The Mathematical Theory of Evidence and Measurement Uncertainty

Expression and Combination of Measurement Results via the Random-Fuzzy Variables

Simona Salicone

In a previous paper [1], it was proved how total ignorance can be effectively represented, in Shafer's theory of evidence [2], by a rectangular possibility distribution. In addition, it was shown how this concept can be usefully employed to mathematically represent situations that are often met in the measurement practice, especially in the industrial world [3]. The aim of this new paper is to show how possibility distributions can be effectively used to represent any kind of knowledge, from total ignorance to total evidence, and combine different contributions, if necessary.

Possibility Distributions, Fuzzy Variables, and Measurement Uncertainty

Possibility Distributions

Possibility distributions (PDs) r have been already defined in [1]. The rectangular PD, which associates a possibility equal to 1 to every value x_i of the universal set X, has been also defined.

This PD represents Shafer's total ignorance [1] that is, "*the situation where we have no evidence about X at all.*" In measurement applications, when X is generally an interval, this means that we have evidence that the measurand value lies in this interval, but we have no evidence at all about how the possible measured values distribute in this interval.

The opposite case of total ignorance is called total evidence. In this case, there is total knowledge about which event will occur, so that a possibility equal to 1 is associated to only one value of the universal set, while a possibility equal to 0 is associated to all other values [4].

While the rectangular PD is the greatest and less informative PD associated to X, this last PD is the smallest and the most informative one. In between total ignorance and total evidence, there are all possible kinds of partial knowledge. Partial knowledge can be represented as well by a PD, whose shape will be different, of course, than the rectangular one, and will depend on the available evidence.

If there is evidence that the value that can be attributed to an event (a measurand, for example) lies within a given interval and that the effects that make it lie in different positions in the interval are of systematic nature then, it can be proved [4] that the possible values assumed by the considered event can be well (and directly) represented by a PD in the given interval, and the PD shape depends on the available information, as shown in [7].

If, on the contrary, there is evidence that the above-considered effects are of random nature, then it can be proved that the possible values assumed by the considered event can be represented, in the given interval, by a probability distribution (pdf) p, whose shape does still depend on the available evidence. According to the relationship between probability and possibility functions briefly recalled in [1], it is possible, and sometimes useful, to convert the obtained probability distribution p into a possibility distribution r that preserves the confidence levels associated with equal intervals subtended by the two distributions [4]. This conversion can be done by applying a suitable probability-possibility transformation [4]–[6] and allows one to represent an effect of random nature with a PD.

As an example, it can be proved that in case the available knowledge shows that an event distributes according to a uniform pdf, then, the corresponding PD is triangular, as shown in Fig. 1a; in case the available knowledge shows that an event distributes according to a Gaussian pdf, then, the corresponding PD is the one reported in Fig. 1b. It is important to underline that the opposite is not possible, from the mathematical point of view. That is, once a probability distribution has been converted into a possibility distribution, it is not possible to convert it back into a single probability distribution. The reason for this is that, as briefly recalled in [1], a possibility distribution represents a family of probability distributions and a single probability distribution of this family cannot be retrieved unless additional information is available to select it. Therefore, it can be stated that both the effects due to random phenomena and the effects due to systematic phenomena can be represented with a single mathematical tool: the possibility distributions. As we will see later, this can be particularly useful to express measurement uncertainty.

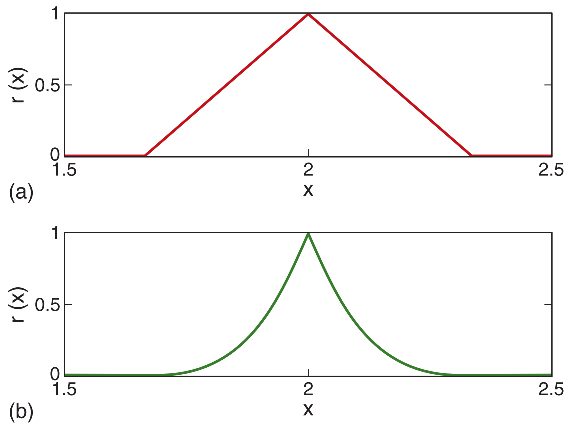

Fig. 1. Example of PDs. The triangular PD (red line) is obtained when the probability-possibility transformation is applied to a uniform pdf. The from-Gaussian PD (green line) is obtained when the probability-possibility transformation is applied to a Gaussian pdf.

Fuzzy Variables

Let us now refer to a new variable: the fuzzy variable (FV). The fuzzy variables have been originally introduced by Zadeh in 1975, when he defined a strict mathematical framework for linguistic variables and human reasoning [8]. In this framework, a fuzzy variable X can be defined both in terms of its membership function $\mu_X(x)$ and in terms of its α-cuts. The membership function of a fuzzy variable is defined as a convex function over the universal set, taking all values in [0,1]. The α-cuts of a fuzzy variable are defined as the cuts of the FV itself at levels α in [0,1] from [4] and [8].

The α-cuts of a fuzzy variable can be hence obtained in a quite immediate way. Fig. 2 shows a fuzzy variable A and some of its α-cuts. It can be easily perceived that each α-cut is an interval, and can be hence represented by its extremes: $A_\alpha = [a_l^\alpha, a_r^\alpha]$. As an example, the α-cut at level $\alpha = 0$ is interval $A_{\alpha=0}$ = [2,6]; the α-cut at level $\alpha = 0.1$ is interval $A_{\alpha=0.1}$ = [2.2,5.8]; the α-cut at level $\alpha = 0.5$ is interval $A_{\alpha=0.5}$ = [3,5]; and so on. Moreover, from the above definitions and the given example, it can be readily seen that the α-cuts of a fuzzy variable are nested intervals. In fact, for every pair of values α' and α'' such that; $\alpha' > \alpha''$, it is $X_{\alpha'} \subset X_{\alpha''}$. This relationship recalls the one among the focal elements in the possibility theory [1], [2], [4], and [9] and suggests that the membership function of a fuzzy variable is, from a strict mathematical point of view, a possibility distribution, that is, $\mu_X(x) = r(x)$, as strictly proved by Zadeh in [10]. This conclusion represents a very interesting bridge between Zadeh's and Shafer's theories, and allows one to employ the fuzzy variables as the natural variables in the possibility theory [4].

It follows that it is possible to associate a degree of belief $Bel(X_\alpha)$ to every α-cut X_α [1], [4]. As also recalled in [1], within the possibility theory, the belief functions are called necessity functions (Nec). It can be proved that [4]:

$$Bel(X_\alpha) = Nec(X_\alpha) = 1 - \alpha \qquad (1)$$

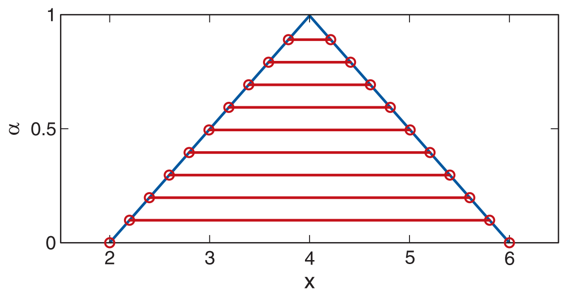

Fig. 2. Example of a fuzzy variable A (blue line) and some of its α-cuts (red lines) at levels [0, 0.1, 0.2, ..., 1]. Each α-cut is an interval, and can be hence represented by its extremes: $A_\alpha = [a_l^\alpha, a_r^\alpha]$.

It follows that all α-cuts associated to a fuzzy variable are nested intervals at the different degrees of belief $1 - \alpha$. Hence, the α-cuts X_α at the different levels α of a fuzzy variable can be considered to extend the probabilistic concept of confidence intervals at the different confidence levels $1 - \alpha$ [4]. If the example in Fig. 2 is considered again, it can be stated that interval [2,6] is the confidence interval at confidence level 1; interval [2.2,5.8] is the confidence interval at confidence level 0.9; and so on.

Measurement Uncertainty

It has been shown that all kinds of knowledge (both of random and systematic nature) can be represented in terms of possibility distributions, whose shape depend on the available knowledge. It has been also shown that the membership functions of the fuzzy variables are possibility distributions. Hence, it can been stated that the fuzzy variables can be employed to represent all kinds of different knowledge (both of random and systematic nature), and the shape of the fuzzy variable is chosen in order to correctly model the available evidence.

How can these concepts be useful in the measurement science? Let us first consider the situation where a measurement procedure is affected by only one kind of uncertainty contributions: many contributions can affect the procedure, but all of them are either random or systematic. The reasons for this simplified assumption will be understood later, when the more general situation where both random and systematic contributions to uncertainty affect the measurement procedure and the final measurement result will be considered. For now, let's stay with this simplified assumption, under which the above considerations lead to state that a measurement result with its associated measurement uncertainty can be represented by a fuzzy variable.

Further on, the above definition of α-cuts allows us to state that, if a measurement result is represented in terms of a fuzzy variable, then the fuzzy variable contains all available, useful metrological information about the measurement result itself, since all confidence intervals and associated levels of confidence are provided.

Let us now remember what the Guide to the Expression of Uncertainty in Measurement (GUM) [11] states: "*the ideal method for evaluating and expressing measurement uncertainty should be capable of readily providing such an interval, in particular, one with a coverage probability or level of confidence that corresponds in a realistic way to that required*." Is the representation of the measurement results in terms of fuzzy variables compliant with the GUM? As a matter of fact, a fuzzy variable is "*capable of providing such an interval*," that is, it is capable of providing the interval associated to the required level of confidence. In fact, the α-cuts of the FVs are the confidence intervals associated to the measurement result. Moreover, a fuzzy variable is "*capable of readily providing such an interval*." In fact, the confidence interval associated to the level of confidence γ is simply the α-cut at level $\alpha = 1 - \gamma$, and the α-cut at level $\alpha = 1 - \gamma$ is simply the cut of the FV at the vertical level $\alpha = 1 - \gamma$. Hence, it is possible to answer to the above question as follows: the representation of the measurement results in terms of FVs is perfectly compliant with the GUM and, to use the GUM words, it really appears to be "*the ideal method for evaluating and expressing measurement uncertainty*" or, if not actually the *ideal* method, something much closer to the ideal one than the purely probabilistic one followed by the GUM, as also highlighted in [9].

In this new approach, hence, a different mathematical theory and a different mathematical variable are employed to express uncertainty in measurements. It is important to underline, however, that, according to the theoretical considerations given in [1], this approach represents a generalization of the standard approach based on probabilities.

Once the contributions to measurement uncertainty have been evaluated and expressed (in terms of fuzzy variables), they must be combined in order to find how they affect the final measurement result. The aim of the following section is to show how to combine FVs, or, in other words, how to combine PDs.

The Combination of the Possibility Distributions

Independent PDs

Since the standard approach for expressing and evaluating measurement uncertainty is based on probabilities, let us first briefly start from the combination of the probability distributions and see whether and how this combination can be generalized for possibilities.

Let us consider two random variables X and Y, which distribute according to the probability distributions $p_X(x)$ and $p_Y(y)$. It is known that the combination of the two random variables X and Y according to a given relationship $f(X,Y)$ goes through the construction of their joint probability distribution $p_{X,Y}(x,y)$; the result of the combination is then obtained through the marginalization of $p_{X,Y}(x,y)$ along a given path, that depends on $f(X,Y)$. (In the probability theory, marginalization is the evaluation of a line integral, that is, an integral evaluated along a given path.) If the random variables X and Y are initially supposed to be independent (the more general case of dependent variables will be considered later), their joint probability distribution is given by the product of $p_X(x)$ and $p_Y(y)$:

$$p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y) \tag{2}$$

If, for instance, their sum is considered, that is $f(X,Y) = X + Y$, then the probability distribution $p_Z(z)$ associated to the sum $Z = X + Y$ is given by marginalizing the joint probability distribution in (2) along the path $y = z - x$, thus obtaining:

$$p_Z(z) = \int_{-\infty}^{+\infty} p_X(x) \cdot p_Y(z-x)dx \tag{3}$$

which corresponds to the convolution product $p_X(x) * p_Y(y)$, as expected.

Let us now consider two FVs X and Y, whose associated possibility distributions are $r_X(x)$ and $r_Y(y)$. Let us recall, for now, the assumption considered above: the nature of the uncertainty contributions affecting the two variables is the same. In other words, if the fuzzy variables X and Y represent two measurement results, the two measurement results are affected by either only systematic contributions to uncertainty or only random contributions to uncertainty. Similarly to probabilities, the combination of the fuzzy variables X and Y according to a certain relationship $f(X,\mathrm{Y})$ goes through the construction of their joint possibility distribution $r_{X,Y}(x,y)$; the result of the combination is then obtained through the marginalization of $r_{X,Y}(x,y)$ along a given path, that depends on $f(X,\mathrm{Y})$. (In the possibility theory, marginalization is the evaluation of the *sup* operator along a given path.) Under the same assumption of independent fuzzy variables, if (2) was simply rewritten by substituting p with r, we would get:

$$r_{X,Y}(x,y) = r_X(x) \cdot r_Y(y) \quad . \tag{4}$$

On the other hand, the marginalization of $r_{X,Y}(x,y)$ is defined by Zadeh's extension principle as [10]:

$$r_Z(z) = \sup_{z=f(x,y)} r_{X,Y}(x,y) \tag{5}$$

which allows one to retrieve, from the joint PD, the PD associated to the fuzzy variable $Z = f(X,Y)$, through the *sup* operator.

Equation (4) does indeed define a joint possibility distribution. However, it does not represent the unique joint PD that can be associated to the independent fuzzy variables X and Y, but only one of the possible joint PDs. In other words, given two fuzzy variables X and Y, their joint possibility distribution is not univocally defined, and (4) only gives one of the possible solutions [12], [13]. This again suggests that the approach based on PDs represents a generalization of the one based on probabilities [1], [4].

Always under the assumption of independent fuzzy variables, the general definition of joint possibility distribution is given by:

$$r_{X,Y}(x,y) = T(r_X(x), r_Y(y)) \text{ for every } x, y \in \Re \tag{6}$$

where T is a class of functions called t-norms.

A t-norm T is an operator that, starting from two values in [0,1], provides a third value again in [0,1] and satisfies the following mathematical properties [4], [14]:

COMMUTATIVITY: $T(a,b) = T(b,a)$ (7)

MONOTONICITY: $T(a,b) \leq T(c,d)$ if $a \leq c$ and $b \leq d$ (8)

ASSOCIATIVITY: $T(a, T(b,c)) = T(T(a,b), c)$ (9)

NUMBER 1 IS THE IDENTITY ELEMENT: $T(a,1) = a$ (10)

It can be readily proven that the product satisfies properties (7) – (10) and hence belongs to the class of t-norm functions. This proves, from a strict mathematical point of view, that (4) represents a particular solution of (6) and hence provides a joint possibility distribution. Of course, many different functions satisfy properties (7) – (10) and, for any particular choice of t-norm in (6), a joint PD with a particular shape is obtained. This is consistent with the fact that, as discussed in the previous section, a PD may be used to represent different kinds of incomplete information. In other words, given that a PD may represent the available knowledge associated to contributions to measurement uncertainty of different nature, two PDs may be jointed in different ways, according to the nature of the contributions by which they are affected. This means, for example, that the joint PD associated to two PDs representing two measurement results affected by systematic contributions to uncertainty has to be different from the joint PD associated to the same two PDs when they represent two measurement results affected by random contributions to uncertainty. In the mathematical framework of the theory of evidence, this is possible thanks to (6), which leaves a degree of freedom in the choice of the t-norm.

It is not the aim of this paper to present and enter the details of all t-norms available in the literature, for which the readers are addressed to [4], [14]. The aim of this paper is instead to show which are the suitable operators to match the specific needs that can be met in the measurement applications. For this reason, only two t-norms are here reported; the *min* t-norm, defined as [14]:

$$T_{min}(r_X(x), r_Y(y)) = \min(r_X(x), r_Y(y)) \tag{11}$$

and the family of Frank t-norms, defined as [14]:

$$T_F^{\gamma}(a,b) = \begin{cases} \min(a,b) & if \;\; \gamma = 0 \\ a \cdot b & if \;\; \gamma = 1 \\ \max(a+b-1, 0) & if \;\; \gamma = +\infty \\ log_{\gamma}\left(1 + \dfrac{(\gamma^a - 1)(\gamma^b - 1)}{\gamma - 1}\right) & otherwise \end{cases} \tag{12}$$

where $a = r_X(x)$ and $b = r_Y(y)$ for the sake of clarity, and γ is a parameter satisfying $\gamma \geq 0$. Eq. (12) provides a family of t-norms since different t-norms are obtained with different γ values. Only to give an example, the Frank t-norm with $\gamma = 0$ is the *min* t-norm, while the Frank t-norm with $\gamma = 1$ is the *product* t-norm. From (12) it follows that, when $0 \leq \gamma \leq 1$, the Frank t-norm provides a result that is in between the ones obtained by applying the *min* and the *product* t-norms [12].

It is not the aim of this paper to enter the mathematical details, let us only state here that the *min* t-norm is the most suitable t-norm to combine systematic effects, and the Frank t-norm with $\gamma = 0.1$ is the most suitable t-norm to combine random effects, as strictly proven in [12], [13] and also intuitively shown by the following examples.

Let us consider the two fuzzy variables X and Y shown in Fig. 3. Then, the joint PD obtained by applying (6) and considering the *min* t-norm is given in Fig. 4a; while the joint PD obtained by applying (6) and considering the Frank t-norm with $\gamma = 0.1$ is given in Fig. 5a. The two figures show the different shape of the two joint PDs, as expected from the above considerations.

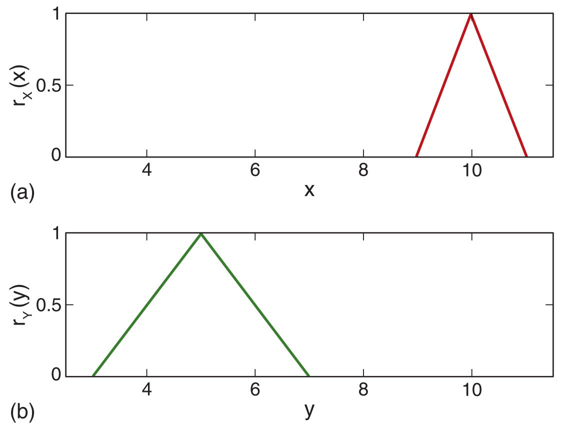

Fig. 3. Triangular fuzzy variables X and Y that must be combined according to the relationship $f(X,Y)$.

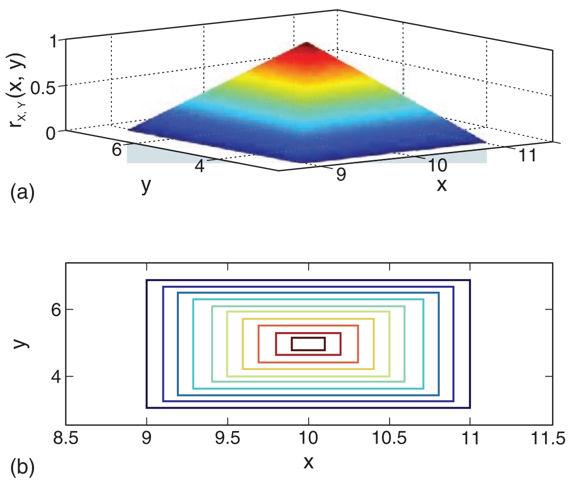

Fig. 4. (a) Joint PD between the fuzzy variables in Fig. 3 under the assumption of independence, obtained by applying the *min* t-norm. (b) Some 2D α-cuts of the joint PD.

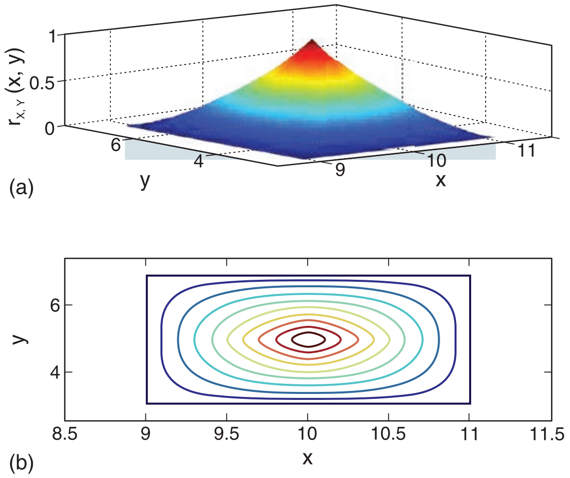

Fig. 5. (a) Joint PD between the fuzzy variables in Fig. 3 under the assumption of independence, obtained by applying the *Frank* *t*-norm. (b) Some 2D α-cuts of the joint PD.

A joint PD is, as a matter of fact, a fuzzy variable of two dimensions (2D-FV). Similarly to a fuzzy variable, a fuzzy variable of two dimensions can be defined both in terms of its membership function $\mu_{X,Y}(x,y)$ and in terms of its α-cuts. The membership function of a 2D-FV is a convex function $\mu_{X,Y}(x,y)$ over the universal set UxV (where U and V are the universal sets associated to X and Y respectively), taking all values in [0,1]. Extending the result obtained in the one-dimensional case, it can be stated that $\mu_{X,Y}(x,y) = r_{X,Y}(x,y)$ and that the α-cuts of a 2D-FV, called 2D α-cuts, are the cuts of the 2D-FV itself at levels α in [0,1] from [12] and [13].

While for an FV the α-cut is an interval, for a 2D-FV the α-cut is a region in the bidimensional xy-plane. Hence, a 2D α-cut is defined by both its shape and its position in the xy-plane. Similarly to the one-dimensional case, it can be proved that the 2D α-cuts associated to a 2D-FV are all nested and that the degree of belief associated to the 2D α-cut at level α is $1-\alpha$. Extending the result obtained in the one-dimensional case, it can be also stated that the α-cuts associated to a 2D-FV are confidence regions at the confidence levels $1-\alpha$ [12], [13].

Figs. 4b and 5b show some of the 2D α-cuts of the corresponding joint PDs in Figs 4a and 5a respectively. It can be noted how different are the shapes of these α-cuts, thought their position in the xy-plane is the same and depends only on the mean values of the two initial triangular PDs. (The mean value of a fuzzy variable is defined as the mean value of its α cut at level $\alpha = 1$.) In particular, the 2D α-cuts obtained by applying the *min* *t*-norm are always rectangular. These means that, for every level α, the α-cuts of the two initial triangular PDs are combined to form a rectangle in the xy-plane. For every level α this kind of combination allows any point belonging to the α-cut X_α to combine with any point belonging to the α-cut Y_α. On the other hand, the 2D α-cuts obtained by applying the *Frank* *t*-norm are more or less ellipsoidal, except for the α-cut at level $\alpha = 0$ which is rectangular. These means that, for every level α (but $\alpha = 0$), some combinations between the points close to the edges of the α-cut X_α and the points close to the edges of the α-cut Y_α are considered not possible and, consequently, are not included in the corresponding 2D α-cut, that cannot be rectangular any longer. A different behavior is given by the 2D α-cut at level $\alpha = 0$, and this is coherent with the fact that, if a confidence level equal to 1 must be provided, it is not possible to discard any possible combination between the points belonging to the α-cut $X_{\alpha=0}$ and the points belonging to the α-cut $Y_{\alpha=0}$.

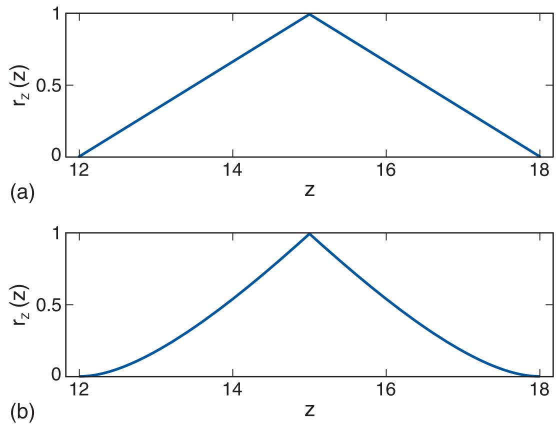

Fig. 6. (a) Fuzzy variable $X + Y$ (where X and Y are the fuzzy variables in Fig. 3) obtained by applying the *min* *t*-norm. (b) Fuzzy variable $X + Y$ obtained by applying the *Frank* *t*-norm.

Once the joint PD is built, it is possible to apply Zadeh's extension principle and obtain the PD associated to fuzzy variable $Z = f(X,Y)$. As an example, Fig. 6 shows the result of the sum $Z = X + Y$, when the *min* *t*-norm (Fig. 6a) and the *Frank* *t*-norm with $\gamma = 0.1$ (Fig. 6b) are applied. This figure shows, once again, which is the impact of the application of the different *t*-norms on the final result: the *min* *t*-norm gives, as the final result, a triangular PD; the *Frank* *t*-norm gives, as the final result, a smaller PD, where the effects of the compensation can be well perceived. This figure also clearly give confirmation of the above considerations, that is, the behavior of the *min* *t*-norm well reflects how the systematic contributions to uncertainty combine with each other and, on the other hand, the behavior of the *Frank* *t*-norm well reflects how the random contributions to uncertainty combine with each other, with their typical probabilistic compensation.

Dependent PDs

Let us now consider the case where fuzzy variables X and Y show some dependence, while the assumption that the nature of the uncertainty contributions affecting the two variables is the same still holds. Similarly to the previous situation, the combination of the fuzzy variables X and Y according to a given relationship $f(X,Y)$ goes through the construction of their joint possibility distribution $r_{X,Y}(x,y)$. The result of the combination is then obtained by marginalizing $r_{X,Y}(x,y)$ along a given path that depends on $f(X,Y)$. In this case, however, the joint possibility distribution $r_{X,Y}(x,y)$ must take into account the dependence

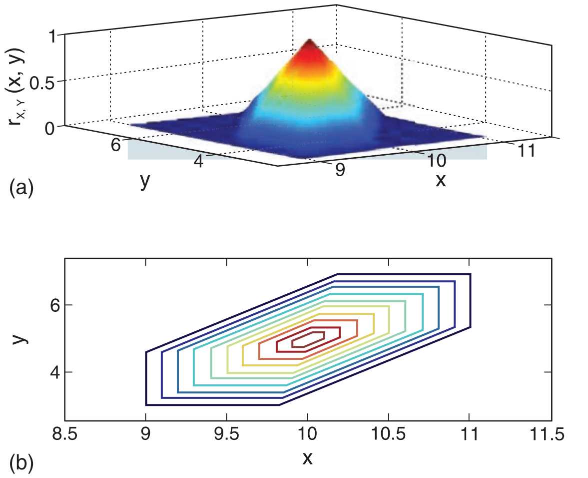

Fig. 7. (a) Joint PD between the fuzzy variables X and Y, obtained by applying the *min t*-norm, under the assumption of a degree of dependence 0.6. (b) Some 2D α-cuts of the joint PD.

between X and Y and (6) has to be slightly modified. The basic concepts are not modified, though the new concept of conditional possibility distributions (not dissimilar from that of conditional probability distribution) has to be considered.

It is not the case, here, to enter the mathematical details, since they are not relevant to perceive the potential of this method. The interested readers are addressed to [12], [13]. However, it is worth while showing, in an intuitive way, how dependence influences the joint PD of two given PDs. Fig. 7a shows the joint PD of the same triangular PDs X and Y in Fig. 3, when they are supposed to represent the effects of two systematic phenomena and have a degree of dependence 0.6, while Fig. 7b shows some of its 2D α-cuts.

As far as the random phenomena, let us consider two random phenomena which distribute according to the two Gaussian pdfs $\mathscr{N}(10, 1/3)$ and $\mathscr{N}(5, 2/3)$ and let us also suppose they are dependent, with a correlation coefficient 0.6. Fig. 8 shows the corresponding PDs (called *from-Gaussian*), while Fig. 9a shows their joint PD under the given assumption. Fig. 9b shows some of the 2D α-cuts of the joint PD in Fig. 9a. With respect to Fig. 4 and 5, it can be perceived how the effect of dependence is to discard some of the possible pairs of points belonging to the initial α-cuts. This is perfectly consistent with the concept of dependence, for which, if one variable behaves in one way, it forces the second one to behave in the same way. Of course, the greater is the dependence, the greater is the number of discarded pairs. In particular, in case of total dependence, the joint PD degenerates into a 2D surface and the 2D α-cuts degenerate into straight lines.

The Practical Implementation

The above theoretical considerations are very useful to fully perceive the meaning of the choice of a particular t-norm and its impact on the final measurement result. However, from the practical point of view, the construction of a joint PD and the marginalization along a certain path is not immediate and can be quite cumbersome. Moreover, when the above equations are applied in practice, the PDs, which are, from the mathematical point of view, continuous functions, must be sampled. This sampling operation, of course, cannot be *a priori* set, since it depends on the possibility distribution function itself. Furthermore, if the sampling operation is not correctly done, numerical problems may arise.

The above method for combining FVs is fully based on their PDs. However, it has been shown that a fuzzy variable can be represented, in a completely equivalent way, both in terms of its PD and in terms of its α-cuts. Hence, it is quite instinctive to wonder whether an equivalent method for combining FVs exists, based on the α-cuts. If this method existed, it would yield a simpler representation and storage of the fuzzy variable, based on the α-cuts. In fact, the α-cuts of a fuzzy variable are intervals, and intervals can be simply represented by their left

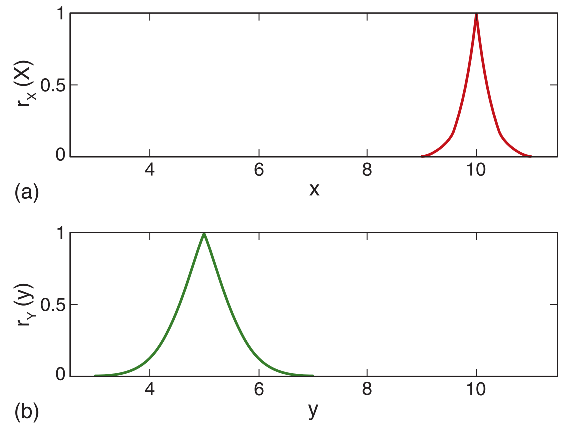

Fig. 8. Fuzzy variables obtained by applying the probability-possibility transformations to the normal pdfs $\mathscr{N}(10, 1/3)$ and $\mathscr{N}(5, 2/3)$.

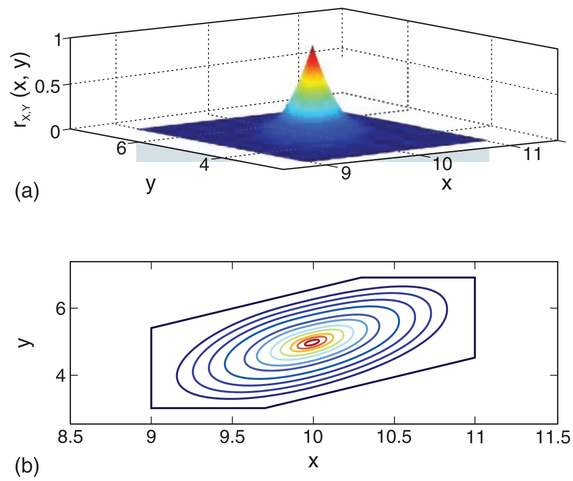

Fig. 9. (a) Joint PD between the fuzzy variables in Fig. 8, obtained by applying the *Frank t*-norm, under the assumption of a correlation coefficient 0.6. (b) Some 2D α-cuts of the joint PD.

and right extremes. If we denote A the fuzzy variable, A_α the generic α-cut and $A_\alpha = [a_l^\alpha, a_r^\alpha]$, then, it is possible to represent the fuzzy variable as a matrix with two columns and a prefixed number of rows. If, for instance, 101 rows are considered, which represents a good compromise between a good resolution and a good execution time, fuzzy variable **A** is represented by the $[101 \times 2]$ matrix:

$$\mathbf{A} = \begin{bmatrix} a_l^{\alpha=0}, a_r^{\alpha=0} \\ a_l^{\alpha=0.01}, a_r^{\alpha=0.01} \\ \vdots \\ a_l^{\alpha=1}, a_r^{\alpha=1} \end{bmatrix} \tag{13}$$

where the i-th raw contains the extremes of the α-cut at level $(i-1)\cdot 0.01$.

It is immediate to perceive the advantages of this representation, which is not affected by the sampling problems above mentioned, and allows all fuzzy variables to be represented in the same way (a $[101 \times 2]$ matrix), so that the corresponding rows in the matrixes always correspond to the same α-level.

Nguyen has proved, in a rigorous mathematical way, that the results obtained by (5) and (6), which are based on the PDs, can be also obtained by considering the α-cuts [15]. This is a very important conclusion, since it yields to consider only algebraic operations among intervals, instead of building the joint PD and marginalizing [12], [13].

Let us consider again fuzzy variables X and Y and relationship $Z = f(X,Y)$; and let us denote X_α, Y_α, and Z_α their α-cuts. Then, an alternative way to combine X and Y is through

$$Z_{\alpha=0} = f(X_{\alpha=0}, Y_{\alpha=0}) \tag{14}$$

and the Nguyen theorem [15, 16]:

$$Z_\alpha = \cup_{T(\xi,\eta)\geq\alpha} f(X_\xi, Y_\eta) \quad \text{for } \alpha \in (0,1] \tag{15}$$

Equation (14) applies the interval mathematics between the two α-cuts of X and Y at the same level $\alpha = 0$. On the contrary, (15) is more complicated, since, for every level α, different α-cuts of X and Y at different levels are selected, according to the considered t-norm; different intervals are hence obtained applying the interval mathematics between each pair of selected α-cuts; and finally, the union operator is applied among these intervals to obtain the α-cut of Z at the considered level α.

Equation (15) is surely less intuitive then the image of a joint PD and its 2D α-cuts, but can be implemented in a very fast way and without numerical problems. Moreover, it has been proven in [12], [13] that it always leads to algebraic operations among intervals.

In the particular case when T is the *min* t-norm and no dependence is present between X and Y, (14)-(15) simplifies to:

$$Z_\alpha = f(X_\alpha, Y_\alpha) \quad \text{for } \alpha \in [0,1] \tag{19}$$

which is the classical mathematics of the intervals. When T is the *Frank* t-norm and when dependence is considered, the final formulation is not as simple, though it is always based on algebraic operations among intervals [12, 13], which can be easily and quickly performed by a PC.

The Random-Fuzzy Variables

In the previous section, it has been shown that, thanks to Zadeh's extension principle and, alternatively, to Nguyen theorem, it is always possible to obtain the combination of two PDs. Moreover, by simply choosing the suitable t-norm, it is possible to model the typical combination of both the effects of two systematic phenomena and the effects of two random phenomena. However, in general, a measurement result is affected by both random and systematic contributions to uncertainty. If an FV is used to model both these kinds of contributions together, then, the combination of FVs would not be able to propagate both these contributions correctly, thus leading to an incorrect propagation of the measurement uncertainty. Therefore, a different solution must be found.

Let us consider an uncompensated systematic contribution. The effect of this contribution on the measured values is that they may lie in any point of a given interval (red line in Fig. 10). Let us now add a random contribution. The additional effect of this contribution on the measured values is that the position of the above given interval is not fixed and can move both on the right and on the left (green lines in Fig. 10). This concept brings to the definition of intervals of confidence of type 2, that is, intervals whose upper and lower bounds are uncertain (blue line in Fig. 10) [4], [14], [17].

Intervals of confidence of type 2 allow us to define fuzzy variables of type 2 [4], [14], [17], which allow us to represent two different concepts of uncertainty together. The definition of the fuzzy variables of type 2 [17] is very general. Therefore, in order to define a variable that perfectly fits with the aim to represent the measurement results and the measurement uncertainty due to both random and systematic contributions, a subclass of the fuzzy variables of type 2 has been defined, called the class of Random-Fuzzy variables (RFVs) [4], [17]. Without entering the mathematical details, Fig. 11 shows an example of RFV. Two possibility distributions define a Random-Fuzzy variable: $r_{int}(x)$ (pink line in Fig. 11b) and $r_{ext}(x)$ (red line in Fig 11b). $r_{int}(x)$ coincides with $r_{sys}(x)$ (pink line in Fig. 11a), that is the possibility distribution representing the effects of the systematic contributions to uncertainty; $r_{ext}(x)$ is obtained by composing linearly $r_{sys}(x)$ with $r_{ran}(x)$ (green line in Fig. 11a), that is, the possibility distribution representing the effects of the random contribution to uncertainty. Hence, $r_{ext}(x)$ shows the global effect of all contributions to uncertainty. As already stated, $r_{sys}(x)$ and $r_{ran}(x)$ are built according to the available metrological information [7].

Similarly to the FVs, it is possible to define the α-cuts of the RFVs, that is, the cuts of PD $r_{ext}(x)$ at levels α in [0,1]. These α-cuts are again nested intervals; however, if also $r_{int}(x)$ is taken into account, intervals of confidence of type 2 are obtained. Extending the definition of the α-cuts given for the FVs, it can be stated that the α-cut at level α of an RFV is the confidence interval to which the confidence level $Nec(X_\alpha) = 1 - \alpha$ is

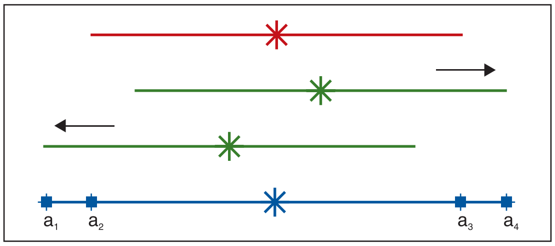

Fig. 10. Red line: interval representing the possible measured values, due to systematic effects. Green lines: possible displacement of the above interval, due to the presence of random contributions. Blue line: interval of confidence of type 2.

associated. Moreover, being the α-cut a confidence interval of type 2, it is possible to know, for every level α, which part is due to the systematic effects and which part is due to the random ones. Each α-cut can be represented by a vector $[a_1^\alpha, a_2^\alpha, a_3^\alpha, a_4^\alpha]$ (Fig. 10) so that, extending the concepts given in the previous section for a fuzzy variable, an RFV $\mathbf{A}$ can be simply represented by a $[101 \times 4]$ matrix [4]:

$$\mathbf{A} = \begin{bmatrix} a_1^{\alpha=0}, a_2^{\alpha=0}, a_3^{\alpha=0}, a_4^{\alpha=0} \\ a_1^{\alpha=0.01}, a_2^{\alpha=0.01}, a_3^{\alpha=0.01}, a_4^{\alpha=0.01} \\ \vdots \\ a_1^{\alpha=1}, a_2^{\alpha=1}, a_3^{\alpha=1}, a_4^{\alpha=1} \end{bmatrix} \qquad (16)$$

where the i-th raw contains the four points defining the α-cut at level $(i-1) \cdot 0.01$.

When two measurement results represented in terms of RFVs must be combined, the combination of the PDs $r_{sys}(x)$ and $r_{ran}(x)$ must be performed according to the methods reported in the previous section. In particular, if A and B are the two variables, $r_{sys,A}(x)$ and $r_{sys,B}(x)$ must be combined taking into account their dependence (if present) and considering the *min* t-norm. On the other hand, $r_{sys,A}(x)$ and $r_{sys,B}(x)$ must be combined taking into account their dependence (if present) and considering the Frank t-norm $T_F^{\gamma=0.1}$. The two obtained possibility distributions represent, respectively, $r_{sys}(x)$ and $r_{ran}(x)$ of the final result and the RFV associated to the final result can be then obtained in a simple way by linearly combining these two PDs (Fig. 11).

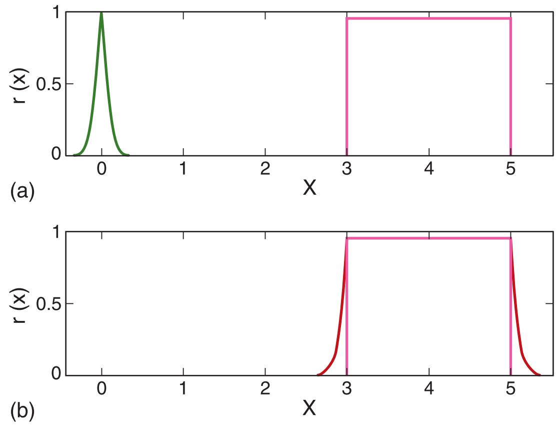

Fig. 11. (a) Green line: PD representing the effects of the random contributions. Magenta line: PD representing the effects of the systematic contributions. (b) RFV obtained by the above *random* and *systematic* PDs.

Discussion and On-Line Example

The advantages of the proposed approach, based on the possibility distributions for the representation and combination of the measurement results and related uncertainty, can be summarized as follows.

- It provides a unique and flexible mathematical tool to represent and combine the uncertainty contributions of different nature.
- It provides a straightforward representation of the final measurement result in terms of an RFV, that is, in terms of a set of confidence intervals of type 2 at the different levels of confidence. Each confidence interval of type 2 shows the effects, on the final measurement result, of both the systematic contributions to uncertainty (internal interval $[a_2^\alpha, a_3^\alpha]$) and the random contributions to uncertainty (external intervals $[a_1^\alpha, a_2^\alpha]$ and $[a_3^\alpha, a_4^\alpha]$), as well as the global effect of all contributions to uncertainty (interval $[a_1^\alpha, a_4^\alpha]$). The confidence level associated to the α-cut at level α is $1-\alpha$.
- The approach provides the same results provided by the probability theory in the case only random contributions are present [12].
- The theoretical complexity of the approach is not reflected into the practical implementation that, thanks to Nguyen theorem [12], [13], yields to combine measurement results through simple algebraic operations, whose implementation is immediate.

The validity of the above statements can be readily checked by the readers by opening this web page, which has been optimized for view in Internet Explorer: http://131.175.120.11:8000/RFVcalculator.html. The front panel of a remotely controlled application is shown, which allows the readers to create two RFVs and combine them according to the four arithmetic operations. This simple example shows that, despite the complex theoretical definitions, the implementation of this new approach is immediate and the results are readily obtained. Even when only random contributions are considered, and the same results as those provided by probability are obtained, much faster computations, than Monte Carlo simulations or complicated convolutions products, are required, as the reader can readily check.

Conclusions

This paper has shown how measurement results can be expressed and combined by means of the Random-Fuzzy variables. In the next paper, the comparison between RFVs will be presented, thus allowing one to perform the final, most important step of each measurement procedure that is, making a decision on the basis of the comparison result.

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