, \\ knA_n + \big(q(n) - p_2(n)\big)B_n + klC_n = 0, \\ ln(k+k_0)A_n + klB_n - p_3(n)C_n = \beta\rho_1, \end{cases} \tag{7.7}$$

having set

$$\begin{aligned} p_1(n) &= \rho_1\lambda_n^2 - kn^2 - k_0 l^2, \\ p_2(n) &= \rho_2\lambda_n^2 - bn^2 - k, \\ p_3(n) &= \rho_1\lambda_n^2 - k_0 n^2 - kl^2, \end{aligned}$$

and

$$q(n) = \frac{\gamma^2 n^2 \lambda_n^2}{\lambda_n^2\rho_3 - k_1 n^2 g(0) + k_1 n^2 F(\lambda_n)}.$$

At this point, we shall consider two cases separately.

Case 1: $k \neq k_0$. Choosing

$$\lambda_n = \sqrt{\frac{k_0 n^2 + kl^2}{\rho_1}} \sim \sqrt{\frac{k_0}{\rho_1}}\,n, \tag{7.8}$$

we find the explicit expressions

[4] Since μ is continuous nonincreasing and summable, it is easy to see that $F(\lambda_n) \neq 0$ for every n.

$$
\begin{aligned}
p_1(n) &= (k_0 - k)n^2 + l^2(k - k_0),\\
p_2(n) &= \left(\frac{\rho_2 k_0}{\rho_1} - b\right)n^2 + \frac{\rho_2 l^2 k}{\rho_1} - k,\\
p_3(n) &= 0,
\end{aligned}
$$

and

$$
q(n) = \frac{\gamma^2 k_0 n^2 + \gamma^2 l^2 k}{\rho_3 k_0 - g(0)k_1\rho_1 + k_1\rho_1 F(\lambda_n) + l^2 k \rho_3 n^{-2}}.
$$

We now claim that, up to a subsequence,

$$
\lim_{n\to\infty} \left|q(n) - p_2(n)\right| = \infty. \tag{7.9}
$$

Indeed, by direct computations and in light of the Riemann–Lebesgue lemma which tells that $F(\lambda_n) \to 0$, we have

$$
q(n) - p_2(n) = \frac{\xi_g n^2 + k_1(b\rho_1 - \rho_2 k_0)n^2 F(\lambda_n) + \mathrm{O}(1)}{\rho_3 k_0 - g(0)k_1\rho_1 + \mathrm{o}(1)}
$$

where

$$
\xi_g = \frac{\gamma^2 k_0 \rho_1 + (b\rho_1 - \rho_2 k_0)(\rho_3 k_0 - g(0)k_1\rho_1)}{\rho_1}.
$$

If $\xi_g \neq 0$ relation (7.9) is plain. On the other hand, when $\xi_g = 0$ it is evident that

$$
b\rho_1 - \rho_2 k_0 \neq 0 \quad \text{and} \quad \rho_3 k_0 - g(0)k_1\rho_1 \neq 0.
$$

Since (7.8) implies that

$$
n^2 \left|F(\lambda_n)\right| \to \infty,
$$

we end up again with (7.9). Setting now $\alpha = 0$ and $\beta = \rho_1^{-1}$, and solving system (7.7) with respect to C_n, we find

$$
C_n = \frac{p_1(n)(q(n) - p_2(n)) + k^2 n^2}{l^2 n^2 (k + k_0)^2 (q(n) - p_2(n)) + \mathrm{O}(n^2)}.
$$

Then, on account of (7.9) and the explicit expression of $p_1(n)$, we get the convergence

$$
\lim_{n\to\infty} C_n = \frac{k_0 - k}{l^2(k + k_0)^2} \neq 0.
$$

As a consequence, for some $c > 0$,

$$
\|z_n\|_{\mathcal{H}} \geq c\|w_{nx}\| = cn|C_n| \left(\int_0^{\pi} \sin^2 nx \, \mathrm{d}x \right)^{\frac{1}{2}} = \frac{c\sqrt{\pi}}{\sqrt{2}} n|C_n| \to \infty,
$$

and the conclusion follows.

Case 2: $k = k_0$ and $\chi_g \neq 0$. Choosing

$$\lambda_n = \sqrt{\frac{kn^2 + k_0 l^2 + 2kln}{\rho_1}} = \sqrt{\frac{k}{\rho_1}}(n+l),$$

we infer that

$$\begin{aligned} p_1(n) &= 2kln, \\ p_2(n) &= \left(\frac{\rho_2 k}{\rho_1} - b\right)n^2 + \mathrm{O}(n), \\ p_3(n) &= 2kln, \end{aligned}$$

and

$$q(n) = \frac{\gamma^2 k(n+l)^2}{\rho_3 \sigma_g + k_1 \rho_1 F(\lambda_n) + \mathrm{o}(1)}$$

where, as before,

$$\sigma_g = k - \frac{g(0) k_1 \rho_1}{\rho_3}.$$

Setting $\alpha = \rho_1^{-1}$ and $\beta = 0$, and solving system (7.7) with respect to A_n, we find

$$A_n = \frac{\rho_2}{k^2}\left(\frac{k}{\rho_1} - \frac{b}{\rho_2}\right)\left(\frac{n}{n+l}\right)^2 - \frac{\gamma^2}{k\rho_3\sigma_g + k\rho_1 k_1 F(\lambda_n) + \mathrm{o}(1)} + \mathrm{o}(1).$$

If $\sigma_g \neq 0$, an application of the Riemann–Lebesgue lemma yields the convergence

$$\lim_{n\to\infty} A_n = \frac{\rho_2}{k^2}\left(\frac{k}{\rho_1} - \frac{b}{\rho_2}\right) - \frac{\gamma^2}{k\rho_3\sigma_g} = \frac{g(0)k_1 b}{\rho_1 \sigma_g}\chi_g \neq 0$$

and hence, for some $c > 0$,

$$\|z_n\|_{\mathcal{H}} \geq c\|\varphi_{nx}\| = cn|A_n|\left(\int_0^\pi \cos^2 nx \,\mathrm{d}x\right)^{\frac{1}{2}} = \frac{c\sqrt{\pi}}{\sqrt{2}} n|A_n| \to \infty.$$

If otherwise $\sigma_g = 0$, exploiting the Riemann–Lebesgue lemma once more we obtain the asymptotic expression as $n \to \infty$

$$A_n \sim -\frac{\gamma^2}{k\rho_1 k_1 F(\lambda_n) + \mathrm{o}(1)},$$

and again

$$\|z_n\|_{\mathcal{H}} \geq \frac{c\sqrt{\pi}}{\sqrt{2}} n|A_n| \to \infty.$$

The proof is finished.

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