The Construction of Joint Possibility Distributions of Random Contributions to Uncertainty

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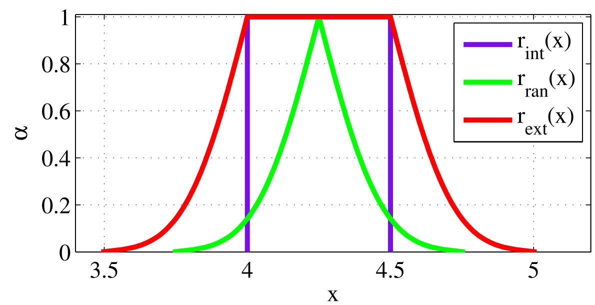

Fig. 1. Example of RFV (red + purple lines) and its PDs: $r_{\mathrm{int}}(x)$ (purple line), $r_{\mathrm{ran}}(x)$ (green line), and $r_{\mathrm{ext}}(x)$ (red line).

I. Introduction

It IS well known, also according to the definitions given in the International Vocabulary of Metrology (VIM) [1] and the Guide to the expression of Uncertainty in Measurement (GUM) [2], that a measurement result is "a set of quantity values being attributed to a measurand together with any other available relevant information" [1]. Consequently, the identification and evaluation of this "other available relevant information" represents one of the fundamental and widely discussed issues in the measurement science [3]–[5]. Generally, as stated again in the VIM, a measurement result is "expressed as a single measured quantity value and a measurement uncertainty" [1]. It is therefore recognized that a measured value provides incomplete knowledge of the measurand, and measurement uncertainty quantifies how incomplete this knowledge is.

The GUM [2] follows a strict probabilistic approach to the definition and evaluation of uncertainty. However, recently, questions have been raised as to whether the probability is always capable of representing incomplete knowledge [6]; and a more modern mathematical approach to represent incomplete knowledge has been proposed [6], [7] in terms of evidence and possibility. This interesting approach has been recently applied to uncertainty in measurement, as a possible way to overcome the limitations of the GUM probabilistic approach [8]–[17] and has already been applied in some interesting cases [18]–[24].

Within this new approach, random-fuzzy variables (RFVs) have been defined to represent and combine measurement results affected by both random and nonrandom contributions to uncertainty [12]–[17]. An RFV is defined by two possibility distributions (PDs) [12]: the internal one $r_{\mathrm{int}}(x)$ considers the nonrandom contributions to uncertainty, whilst the external one $r_{\mathrm{ext}}(x)$ considers also the random contributions. The external PD is obtained by combining the internal PD $r_{\mathrm{int}}(x)$ with a PD $r_{\mathrm{ran}}(x)$ (random PD in the following), which considers only the random contributions to uncertainty [12]. An example of RFV and its PDs is shown in Fig. 1.

Generally, in measurement practice, a random contribution is expressed by means of a probability distribution function (PDF). This PDF can be experimentally evaluated, or assumed on the basis of *a priori* knowledge. When the random contribution is expressed in terms of a PDF, it is possible to find its corresponding PD $r_{\mathrm{ran}}(x)$ by applying a suitable 1-D probability–possibility transformation [25]–[28]. On the other hand, the internal PD is directly built from the available metrological information [15].

When measurement results, expressed in terms of RFVs, have to be combined, different operators have to be applied to the internal and random PDs, in order to preserve the specificity of the different ways random and nonrandom contributions propagate through the measurement process.

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The combination of internal PDs is quite straightforward in the possibility theory [29], and is not covered by this paper. On the contrary, the combination of the random PDs is not straightforward. Indeed, random effects are more effectively represented and propagated within the probability theory, which remains the most effective mathematical tool to handle random contributions to measurement uncertainty. On the other hand, when multiple random and nonrandom effects have to be taken into account, probability is not the most effective tool any longer and possibility appears to be more promising [6]. To cover these cases, a method should be defined to propagate random effects also in the possibility domain, while preserving their random nature.

In the recent literature, the use of t-norms is suggested for combining the random PDs [10], [30], [31]. In fact, t-norms are associative functions [32] that can be used to build a joint PD with given marginal PDs, and, once the joint PD is built, the combination of the marginal PDs is easily performed in the possibility theory by means of Zadeh's extension principle [33].

Two problems still exist in the construction of a joint PD with given marginals by means of t-norms. First, the resulting joint PD strongly depends on the considered t-norm, and, therefore, the choice of one particular t-norm instead of another should be justified. A first attempt was made in [31], where the results of the combination of random PDs induced by a particular t-norm have been compared with those obtained with Monte Carlo simulations, which are themselves approximations of the true results.

In this paper, a more in-depth comparison is performed, since the joint PD induced by a particular t-norm is directly compared with a reference joint PD. The reference joint PD is provided by a 2-D probability–possibility transformation, which was recently proposed by the authors [28], and allows one to build the corresponding joint PD of a given joint PDF. This joint PD can be taken as the reference joint PD, since it expresses the same information content as the original joint PDF [28]. Therefore, the more the joint PD provided by a t-norm is similar to the reference joint PD, the more the propagation of random contributions in the possibility domain is similar to the propagation of the same contributions in the probability domain.

An additional problem is related to the construction of the joint PD when the two random contributions show a dependence, i.e., in the case of correlated random contributions. Also this problem is addressed in this paper, with specific reference to a particular class of possibility distributions.

This paper covers the problem of identifying the "best" t-norm for the construction of a joint PD starting from two marginal PDs induced by probability distributions. It does not cover any specific practical example since it does not add any additional relevant information to the theoretical analysis. However, a simple but significant application of the identified "best" t-norm can be found in [34].

II. Construction of Joint Distributions

Both in the probability and possibility frameworks, joint distributions express the same type of information content. From joint distributions, it is easy to obtain information about the two marginal distributions, which express the probability of each random contribution independently of the other, and the dependence (correlation) of the two contributions. The problem arises when the inverse problem is considered: i.e., the construction of joint distributions starting from their marginal distributions. This is not immediate, not even in the probability domain, except for uncorrelated distributions or correlated normal distributions. In this section, this problem is discussed separately for the probabilistic and possibilistic frameworks.

A. Construction of Joint PDFs

In probability theory, the joint PDF $p_{X,Y}$ of two generically correlated random variables X and Y is obtained as

$$p_{X,Y}(x, y) = p_X(x) \cdot p_{Y|X=x}(y) = p_Y(y) \cdot p_{X|Y=y}(x) \quad (1)$$

where p_X and p_Y are the marginal distributions of $p_{X,Y}$, and $p_{Y|X=x}$ and $p_{X|Y=y}$ are the conditional probability distributions of Y, given $X = x$, and X, given $Y = y$, respectively. Therefore, the joint PDF $p_{X,Y}$ is uniquely identified only if the two distributions p_X and $p_{Y|X=x}$ (or the other two corresponding distributions) are known. Unfortunately, starting from the distributions of X and Y, it is possible to obtain $p_{Y|X=x}$ only in two cases: i.e., when X and Y are independent (uncorrelated) random variables, since in this case $p_{Y|X=x} = p_Y$ holds; or when X and Y are normally distributed and their covariance $\sigma_{X,Y}$ is known. In fact, if $X \sim \mathcal{N}(\mu_x, \sigma_x)$ and $Y \sim \mathcal{N}(\mu_y, \sigma_y)$, $p_{Y|X=x}$ is a normal distribution, too, with mean and standard deviation given by

$$\begin{aligned} \mu_{Y|X=x} &= \mu_Y + \rho_{X,Y}\frac{\sigma_Y}{\sigma_X}(x - \mu_X) \\ \sigma_{Y|X=x} &= \sigma_Y\sqrt{1 - \rho_{X,Y}^2} \end{aligned} \quad (2)$$

where $\rho_{X,Y}$ is the linear correlation coefficient between X and Y, given by $\rho_{X,Y} = \sigma_{X,Y}/(\sigma_X\sigma_Y)$.

In all other cases, the information about X and Y probability distributions and their correlation is not sufficient to define a specific $p_{Y|X=x}$, and, therefore, the joint PDF $p_{X,Y}$ cannot be obtained.

B. 2-D Probability–Possibility Transformation

A first way to obtain a joint PD is to apply a 2-D probability–possibility transformation [28] to a given joint PDF. As an example, Fig. 2 shows a normal joint PDF (left plot) and the corresponding joint PD induced by this transformation (right plot).

Following this transformation, a joint PD expressing the same information content as the original joint PDF is obtained. In fact, this transformation is aimed at preserving the maximum amount of information associated with the marginal distributions and the information about their correlation [28]. For this reason, the joint PD induced by the 2-D transformation can be considered as the reference joint PD in the comparison of the methods for the construction of a joint PD with given marginals.

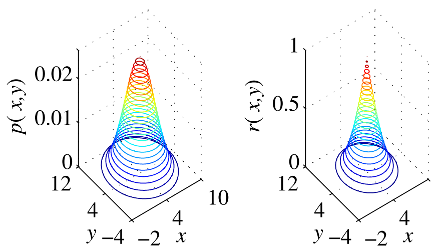

Fig. 2. Example of application of the 2-D probability–possibility transformation.

C. Construction of Joint PDs

To build a joint PD $r_{X,Y}$ of given marginal distributions r_X and r_Y, a similar equation as (1) can be invoked

$$r_{X,Y}(x,y) = T(r_X(x), r_{Y|X=x}(y)) = T(r_Y(y), r_{X|Y=y}(x)) \tag{3}$$

where $r_{Y|X=x}$ and $r_{X|Y=y}$ are the conditional possibility distributions of Y given $X = x$, and X given $Y = y$, respectively, and T is a t-norm. Therefore, the joint PD $r_{X,Y}$ is uniquely identified only if the two distributions r_X and $r_{Y|X=x}$ (or the other two corresponding distributions) are known and a specific t-norm is chosen.

A t-norm [35]–[37] is a function $T : [0,1] \times [0,1] \to [0,1]$ which satisfies the following mathematical properties.

1) *Commutativity:* $T(a,b) = T(b,a)$.
2) *Monotonicity:* $T(a,b) \le T(c,d)$ if $a \le c$ and $b \le d$.
3) *Associativity:* $T(a, T(b,c)) = T(T(a,b), c)$.
4) *Number 1 as the identity element:* $T(a,1) = a$.

A more general operator than the product in (1) is considered here, because, in possibility theory, more degrees of freedom can be exploited than in probability theory, to combine incomplete knowledge [36]. t-Norms represent an effective mathematical tool to combine possibility distributions and generalize the product operation, which can be indeed defined as a particular t-norm [35]. In this respect, (3) results in a generalization of (1), since a t-norm has been used instead of the product operator.

All t-norms differ from each other in the way they associate, in the 2-D space, the information contained in the marginal PDs, i.e., in the shape of the resulting joint possibility distribution. Therefore, the choice of a specific t-norm in (3) can deeply affect the shape of $r_{X,Y}$ and, hence, the way random contributions propagate in the possibility domain.

To investigate which t-norm provides the most similar joint PD to the reference joint PD, the most widely employed t-norms are considered in this paper to build the joint PD according to (3), and are reported here for the convenience of the reader, starting from the following fundamental ones:

$$\begin{aligned} T_{\min}(a,b) &= \min\{a,b\} \\ T_{\text{prod}}(a,b) &= a \cdot b \\ T_L(a,b) &= \max\{a+b-1, 0\} \\ T_D(a,b) &= \begin{cases} b, & \text{if } a = 1 \\ a, & \text{if } b = 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{4}$$

where $T_{\min}$ is the min t-norm, T_{prod} is the prod t-norm, T_L is the Lukasiewicz's t-norm, and T_D is the drastic t-norm [35], [36]. These t-norms show important properties. Without entering into the mathematical details, $T_{\min}$ is the largest possible t-norm, T_D is the smallest possible t-norm, T_{prod} is the prototype of the subclass of strict t-norms (continuous and strictly monotone t-norms), and T_L is the prototype of the subclass of nilpotent t-norms [35], [37]. Moreover, these four t-norms are strictly ordered according to the inequality $T_{\min} > T_{\text{prod}} > T_L > T_D$. An in-depth survey on t-norms and their properties can be found in [35] and [37].

A further generalization of the concept of t-norm is provided by the class of parameterized families of t-norms [35], [38]. In this class, a single equation identifies every member of one family: each single member can be obtained by changing the value of a parameter (γ) in the equation. Using this class in (3), $r_{X,Y}$ is obtained as a function of γ, and, therefore, it is possible to vary the γ value to change the shape of $r_{X,Y}$, so that it better approximates that of the reference joint PD. For this reason, the most important parameterized families of t-norms are also considered in this paper, and are here reported for the convenience of the reader and the sake of completeness

$$\begin{aligned}
T_\gamma^{\text{DP}}(a,b) &= \frac{a \cdot b}{\max\{a,b,\gamma\}}, \quad \gamma \in [0,1] \\
T_\gamma^{F}(a,b) &= \begin{cases} T_{\min}(a,b), & \text{if } \gamma = 0 \\ T_{\text{prod}}(a,b), & \text{if } \gamma = 1 \\ T_L(a,b), & \text{if } \gamma = +\infty \\ \log_\gamma\left(1 + \frac{(\gamma^a - 1)\cdot(\gamma^b - 1)}{\gamma - 1}\right), & \text{otherwise} \end{cases} \\
T_\gamma^{H}(a,b) &= \begin{cases} T_D(a,b), & \text{if } \gamma = +\infty \\ 0, & \text{if } \gamma = a = b = 0 \\ \frac{a\cdot b}{\gamma + (1-\gamma)\cdot(a+b-a\cdot b)}, & \text{otherwise} \end{cases} \\
T_\gamma^{\text{SS}}(a,b) &= \begin{cases} T_{\min}(a,b), & \text{if } \gamma = -\infty \\ (a^\gamma + b^\gamma - 1)^{1/\gamma}, & \text{if } \gamma < 0 \\ T_{\text{prod}}(a,b), & \text{if } \gamma = 0 \\ (\max\{0, a^\gamma + b^\gamma - 1\})^{1/\gamma}, & \text{if } \gamma > 0 \\ T_D(a,b), & \text{if } \gamma = +\infty \end{cases} \\
T_\gamma^{D}(a,b) &= \begin{cases} 0, & \text{if } a = 0 \text{ or } b = 0 \\ T_D(a,b), & \text{if } \gamma = 0 \\ T_{\min}(a,b), & \text{if } \gamma = +\infty \\ \frac{1}{1 + \left[\left(\frac{1-a}{a}\right)^\gamma + \left(\frac{1-b}{b}\right)^\gamma\right]^{1/\gamma}}, & \text{otherwise} \end{cases} \\
T_\gamma^{Y}(a,b) &= \begin{cases} T_D(a,b), & \text{if } \gamma = 0 \\ T_{\min}(a,b), & \text{if } \gamma = +\infty \\ \max\left\{0, 1 - [(1-a)^\gamma + (1-b)^\gamma]^{\frac{1}{\gamma}}\right\}, & \text{otherwise} \end{cases}
\end{aligned} \tag{5}$$

where T_γ^{DP} is the Dubois and Prade t-norm family [39], T_γ^F is the Frank t-norm family [40], T_γ^H is the Hamacher t-norm family [41], T_γ^{SS} is the Schweizer and Sklar t-norm family [42], T_γ^D is the Dombi t-norm family[1] [44], and T_γ^Y is the Yager t-norm family [45]. In the above equations, γ values range in $[0, \infty]$, if not specified otherwise.

[1]A further generalization of the Dombi t-norm family has been recently proposed in [43], yielding a two-parameter operator class. However, this class of operators has not been considered in this paper because the considered one-parameter t-norms appear to be a satisfactory tradeoff between performance and complexity in parameter optimization.

Each t-norm family reported in (5) satisfies specific mathematical properties that have been widely discussed in [35] and [38]. The reader is referred to the same for further details, which are not essential to fully perceive the following considerations. According to the discussion reported in the literature [35], [46]–[48], the t-norm family defined by Frank [40] appears to be the most suitable one to be employed in the construction of the joint PD (3). The first interesting mathematical property is that the Frank family represents also a family of copulas [46] in the probability domain. Copulas are mathematical tools used for the construction of joint PDFs with fixed marginals [46]. Therefore, as a family of t-norms in the possibility domain, they are expected to be also suitable tools to build joint PDs with fixed marginals, in a similar way as joint PDFs are built. Moreover, the Frank family is a continuous and strictly decreasing family with respect to γ [35] and its bounds (obtained for $\gamma = 0$ and $\gamma = \infty$) are the largest and smallest allowed for copulas (Fréchet bounds [46]). Therefore, very different shapes for $r_{X,Y}$ can be obtained by varying γ. Finally, as a family of copulas, the Frank family shows interesting statistical properties in the probability domain [47], [48], and is expected to show the same properties also in the possibility domain, as a t-norm family. In the next section, its performances in building a joint PD similar to the reference joint PD will be compared with the performances of all other reported t-norms and families of t-norms.

As stated at the beginning of this section, the joint PD in (3) is uniquely identified only if a specific t-norm is chosen and the two distributions r_X and $r_{Y|X=x}$ are known. To our knowledge, the definition of $r_{Y|X=x}$ is straightforward only in the simplest case in which X and Y are independent (uncorrelated), since $r_{Y|X=x} = r_Y$ holds. In the general case of correlated random variables X and Y, the information about X and Y possibility distributions and their correlation is not sufficient to define a specific $r_{Y|X=x}$, and, therefore, the joint PD $r_{X,Y}$ cannot be obtained.

To extend the applicability of (3) to the case of correlated random variables, a method is presented to build $r_{Y|X=x}$ when the PDs r_X and r_Y are induced by normal distributions p_X and p_Y through the 1-D probability–possibility transformation [26]–[28]. If r_X and r_Y are induced by normal distributions, it is possible to assume that the construction of $r_{Y|X=x}$ follows similar rules as the construction of $p_{Y|X=x}$ when p_X and p_Y are normal distributions [see (2)]. In particular, an equation similar to (2) can be obtained for the possibility domain, considering that the mean values of p_X and p_Y directly translate, in the possibility domain, into the mean values μ_X and μ_Y of the α-cuts of r_X and r_Y, and the ratio between p_Y and p_X variances equals the ratio of r_Y and r_X α-cuts amplitudes, i.e., $\sigma_Y/\sigma_X = \bar{A}_Y/\bar{A}_X$. When r_X and r_Y are induced by normal distributions, μ_X, μ_Y, and $\bar{A}_X/\bar{A}_Y$ assume the same values for each level α, and therefore they do not depend on α.[2]

[2]It can be readily proved that μ_X and μ_Y do not depend on α if r_Y and r_X are symmetric distributions. Similarly, it can be readily proved that the ratio $\bar{A}_X/\bar{A}_Y$ does not depend on α if r_Y and r_X distributions have the same shape. This can be proved considering the geometrical properties of such distributions.

On the basis of these considerations and taking into account (2), the mean value $\mu_{Y|X=x}$ and amplitude $\bar{A}_{Y|X=x}$ of the α-cuts of $r_{Y|X=x}$ can be defined as follows:

$$\mu_{Y|X=x} = \mu_Y + \rho_{X,Y}\frac{\bar{A}_Y}{\bar{A}_X}(x-\mu_X)$$
$$\frac{\bar{A}_{Y|X=x}}{\bar{A}_Y} = \sqrt{1-\rho_{X,Y}^2} \tag{6}$$

where $\rho_{X,Y}$ is the linear correlation coefficient between X and Y.

Following (6) and (3), it is hence possible to build the joint PD of correlated random variables X and Y if the associated PDs r_X and r_Y are induced by normal PDFs. Also, this method will be numerically validated in the next section.

III. Comparison of the Different t-Norms

A. Comparison Methodology

The following comparison methodology has been devised to evaluate the performances of each t-norm in the construction of the joint PD. First of all, a joint PDF $p_{X,Y}$ of given marginal distributions p_X and p_Y is obtained according to (1), and is transformed into its equivalent join PD $\hat{r}_{X,Y}$ according to the 2-D probability–possibility transformation [28]. This joint PD, having been obtained by direct transformation of the joint PDF, will serve as the reference PD in evaluating how good the considered t-norm is in approximating it.

In a second step, the 1-D probability–possibility transformation [26]–[28] is applied to transform distributions p_X and p_Y into their equivalent possibility distributions r_X and r_Y, and their joint PD $r_{X,Y}$ is built according to (3) for all considered t-norms. Finally, every $r_{X,Y}$ is compared with the reference joint PD $\hat{r}_{X,Y}$ by evaluating the differences among the volumes bounded by the $r_{X,Y}$ and $\hat{r}_{X,Y}$ surfaces and the x, y-plane. In particular, the following volume error is used:

$$e = \sqrt{\frac{\iint\left(\hat{r}_{X,Y}(x,y) - r_{X,Y}(x,y)\right)^2 dxdy}{\iint \hat{r}_{X,Y}(x,y)^2 dxdy}}. \tag{7}$$

In this definition, the difference $\hat{r}_{X,Y}(x,y) - r_{X,Y}(x,y)$ is squared to avoid compensating the negative differences with the positive ones throughout the integration process. Therefore, $e = 0$ only when $\hat{r}_{X,Y} = r_{X,Y}$. Moreover, the denominator term in (7) leads to a normalized volume error, since $e = 1$ when $r_{X,Y} = 0$.

Six case studies are considered in the comparison of the volume errors introduced by the t-norms, composed by the possible two-to-two combinations between the most common types of PDFs to be considered in (1), i.e., uniform PDFs, normal PDFs, and triangular PDFs. Different types of marginal probability distributions p_X and p_Y, and consequently the different equivalent r_X and r_Y distributions, have to be considered because the volume error introduced by a specific t-norm depends on the type of r_X and r_Y distributions.

B. Case of Uncorrelated Random Contributions

Table I shows the volume errors e when $T_{\min}$, T_{prod}, and T_L are considered in (3). For each t-norm, the table lists

TABLE I

Volume Errors Induced by $T_{\min}$, T_{prod}, and T_L

p_X	unif.	unif.	unif.	norm.	norm.	tri.	
p_Y	tri.	norm.	unif.	tri.	norm.	tri.	e_{AVG}
T_{min}	27.5%	26.1%	31.2%	23.5%	22.0%	24.8%	25.9%
T_{prod}	6.9%	8.0%	0%	14.4%	15.7%	13.3%	9.7%
T_L	39.1%	40.4%	31.4%	48.3%	49.4%	47.2%	42.6%

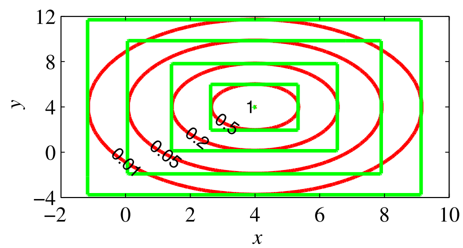

Fig. 3. Comparison of the α-cuts of the reference joint PD in Fig. 2 (red lines) and the α-cuts of the corresponding joint PD induced by $T_{\min}$ (green lines). Their levels α are also shown. The initial considered PDFs are both normal.

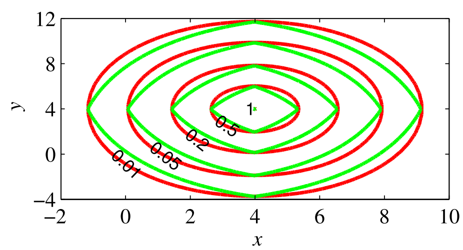

Fig. 4. Comparison of the α-cuts of the reference joint PD in Fig. 2 (red lines) and the α-cuts of the corresponding joint PD induced by T_{prod} (green lines). Their levels α are also shown. The initial considered PDFs are both normal.

also the average value e_{AVG} of the volume errors over the considered case studies. This value represents a rough index of the overall performance of the considered t-norm. The volume errors induced by the drastic t-norm T_D are not reported since their values are much higher than the others, and hence meaningless. These errors are due to the drastic method with which T_D associates r_X and r_Y in the 2-D space, according to its definition in (4).

It can be seen that $T_{\min}$ introduces large volume errors for all considered types of probability distributions, leading to an average volume error of about 26%. Moreover, the simulations have shown that the rectangular α-cuts of $r_{X,Y}$ provided by the min t-norm include the α-cuts of $\hat{r}_{X,Y}$ for all the considered case studies. Therefore, they are always larger than the reference ones, in accordance with the definition of the min t-norm as the largest t-norm. This is shown in Fig. 3 for the two normal PDFs case study, where the α-cuts of the reference joint PD in Fig. 2 and the α-cuts of the corresponding joint PD induced by $T_{\min}$ are plotted.

T_{prod} is the t-norm introducing the smallest errors and provides even zero errors when two uniform PDFs are considered. This means that the same operator used in the two

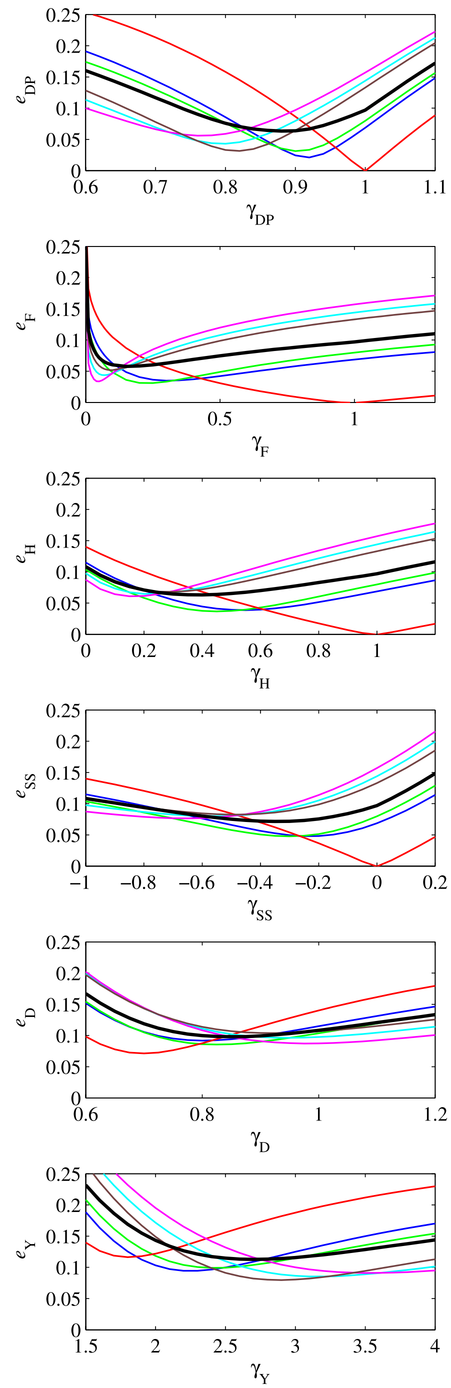

Fig. 5. Volume errors versus the variations of γ for t-norms T_γ^{DP}, T_γ^F, T_γ^H, T_γ^{SS}, T_γ^D, and T_γ^Y. p_X and p_Y are, respectively, uniform and triangular (blue line); uniform and normal (green line); uniform and uniform (red line); normal and triangular (cyan line); normal and normal (magenta line); triangular and triangular (brown line). Average volume error: e_{AVG} (black line).

different probability and possibility contexts leads to the same joint PDs when uniform PDF are considered. For all other case studies, the joint PDs $r_{X,Y}$ and $\hat{r}_{X,Y}$ are significantly different, leading to an average volume error of about 10%. Moreover, the simulations have shown that the α-cuts of $r_{X,Y}$ provided by the prod t-norm are included in the α-cuts of $\hat{r}_{X,Y}$ for all the considered case studies, except for the uniform–uniform case study for which the α-cuts of $r_{X,Y}$ and $\hat{r}_{X,Y}$ are

TABLE II
VOLUME ERRORS INTRODUCED BY t-NORM FAMILIES

p_X	unif.	unif.	unif.	norm.	norm.	tri.
p_Y	tri.	norm.	unif.	tri.	norm.	tri.
T_γ^{DP}	2.1% ($\gamma = 0.92$)	3.1% ($\gamma = 0.9$)	0% ($\gamma = 1$)	4.3% ($\gamma = 0.8$)	5.6% ($\gamma = 0.76$)	3.1% ($\gamma = 0.82$)
T_γ^{F}	3.5% ($\gamma = 0.3$)	3.1% ($\gamma = 0.25$)	0% ($\gamma = 1$)	4.4% ($\gamma = 0.07$)	3.4% ($\gamma = 0.05$)	5.2% ($\gamma = 0.1$)
T_γ^{H}	3.9% ($\gamma = 0.55$)	3.7% ($\gamma = 0.45$)	0% ($\gamma = 1$)	6.5% ($\gamma = 0.25$)	6.1% ($\gamma = 0.15$)	6.8% ($\gamma = 0.3$)
T_γ^{SS}	4.8% ($\gamma = -0.25$)	4.8% ($\gamma = -0.3$)	0% ($\gamma = 0$)	8.0% ($\gamma = -0.55$)	7.7% ($\gamma = -0.65$)	8.3% ($\gamma = -0.5$)
T_γ^{D}	9.2% ($\gamma = 0.8$)	8.6% ($\gamma = 0.83$)	7.1% ($\gamma = 0.7$)	9.6% ($\gamma = 0.95$)	8.7% ($\gamma = 0.98$)	10.4% ($\gamma = 0.93$)
T_γ^{Y}	9.5% ($\gamma = 2.3$)	9.9% ($\gamma = 2.4$)	11.6% ($\gamma = 1.8$)	8.5% ($\gamma = 3.2$)	9.1% ($\gamma = 3.5$)	8.0% ($\gamma = 2.9$)

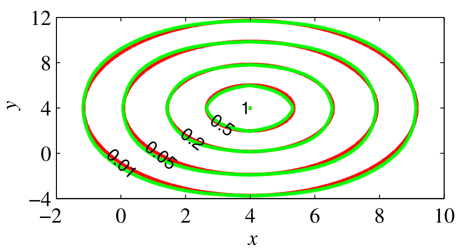

Fig. 6. Comparison of the α-cuts of the reference joint PD in Fig. 2 (red lines) and the corresponding joint PD induced by $T_{\gamma=0.05}^{F}$ (green lines). Their levels α are also shown. The initial considered PDFs are both normal.

equal. Therefore, they are always smaller than or equal to the reference ones. This is shown in Fig. 4 for the two normal PDFs case study, where the α-cuts of the reference joint PD in Fig. 2 and the α-cuts of the corresponding joint PD induced by T_{prod} are plotted.

The errors introduced by T_L are always greater than those introduced by $T_{\min}$ and T_{prod}. This is in agreement with the ordering $T_L < T_{\text{prod}} < T_{\min}$, which implies that the α-cuts of the joint PD built by T_L are even smaller than those built by T_{prod}.

To obtain smaller values for the average volume error, a t-norm should be employed such that the resulting α-cuts of $r_{X,Y}$ are smaller than those provided by $T_{\min}$ and larger than those provided by T_{prod}. Therefore, the desired t-norm T^* shall satisfy $T_{\text{prod}} < T^* < T_{\min}$. In this respect, the Frank parametric family of t-norms is expected to show better performances in the construction of the joint PD than the ordinary t-norms. In fact, by varying the value of its parameter γ in the range [0, 1], several t-norms can be obtained showing an intermediate behavior between $T_{\min}$ and T_{prod}.

This is confirmed by the simulation results shown in Fig. 5, where the volume errors e are plotted versus the values of γ for the t-norm families defined in (5). For each t-norm family, the volume errors associated with the six case studies are reported with different colors. All reported volume errors vary continuously with γ variations and show a relative minimum. Therefore, for each t-norm family and case study, it is possible to find the optimum γ value that minimizes e.

The optimum γ values and the resulting errors are listed in Table II. The first two families, i.e., T_γ^{DP} and T_γ^{F}, introduce the smallest volume errors, which range in the interval 0%–5%. Therefore, with T_γ^{DP} and T_γ^{F}, joint PDs $r_{X,Y}$ very similar to the reference joint PDs $\hat{r}_{X,Y}$ are obtained. This is also confirmed, in a visual perspective, by Fig. 6, where the α-cuts of the reference joint PD in Fig. 2 and the α-cuts of the corresponding joint PD induced by $T_{\gamma=0.05}^{F}$ are shown, when the considered initial PDFs are both normal. Similar figures as Fig. 6 are obtained for the other case studies, but are not reported here for the sake of brevity. The last four families show volume errors in the range 0%–12%, but still lower than the errors introduced by T_{prod}. Moreover, the first four families show zero errors when two uniform PDFs are considered, because, for particular γ values, they degenerate into T_{prod}.

Since it is not always convenient, or even possible, to change the γ value on the basis of the considered PDF type, it is possible to find a unique γ value for each t-norm family that minimizes the average of the volume errors over the considered case studies (e_{AVG}). In this respect, each black line in Fig. 5 represents the e_{AVG} values versus the γ values for a specific t-norm family. Also these lines show a relative minimum. The γ values minimizing e_{AVG}, as well as the resulting volume errors, are listed in Table III.

The comparison of Tables II and III allows one to state that, by choosing only one γ value for all considered case studies, larger volume errors are obtained, as expected. The first three t-norm families yield, in this case, similar average errors (about 6%). In particular, T_γ^{F} yields the smallest e_{AVG}, meaning that it provides, on average, the most similar joint PD to the reference one.

Further considerations can be made by recalling that some t-norm families (T_γ^{DP}, T_γ^{F}, T_γ^{H}, and T_γ^{SS}) degenerate into T_{prod} for some γ values and, hence, can lead to zero

TABLE III

VOLUME ERRORS INTRODUCED BY t-NORM FAMILIES WITH THE γ VALUES MINIMIZING THE AVERAGE ERRORS

p_X	unif.	unif.	unif.	norm.	norm.	tri.	
p_Y	tri.	norm.	unif.	tri.	norm.	tri.	e_{AVG}
T_γ^{DP} $(\gamma = 0.88)$	3.5%	3.6%	10.2%	6.9%	8.7%	5.2%	6.3%
T_γ^{F} $(\gamma = 0.15)$	4.9%	3.6%	8.7%	5.7%	6.2%	5.6%	5.8%
T_γ^{H} $(\gamma = 0.4)$	4.5%	3.8%	6.8%	7.5%	8.2%	7.2%	6.3%
T_γ^{SS} $(\gamma = -0.35)$	5.3%	4.9%	6.2%	8.8%	9.2%	8.6%	7.2%
T_γ^{D} $(\gamma = 0.85)$	9.4%	8.6%	10.1%	10.2%	9.8%	10.7%	9.8%
T_γ^{Y} $(\gamma = 2.7)$	10.9%	10.4%	17.0%	9.6%	11.6%	8.2%	11.3%

TABLE IV

VOLUME ERRORS INTRODUCED BY t-NORM FAMILIES WHEN A DIFFERENT γ VALUE IS CHOSEN IN THE CASE OF UNIFORM PDFS

p_X	unif.	unif.	norm.	norm.	tri.		p_X	unif.
p_Y	tri.	norm.	tri.	norm.	tri.	e_{AVG}	p_Y	unif.
T_γ^{DP} $(\gamma = 0.84)$	6.0%	5.4%	5.1%	7.0%	3.5%	5.4%	T_γ^{DP} $(\gamma = 1)$	0%
T_γ^{F} $(\gamma = 0.1)$	6.3%	4.8%	4.7%	4.7%	5.2%	5.1%	T_γ^{F} $(\gamma = 1)$	0%
T_γ^{H} $(\gamma = 0.3)$	5.5%	4.5%	6.7%	6.9%	6.8%	6.1%	T_γ^{H} $(\gamma = 1)$	0%
T_γ^{SS} $(\gamma = -0.4)$	5.7%	5.1%	8.4%	8.7%	8.4%	7.3%	T_γ^{SS} $(\gamma = 0)$	0%

TABLE V

VOLUME ERRORS INTRODUCED BY t-NORM FAMILIES WHEN CORRELATED RANDOM VARIABLES ARE CONSIDERED

p_X	norm.
p_Y	norm.
T_γ^{DP}	5.7% $(\gamma = 0.76)$
T_γ^{F}	3.6% $(\gamma = 0.05)$
T_γ^{H}	6.2% $(\gamma = 0.15)$
T_γ^{SS}	7.7% $(\gamma = -0.65)$
T_γ^{D}	8.7% $(\gamma = 0.98)$
T_γ^{Y}	9.1% $(\gamma = 3.5)$

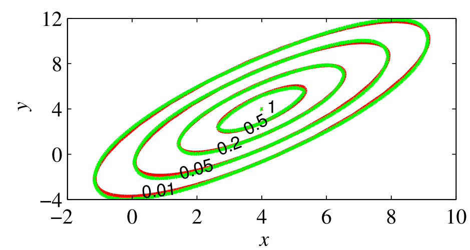

Fig. 7. Comparison of the α-cuts of a reference joint PD of correlated random variables ($\rho = 0.8$) (red lines) and the corresponding joint PD induced by $T^F_{\gamma=0.05}$ (green lines). Their levels α are also shown. The initial considered PDFs are both normal.

errors when two uniform PDFs are considered (see Table II and Fig. 5). Therefore, whenever it is known that the initial PDFs are uniform, these t-norm families can be used, with their appropriate values of γ, to obtain the correct joint PD. For all other case studies, it is possible to find different optimum γ values minimizing the average volume errors, as shown in Table IV. Once again, T_γ^F appears to be the best t-norm family to approximate the reference joint PD because of its small volume error. T_γ^{DP} provides a small average volume error, too.

C. Case of Correlated Random Contributions

If the two considered random contributions are not independent, it is assumed here that they are normally distributed. In fact, only if p_X and p_Y are normal distributions and their correlation is expressed by the linear correlation coefficient ρ, it is possible to find $p_{X,Y}$ according to (1) and (2) and, hence, to transform it into the reference joint PD. On the other side, thanks to (6) and (3), it is possible to evaluate the approximating joint PD induced by a specific t-norm when the initial possibility distributions are induced by normal PDFs.

Assuming a linear correlation coefficient $\rho = 0.8$,[3] the t-norms $T_{\min}$, T_{prod}, and T_L lead to volume errors of 21.7%, 15.9%, and 49.5%, respectively. Analyzing these values and comparing them with those in the norm–norm column in Table I, it can be stated that the correlation of the marginal distributions does not significantly affect the volume errors and, therefore, the considerations made in Section III-B are still valid. Moreover, the fact that the volume errors are the same for the cases of uncorrelated and correlated contributions suggests that the method to build joint PDs of correlated random contributions (6) can represent an effective transla-

[3]The numerical simulations have been performed for different ρ values in the range $[-1, 1]$. Since it has been checked that the value of ρ does not significantly affect the volume errors, only the results corresponding to $\rho = 0.8$ are reported here for the sake of brevity.

tion to the possibility domain of the method to build joint PDFs of correlated random contributions (2) in the probability domain.

The minimum volume errors induced by the t-norm families for the correlated case are reported in Table V together with the optimum γ values that lead to these errors. Comparing these values with the norm–norm column in Table II, it can be checked that the correlation of the marginal distributions does not significantly affect neither the volume errors nor the optimum γ values. Therefore, the comments made in Section III-B about the t-norm families and their performances are still valid. This is also confirmed, from a visual perspective, by Fig. 7, where the α-cuts of a reference joint PD of correlated random variables ($\rho = 0.8$) and the α-cuts of the corresponding joint PD induced by $T^F_{\gamma=0.05}$ are shown. Fig. 7 confirms also the validity of the proposed method to build joint PDs of normally distributed correlated random contributions.

IV. Conclusion

In this paper, a method to build the joint possibility distribution of random contributions to uncertainty has been discussed. Since the joint PD strongly depends on the choice of the particular employed t-norm, the performances of the most popular t-norms and t-norm families have been evaluated by comparing the resulting joint PD with a reference joint PD. According to theoretical considerations and numerical analysis, the Frank t-norm family appears to be particularly suitable for the construction of the joint PD.

Moreover, the optimum γ values to be used with this family, and the other families, have been defined for the most common types of marginal distributions. From a practical point of view, the choice of the optimum γ value depends on the available information about the type of the original marginal PDFs. If this information is available, a different optimum γ value can be chosen for the different types of marginal PDFs. On the contrary, if this information is not available, a γ value can be chosen that is, on average, the optimum value with respect to the most common types of marginal PDFs, as proposed in Section III-B.

Finally, the method to build the joint PDF of correlated random contributions, when their distribution is normal, has been translated to the possibility domain. The effectiveness of the possibility-domain method has been confirmed, once again, by numerical analysis.

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