Switching Gains for Semiactive Damping via Nonconvex Lyapunov Functions

Franco Blanchini, Patrizio Colaneri, Daniele Casagrande, Paolo Gardonio, and Stefano Miani

I. Introduction and Preliminaries

THE control of mechanical vibrations is a challenging problem of great interest in many applications [1]–[3]. In recent years, solutions based on decentralized velocity feedback were adopted, which are unconditionally stable [4], [5]. Optimal tuning techniques of decentralized velocity feedback control loops with constant gain are suggested [6], [7]. Adapting gain [8] and extremum seeking control [9] techniques are also investigated. A survey of these approaches was provided in [10].

Many practical applications such as, for example, seismic vibration isolation of delicate equipment [2], [8], running machinery [2], [11] and civil constructions [12], vehicle suspensions [13], and vibration control of distributed flexible structures [1], [14], [15] can be efficiently tackled with semiactive control systems. In particular, among other techniques, the well known $\mathcal{H}_\infty$ approach [16]–[20] and the model predictive control [21] are successfully applied.

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F. Blanchini is with the Department of Mathematics and Computer Science, University of Udine, Udine 33100, Italy (e-mail: blanchini@uniud.it).

P. Colaneri is with the Dipartimento di Elettronica e Informazione, Politecnico di Milano, Milano 20133, Italy (e-mail: colaneri@elet.polimi.it).

D. Casagrande, P. Gardonio, and S. Miani are with the Department of Electrical, Managerial, and Mechanical Engineering, University of Udine, Udine 33100, Italy (e-mail: daniele.casagrande@uniud.it; paolo.gardonio@uniud.it; of the figures in this paper are available online at http://ieeexplore.ieee.org.

The idea of implementing active damping with switching systems is established in the literature [22]. Semiactive vehicle suspension systems are developed where the damping of the suspension is switched from low to high values in such a way as to synthesize a sky-hook damping effect, which more efficiently dissipates energy and thus reduces vibration transmission [23], [24]. In particular, the use of magneto-rheological fluid dampers is extensively investigated [25]. Switching techniques are developed, which provide good damping performance over wide frequency bands, good robustness, and very low power requirements [26], [27]. From a technological point of view, switching strategies are suitable for on-line retuning and can be easily implemented at a reasonable cost.

The ideas presented are in line with recent literature [28], [29]. In particular the concept of consistency proposed in [28] plays a fundamental role. A state feedback switching control strategy is strictly consistent whenever it improves performance of all isolated subsystems. In contrast with our previous paper [29], where consistency is pursued by means of a single quadratic function, we exploit a theory recently reported in [30]–[32]. This fact supports recent switching control schemes based on composite functions [30] and Lyapunov-Metzler (LM) inequalities [33]. The main results of this brief are detailed below.

1) We provide a user friendly tutorial concise explanation of the basic properties of the min–of–quadratic functions and their generalized Lyapunov derivative.
2) We propose schemes based on functions that are easy to implement as they are based on standard tools such as Lyapunov and Riccati equations.
3) We show that the switching semiactive damping scheme outperforms the optimal constant switching approach under the $\mathcal{L}_2$ and $\mathcal{L}_2$ induced performance criteria.
4) We provide a robust version of the scheme by just replacing equalities by inequalities hence good performance of the overall system can be guaranteed even under parameter changes.
5) We consider the case in which the stiffness coefficient may also be switched. In this case, the overall stability is not assured under any switching rule, as in the case of switched dampers. Still the adopted function scheme assures robust stability.
6) We provide two realistic examples to show how the considered strategy in general is highly preferable to the constant optimal one. In particular, one of them considers a realistic building structure under seismic action, with base acceleration from recorded data of the El centro earthquake, which are available on–line [34].

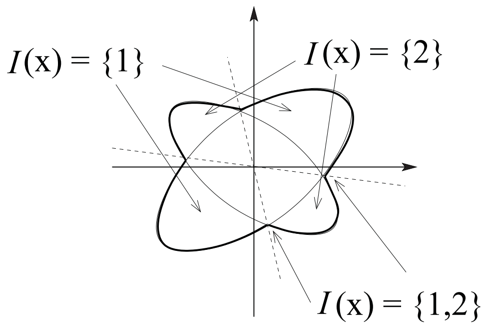

Fig. 1. Graphical representation of a level set of (1) when $i \in \{1, 2\}$ and V_1 and V_2 are quadratic functions.

II. Preliminaries About Nonconvex Min–Type Functions

This section presents a short overview about nonconvex min-type functions, which are extensively discussed, for example, in [31] and [30]. Given a set of smooth positively definite radially unbounded functions $V_i(x)$, $i = 1, 2, \ldots, M$, consider the function

$$V(x) \doteq \min_i \ V_i(x). \tag{1}$$

The resulting function V is positive definite, radially unbounded and locally Lipschitz, but nonsmooth in general. In addition, even if the original functions V_i are convex, the compounded Lyapunov function may not be convex. In the sequel, we consider the case in which the compounding functions are positively homogeneous hence also the compounded function has this property. From (1), it is clear that each level set of the resulting function is the union of the corresponding level sets of the components. As an example, in Fig. 1 the case of quadratic functions is considered; the level sets of the components are ellipses while the level set of the compound function is the region enclosed by the bold line.

Consider a dynamical system of the form

$$\dot{x}(t) = f\,(x(t), u(t)) \tag{2}$$

where $u(t)$ is a function taking values in an assigned compact set $\mathcal{U}$. Assume that 0 is an equilibrium state, namely that, for all $u \in \mathcal{U}$, $f(0, u) = 0$ and assume that the set

$$\mathcal{F}(x) = \{f(x, u)| \ \ u \in \mathcal{U}\}$$

is convex. As the function defined by (1) is nondifferentiable, to apply the Lyapunov theory, we must resort to the Dini derivative in direction v which is defined as follows:[1]

$$DV(x, v) = \liminf_{h \to 0^+} \frac{V(x + hv) - V(x)}{h}.$$

It is known that, for any input $u(t)$, the solution $x(t)$ of the differential equation (2) satisfies the condition

$$\frac{d}{dt} V\,(x(t)) = DV\,(x(t), f(x(t), u(t)))$$

almost everywhere. To simplify the notation, we define

$$\dot{V}(x, u) \doteq DV\,(x, f(x, u))\,.$$

[1]If the functions V_i are quadratic, as assumed in the following, the limit and the inferior limit coincide.

For a given feedback $u(x)$, the condition on $\dot{V}(x, u)$ to be negative definite implies asymptotic stability of $x = 0$. To derive an expression for the derivative $DV(x, f(x, u))$ consider the set

$$\mathcal{I}(x) = \{i : \quad V(x) = V_i(x)\} \tag{3}$$

namely the set of indexes that minimize (1), for a given x. It is reasonable to assume that in a real case this set is a singleton in all the state–space apart from a subset (of the state–space) of zero measure. In the case of the two quadratic functions of Fig. 1, $\mathcal{I}$ includes two indexes only in the points where the two components are equal. In the figure all these points are supposed to lie on the straight dashed line. In the points where $\mathcal{I}$ is not a singleton, smoothness of the compound function is not guaranteed. However, (see for instance [30])

$$\dot{V}(x, u) = \min_{i \in \mathcal{I}(x)} \nabla V_i(x)^T f\,(x, u(x))\,. \tag{4}$$

Once a function of the min–type (1) is assigned, the problem of choosing, among all possible values of $u \in \mathcal{U}$, the value that minimizes the derivative is of great interest. A fundamental preliminary observation is that in the case in which $\mathcal{U}$ is a polytope and f is affine in u the minimum is achieved on the vertices of $\mathcal{U}$ [30].

Lemma 2.1: Assume that $\mathcal{U}$ is a polytope identified by the set of vertices vert$\{\mathcal{U}\} = \{\eta_1, \ldots, \eta_{n_v}\}$ and that f is affine in u. Then

$$\min_{u \in \mathcal{U}} \ \ DV(x, f(x, u)) = \min_{u \in \mathrm{vert}\{\mathcal{U}\}} \ \ DV(x, f(x, u)).$$

We anticipate that the above lemma has the quite important consequence that, no matter how the functions $V_i(x)$ are found, the best convergence and performance is achieved by switching between the dynamics associated with the vertices $u \in$ vert$\{\mathcal{U}\}$ even if, as it will be explained, some of the $V_i(x)$ are Lyapunov functions associated with constant internal values of $\mathcal{U}$.

Example: Assume $V(x) = \min_i \ x^T P_i x$, where P_i are a family of positive-definite matrices and assume $f(x, u) = A(u)x = \left[\sum_h \ u_h A_h\right] x$, with $\sum_h \ u_h = 1$ and $u_h \geq 0$. Then to compute and minimize $DV(x, u)$, given x, one needs to:

1) evaluate $\mathcal{I}(x)$, namely the set of the indexes k for which

$$x^T P_k x = V(x) = \min_i \ x^T P_i x;$$

2) compute

$$\dot{V}(x, u) = \min_{k \in \mathcal{I}(x)} \ 2x^T P_k A(u) x;$$

3) find the minimizer $u(x)$ by computing

$$\bar{g} = \arg\min_h \quad \min_{k \in \mathcal{I}(x)} \ 2x^T P_k A_h x.$$

Then the minimizing control law is

$$u(x) = [0 \ \ldots \underbrace{1}_{\bar{g}} \ \ldots \ 0]$$

which is associated with $A_{\bar{g}}$. ■

Another preliminary lemma (whose proof is straightforward), important in the sequel, establishes that stability may

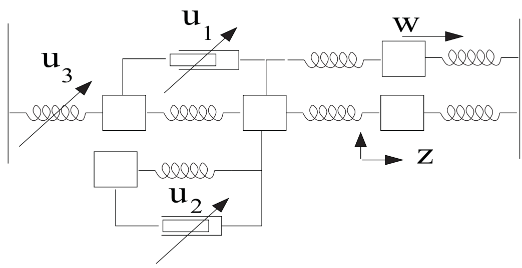

Fig. 2. Oscillating system with two devices with variable damping value and one device with variable stiffness value.

be easily proven as long as each of the component functions $V_i = x^T P_i x$ is itself a Lyapunov function.

Lemma 2.2: Assume that for all $i \in \{1, \ldots, M\}$ there exist a constant value $\hat{u}_i \in \mathcal{U}$ and a constant $\alpha_i \in \mathbb{R}^+$ such that for all x

$$\dot{V}_i(x, \hat{u}_i) = 2x^T P_i A(\hat{u}_i)x \leq -\alpha_i \|x\|^2.$$

In this case the switching strategy

$$u(x) = \arg\min_{u \in \mathcal{U}} \quad \dot{V}(x, u) \tag{5}$$

assures stability. More precisely, if $\alpha = \min_i \alpha_i$ then $\dot{V}(x, u(x)) \leq -\alpha\|x\|^2$.

III. Problem Setup

Consider a mechanical vibrating system modeled by

$$\begin{cases} M\ddot{q}(t) = -\bar{K}(u)q(t) - \bar{D}(u)\dot{q}(t) + \bar{E}_2 w(t) \\ z(t) = H_1 q(t) + H_2 \dot{q}(t) \end{cases} \tag{6}$$

where q is the state, w is the primary excitation, M and $\bar{K}$ are the mass and stiffness positive definite matrices, $\bar{D}$ is the damping matrix, $\bar{E}_2$ is the primary excitation matrix, z is the performance output, and H_1 and H_2 are constant matrices. We assume that

1) $u \in \mathbb{R}^p$ is a vector parameter that belongs to a polytope $\mathcal{U}$;
2) matrices $\bar{K}(u)$ and $\bar{D}(u)$ are affine in the parameter u.

For instance in Fig. 2, the set $\mathcal{U}$ is a box

$$\mathcal{U} = [u_1^-, u_1^+] \times [u_2^-, u_2^+] \times [u_3^-, u_3^+]$$

where u_i^- and u_i^+ are the minimum and maximum values, respectively, of the damper coefficients and of the stiffness.

By introducing the vector variables $x_1 = q$ and $x_2 = \dot{q}$, we obtain the state space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K(u) & -D(u) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ E_2 \end{bmatrix} w \tag{7}$$

$$z = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8}$$

where $K = M^{-1}\bar{K}$, $E_2 = M^{-1}\bar{E}_2$, and $D = M^{-1}\bar{D}$. The closed-loop system, written in a compact form, becomes

$$\begin{aligned} \dot{x}(t) &= A(u)x(t) + Ew(t) \\ z(t) &= Hx(t) \end{aligned} \tag{9}$$

with obvious meaning of the terms x, A, E, and H.

As previously mentioned, we are interested in performance so we introduce the following assumption that is well known to be satisfied in the context of vibrating systems.

Assumption 1: For any fixed value $u \in \mathcal{U}$ the system (6) is asymptotically stable.

Remark 3.1: If only the matrix D is a function of u, the system is dissipative and stability is guaranteed under arbitrary switching. Conversely, switching could destabilize the system if K depends on u.

The problem considered in this brief is how to minimize the effect of the single input w on the output z by switching u. The resulting performance can be measured by using any of the following performance indexes, defined for $x(0) = 0$.

1) Impulse response energy:

$$J_2 = \int_0^\infty \quad \|z_D(t)\|_2^2 dt \tag{10}$$

where $z_D(t)$ is the impulse response of system (9).

2) Energy-to-energy gain

$$J_\infty = \sup_{w \in \mathcal{L}_2, w \neq 0} \frac{\int_0^\infty \|z(t)\|_2^2 dt}{\int_0^\infty \|w(t)\|_2^2 dt} \tag{11}$$

where $\|w\|_2 = \sqrt{w^T w}$ is the Euclidean norm of w and $\mathcal{L}_2$ is the set of all signals $w(t)$ such that $\int_0^\infty \|w(t)\|_2^2 \; dt < +\infty$. As we will explain later the basic goal is to choose $\gamma > 0$ as small as possible to assure the condition $J_\infty < \gamma^2$.

A. Switching Versus Constant Control

In the sequel, given the polytope $\mathcal{U}$ we denote by

1) $g \in \mathcal{U}$, when g is a constant value;
2) $u \in \mathcal{U}$, when u is controlled as $u = u(x)$;

and we compare the two cases.

According to Lemma 2.1, we will derive strategies of the form

$$u(x) \in \mathcal{V}$$

where $\mathcal{V}$ is the set of vertices of $\mathcal{U}$. The following fundamental points are worth mentioning.

1) It is reasonable-and almost obvious-that a control $u(x) \in \mathcal{U}$ can perform better than a constant control $g \in \mathcal{U}$. It is not so obvious, but, in view of Lemma 2.1 still true, that on the vertices $u(x) \in \mathcal{V}$ can perform as well as $u(x) \in \mathcal{U}$.
2) Although the adopted control assumes values on the vertices, the internal values g that provide optimal constant gains are of fundamental importance, because their cost function contributes to the overall performance even if the condition $u = g$ never holds.
3) From the practical standpoint, assuming $u = g$ is purely theoretical because in general tuning the damping coefficient is a difficult task. Conversely, switching among the vertices can be done efficiently at a very low cost.

IV. Impulse Response Energy

To simplify the reasoning, assume for a moment that the disturbance w is a scalar. For a fixed value of the parameter $g \in \mathcal{U}$, the impulse response energy is given by

$$J_2 = E^T P(g) E$$

where $P(g)$ solves the equation

$$A^T(g)P(g) + P(g)A(g) = -H^T H. \tag{12}$$

Hence we may define the minimum value of the energy, for all possible constant values of the input, as

$$\bar{J}_2 = \min_{g \in \mathcal{U}} \{E^T P(g) E : \quad (12) \quad \text{holds}\}.$$

The optimal constant value $\bar{g}$, namely the value such that $\bar{J}_2 = E^T P(\bar{g}) E$ can be found by standard optimization procedures (see for instance [29] and the references therein).

On the other hand, the switching technique consists in computing the solution $P(g_i)$ on a certain number of values $g_i \in \mathcal{U}$

$$g_i \in \mathcal{G} = \{g_1, g_2, \ldots, g_M\}.$$

To prove the benefit of this approach the following assumption is needed.

Assumption 2: The set $\mathcal{G}$ includes the optimal value $\bar{g}$.

The candidate control Lyapunov function is

$$V(x) = \min_{g_i} \{x^T P(g_i) x\} \tag{13}$$

while the associated switching strategy, of the type (5), is

$$u(x) = \arg \min_{u \in \mathcal{U}} \quad \dot{V}(x, u). \tag{14}$$

The advantage of the switching is justified by the following proposition.

Proposition 4.1: If Assumption 2 holds and each $P(g_i)$ is positive definite then

i) the minimum of (14) is achieved on the vertices of $\mathcal{U}$;
ii) the control (14) assures asymptotic stability if H has full column rank;
iii) the control (14) outperforms the optimal constant one in the sense that

$$\tilde{J}_2 = \int_0^\infty z(t)^T z(t) dt \le V(E) \le E^T P(\bar{g}) E = \bar{J}_2. \tag{15}$$

Proof: The first two claims follow from Lemma 2.1 and Lemma 2.2, respectively. Only the third claim has to be proven (using again Lemma 2.1). To this purpose, note that

$$\begin{aligned} \dot{V}(x, u(x)) &= \min_{u \in \mathcal{U}} \quad \min_{i \in \mathcal{I}(x)} \nabla V_i(x)^T A(u) x \\ &\le \min_{i \in \mathcal{I}(x)} \nabla V_i(x)^T A(g_i) x. \end{aligned}$$

Since

$$\begin{aligned} &\min_{i \in \mathcal{I}(x)} \nabla V_i(x)^T A(g_i) x \\ &\quad = 2 \min_{i \in \mathcal{I}(x)} x^T P(g_i) A(g_i) x \\ &\quad = \min_{i \in \mathcal{I}(x)} x^T [A(g_i)^T P(g_i) + P(g_i) A(g_i)] x \\ &\quad = -x^T H^T H x \end{aligned}$$

we obtain

$$\begin{aligned} \tilde{J}_2 &= \int_0^\infty z^T(t) z(t) \quad dt \\ &= \int_0^\infty x^T(t) H^T H x(t) \quad dt \\ &\le \int_0^\infty -\dot{V}(x, u(x)) dt. \end{aligned}$$

Hence, by integrating and assuming $x(0)$ as initial condition, we have

$$\tilde{J}_2 \le V(x(0)) \le x(0)^T P(\bar{g}) x(0).$$

Since the impulse response is the free response with initial condition $x(0) = E$, (15) is proven. ■

Remark 4.1: The previous proof requires H to have full column rank, hence $-x^T H^T H x$ is negative definite and Lemma 2.2 can be applied. In the opposite case, $\dot{V}(x, u)$ would be negative semidefinite only. In practice, the assumption is not a restriction since, in place of the performance signal $z = Hz$, the modified output

$$\hat{z} = \begin{bmatrix} z \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} H \\ \epsilon I \end{bmatrix} x = \hat{H} x$$

can be used, with ϵ small such that there is no change in the problem.

Let us now consider the multi–input case, in which an impulse can excite the system from any of several input channels E_j. To solve this case, we define the set G as the set of all the optimal values $\bar{g}_i$ corresponding to impulses on the existing input channels. The cost function associated with the jth input is

$$\bar{J}_2^{(i)} = \min_{g \in \mathcal{U}} \{E_j^T P(g) E_j : \quad (12) \quad \text{holds}\} = E_j^T P(\bar{g}_i) E_j.$$

Then, by construction (assuming V defined as above), we obtain

$$V(E_j) \le E_j^T P(\bar{g}_i) E_j.$$

Thus, yet again, the switching strategy outperforms the constant one no matter from which input channel the impulse affects the system.

Along the lines suggested in [33], it is possible to improve the results by determining the generating functions by adding degrees of freedom (DFs) as follows. Let N be the number of points considered in the polytope; for all $i = 1, \ldots, N$ and $g_i \in \mathcal{G}$, let $A_i = A(g_i)$ and find the unique solutions P_i of the cross linear equation

$$A_i^T P_i + P_i A_i + \sum_{j=1}^{N} \lambda_{ij} P_j + H^T H = 0 \tag{16}$$

where the parameters λ_{ij} are such that $\lambda_{ij} \ge 0$ for all $i \ne j$ and, for all i

$$\sum_{j=1}^{N} \lambda_{ij} = 0.$$

The resulting switching strategy, in view of the results in [33], is such that (15) is satisfied.

Remark 4.2: The solutions can be computed with Kronecker calculus, via the *vec* operator[2] and the Kronecker sum of matrices.[3] Letting

$$p \triangleq \begin{bmatrix} \mathrm{vec}(P_1) \\ \mathrm{vec}(P_2) \\ \vdots \\ \mathrm{vec}(P_N) \end{bmatrix}$$

$$h \triangleq \begin{bmatrix} \mathrm{vec}(H^T H) \\ \mathrm{vec}(H^T H) \\ \vdots \\ \mathrm{vec}(H^T H) \end{bmatrix}$$

$$\Lambda \triangleq \begin{bmatrix} \lambda_{11} I & \cdots & \lambda_{1N} I \\ \vdots & \ddots & \vdots \\ \lambda_{N1} I & \cdots & \lambda_{NN} I \end{bmatrix}$$

and $\mathcal{A} = \mathrm{diag}\{A_i^T \oplus A_i^T\} + \Lambda$, the cross linear equation can be equivalently rewritten as follows:

$$\mathcal{A} p + h = 0. \tag{17}$$

A solution p such that $P_i > 0$, for all $i = 1, \ldots, N$ exists if and only if $\mathcal{A}$ is Hurwitz which happens, when $\lambda_{ij} = 0$ for all i and all j. When $\mathcal{A}$ is a Hurwitz matrix, $p = -\mathcal{A}^{-1} h$ and $P_i > 0$ can be recovered from p by reshaping. Note that (17) is nonlinear in the parameters p and Λ and finding the solution might be difficult. However, for a small number of modes (subsystems) it can be easily solved numerically.

A. Parametric Uncertainty

In the case of an uncertain representation of the form

$$\dot{x}(t) = A(\Delta, u) x(t).$$

Equation (12) cannot be used and must be replaced by the inequality [35]

$$A^T(\Delta, g) P(g) + P(g) A(\Delta, g) + H^T H \leq 0. \tag{18}$$

This inequality assures that for fixed g

$$\begin{aligned} \dot{V}_g(x) &= x^T [A^T(\Delta, g) P(g) + P(g) A(\Delta, g)] x \\ &\leq -x^T H^T H x \end{aligned}$$

where $V_g(x) = x^T P(g) x$. Note that $V_g(x)$ is the guaranteed performance for fixed g. If the inequalities (18) are satisfied, the nonconvex function (13) along with the switching strategy is such that $V(x) \leq V_g(x)$ and $\dot{V}(x) \leq -x^T H^T H x$. Then by integration we obtain

$$\int_0^\infty z^T(t) z(t) dt \leq V(x(0)) \leq V_g(x(0))$$

which shows that the switching strategy yields a better performance.

Remark 4.3: It is well know that if $A(\Delta, g)$ has a polytopic structure, conditions (18) are equivalent to a set of LMIs [see [35] subsection 6.2.1)].

[2]For a matrix with columns $a_1, \ldots, a_n$ the *vec* operator is defined by $vec\,([a_1 \cdots a_n]) = \left[a_1^T \ldots a_n^T\right]^T$.

[3]The Kronecker sum of two matrices B and C is defined by $B \oplus C = B \otimes I + I \otimes C$, where $\otimes$ is the Kronecker product.

B. Energy to Energy Amplification

We start this section by recalling some well–known facts. Let g be fixed and assume that $P(g)$ is a positive definite and stabilizing solution of the Riccati equation

$$A^T(g) P(g) + P(g) A(g) + P(g) \frac{E E^T}{\gamma^2} P(g) + H^T H = 0 \tag{19}$$

for some $\gamma > 0$. Let $V_g(x) = x^T P(g) x$; it is known that, for all $w \in \mathcal{L}_2$

$$\dot{V}_g(x) = 2x^T P(g)(A(g)x + Ew) < -\|z\|_2^2 + \gamma^2 \|w\|_2^2.$$

Given the initial condition x_0, after integration and recalling that $z = Hx$, one obtains that, for all $w \in \mathcal{L}_2$

$$\int_{t_0}^\infty \left(\|z(t)\|_2^2 - \gamma^2 \|w(t)\|_2^2\right) dt \leq x_0^T P(g) x_0. \tag{20}$$

In the worst case, $w = \frac{E^T P(g)}{\gamma^2} x \in \mathcal{L}_2$ and (20) is indeed an equality.

Once we have taken the parameter γ as small as possible with the constraint that (20) admits a positive definite solution, we get the tightest upper bound for the output energy

$$\int_{t_0}^\infty \|z(t)\|_2^2 dt \leq \gamma^2 \int_{t_0}^\infty \|w(t)\|_2^2 dt + x_0^T P(g) x_0.$$

Therefore the quantity (20) can be taken as an index for the transient performance, starting from x_0 and after t_0 time units. The goal is to render it as small as possible. Notice that the input output performance $J_\infty < \gamma^2$ is achieved [see (11)] for $x_0 = 0$.

Assume now that values γ_g are evaluated for each $g \in \mathcal{G}$. Also assume that $u(t_0) = g_0$ for some initial time t_0. This value is efficient until the time in which switching to another value allows for a better worst–case transient for the future. If at time t_1 a new mode g_1 is selected on the basis of the functions $x^T P(g) x$, for $g \in \mathcal{G}$, the transient is improved. To formalize this reasoning along the line suggested in [33], consider the control Lyapunov function

$$V(x) = \min_{g \in \mathcal{G}} \quad V_g(x) = \min_{g \in \mathcal{G}} \quad x^T P(g) x$$

and, correspondingly, the control law

$$u = \arg\min_{g \in \mathcal{G}} \; x^T P(g) x. \tag{21}$$

Note that stability under (21) is not an issue, as long as the solutions $P(g) > 0$ exist. As a matter of fact, $P(g)$ also satisfies (assuming H full column rank for simplicity) the inequality $A^T(g) P(g) + P(g) A(g) < 0$ hence stability under the switching law is ensured [see Proposition 4.1, point ii)]. In addition, note that each time there is a commutation from the current value, say $u = g_0$, to a new value $u = g_1$ such that $x(t_1)^T P(g_1) x(t_1) < x(t_1)^T P(g_0) x(t_1)$, the worst–case future transient after t_1 is necessarily improved. In particular

$$\begin{aligned} \sup_{w \in \mathcal{L}_2} \int_{t_1}^\infty & \left(\|z(t)\|_2^2 - \gamma_1^2 \|w(t)\|_2^2\right) dt \\ &= x(t_1)^T P(g_1) x(t_1) < x(t_1)^T P(g_0) x(t_1). \end{aligned}$$

Hence the switching strategy in general provides a better performance, in terms of the criterion (20), if compared with the (possibly optimal) gain $\hat{g}$.

All the reasoning above is summarized in the following statement:

Proposition 4.2: Assume that (19) admits a positive stabilizing solution $P(g)$, for all $g \in \mathcal{G}$, and assume that H is full column rank. Then, the control law (21) assures stability and $J_\infty < \gamma^2$.

Proof: We have already discussed the issue of stability where the positive definite function $V(x)$ acts as a Lyapunov function for the unforced system ($w = 0$). In addition, for all $w \in \mathcal{L}_2$

$$\begin{aligned}\dot{V}(x,u) &= \min_{g \in \mathcal{G}} \nabla V_g(x)^T A(u) + Ew \\ &= 2x^T P(g)(A(u)x + Ew) \\ &< -\|z\|_2^2 + \gamma^2 \|w\|_2^2.\end{aligned}$$

Integrating both sides from 0 to ∞, and recalling that $x_0 = 0$ we obtain that with the given switching law $J_\infty < \gamma^2$. ■

Remark 4.4: As done for the Lyapunov equations in (16), also the Riccati equations can be extended by adding more design parameters, thus obtaining the matrix inequalities

$$\begin{bmatrix} A_i^T P_i + P_i A_i + \sum_{j=1}^{N} \lambda_{ij} P_j + H^T H & P_i E \\ E^T P_i & -\gamma^2 I \end{bmatrix} < 0$$

with, again, $\sum_{j=1}^{N} \lambda_{ij} = 0$, for all i, and $\lambda_{ij} \geq 0$, for all $i \neq j$. If $P_i > 0$ exist, then the control law $\sigma = \arg\min_{i \in \mathcal{I}(x)} x' P_i x$ is stabilizing and such that $J_\infty < \gamma^2$ [36].

V. Implementation and Examples

The implementation of our strategy has several advantages

1) Simple theoretical tools are required such as Lyapunov equation, Riccati equations, or LMI solvers are needed.
2) The strategy requires only the evaluation of (1) and (4), given a certain number of functions.
3) The scheme is robust under bounds variations, provided that $g_k \in \mathcal{U}$. In practice, this requirement means that, if the parameters have nominal upper and lower bounds that are stricter than the actual one.
4) The scheme is amenable for on–line adaptation. Any parameter change, can be compensated by the fast re–computation of the function.

We provide two examples to support the proposed control.

A. Example A: Four DF System

Consider the case of the four DF system shown in Fig. 3 that is equipped with a single tunable damper with value $g \in [0.5, 10]$ in parallel with the spring k_3. The input is a force on the fourth mass and the output is the displacement of the third mass. With reference to (7) and (8), the matrices H and E are

$$H = [0\ 0\ 1\ 0\ 0\ 0\ 0\ 0], \quad E = [0\ 0\ 0\ 0\ 1\ 0\ 0\ 0]^T$$

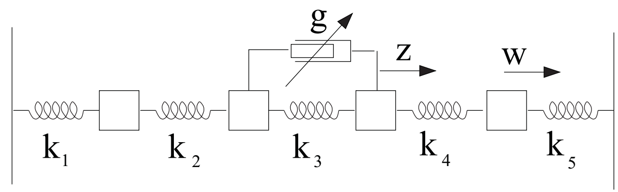

Fig. 3. Mechanical system with four DFs and one damper.

while the matrices K and D are

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1.2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & g & -g & 0 \\ 0 & -g & g & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The optimal value of the damping coefficient, computed numerically, is $\bar{g} = 2.8$ The corresponding (minimum) value of the energy is, approximately, $\bar{J}_2 = 16.8$.

We have computed the positive definite solutions P_i of the Lyapunov equations (12) for $g = g^-$, $g = g^+$, and $g = \bar{g}$, hence $V(x) = \min_{i=1,2,3}\ x^T P_i x$ according to (1). Then we have implemented the Lyapunov switching (L-switching) strategy and compared it with the constant gain strategy. The ratio of the L-switching cost over the constant gain cost is

$$J_{\text{sw}}/\bar{J}_2 = 0.6067.$$

In addition, we solved the LM equations (16) with $N = 3$, corresponding to the three dynamics associated with $g = g^-$, $g = g^+$, and $g = \bar{g}$, respectively. The solution is optimized with respect to six free positive parameters λ_{12}, λ_{13}, λ_{21}, λ_{23}, λ_{31}, and λ_{32}. Then we implemented the LM-switching strategy and compared it with the constant gain strategy and the previous L-switching strategy. Simulations show that LM-switching strategy can potentially outperform the L-switching strategy. Indeed, the ratio of the LM-switching cost over the constant gain cost is

$$J_{\text{sw}}/\bar{J}_2 = 0.5455.$$

When adopting the state-feedback switching strategy, the transient is clearly shorter than that achieved with the optimal constant gain (see again Fig. 4).

We have also assumed a change in the extreme value by taking $g^- = 0.5$ and $g^+ = 20$ (while keeping the same function). The scheme works properly, actually slightly better, since $J_{\text{sw}}/\bar{J}_2 = 0.5647$ for the L-strategy and $J_{\text{sw}}/\bar{J}_2 = 0.5207$ for the LM-strategy. This is not surprising since, according to Lemma 2.1, the new interval includes the old one.

B. Example B: A Four Floor Building

Consider the four–storeys building in Fig. 5. Each floor is equipped with a dynamic absorber that is formed by an additional floating mass suspended on a spring and a damper in parallel [2]. The floor masses are $M_i = 2 \times 10^4$ kg, for $i = 1, \ldots, 4$, while the absorber masses are $m_i = 600$ kg, for $i = 5, \ldots, 8$. The stiffness coefficients of the floor pillars are $K_i = 727$ kN/m, for $i = 1, \ldots, 4$, while the absorber

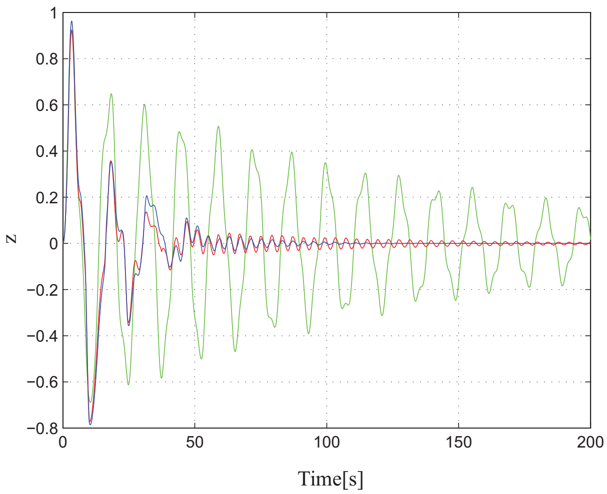

Fig. 4. (Green line) impulse response with constant, (blue line) L-switching, and (red line) LM-switching strategies with $g^- = 0.5$ and $g^+ = 10$.

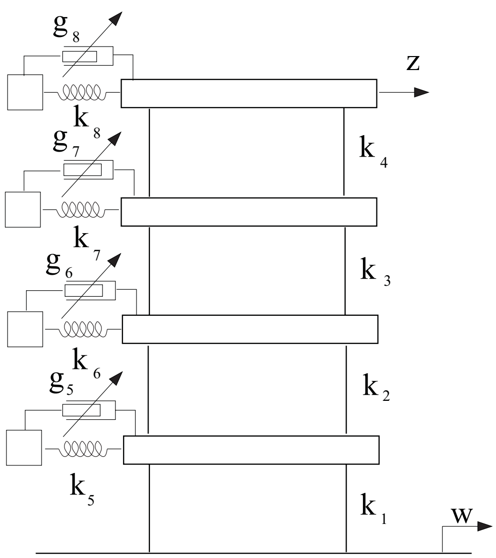

Fig. 5. Model of a building with four floors and vibration absorbers with switching dampers.

spring stiffnesses are $k_i = 2900\,\text{kN/m}$, for $i = 5, \ldots, 8$. These values are chosen in such a way that the natural frequencies of the four absorbers are tuned to the fundamental natural frequency relative to the transverse oscillations of the building. Finally, we assume the damping factor at the pillars to be $g_i = 12\ \text{kN}/(\text{ms}^{-1})$, for $i = 1, \ldots, 4$, so that the damping ratio of the two lower natural frequencies, $\omega_1^n = 2.09$ rad/s and $\omega_2^n = 6.03$ rad/s, are $\xi_1 = 0.017$ and $\xi_2 = 0.049$, respectively. The damping parameters g_i, for $i = 5, \ldots, 8$, can vary in the interval $[52.7, 5276]\ \text{N}/(\text{ms}^{-1})$.

We simulated a lateral excitation at the base of the building according to the acceleration data retrieved from the record of an earthquake occurred in 1940 (El Centro [34], Fig. 6). In Fig. 7 we reported the displacement[4] of the fourth floor for the constant gain strategy and for the L switching strategy. It is apparent that after an initial stage of about three period (8 s), in which the constant and switched strategy perform almost identically, there is a consistent reduction in the amplitude of oscillations because of a faster damping action produced by the proposed switching approach.

[4]The code that generated these data, with the displacements of all floors is available on–line http://www.diegm.uniud.it/smiani/Ongoing/Ongoing.html.

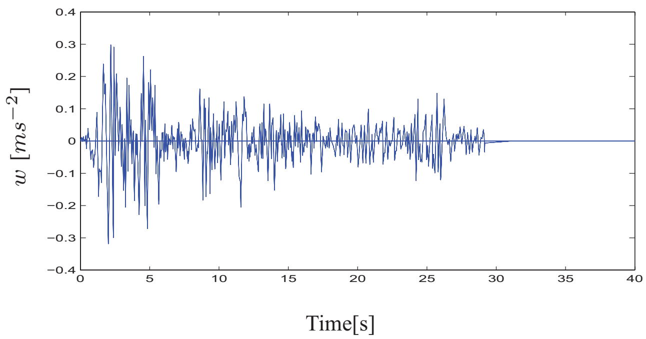

Fig. 6. Recorded acceleration data of the El Centro earthquake.

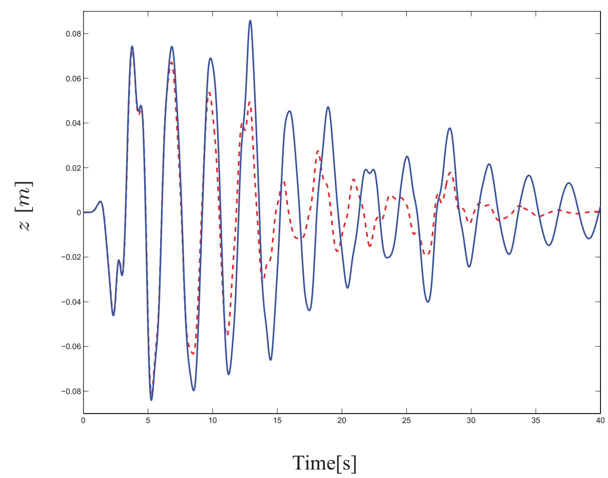

Fig. 7. (Blue-plain line) fourth floor displacement with constant damping and (red-dashed line) with switched damping.

This graph confirms that for such a realistic problem, the proposed switching control approach improves the vibration control effects both in terms of peak response reduction and in terms of reduction of transient response.

Fig. 8 shows the switching pattern. Note that since there are four variable parameters and each of them may assume value in an interval $[g_i^-, g_i^+]$, for $i = 1, \ldots, 4$, the total number of vertices of the set of admissible input signals is $4^2 = 16$. In Fig. 8, the binary convention is adopted, assigning 0 with the value g^- and 1 with the value g^+. Hence, each vertex is associated with a binary number $k = g_5 g_6 g_7 g_8$ in which g_5 is the most significant digit (e.g. $g_5^-\ g_6^+\ g_7^-\ g_8^+$ corresponds to $k = [0\ 1\ 0\ 1] = 5$). The integer $k + 1$ is shown in Fig. 8. It can be noted that there are preferred vertices although all of them are involved in the strategy at some point.

VI. Conclusion

We gave constructive techniques for active damping based on nonconvex Lyapunov functions generated by simple tools such as Lyapunov and Riccati equations and inequalities or LM inequalities. We showed by realistic simulations that a strong performance improvement was in general assured. The proposed technique required full state feedback that was quite

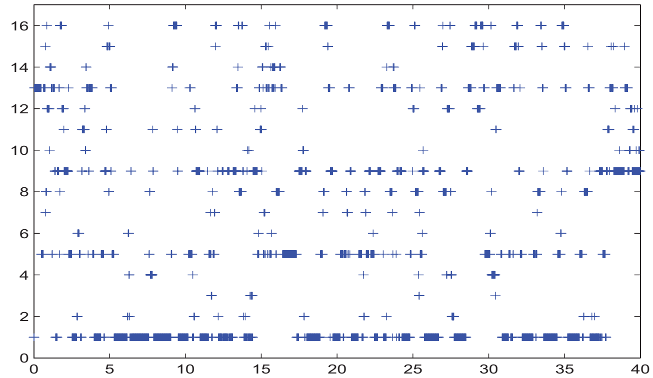

Fig. 8. Active damper configuration time evolution: index g versus time.

reasonable for simple systems but a challenge for high dimensional ones. Therefore, future work along this line includes developing control strategies that require only partial state feedback or, even better, distributed or decentralized feedback laws. We also believe that the strategy can be extended to nonlinear oscillatory and systems with dwell time.

References

[1] R. Clark, R. Saunders, and G. Gibbs, *Adaptive Structures: Dynamics and Control*. New York, NY, USA: Wiley, 1978.

[2] A. Preumont, *Vibration Control of Active Structures*. London, U.K.: Kluwer, 2002.

[3] A. Cavallo, G. Maria, and C. Natale, *Active Control of Flexible Structures: From Modeling to Implementation* (Advances in Industrial Control). New York, NY, USA: Springer-Verlag, 2010.

[4] M. Balas, "Direct velocity feedback control of large space structures," *J. Guid. Control Dynamics*, vol. 2, no. 3, pp. 252–253, 1979.

[5] P. Gardonio, S. Miani, F. Blanchini, D. Casagrande, and S. J. Elliott, "Plates with decentralised velocity feedback loops. Power absorption and kinetic energy considerations," *J. Sound Vibrat.*, vol. 331, no. 8, pp. 1722–1741, 2012.

[6] H. Toivonen, "A globally convergent algorithm for the optimal constant output feedback problem," *Int. J. Control*, vol. 41, no. 6, pp. 1589–1599, 1985.

[7] S. S. I. Postlethwaite, *Multivariable Feedback Control—Analysis and Design*. New York, NY, USA: Wiley, 1996.

[8] S. Kim, S. Elliott, and M. Brennan, "Decentralized control for multichannel active vibration isolation," *IEEE Trans. Control Syst. Technol.*, vol. 9, no. 1, pp. 93–100, Jan. 2001.

[9] K. Hayakawa, K. Matsumoto, M. Yamashita, Y. Suzuki, K. Fujimori, and H. Kimura, "Robust H_∞ output feedback control of decoupled automobile active suspension systems," *IEEE Trans. Autom. Control*, vol. 44, no. 2, pp. 392–396, Feb. 1999.

[10] P. Mkula and H. Toivonen, "Computational methods for parametric LQ problems—A survey," *IEEE Trans. Autom. Control*, vol. 32, no. 8, pp. 658–671, Aug. 1987.

[11] K. B. Scribner, L. A. Sievers, and A. H. Vonflotow, "Active narrow band vibration isolation of machinery noise from resonant substructures," *J. Sound Vibrat.*, vol. 167, no. 1, pp. 17–40, 1990.

[12] M. D. Symans and M. C. Constantinou, "Semi-active control systems for seismic protection of structures: A state-of-the-art review," *Eng. Struct.*, vol. 21, no. 6, pp. 469–487, 1999.

[13] D. Karnopp, "Active and semi-active vibration isolation," *ASME J. Mech. Design*, vol. 117, pp. 177–185, Jun. 1995.

[14] C. Fuller, S. Elliott, and P. Nelson, *Active Control of Vibration*. London, U.K.: Academic, 1996.

[15] B. Spencer and S. Nagarajaiah, "State of the art of structural control," *J. Struct. Eng.*, vol. 129, no. 7, pp. 845–856, 2003.

[16] H. Du, K. Sze, and J. Lam, "Semi-active H_∞ control of vehicle suspension with magnetorheological dampers," *J. Sound Vibrat.*, vol. 283, pp. 981–996, May 2005.

[17] M. Fallah, R. Bhat, and W. Xie, "H_∞ robust control of semi-active Macpherson suspension system: New applied design," *Veh. Syst. Dynamics*, vol. 48, no. 3, pp. 339–360, 2010.

[18] H. Karimi, M. Zapateiro, and N. Luo, "Vibration control of base-isolated structures using mixed H_2/H_∞ output-feedback control," *Proc. Inst. Mech. Eng. I, J. Syst. Control Eng.*, vol. 223, no. 6, pp. 809–820, 2009.

[19] H. Karimi, M. Zapateiro, and N. Luo, "Semiactive vibration control of offshore wind turbine towers with tuned liquid column dampers using H_∞ output feedback control," in *Proc. IEEE Int. Conf. Control Appl.*, Sep. 2010, pp. 2245–2249.

[20] R. Prabakar, C. Sujatha, and S. Narayanan, "Optimal semi-active preview control response of a half car vehicle model with magnetorheological damper," *J. Sound Vibrat.*, vol. 326, nos. 3–5, pp. 400–420, 2009.

[21] M. Canale, M. Milanese, and C. Novara, "Semi-active suspension control using 'fast' model predictive techniques," *IEEE Trans. Control Syst. Technol.*, vol. 14, no. 6, pp. 1034–1046, Nov. 2006.

[22] C.-T. Wu, C.-H. Yang, W.-J. Wu, and C.-K. Lee, "Design of switching damping control for smart structure self-powered adaptive damping control," in *Proc. ASME Conf.*, 2010, pp. 431–436.

[23] D. Karnopp, M. Crosby, and R. Harwood, "Vibration control using semi-activeforce generators," *J. Eng. Ind.*, vol. 96, no. 2, pp. 619–626, 1974.

[24] D. Fischer and R. Isermann, "Mechatronic semi-active and active vehicle suspensions," *Control Eng. Pract.*, vol. 12, no. 11, pp. 1353–1367, 2003.

[25] P. Jean, R. Ohayon, B. Mace, and P. Gardonio, "Effects of magneto-rheological damper performance limitations on semi-active isolation of tonal and broadband vibration," in *Proc. 9th Int. Conf. Recent Adv. Struct. Dynamics*, 2006, no. 153.

[26] M. Lallart and D. Guyomar, "Self-powered and low-power piezoelectric vibration control using nonlinear approaches," in *Vibration Control*, M. Lallart, Ed. Rijeka, Croatia: In Tech, 2010, pp. 265–292.

[27] M. Lallart, E. Lefeuvre, C. Richard, and D. Guyomar, "Self-powered circuit for broadband, multimodal piezoelectric vibration control," *Sens. Actuators A, Phys.*, vol. 143, no. 2, pp. 377–382, 2008.

[28] J. C. Geromel, G. S. Deaecto, and J. Daafouz, "Suboptimal switching state feedback control consistency analysis for switched linear systemson optimal damping of vibrating structures," in *Proc. 18th IFAC World Congr.*, Milan, Italy, Sep. 2011, pp. 1–22.

[29] F. Blanchini, D. Casagrande, P. Gardonio, and S. Miani, "Constant and switching gains in semi-active damping of vibrating structures," *Int. J. Control*, vol. 85, no. 12, pp. 1886–1897, 2012.

[30] T. Hu and Z. Lin, "Stabilization of switched systems via composite quadratic functions," *IEEE Trans. Autom. Control*, vol. 53, no. 11, pp. 2571–2585, Dec. 2008.

[31] Z. Sun and S. Ge, *Stability Theory of Switched Dynamical Systems*. New York, NY, USA: Springer-Verlag, 2011.

[32] F. Blanchini and C. Savorgnan, "Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions," *Automatica*, vol. 44, no. 4, pp. 1166–1170, 2008.

[33] J. Geromel, P. Colaneri, and P. Bolzern, "Dynamic output feedback control of switched linear systems," *IEEE Trans. Autom. Control*, vol. 53, no. 3, pp. 720–733, Apr. 2008.

[34] T. Irvine. (1940, May 18). *El Centrol Earthquake: North-South Component* [Online]. Available: http://www.vibrationdata.com/elcentro.dat

[35] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: SIAM, 2004.

[36] G. S. Daecto, J. C. Geromel, and J. Daafouz, "Dynamic output feedback H_∞ control of switched linear system," *Automatica*, vol. 47, no. 8, pp. 1713–1720, 2001.