Backward stochastic differential equations associated to jump Markov processes and applications

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1. Introduction

In this paper we introduce and solve a class of backward stochastic differential equations (BSDEs for short) driven by a random measure associated to a given jump Markov process. We apply the results to study nonlinear variants of the Kolmogorov equation of the Markov process and to solve optimal control problems.

Let us briefly describe our framework. Our starting point is a pure jump Markov process X on a general state space K. It is constructed in a usual way starting from a positive measure

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$A \mapsto \nu(t, x, A)$ on K, depending on $t \geq 0$ and $x \in K$ and called rate measure, that specifies the jump rate function $\lambda(t, x) = \nu(t, x, K)$ and the jump measure $\pi(t, x, A) = \nu(t, x, A)/\lambda(t, x)$. If the process starts at time t from x then the distribution of its first jump time T_1 is described by the formula

$$\mathbb{P}(T_1 > s) = \exp\left(-\int_t^s \lambda(r, x)\, dr\right), \tag{1.1}$$

and the conditional probability that the process is in A immediately after a jump at time $T_1 = s$ is

$$\mathbb{P}(X_{T_1} \in A \mid T_1 = s) = \pi(s, x, A),$$

see below for precise statements. We denote by $\mathbb{F}$ the natural filtration of the process X. Denoting by T_n the jump times of X, we consider the marked point process (T_n, X_{T_n}) and the associated random measure $p(dt\, dy) = \sum_n \delta_{(T_n, X_{T_n})}$ on $(0, \infty) \times K$, where δ denotes the Dirac measure. In the Markovian case the dual predictable projection $\tilde{p}$ of p (shortly, the compensator) has the following explicit expression

$$\tilde{p}(dt\, dy) = \nu(t, X_{t-}, dy)\, dt.$$

In the first part of the paper we introduce a class of BSDEs driven by the compensated random measure $q(dt\, dy) := p(dt\, dy) - \tilde{p}(dt\, dy)$ and having the following form

$$Y_t + \int_t^T \int_K Z_r(y)\, q(dr\, dy) = g(X_T) + \int_t^T f(r, X_r, Y_r, Z_r(\cdot))\, dr, \quad t \in [0, T], \tag{1.2}$$

for given generator f and terminal condition g. Here Y is real-valued, while Z is indexed by $y \in K$, i.e. it is a random field on K, with appropriate measurability conditions, and the generator depends on Z in a general functional way. Relying upon the representation theorem for the $\mathbb{F}$-martingales by means of stochastic integrals with respect to q we can prove several results on (1.2), including existence, uniqueness and continuous dependence on the data.

In spite of the large literature devoted to random measures (or equivalently to marked point processes) there are relatively few results on their connections with BSDEs. General nonlinear BSDEs driven by the Wiener process were first solved in [18]. Since then, many generalizations have been considered where the Wiener process was replaced by more general processes. Backward equations driven by random measures have been studied in [21,2,20,17], in view of various applications including stochastic maximum principle, partial differential equations of nonlocal type, quasi-variational inequalities and impulse control. The stochastic equations addressed in these papers are driven by a Wiener process and by a jump process, but the latter is only considered in the Poisson case. More general results on BSDEs driven by random measures can be found in the paper [22], but they require a more involved formulation; moreover, in contrast to [21] or [2], the generator f depends on the process Z in a specific way (namely as an integral of a Nemytskii operator) that prevents some of applications that we wish to address, for instance optimal control problems.

In this paper X is not defined as a solution of a stochastic equation, but rather constructed as described above. While we limit ourselves to the case of a pure jump process X, we can allow great generality. Roughly speaking, we can treat all strong Markov jump processes such that the distribution of holding times admits a rate function $\lambda(t, x)$ as in (1.1): compare Remark 2.1-3. The process X is not required to be time-homogeneous, the holding times are not necessarily exponentially distributed and can be infinite with positive probability. Our main restriction is that the rate measure ν is uniformly bounded, which implies that the process X is non explosive. Our results hold for an arbitrary measurable state space K (provided one-point sets are measurable) and in particular they can be directly applied to Markov processes with discrete state space. We note that a different formulation of the BSDE is possible for the case of finite or countable Markov chains and has been studied in [6,7]. In the paper [8] we address a class of BSDEs driven by more general random measures, not necessarily related to a Markov process, but the formulation is different and more involved, and the corresponding results are less complete. The results described so far are presented in Section 3, after an introductory section devoted to notation and preliminaries.

In Sections 4 and 5 we present two main applications of the general results on the BSDE (1.2). In Section 4 we consider a class of parabolic differential equations on the state space K, of the form

$$\begin{cases} \partial_t v(t,x) + \mathcal{L}_t v(t,x) + f\Big(t, x, v(t,x), v(t,\cdot) - v(t,x)\Big) = 0, \\ \quad t \in [0,T],\ x \in K, \\ v(T,x) = g(x), \end{cases} \tag{1.3}$$

where $\mathcal{L}_t$ denotes the generator of X and f, g are given functions. Eq. (1.3) is a nonlinear variant of the Kolmogorov equation for the process X, the classical equation corresponding to the case $f = 0$. While it is easy to prove well-posedness of (1.3) under boundedness assumptions, we achieve the purpose of finding a unique solution under much weaker conditions related to the distribution of the process X: see Theorem 4.4. We construct the solution v by means of a family of BSDEs parametrized by $(t,x) \in [0,T] \times K$:

$$\begin{aligned} &Y_s^{t,x} + \int_s^T \int_K Z_r^{t,x}(y)\, q^t(dr\, dy) = g(X_T) + \int_s^T f(r, X_r, Y_r^{t,x}, Z_r^{t,x}(\cdot))\, dr, \\ &\quad s \in [t,T]. \end{aligned} \tag{1.4}$$

By the results above there exists a unique solution $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$ and previous estimates on the BSDEs are used to prove well-posedness of (1.3). As a by-product we also obtain the representation formulae

$$v(t,x) = Y_t^{t,x}, \qquad Y_s^{t,x} = v(s, X_s), \qquad Z_s^{t,x}(y) = v(s,y) - v(s, X_{s-}),$$

which are sometimes called, at least in the diffusive case, nonlinear Feynman–Kac formulae.

The second application, that we present in Section 5 is an optimal control problem. This is formulated in a classical way by means of a change of probability measure, see e.g. [11,12,5]. For every fixed $(t,x) \in [0,T] \times K$, we define a class $\mathcal{A}^t$ of admissible control processes u, and the cost to be minimized and the corresponding value function are

$$J(t,x,u(\cdot)) = \mathbb{E}_u^{t,x}\left[\int_t^T l(s, X_s, u_s)\, ds + g(X_T)\right], \qquad v(t,x) = \inf_{u(\cdot)\in\mathcal{A}^t} J(t,x,u(\cdot)),$$

where g, l are given real functions. Here $\mathbb{E}_u^{t,x}$ denotes the expectation with respect to another probability $\mathbb{P}_u^{t,x}$, depending on the control process u and constructed in such a way that the compensator under $\mathbb{P}_u^{t,x}$ equals $r(s, X_{s-}, y, u_s)\, \nu(s, X_{s-}, dy)\, ds$ for some function r given in advance as another datum of the control problem. The Hamilton–Jacobi–Bellman equation for this problem has the form (1.3) where the generator is the Hamiltonian function

$$f(s,x,z(\cdot)) = \inf_{u \in U} \left\{ l(s,x,u) + \int_K z(y)\,(r(s,x,y,u)-1)\,\nu(s,x,dy) \right\}. \tag{1.5}$$

Optimal control of jump Markov processes is a classical topic in stochastic optimization, and some of the first main results date back several decades: among the earliest contributions we mention the papers [3,4] where, following the dynamic programming approach, the value function of the optimal control problem is characterized as the solution of Hamilton–Jacobi–Bellman, whenever it exists. The results are given under boundedness assumptions on the coefficients. We refer the reader to the treatise [14] for a modern account of the existing theory; in this book optimal control problems for continuous time Markov chain are studied in the case of discrete state space and infinite time horizon.

Our approach to this control problem consists in introducing a BSDE of the form (1.4), where the generator is given by (1.5). Under appropriate assumptions and making use of the previous results we prove that the optimal control problem has a solution, that the value function is the unique solution to the Hamilton–Jacobi–Bellman equation and that the value function and the optimal control can be represented by means of the solution to the BSDE. This approach based on BSDEs equations allows to treat in a unified way a large class of control problems, where the state space is general and the running and final cost are not necessarily bounded; moreover it allows to construct probabilistically a solution of the Hamilton–Jacobi–Bellman equation and to identify it with the value function. As in optimal control for diffusive processes (perhaps with the exception of some recent results) it seems that the approach via BSDEs is limited to the case when the controlled processes have laws that are all absolutely continuous with respect to a given, uncontrolled process. More general cases can be found for instance in [14] or, for more general classes of Markov processes, in [9]: see also a more detailed comment in Remark 5.4 below.

We finally mention that the results of this paper admit several variants and generalizations: some of them are not included here for reasons of brevity and some are presently in preparation. For instance, the Lipschitz assumptions on the generator of the BSDE can be relaxed, along the lines of the many results available in the diffusive case, or extensions to the case of vector-valued process Y or of random time interval can be considered.

2. Notations, preliminaries and basic assumptions

2.1. Jump Markov processes

We recall the definition of a Markov process as given, for instance, in [13]. More precisely we will consider a normal, jump Markov process, with respect to the natural filtration, with infinite lifetime (i.e. non explosive), in general non homogeneous in time.

Suppose we are given a measurable space $(K, \mathcal{K})$, a set Ω and a function $X : \Omega \times [0,\infty) \to K$. For every $I \subset [0,\infty)$ we denote $\mathcal{F}_I = \sigma(X_t,\ t \in I)$. We suppose that for every $t \in [0,\infty)$, $x \in K$ a probability $\mathbb{P}^{t,x}$ is given on $(\Omega, \mathcal{F}_{[t,\infty)})$ and that the following conditions hold.

1. $\mathcal{K}$ contains all one-point sets. Δ denotes a point not included in K.
2. $\mathbb{P}^{t,x}(X_t = x) = 1$ for every $t \in [0,\infty)$, $x \in K$.
3. For every $0 \le t \le s$ and $A \in \mathcal{K}$ the function $x \mapsto \mathbb{P}^{t,x}(X_s \in A)$ is $\mathcal{K}$-measurable.
4. For every $0 \le u \le t \le s$, $A \in \mathcal{K}$ we have $\mathbb{P}^{u,x}(X_s \in A|\mathcal{F}_{[u,t]}) = \mathbb{P}^{t,X_t}(X_s \in A)$, $\mathbb{P}^{u,x}$-a.s.

5. For every $\omega \in \Omega$ and $t \geq 0$ there exists $\delta > 0$ such that $X_s(\omega) = X_t(\omega)$ for $s \in [t, t+\delta]$; this is equivalent to requiring that all the trajectories of X have right limits when K is given the discrete topology (the one where all subsets are open).
6. For every $\omega \in \Omega$ the number of jumps of the trajectory $t \mapsto X_t(\omega)$ is finite on every bounded interval.

X is called a (pure) jump process because of condition 5, and a non explosive process because of condition 6.

The class of Markov processes we will consider in this paper will be described by means of a special form of the joint law Q of the first jump time T_1 and the corresponding position X_{T_1}. To proceed formally, we first fix $t \geq 0$ and $x \in K$ and define the first jump time $T_1(\omega) = \inf\{s > t : X_s(\omega) \neq X_t(\omega)\}$, with the convention that $T_1(\omega) = \infty$ if the indicated set is empty. Clearly, T_1 depends on t. Take the extra point $\Delta \notin K$ and define $X_\infty(\omega) = \Delta$ for all $\omega \in \Omega$, so that $X_{T_1} : \Omega \to K \cup \{\Delta\}$ is well defined. On the extended space $S := ([0,\infty) \times K) \cup \{(\infty, \Delta)\}$ we consider the smallest σ-algebra, denoted $\mathcal{S}$, containing $\{(\infty, \Delta)\}$ and all sets of $\mathcal{B}([0,\infty)) \otimes \mathcal{K}$ (here and in the following $\mathcal{B}(\Lambda)$ denotes the Borel σ-algebra of a topological space Λ). Then (T_1, X_{T_1}) is a random variable with values in $(S, \mathcal{S})$. Its law under $\mathbb{P}^{t,x}$ will be denoted by $Q(t, x, \cdot)$.

We will assume that Q is constructed starting from a given transition measure from $[0,\infty) \times K$ to K, called *rate measure* and denoted $\nu(t, x, A)$, $t \in [0, T]$, $x \in K$, $A \in \mathcal{K}$. Thus, we require that $A \mapsto \nu(t, x, A)$ is a positive measure on $\mathcal{K}$ for all $t \in [0, T]$ and $x \in K$, and $(t, x) \mapsto \nu(t, x, A)$ is $\mathcal{B}([0,\infty)) \otimes \mathcal{K}$-measurable for all $A \in \mathcal{K}$. We also assume

$$\sup_{t\in[0,T],x\in K} \nu(t, x, K) < \infty, \qquad \nu(t, x, \{x\}) = 0, \quad t \in [0,\infty), x \in K. \tag{2.1}$$

Define

$$\lambda(t, x) = \nu(t, x, K), \qquad \pi(t, x, A) = \begin{cases} \dfrac{\nu(t, x, A)}{\lambda(t, x)}, & \text{if } \lambda(t, x) > 0, \\ 1_A(x), & \text{if } \lambda(t, x) = 0. \end{cases}$$

Therefore λ is a nonnegative bounded measurable function and π is a transition probability on K satisfying $\pi(t, x, \{x\}) = 0$ if $\lambda(t, x) > 0$, and $\pi(t, x, \cdot) = \delta_x$ (the Dirac measure at x) if $\lambda(t, x) = 0$. λ is called *jump rate function* and π *jump measure*. Note that we have $\nu(t, x, A) = \lambda(t, x)\pi(t, x, A)$ for all $t \in [0, T]$, $x \in K$, $A \in \mathcal{K}$.

Given ν, we will require that for the Markov process X we have, for $0 \leq t \leq a < b \leq \infty$, $x \in K$, $A \in \mathcal{K}$,

$$Q(t, x, (a, b) \times A) = \int_a^b \pi(s, x, A)\, \lambda(s, x)\, \exp\left(-\int_t^s \lambda(r, x)\, dr\right) ds, \tag{2.2}$$

where Q was described above as the law of (T_1, X_{T_1}) under $\mathbb{P}^{t,x}$. Note that (2.2) completely specifies the probability measure $Q(t, x, \cdot)$ on $(S, \mathcal{S})$: indeed simple computations show that, for $s \geq t$,

$$\mathbb{P}^{t,x}(T_1 \in (s, \infty]) = 1 - Q(t, x, (t, s] \times K) = \exp\left(-\int_t^s \lambda(r, x)\, dr\right), \tag{2.3}$$

and we clearly have

$$\begin{aligned}&\mathbb{P}^{t,x}(T_1=\infty)=Q(t,x,\{(\infty,\Delta)\})=\exp\left(-\int_t^\infty \lambda(r,x)\,dr\right),\\&\mathbb{P}^{t,x}(T_1\le t)=Q(t,x,[0,t]\times K)=0.\end{aligned}\tag{2.4}$$

We may interpret (2.3) as the statement that T_1 has exponential distribution on $[t,\infty]$ with variable rate $\lambda(r,x)$. Moreover, the probability $\pi(s,x,\cdot)$ can be interpreted as the conditional probability that X_{T_1} is in $A\in\mathcal{K}$ given that $T_1=s$; more precisely,

$$\mathbb{P}^{t,x}(X_{T_1}\in A, T_1<\infty \mid T_1)=\pi(T_1,x,A)\,1_{T_1<\infty},\quad \mathbb{P}^{t,x}\text{-}a.s.$$

Remark 2.1. 1. The existence of a jump Markov process satisfying (2.2) is a well known fact, see for instance [13] (Chapter III, Section 1, Theorems 3 and 4) where it is proved that X is in addition a strong Markov process. The nonexplosive character of X ($T_n\to\infty$) is made possible by our assumption (2.1).

We note that our data only consist initially in a measurable space $(K,\mathcal{K})$ and a transition measure ν satisfying (2.1). The Markov process $(\Omega, X, \mathbb{P}^{t,x})$ can be constructed in an arbitrary way provided (2.2) holds.

2. In [13] (Chapter III, Section 1, Theorem 2) the following is also proved: starting from $T_0=t$ define inductively $T_{n+1}=\inf\{s>T_n : X_s\neq X_{T_n}\}$, with the convention that $T_{n+1}=\infty$ if the indicated set is empty; then, under the probability $\mathbb{P}^{t,x}$, the sequence $(T_n,X_{T_n})_{n\ge 0}$ is a discrete-time Markov process in $(S,\mathcal{S})$ with transition kernel Q, provided we extend the definition of Q making the state (∞,Δ) absorbing, i.e. we define

$$Q(\infty,\Delta,[0,\infty)\times K)=0,\qquad Q(\infty,\Delta,\{(\infty,\Delta)\})=1.$$

Note that $(T_n,X_{T_n})_{n\ge 0}$ is time-homogeneous although X is not, in general.

This fact allows for a simple description of the process X. Suppose one starts with a discrete-time Markov process $(\tau_n,\xi_n)_{n\ge 0}$ in S with transition probability kernel Q and a given starting point $(t,x)\in[0,\infty)\times K$ (conceptually, trajectories of such a process are easy to simulate). One can then define a process Y in K setting $Y_t=\sum_{n=0}^N \xi_n 1_{[\tau_n,\tau_{n+1})}(t)$, where $N=\sup\{n\ge 0 : \tau_n<\infty\}$. Then Y has the same law as the process X under $\mathbb{P}^{t,x}$.

3. We comment on the special form (2.2) of the kernel Q, that may seem somehow strange at first sight. In [13] (Chapter III, Section 1) it is proved that for a general jump Markov process with the strong Markov property the kernel Q must have the form

$$\begin{aligned}&Q(t,x,(a,b)\times A)=-\int_a^b \pi(s,x,A)\,q(x,t,ds),\\&\quad 0\le t\le a<b\le\infty, x\in K, A\in\mathcal{K},\end{aligned}$$

where $q(x,t,s)=\mathbb{P}^{t,x}(T_1>s)$ is the survivor function of T_1 under $\mathbb{P}^{t,x}$ and each $\pi(s,x,\cdot)$ is a suitable probability on K. Therefore our assumption (2.2) is basically equivalent to the requirement that $q(x,t,\cdot)$ admits a hazard rate function $\lambda(s,x)$ (which turns out to be independent of t because of the Markov property). Because of the clear probabilistic interpretation of λ and π, or equivalently of ν, we have preferred to start with the measure ν as our basic object.

4. Clearly, the class of processes we consider includes as a very special case all the time-homogeneous, nonexplosive, jump Markov processes, which correspond to the function ν not depending on t. In this time-homogeneous case the only restriction we retain is the boundedness assumption (2.1) on the rate function.

In the time-homogeneous case with K a finite or countable set, the matrix $\nu(x,\{y\})_{x,y\in K}$ is the usual matrix of transition rates (or Q-matrix) and $\pi(x,\{y\})_{x,y\in K}$ is the stochastic transition matrix of the embedded discrete-time Markov chain.

2.2. *Marked point processes and the associated martingales*

In this subsection we recall some basic facts following [16]. In the following we fix a pair $(t,x)\in[0,\infty)\times K$ and look at the process X under the probability $\mathbb{P}^{t,x}$. For every $t\geq 0$ we denote by $\mathbb{F}^t$ the filtration $(\mathcal{F}_{[t,s]})_{s\in[t,\infty)}$. We recall that condition 5 above implies that for every $t\geq 0$ the filtration $\mathbb{F}^t$ is right-continuous (see [5], Appendix A2, Theorem T26).

The predictable σ-algebra (respectively, the progressive σ-algebra) on $\Omega\times[t,\infty)$ will be denoted by $\mathcal{P}^t$ (respectively, by $Prog^t$). The same symbols will also denote the restriction to $\Omega\times[t,T]$ for some $T>t$.

For every $t\geq 0$ we define a sequence $(T_n^t)_{n\geq 0}$ of random variables with values in $[0,\infty]$ setting

$$T_0^t(\omega)=t,\qquad T_{n+1}^t(\omega)=\inf\{s>T_n^t(\omega)\ :\ X_s(\omega)\neq X_{T_n^t(\omega)}(\omega)\},\tag{2.5}$$

with the convention that $T_{n+1}^t(\omega)=\infty$ if the indicated set is empty. Since X is a jump process we have $T_n^t(\omega)<T_{n+1}^t(\omega)$ if $T_n^t(\omega)<\infty$. Since X is non explosive we have $T_n^t(\omega)\to\infty$.

For $\omega\in\Omega$ we define a random measure on $((t,\infty)\times K,\mathcal{B}((0,\infty))\otimes\mathcal{K})$ setting

$$p^t(\omega,C)=\sum_{n\geq 1}1\Big((T_n^t(\omega),X_{T_n^t}(\omega))\in C\Big),\quad C\in\mathcal{B}((t,\infty))\otimes\mathcal{K},$$

where $1(\ldots)$ is the indicator function. We also use the notation $p^t(ds\,dy)$ or simply $p(ds\,dy)$. Note that

$$p^t((t,s]\times A)=\sum_{n\geq 1}1(T_n^t\leq s)\,1(X_{T_n^t}\in A),\quad s\geq t,\ A\in\mathcal{K}.$$

By general results (see [16]) it turns out that for every nonnegative $\mathcal{P}^t\otimes\mathcal{K}$-measurable function $H_s(\omega,y)$ defined on $\Omega\times[t,\infty)\times K$ we have

$$\mathbb{E}^{t,x}\int_t^\infty\int_K H_s(y)\ p^t(ds\,dy)=\mathbb{E}^{t,x}\int_t^\infty\int_K H_s(y)\ \nu(s,X_s,dy)\,ds.\tag{2.6}$$

Note that in this equality we may replace $\nu(s,X_s,dy)\,ds$ by $\nu(s,X_{s-},dy)\,ds$. The random measure $\nu(s,X_{s-},dy)\,ds$ is called the compensator, or the dual predictable projection, of $p^t(ds\,dy)$.

Now fix $T>t$. If a real function $H_s(\omega,y)$, defined on $\Omega\times[t,\infty)\times K$, is $\mathcal{P}^t\otimes\mathcal{K}$-measurable and satisfies

$$\int_t^T\int_K|H_s(y)|\ \nu(s,X_s,dy)\,ds<\infty,\quad \mathbb{P}^{t,x}\text{-}a.s.$$

then the following stochastic integral can be defined

$$\begin{aligned}\int_t^s\int_K H_r(y)\ q^t(dr\,dy):=&\int_t^s\int_K H_r(y)\ p^t(dr\,dy)\\&-\int_t^s\int_K H_r(y)\ \nu(r,X_r,dy)\,dr,\quad s\in[t,T],\end{aligned}\tag{2.7}$$

as the difference of ordinary integrals with respect to $p^t(ds\,dy)$ and $\nu(s, X_{s-}^{t,x}, dy)\,ds$. Here and in the following the symbol $\int_a^b$ is to be understood as an integral over the interval $(a, b]$. We shorten this identity writing $q^t(ds\,dy) = p^t(ds\,dy) - \nu(s, X_{s-}^{t,x}, dy)\,ds$. Note that

$$\int_t^s \int_K H_r(y)\; p^t(dr\,dy) = \sum_{n\geq 1, T_n^t \leq s} H_{T_n^t}(X_{T_n^t}), \quad s \in [t, T],$$

is always well defined since $T_n^t \to \infty$.

For $m \in [1, \infty)$ we define $\mathcal{L}^m(p^t)$ as the space of $\mathcal{P}^t \otimes \mathcal{K}$-measurable real functions $H_s(\omega, y)$ on $\Omega \times [t, T] \times K$ such that

$$\mathbb{E}^{t,x} \int_t^T \int_K |H_s(y)|^m \; p^t(ds\,dy) = \mathbb{E}^{t,x} \int_t^T \int_K |H_s(y)|^m \; \nu(s, X_s, dy)\,ds < \infty$$

(the equality of the integrals follows from (2.6)). Given an element H of $\mathcal{L}^1(p^{t,x})$, the stochastic integral (2.7) turns out to be a finite variation martingale.

We define the space $\mathcal{L}^1_{loc}(p^t)$ as the space of those elements H such that $H\, 1_{(t,S_n]} \in \mathcal{L}^1(p^t)$ for some increasing sequence of $\mathbb{F}^t$-stopping times S_n diverging to $+\infty$.

The key result used in the construction of a solution to BSDEs is the integral representation theorem of marked point process martingales, which is a counterpart of the well known representation result for Brownian martingales (see e.g. [19] Ch V.3 or [12] Thm 12.33).

Theorem 2.2. *Given $(t, x) \in [0, T] \times K$, let M be an $\mathbb{F}^t$-martingale on $[t, T]$ with respect to $\mathbb{P}^{t,x}$. Then there exists a process $H \in \mathcal{L}^1(p^t)$ such that*

$$M_s = M_t + \int_t^s \int_K H_r(y)\, q^t(dr\; dy), \quad s \in [t, T]. \tag{2.8}$$

Proof. When M is right-continuous the result is well known: see e.g. [10,9]. The general case reduces to this one by standard arguments that we only sketch: one first introduces the completion $\overline{\mathbb{F}}^t$ of the filtration $\mathbb{F}^t$ with respect to $\mathbb{P}^{t,x}$. Then $\overline{\mathbb{F}}^t$ satisfies the usual assumptions, so that M admits a right-continuous modification $\bar{M}$, that can be represented as in (2.8) by means of a process $\bar{H} \in \mathcal{L}^1(p^t)$ and $\overline{\mathcal{P}}^t \otimes \mathcal{K}$-measurable, where $\overline{\mathcal{P}}^t$ denotes the $\overline{\mathbb{F}}^t$-predictable σ-field. By monotone class arguments, starting from a simple set of generators of $\overline{\mathcal{P}}^t \otimes \mathcal{K}$, one finally proves that $\bar{H}$ has a modification H such that (2.8) holds. □

Let us define the generator of the Markov process X setting

$$\mathcal{L}_t \psi(x) = \int_K \Big(\psi(y) - \psi(x)\Big)\, \nu(t, x, dy), \quad t \geq 0, x \in K,$$

for every measurable function $\psi : K \to \mathbb{R}$ for which the integral is well defined.

We recall the Ito formula for the process X, see e.g. [9] or [15]. Suppose $0 \leq t < T$ and let $v : [t, T] \times K \to \mathbb{R}$ be a measurable function such that

1. $s \mapsto v(s, x)$ is absolutely continuous for every $x \in K$, with time derivative denoted by $\partial_s v(s, x)$;
2. $\{v(s, y) - v(s, X_{s-}),\; s \in [t, T], y \in K\}$ belongs to $\mathcal{L}^1_{loc}(p^t)$;

then, $\mathbb{P}^{t,x}$-a.s.

$$v(s, X_s) = v(t,x) + \int_t^s \Big(\partial_r v(r, X_r) + \mathcal{L}_r v(r, X_r)\Big)\, dr$$
$$+ \int_t^s \int_K \Big(v(r,y) - v(r, X_{r-})\Big)\, q^t(dr\, dy), \quad s \in [t,T] \tag{2.9}$$

where the stochastic integral is a local martingale. In differential notation:

$$dv(s, X_s) = \partial_s v(s, X_s)\, ds + \mathcal{L}_s v(s, X_s^{t,x})\, ds + \int_K \Big(v(s,y) - v(s, X_{s-})\Big)\, q^t(ds\, dy).$$

3. The backward equation

Let us assume that ν is a transition measure on K satisfying (2.1). X denotes the Markov process constructed in Section 2, satisfying conditions 1–6 in Section 2.1 as well as (2.2).

Throughout this section we fix a deterministic terminal time $T > 0$ and a pair $(t,x) \in [0,T] \times K$. We look at all processes under the probability $\mathbb{P}^{t,x}$. In the following, especially in the proofs, we will omit the superscript t and write $\mathbb{F}$, $\mathcal{P}$, $Prog$, T_n, $p(ds\, dy)$, $q(ds\, dy)$, $\mathcal{L}^2(p)$ instead of T_n^t, $\mathbb{F}^t$, $\mathcal{P}^t$, $Prog^t$, $p^t(ds\, dy)$, $q^t(ds\, dy)$, $\mathcal{L}^2(p^t)$.

We are interested in studying the following family of backward equations parametrized by (t,x): $\mathbb{P}^{t,x}$-a.s.

$$Y_s + \int_s^T \int_K Z_r(y)\, q(dr\, dy) = g(X_T) + \int_s^T f(r, X_r, Y_r, Z_r(\cdot))\, dr, \quad s \in [t,T], \tag{3.1}$$

under the following assumptions on the data f and g:

Hypothesis 3.1. 1. The final condition $g : K \to \mathbb{R}$ is $\mathcal{K}$-measurable and $\mathbb{E}^{t,x}|g(X_T)|^2 < \infty$.
2. The generator f is such that

(i) for every $s \in [0,T]$, $x \in K$, $r \in \mathbb{R}$, $f(s,x,r,\cdot)$ is a mapping $L^2(K, \mathcal{K}, \nu(s,x,dy)) \to \mathbb{R}$;

(ii) for every bounded and $\mathcal{K}$-measurable function $z : K \to \mathbb{R}$, the mapping
$$(s,x,r) \mapsto f(s,x,r,z(\cdot)) \tag{3.2}$$
is $\mathcal{B}([0,T]) \otimes \mathcal{K} \otimes \mathcal{B}(\mathbb{R})$-measurable;

(iii) there exist $L \geq 0$, $L' \geq 0$ such that for every $s \in [0,T]$, $x \in K$, $r, r' \in \mathbb{R}$, $z, z' \in L^2(K, \mathcal{K}, \nu(s,x,dy))$,
$$|f(s,x,r,z(\cdot)) - f(s,x,r',z'(\cdot))|$$
$$\leq L'|r - r'| + L\left(\int_K |z(y) - z'(y)|^2 \nu(s,x,dy)\right)^{1/2}. \tag{3.3}$$

(iv) $\mathbb{E}^{t,x} \int_t^T |f(s, X_s, 0, 0)|^2 ds < \infty$.

In order to study the backward equation (3.1) we need to check the following measurability property of $f(s, X_s, Y_s, Z_s(\cdot))$.

Lemma 3.2. *Let f be a generator satisfying assumptions* (i), (ii) *and* (iii).

If $Z \in \mathcal{L}^2(p^t)$, then the mapping

$$(\omega, s, y) \mapsto f(s, X_{s-}(\omega), y, Z_s(\omega, \cdot)) \tag{3.4}$$

is $\mathcal{P}^t \otimes \mathcal{B}(\mathbb{R})$-measurable.

If, in addition, Y is a $Prog^t$-measurable process, then

$$(\omega, s) \mapsto f(s, X_{s-}(\omega), Y_s(\omega), Z_s(\omega, \cdot))$$

is $Prog^t$-measurable.

Proof. It is enough to prove the required measurability of the mapping (3.4), since the other statement of the lemma follows by composition.

Let $B(K)$ denote the space of $\mathcal{K}$-measurable and bounded maps $z : K \to \mathbb{R}$, endowed with the supremum norm and the corresponding Borel σ-algebra $\mathcal{B}(B(K))$. Note that $B(K) \subset L^2(K, \mathcal{K}, \nu(s, x, dy))$ by (2.1). Consider the restriction of the generator f to $[0, T] \times K \times \mathbb{R} \times B(K)$. By (ii) we have that for all $z \in B(K)$ the function $f(\cdot, \cdot, \cdot, z)$ is $\mathcal{B}([0, T]) \otimes \mathcal{K} \otimes \mathcal{B}(\mathbb{R})$-measurable. Moreover, by (3.3) and (2.1) it follows that for all $(s, x, y) \in [0, T] \times K \times (s, x, y, \cdot)$ is continuous. This means that the mapping $f : [0, T] \times K \times \mathbb{R} \times B(K) \to \mathbb{R}$ is a Carathéodory function so, in particular, it is $\mathcal{B}([0, T]) \otimes \mathcal{K} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(B(K))$-measurable.

Now let Z be a bounded $\mathcal{P}^t \otimes \mathcal{K}$-measurable real function. Then, for all $s \in [t, T]$ and $\omega \in \Omega$, $Z_s(\omega, \cdot)$ belongs to $B(K)$ and, using a monotone class argument, it easy to verify that the map $(s, \omega) \mapsto Z_s(\omega, \cdot)$ is measurable with respect to $\mathcal{P}^t$ and $\mathcal{B}(B(K))$. By composition it follows that the mapping

$$(\omega, s, y) \mapsto f(s, X_{s-}(\omega), y, Z_s(\omega, \cdot)),$$

is $\mathcal{P}^t \otimes \mathcal{B}(\mathbb{R})$-measurable.

Finally, for general $Z \in \mathcal{L}^2(p^t)$, thanks to the Lipschitz condition (iii), it is possible to write

$$f(t, X_{t-}(\omega), y, Z_t(\omega)) = \lim_{n\to\infty} f(t, X_{t^-}(\omega), y, Z_t^n(\omega)),$$

where Z^n is a sequence of bounded and $\mathcal{P}^t \otimes \mathcal{K}$-measurable real functions converging to Z in $\mathcal{L}^2(p^t)$. The required measurability follows. $\square$

We introduce the space $\mathbb{M}^{t,x}$ of the processes (Y, Z) on $[t, T]$ such that Y is real-valued and $Prog^t$-measurable, $Z : \Omega \times [t, T] \times K \to \mathbb{R}$ is $\mathcal{P}^t \otimes \mathcal{K}$-measurable and

$$\|(Y, Z)\|^2_{\mathbb{M}^{t,x}} := \mathbb{E}^{t,x} \int_t^T |Y_s|^2 ds + \mathbb{E}^{t,x} \int_t^T \int_K |Z_s(y)|^2 \nu(s, X_s, dy)\, ds < \infty.$$

The space $\mathbb{M}^{t,x}$, endowed with this norm, is a Banach space, provided we identify pairs of processes whose difference has norm zero.

Lemma 3.3. *Suppose that $f : \Omega \times [t, T] \to \mathbb{R}$ is $Prog^t$-measurable, $\xi : \Omega \to \mathbb{R}$ is $\mathcal{F}_{[t,T]}$-measurable and*

$$\mathbb{E}^{t,x}|\xi|^2 + \mathbb{E}^{t,x} \int_t^T |f_s|^2 ds < \infty.$$

Then there exists a unique pair (Y, Z) in $\mathbb{M}^{t,x}$ solution to the BSDE

$$Y_s + \int_s^T \int_K Z_r(y)\, q^t(dr\, dy) = \xi + \int_s^T f_r\, dr, \quad s \in [t, T]. \tag{3.5}$$

Moreover for all $\beta \in \mathbb{R}$ we have

$$\mathbb{E}^{t,x} e^{\beta s}|Y_s|^2 + \beta\, \mathbb{E}^{t,x} \int_s^T e^{\beta r}|Y_r|^2 dr + \mathbb{E}^{t,x} \int_s^T \int_K e^{\beta r}|Z_r(y)|^2 \nu(r, X_r, dy)\, dr$$
$$= \mathbb{E}^{t,x} e^{\beta T}|\xi|^2 + 2\mathbb{E}^{t,x} \int_s^T e^{\beta r} Y_r\, f_r\, dr, \quad s \in [t, T], \tag{3.6}$$

and

$$\mathbb{E}^{t,x} \int_t^T |Y_r|^2 dr + \mathbb{E}^{t,x} \int_t^T \int_K |Z_r(y)|^2 \nu(r, X_r, dy)\, dr$$
$$\leq 8\, \mathbb{E}^{t,x}\, |\xi|^2 + 8(T+1)\mathbb{E}^{t,x} \left[\int_t^T |f_r|^2\, dr\right]. \tag{3.7}$$

Proof. To simplify notation we will drop the superscripts t, x and we write the proof in the case $t = 0$.

We start by proving the uniqueness. The equation for the difference of two solutions $\overline{Y} = Y^1 - Y^2$ and $\overline{Z} = Z^1 - Z^2$ becomes

$$\overline{Y}_s + \int_s^T \int_K \overline{Z}_r(y)\, q(dr\ dy) = 0 \quad s \in [t, T]. \tag{3.8}$$

Taking the conditional expectation given $\mathcal{F}_{[0,s]}$ it follows that $\overline{Y}_s = 0$ d$s \otimes$ d$\mathbb{P}$-a.s. By consequence, also the process $Q_s =: \int_s^T \int_K \overline{Z}_r(y)\, q(dr\ dy) = 0$ d$s \otimes$ d$\mathbb{P}$-a.s. and then indistinguishable from zero, since it is cadlag. It follows that, for $s \in [t, T]$, $\triangle Q_s = \overline{Z}_s(X_s) 1_{\{s=T_n\}} = 0$ and so $\mathbb{E} \int_t^T \int_K |Z_r(y)|^2\, p(dr\ dy) = 0$. Since $\overline{Z}$ is $\mathcal{P} \otimes \mathcal{K}$-measurable the last equality implies that $\mathbb{E} \int_t^T \int_K |\overline{Z}_r(y)|^2\, \nu(r, X_r, dy) dr = 0$.

Now, assuming that $(Y, Z) \in \mathbb{M}$ is a solution, we prove the identity (3.6). From the Ito formula applied to $e^{\beta s}|Y_s|^2$ it follows that

$$d(e^{\beta s}|Y_s|^2) = \beta e^{\beta s}|Y_s|^2\, ds + 2e^{\beta s} Y_{s-} dY_s + e^{\beta s}|\Delta Y_s|^2.$$

So integrating on $[s, T]$

$$e^{\beta s}|Y_s|^2 = -\int_s^T \beta e^{\beta r}|Y_r|^2\, dr - 2\int_s^T \int_K e^{\beta r} Y_{r-} Z_r(y) q(dr\, dy) - \sum_{s<r\leq T} e^{\beta r}|\Delta Y_r|^2$$
$$+ e^{\beta T}|\xi|^2 + 2\int_s^T e^{\beta r} Y_r\, f_r\, dr. \tag{3.9}$$

The process $\int_0^s \int_K e^{\beta r} Y_{r-} Z_r(y) q(dr\, dy)$ is a martingale, because the integrand process $e^{\beta r} Y_{r-} Z_r(y)$ is in $\mathcal{L}^1(p)$: in fact from the Young inequality and (2.1) we get

$$\mathbb{E} \int_0^T \int_K e^{\beta r}|Y_{r-}||Z_r(y)|\nu(r, X_r, dy)\, dr$$
$$\leq \frac{1}{2}\mathbb{E} \int_0^T \int_K e^{\beta r}|Y_{r-}|^2 \nu(r, X_r, dy)\, dr + \frac{1}{2}\mathbb{E} \int_0^T \int_K e^{\beta r}|Z_r(y)|^2 \nu(r, X_r, dy)\, dr$$
$$\leq \sup_{t,x} \nu(t, x, K)\, \frac{e^{\beta T}}{2}\mathbb{E} \int_0^T |Y_r|^2\, dr + \frac{e^{\beta T}}{2}\mathbb{E} \int_0^T |Z_r(y)|^2 \nu(r, X_r, dy)\, dr < \infty.$$

Moreover we have

$$\sum_{0<r\le s} e^{\beta r}|\Delta Y_r|^2 = \int_0^t \int_K e^{\beta r}|Z_r(y)|^2\, p(dr\, dy)$$
$$= \int_0^s \int_K e^{\beta r}|Z_r(y)|^2\, q(dr\, dy) + \int_0^s \int_K e^{\beta r}|Z_r(y)|^2 \nu(r, X_r, dy)\, dr,$$

where the stochastic integral with respect to q is a martingale. Taking the expectation in (3.9) we obtain (3.6).

We now pass to the proof of existence. The solution (Y, Z) is defined by considering the martingale $M_s = \mathbb{E}^{\mathcal{F}_{[0,s]}}[\xi + \int_0^T f_r\, dr]$. By the martingale representation Theorem 2.2, there exists a process $Z \in \mathcal{L}^1(p)$ such that

$$M_s = M_0 + \int_0^s \int_K Z_r(y)\, q(dy\, dr), \quad s \in [0, T].$$

Define the process Y by the formula

$$Y_s = M_s - \int_0^s f_r\, dr, \quad s \in [0, T].$$

Noting that $Y_T = \xi$, we easily deduce that Eq. (3.5) is satisfied.

It remains to show that $(Y, Z) \in \mathbb{M}$. Taking the conditional expectation, it follows from (3.5) that $Y_s = \mathbb{E}^{\mathcal{F}_{[0,s]}}[\xi + \int_s^T f_r\, dr]$ so that we obtain

$$|Y_s|^2 \le 2|\mathbb{E}^{\mathcal{F}_{[0,s]}}\xi|^2 + 2\left|\mathbb{E}^{\mathcal{F}_{[0,s]}} \int_s^T f_r\, dr\right|^2$$
$$\le 2\mathbb{E}^{\mathcal{F}_{[0,s]}}\left[|\xi|^2 + T\int_0^T |f_r|^2\, dr\right]. \tag{3.10}$$

Denoting by m_s the right-hand side of (3.10), we see that m is a martingale by the assumptions of the lemma. In particular, for every stopping time S with values in $[0, T]$, we have

$$\mathbb{E}|Y_S|^2 \le \mathbb{E}\, m_S = \mathbb{E}\, m_T < \infty \tag{3.11}$$

by the optional stopping theorem. Next we define the increasing sequence of stopping times

$$S_n = \inf\left\{s \in [0, T] \,:\, \int_0^s |Y_r|^2 dr + \int_0^s \int_K |Z_r(y)|^2 \nu(r, X_r, dy)\, dr > n\right\},$$

with the convention $\inf \emptyset = T$. Computing the Itô differential $d(|Y_s|^2)$ on the interval $[0, S_n]$ and proceeding as before we deduce

$$\mathbb{E}\int_0^{S_n} |Y_r|^2\, dr + \mathbb{E}\int_0^{S_n}\int_K |Z_r(y)|^2 \nu(r, X_r, dy)\, dr \le \mathbb{E}\,|Y_{S_n}|^2 + 2\mathbb{E}\int_0^{S_n} Y_r f_r\, dr.$$

Using the inequalities $2Y_r f_r \le (1/2)|Y_r|^2 + 2|f_r|^2$ and (3.11) (with $S = S_n$) we find the following estimates

$$\mathbb{E}\int_0^{S_n} |Y_r|^2\, dr \le 4\,\mathbb{E}|\xi|^2 + 4(T+1)\mathbb{E}\int_0^T |f_r|^2\, dr \tag{3.12}$$

$$\mathbb{E}\int_0^{S_n}\int_K |Z_r(y)|^2\nu(r,X_r,dy)\,dr \le 4\mathbb{E}|\xi|^2 + 4(T+1)\mathbb{E}\int_0^T |f_r|^2\,dr. \tag{3.13}$$

Setting $S=\lim_n S_n$ we obtain

$$\int_0^S |Y_r|^2\,dr + \int_0^S\int_K |Z_r(y)|^2\nu(r,X_r,dy)\,dr < \infty, \qquad \mathbb{P}\text{-}a.s.$$

which implies $S=T$, $\mathbb{P}$-a.s., by the definition of S_n. Letting $n\to\infty$ in (3.12) and (3.13) we conclude that (3.7) holds, so that $(Y,Z)\in\mathbb{M}$. □

Theorem 3.4. *Suppose that Hypothesis 3.1 holds for some $(t,x)\in[0,T]\times K$.*
Then there exists a unique pair (Y,Z) in $\mathbb{M}^{t,x}$ which solves the BSDE (3.1).

Proof. To simplify notation we drop the superscripts t,x and we write the proof in the case $t=0$. We use a fixed point argument. Define the map $\Gamma:\mathbb{M}\to\mathbb{M}$ as follows: for $(U,V)\in\mathbb{M}$, $(Y,Z)=\Gamma(U,V)$ is defined as the unique solution in $\mathbb{M}$ to the equation

$$Y_s + \int_s^T\int_K Z_r(y)\,q(dr\,dy) = g(X_T) + \int_s^T f(r,X_r,U_r,V_r)\,dr, \qquad s\in[0,T]. \tag{3.14}$$

From the assumptions on f it follows that $\mathbb{E}\int_0^T |f(s,X_s,U_s,V_s)|^2ds<\infty$, so by Lemma 3.3 there exists a unique $(Y,Z)\in\mathbb{M}$ satisfying (3.14) and Γ is a well defined map.

We show that Γ is a contraction if $\mathbb{M}$ is endowed with the equivalent norm

$$\|(Y,Z)\|^2_{\mathbb{M}} := C|Y|^2_\beta + \|Z\|^2_\beta, \tag{3.15}$$

where

$$|Y|^2_\beta := \mathbb{E}\int_0^T e^{\beta s}|Y_s|^2ds, \qquad \|Z\|^2_\beta := \mathbb{E}\int_0^T\int_K e^{\beta s}|Z_s(y)|^2\nu(s,X_s,dy)\,ds,$$

for some constants $C>0$ and $\beta>0$ sufficiently large, that will be determined in the sequel.

Let (U^1,V^1), (U^2,V^2) be two elements of $\mathbb{M}$ and let (Y^1,Z^1), (Y^2,Z^2) the associated solutions. Lemma 3.3 applies to the difference $\overline{Y}=Y^1-Y^2$, $\overline{Z}=Z^1-Z^2$, $\overline{f}_s=f(s,X_s,U^1_s,V^1_s)-f(s,X_s,U^2_s,V^2_s)$ and (3.6) yields, noting that $\overline{Y}_T=0$,

$$\begin{aligned}&\mathbb{E}e^{\beta s}|\overline{Y}_s|^2 + \beta\,\mathbb{E}\int_s^T e^{\beta r}|\overline{Y}_r|^2dr + \mathbb{E}\int_s^T\int_K e^{\beta r}|\overline{Z}_r(y)|^2\nu(r,X_r,dy)\,dr\\ &\quad = 2\mathbb{E}\int_s^T e^{\beta r}\overline{Y}_r\,\overline{f}_r dr, \qquad s\in[0,T].\end{aligned}$$

From the Lipschitz conditions of f and elementary inequalities, putting $\overline{U}=U^1-U^2$, $\overline{V}=V^1-V^2$, it follows that

$$\begin{aligned}&\beta\,\mathbb{E}\int_0^T e^{\beta s}|\overline{Y}_s|^2ds + \mathbb{E}\int_0^T\int_K e^{\beta s}|\overline{Z}_s(y)|^2\nu(s,X_s,dy)\,ds\\ &\quad\le 2L\mathbb{E}\int_0^T e^{\beta s}|\overline{Y}_s|\left(\int_K |\overline{V}_s(y)|^2\nu(s,X_s,dy)\right)^{1/2}ds + 2L'\mathbb{E}\int_0^T e^{\beta s}|\overline{Y}_s|\,|\overline{U}_s|\,ds\\ &\quad\le \alpha\mathbb{E}\int_0^T\int_K e^{\beta s}|\overline{V}_s(y)|^2\nu(s,X_s,dy)\,ds + \frac{L^2}{\alpha}\mathbb{E}\int_0^T e^{\beta s}|\overline{Y}_s|^2\,ds\end{aligned}$$

$$+\gamma L'\mathbb{E}\int_0^T e^{\beta s}|\overline{Y}_s|^2\,ds+\frac{L'}{\gamma}\mathbb{E}\int_0^T e^{\beta s}|\overline{U}_s|^2\,ds$$

for every $\alpha>0$, $\gamma>0$. This can be written as

$$\left(\beta-\frac{L^2}{\alpha}-\gamma L'\right)|\overline{Y}|_\beta^2+\|\overline{Z}\|_\beta^2\le\alpha\|\overline{V}\|_\beta^2+\frac{L'}{\gamma}|\overline{U}|_\beta^2.$$

If we choose $\beta>L^2+2L'$, it is possible to find $\alpha\in(0,1)$ such that

$$\beta>\frac{L^2}{\alpha}+\frac{2L'}{\sqrt{\alpha}}.$$

If $L'=0$ we see that Γ is an α-contraction on $\mathbb{M}$ endowed with the norm (3.15) for $C=\beta-(L^2/\alpha)$. If $L'>0$ we choose $\gamma=1/\sqrt{\alpha}$ and obtain

$$\frac{L'}{\sqrt{\alpha}}|\overline{Y}|_\beta^2+\|\overline{Z}\|_\beta^2\le\alpha\|\overline{V}\|_\beta^2+L'\sqrt{\alpha}|\overline{U}|_\beta^2=\alpha\left(\frac{L'}{\sqrt{\alpha}}|\overline{U}|_\beta^2+\|\overline{V}\|_\beta^2\right),$$

so that Γ is an α-contraction on $\mathbb{M}$ endowed with the norm (3.15) for $C=(L'/\sqrt{\alpha})$. In all cases there exists a unique fixed point which is the required unique solution to the BSDE (3.1). □

Next we prove some estimates on the solutions of the BSDE, which show in particular the continuous dependence upon the data. Let us consider two solutions $(Y^1,Z^1),(Y^2,Z^2)\in\mathbb{M}^{t,x}$ to the BSDE (3.1) associated with the drivers f^1 and f^2 and final data g^1 and g^2, respectively, which are assumed to satisfy Hypothesis 3.1. Denote $\overline{Y}=Y^1-Y^2$, $\overline{Z}=Z^1-Z^2$, $\overline{g}_T=g^1(X_T^{t,x})-g^2(X_T^{t,x})$, $\overline{f}_s=f^1(s,X_s^{t,x},Y_s^2,Z_s^2(\cdot))-f^2(s,X_s^{t,x},Y_s^2,Z_s^2(\cdot))$.

Proposition 3.5. *Suppose that Hypothesis 3.1 holds for some t,x. Let $(\overline{Y},\overline{Z})$ be the processes defined above. Then the a priori estimates hold:*

$$\begin{aligned}&\sup_{s\in[t,T]}\mathbb{E}^{t,x}|\overline{Y}_s|^2+\mathbb{E}^{t,x}\int_t^T|\overline{Y}|_s^2\,ds+\mathbb{E}^{t,x}\int_t^T\int_K|\overline{Z}|_s^2\nu(s,X_s,dy)\,ds\\&\quad\le C\left(\mathbb{E}^{t,x}|\overline{g}_T|^2+\mathbb{E}^{t,x}\int_t^T|\overline{f}_s|^2\,ds\right),\end{aligned}\tag{3.16}$$

where C is a constant depending only on T,L,L'.

Proof. Again we drop the superscripts t,x. Arguing as in the proof of (3.6) we obtain

$$\begin{aligned}&\mathbb{E}|\overline{Y}_s|^2+\mathbb{E}\int_s^T\int_K|\overline{Z}_r(y)|^2\nu(r,X_r,dy)\,dr\\&\quad=\mathbb{E}|\overline{g}_T|^2+2\mathbb{E}\int_s^T\overline{Y}_rf^1(r,X_r,Y_r^1,Z_r^1)-f^2(r,X_r,Y_r^2,Z_r^2)\,dr.\end{aligned}$$

By the Lipschitz property of the driver f^1 we get

$$\begin{aligned}\mathbb{E}|\overline{Y}_s|^2&\le\mathbb{E}|\overline{g}_T|^2+2\mathbb{E}\int_s^T|\overline{Y}_r|(|f^1(r,X_r,Y_r^1,Z_r^1)-f^1(r,X_r,Y_r^2,Z_r^2)|+|\overline{f}_r|)\,dr\\&\le\mathbb{E}|\overline{g}_T|^2+2L'\mathbb{E}\int_s^T|\overline{Y}_r|^2\,dr+2L\mathbb{E}\int_s^T|\overline{Y}_r|\left\{\int_K|\overline{Z}_r|^2\nu(r,X_r,dy)\right\}^{\frac{1}{2}}\,dr\end{aligned}$$

$$+2\mathbb{E}\int_s^T |\overline{Y}_r||\overline{f}_r|\,dr,$$

$$\leq \mathbb{E}|\overline{g}_T|^2 + C\mathbb{E}\int_s^T |\overline{Y}_r|^2\,dr + \frac{1}{2}\mathbb{E}\int_s^T\int_K |\overline{Z}_r|^2\nu(r, X_r, dy)\,dr$$

$$+\mathbb{E}\int_s^T |\overline{f}_r|^2\,dr,$$

for some constant C. Hence we deduce

$$\mathbb{E}|\overline{Y}_s|^2 + \frac{1}{2}\mathbb{E}\int_s^T\int_K |\overline{Z}_r|^2\nu(r, X_r, dy)\,dr$$

$$\leq \mathbb{E}|\overline{g}_T|^2 + \mathbb{E}\int_s^T |\overline{f}_r|^2\,dr + C\,\mathbb{E}\int_s^T |\overline{Y}_r|^2\,dr \tag{3.17}$$

and by Gronwall's lemma we get

$$\mathbb{E}|\overline{Y}_s|^2 \leq e^{C(T-s)}\left(\mathbb{E}|\overline{g}_T|^2 + \mathbb{E}\int_t^T |\overline{f}_r|^2\,dr\right)$$

and the conclusion follows from (3.17). □

From the continuous dependence of the solution upon the data we deduce the following a priori estimates:

Corollary 3.6. *Suppose that Hypothesis 3.1 holds for some t, x. Let (Y, Z) be the unique solution in $\mathbb{M}^{t,x}$ of the BSDE (3.1). Then there exists a positive constant C, depending only on T, L, L', such that*

$$\mathbb{E}^{t,x}\int_t^T |Y_s|^2\,ds + \mathbb{E}^{t,x}\int_t^T\int_K |Z_s(y)|^2\nu(s, X_s, dy)\,ds$$

$$\leq C\mathbb{E}^{t,x}\left[|g(X_T)|^2 + \int_t^T |f(s, X_s, 0, 0)|^2\,ds\right]. \tag{3.18}$$

Proof. The thesis follows from Proposition 3.5 setting $f^1 = f$, $g^1 = g$, $f^2 = 0$ and $g^2 = 0$. □

4. Nonlinear variants of the Kolmogorov equation

Let us assume that ν is a transition measure on K satisfying (2.1). X denotes the Markov process constructed in Section 2, satisfying conditions 1–6 in Section 2.1 as well as (2.2).

In this section it is our purpose to present some nonlinear variants of the classical backward Kolmogorov equation associated to the Markov process X and to show that their solution can be represented probabilistically by means of an appropriate BSDE of the type considered above.

Suppose that two functions f, g are given, satisfying the assumptions of Hypothesis 3.1 for every $t \in [0, T]$, $x \in K$. The equation

$$v(t,x) = g(x) + \int_t^T \mathcal{L}_s v(s,x)\,ds + \int_t^T f\Big(s, x, v(s,x), v(s,\cdot) - v(s,x)\Big)\,ds,$$
$$t \in [0,T],\ x \in K, \tag{4.1}$$

with unknown function $v : [0, T] \times K \to \mathbb{R}$, will be called the nonlinear Kolmogorov equation. Equivalently, one requires that for every $x \in K$ the map $t \mapsto v(t, x)$ is absolutely continuous on $[0, T]$ and

$$\begin{cases} \partial_t v(t,x) + \mathcal{L}_t v(t,x) + f\Big(t, x, v(t,x), v(t,\cdot) - v(t,x)\Big) = 0, \\ v(T,x) = g(x), \end{cases} \tag{4.2}$$

where the first equality is understood to hold almost everywhere on $[0, T]$, outside of a dt-null set of points which can depend on x.

The classical Kolmogorov equation corresponds to the case $f = 0$.

Under appropriate boundedness assumptions we have the following immediate result:

Lemma 4.1. *Suppose that f, g verify Hypothesis 3.1 and, in addition,*

$$\sup_{t\in[0,T],\ x\in K} (|g(x)| + |f(t, x, 0, 0)|) < \infty. \tag{4.3}$$

Then the nonlinear Kolmogorov equation has a unique solution in the class of measurable bounded functions.

Proof. The result is essentially known (see for instance [5], Chapter VII, Theorem T3), so we only sketch the proof. In the space of bounded measurable real functions on $[0, T] \times K$ endowed with the supremum norm one can define a map Γ setting $v = \Gamma(u)$ where

$$v(t,x) = g(x) + \int_t^T \mathcal{L}_s u(s,x)\, ds + \int_t^T f\Big(s, x, u(s,x), u(s,\cdot) - u(s,x)\Big)\, ds,$$
$$t \in [0, T],\ x \in K.$$

Using the boundedness condition (2.1) and the Lipschitz character of f, by standard estimates one can prove that Γ has a unique fixed point, which is the required solution. □

Now we plan to remove the boundedness assumption (4.3). On the functions f, g we will only impose the conditions required in Hypothesis 3.1, for every $t \in [0, T]$, $x \in K$.

Definition 4.1. We say that a measurable function $v : [0, T] \times K \to \mathbb{R}$ is a solution of the nonlinear Kolmogorov equation (4.1) if, for every $t \in [0, T]$, $x \in K$,

1. $\mathbb{E}^{t,x} \int_t^T \int_K |v(s, y) - v(s, X_s)|^2 \nu(s, X_s, dy)\, ds < \infty$;
2. $\mathbb{E}^{t,x} \int_t^T |v(s, X_s)|^2\, ds < \infty$;
3. (4.1) is satisfied.

Remark 4.2. Condition 1 is equivalent to the fact that $v(s, y) - v(s, X_{s-})$ belongs to $\mathcal{L}^2(p^t)$. Conditions 1 and 2 together are equivalent to the fact that the pair $\{v(s, X_s), v(s, y) - v(s, X_{s-});\ s \in [t, T], y \in K\}$ belongs to the space $\mathbb{M}^{t,x}$; in particular they hold true for every measurable bounded function v.

Remark 4.3. We need to verify that for a function v satisfying the conditions 1 and 2 above Eq. (4.1) is well defined.

We first note that for every $x \in K$ we have, $\mathbb{P}^{0,x}$-a.s.,

$$\int_0^T \int_K |v(s, y) - v(s, X_s)|^2 \nu(s, X_s, dy)\, ds + \int_0^T |v(s, X_s)|^2\, ds < \infty.$$

We recall that the law of the first jump time T_1 is exponential with variable rate, according to (2.3). It follows that the set $\{\omega \in \Omega : T_1(\omega) > T\}$ has positive $\mathbb{P}^{0,x}$ probability, and on this set we have $X_s(\omega) = x$. Taking such an ω we conclude that

$$\int_0^T \int_K |v(s,y) - v(s,x)|^2 \nu(s,x,dy)\, ds + \int_0^T |v(s,x)|^2\, ds < \infty, \quad x \in K.$$

Since we are assuming $\sup_{t,x} \nu(t,x,K) < \infty$, it follows from Hölder's inequality that

$$\begin{aligned}\int_0^T |\mathcal{L}_s v(s,x)|\, ds &\le \int_0^T \int_K |v(s,y) - v(s,x)| \nu(s,x,dy)\, ds \\ &\le c \left(\int_0^T \int_K |v(s,y) - v(s,x)|^2 \nu(s,x,dy)\, ds \right)^{1/2} < \infty\end{aligned}$$

for some constant c and for all $x \in K$.

Similarly, from our assumption $\mathbb{E}^{t,x} \int_0^T |f(s,X_s,0,0)|^2 ds < \infty$ we deduce, arguing again on the jump time T_1, that

$$\int_0^T |f(s,x,0,0)|^2\, ds < \infty, \quad x \in K,$$

and from the Lipschitz conditions on f we conclude that

$$\begin{aligned}&\int_0^T |f(s,x,v(s,x),v(s,\cdot) - v(s,x))|\, ds \\ &\quad\le c_1 \left(\int_0^T |f(s,x,0,0)|^2\, ds \right)^{\frac{1}{2}} + c_2 \left(\int_0^T |v(s,x)|^2\, ds \right)^{\frac{1}{2}} \\ &\qquad + c_3 \left(\int_0^T \int_K |v(s,y) - v(s,x)|^2 \nu(s,x,dy)\, ds \right)^{\frac{1}{2}} < \infty\end{aligned}$$

for some constants c_i and for all $x \in K$.

We have thus verified that all the terms occurring in Eq. (4.1) are well defined.

In the following, a basic role will be played by the BSDEs: $\mathbb{P}^{t,x}$-a.s.

$$\begin{aligned}&Y_s^{t,x} + \int_s^T \int_K Z_r^{t,x}(y)\, q^t(dr\, dy) = g(X_T) + \int_s^T f(r, X_r, Y_r^{t,x}, Z_r^{t,x}(\cdot))\, dr, \\ &s \in [t,T],\end{aligned} \tag{4.4}$$

with unknown processes $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$. For every $(t,x) \in [0,T] \times K$ there exists a unique solution in the sense of Theorem 3.4. Note that $Y_t^{t,x}$ is deterministic.

We are ready to state the main result of this section.

Theorem 4.4. *Suppose that Hypothesis 3.1 holds for every $(t,x) \in [0,T] \times K$. Then the nonlinear Kolmogorov equation (4.1) has a unique solution v.*

Moreover, for every $(t,x) \in [0,T] \times K$ we have

$$Y_s^{t,x} = v(s, X_s), \tag{4.5}$$

$$Z_s^{t,x}(y) = v(s,y) - v(s, X_{s-}), \tag{4.6}$$

so that in particular $v(t,x) = Y_t^{t,x}$.

Remark 4.5. The equalities (4.5) and (4.6) are understood as follows.

- $\mathbb{P}^{t,x}$-a.s., equality (4.5) holds for all $s \in [t, T]$.
  Since the trajectories of X are piecewise constant and cadlag this is equivalent to the condition $\mathbb{E}^{t,x} \int_t^T |Y_s^{t,x} - v(s, X_s)|^2 ds = 0$.
- The equality (4.6) holds for almost all (ω, s, y) with respect to the measure $\nu(s, X_{s-}^{t,x}(\omega), dy)\, \mathbb{P}^{t,x}(d\omega)\, ds$, i.e.

$$\mathbb{E}^{t,x} \int_t^T |Z_s^{t,x}(y) - v(s, y) + v(s, X_{s-})|^2 \nu(s, X_{s-}, dy)\, ds = 0.$$

Proof (*Uniqueness*)**.** Let v be a solution. It follows from equality (4.1) itself that $t \mapsto v(t, x)$ is absolutely continuous on $[0, T]$ for every $x \in K$. Since we assume that the process $v(s, y) - v(s, X_{s-})$ belongs to $\mathcal{L}^2(p^t)$, we are in a position to apply the Ito formula (2.9) to the process $v(s, X_s)$, $s \in [t, T]$, obtaining, $\mathbb{P}^{t,x}$-a.s.,

$$\begin{aligned} v(s, X_s) &= v(t, x) + \int_t^s \Big(\partial_r v(r, X_r) + \mathcal{L}_r v(r, X_r)\Big)\, dr \\ &\quad + \int_t^s \int_K \Big(v(r, y) - v(r, X_{r-})\Big)\, q^t(dr\, dy), \qquad s \in [t, T]. \end{aligned}$$

Taking into account that v satisfies (4.2) and that X has piecewise constant trajectories we obtain, $\mathbb{P}^{t,x}$-a.s.,

$$\partial_r v(r, X_r) + \mathcal{L}_r v(r, X_r) + f\Big(r, X_r, v(r, X_r), v(r, \cdot) - v(r, X_r)\Big) = 0,$$

for almost all $r \in [t, T]$. It follows that, $\mathbb{P}^{t,x}$-a.s.,

$$\begin{aligned} v(s, X_s) &= v(t, x) - \int_t^s f\Big(r, X_r, v(r, X_r), v(r, \cdot) - v(r, X_r)\Big)\, dr \\ &\quad + \int_t^s \int_K \Big(v(r, y) - v(r, X_{r-})\Big)\, q^t(dr\, dy), \qquad s \in [t, T]. \end{aligned} \tag{4.7}$$

Since $v(T, x) = g(x)$ for all $x \in K$, simple passages show that

$$\begin{aligned} &v(s, X_s) + \int_s^T \int_K \Big(v(r, y) - v(r, X_{r-})\Big)\, q^t(dr\, dy) \\ &\quad = g(X_T) + \int_s^T f\Big(r, X_r, v(r, X_r), v(r, \cdot) - v(r, X_r)\Big)\, dr, \qquad s \in [t, T]. \end{aligned}$$

Therefore the pairs $(Y_s^{t,x}, Z_s^{t,x}(y))$ and $v(s, X_s), v(s, y) - v(s, X_{s-})$ are both solutions to the same BSDE under $\mathbb{P}^{t,x}$, and therefore they coincide as members of the space $\mathbb{M}^{t,x}$. The required equalities (4.5) and (4.6) follow. In particular we have $v(t, x) = Y_t^{t,x}$, which proves uniqueness of the solution.

Existence. By Theorem 3.4, for every $(t, x) \in [0, T] \times K$ the BSDE (4.4) has a unique solution $(Y_s^{t,x}, Z_s^{t,x})_{s \in [t,T]}$ and, moreover, $Y_t^{t,x}$ is deterministic, i.e. there exists a real number, denoted $v(t, x)$, such that $\mathbb{P}^{t,x}(Y_t^{t,x} = v(t, x)) = 1$.

We proceed by an approximation argument. Let $f^n = (f \wedge n) \vee (-n)$, $g^n = (g \wedge n) \vee (-n)$ denote the truncations of f and g at level n. By Lemma 4.1 there exists a unique bounded measurable solution v^n to the equation: for $t \in [0,T]$, $x \in K$,

$$v^n(t,x) = g^n(x) + \int_t^T \mathcal{L}_s v^n(s,x)\,ds + \int_t^T f^n\Big(s,x,v^n(s,x),v^n(s,\cdot)-v^n(s,x)\Big)\,ds. \tag{4.8}$$

By the first part of the proof, we known that

$$v^n(t,x) = Y_t^{t,x,n}, \qquad v^n(s,X_s) = Y_s^{t,x,n}, \qquad v^n(s,y) - v^n(s,X_{s-}) = Z_s^{t,x,n}(y),$$

in the sense of Remark 4.5, where $(Y_s^{t,x,n}, Z_s^{t,x,n})_{s\in[t,T]}$ is the unique solution to the BSDE

$$Y_s^{t,x,n} + \int_s^T \int_K Z_r^{t,x,n}(y)\,q^t(dr\,dy) = g^n(X_T) + \int_s^T f^n(r,X_r,Y_r^{t,x,n},Z_r^{t,x,n}(\cdot))\,dr,$$
$$s \in [t,T].$$

Comparing with (4.4) and applying Proposition 3.5 we deduce that for some constant c

$$\begin{aligned}
&\sup_{s\in[t,T]} \mathbb{E}^{t,x}|Y_s^{t,x} - Y_s^{t,x,n}|^2 + \mathbb{E}^{t,x}\int_t^T |Y_s^{t,x} - Y_s^{t,x,n}|^2 ds \\
&\quad + \mathbb{E}^{t,x}\int_t^T \int_K |Z_s^{t,x} - Z_s^{t,x,n}|^2 \nu(s,X_s,dy)\,ds \\
&\le c\mathbb{E}^{t,x}|g(X_T) - g^n(X_T)|^2 + c\mathbb{E}^{t,x}\int_t^T |f(s,X_s,Y_s^{t,x},Z_s^{t,x}(\cdot)) \\
&\quad - f^n(s,X_s,Y_s^{t,x},Z_s^{t,x}(\cdot))|^2\,ds \to 0
\end{aligned} \tag{4.9}$$

where the right-hand side tends to zero by monotone convergence.

In particular it follows that

$$|v(t,x) - v^n(t,x)|^2 = |Y_t^{t,x} - Y_t^{t,x,n}|^2 \le \sup_{s\in[t,T]} \mathbb{E}|Y_s^{t,x} - Y_s^{t,x,n}|^2 \to 0,$$

which shows that v is a measurable function. An application of the Fatou lemma gives

$$\begin{aligned}
&\mathbb{E}^{t,x}\int_t^T |Y_s^{t,x} - v(s,X_s)|^2 ds + \mathbb{E}^{t,x}\int_t^T |Z_s^{t,x}(y) - v(s,y) \\
&\quad + v(s,X_{s-})|^2 \nu(s,X_{s-},dy)\,ds \\
&\le \liminf_{n\to\infty} \mathbb{E}^{t,x}\int_t^T |Y_s^{t,x} - v^n(s,X_s)|^2 ds \\
&\quad + \liminf_{n\to\infty} \mathbb{E}^{t,x}\int_t^T |Z_s^{t,x}(y) - v^n(s,y) + v^n(s,X_{s-})|^2 \nu(s,X_{s-},dy)\,ds \\
&= \liminf_{n\to\infty} \mathbb{E}^{t,x}\int_t^T |Y_s^{t,x} - Y_s^{n,t,x}|^2 ds \\
&\quad + \liminf_{n\to\infty} \mathbb{E}^{t,x}\int_t^T |Z_s^{t,x}(y) - Z_s^{n,t,x}(y)|^2 \nu(s,X_{s-},dy)\,ds = 0
\end{aligned}$$

by (4.9). This proves that (4.5) and (4.6) hold. These formulae also imply that

$$\mathbb{E}^{t,x}\int_t^T\int_K |v(s,y)-v(s,X_s)|^2\nu(s,X_s,dy)\,ds+\mathbb{E}^{t,x}\int_t^T |v(s,X_s)|^2\,ds$$
$$=\mathbb{E}^{t,x}\int_t^T\int_K |Z_s^{t,x}|^2\nu(s,X_s,dy)\,ds+\mathbb{E}^{t,x}\int_t^T |Y_s^{t,x}|^2\,ds<\infty,$$

according to the requirements of Definition 4.1. It only remains to show that v satisfies (4.1). This will follow from a passage to the limit in (4.8), provided we can show that

$$\int_t^T \mathcal{L}_s v^n(s,x)\,ds\to\int_t^T \mathcal{L}_s v(s,x)\,ds,$$
$$\int_t^T f^n\Big(s,x,v^n(s,x),v^n(s,\cdot)-v^n(s,x)\Big)\,ds$$
$$\to\int_t^T f\Big(s,x,v(s,x),v(s,\cdot)-v(s,x)\Big)\,ds. \tag{4.10}$$

We first consider

$$\mathbb{E}^{t,x}\left|\int_t^T \mathcal{L}_s v(s,X_s)\,ds-\int_t^T \mathcal{L}_s v^n(s,X_s)\,ds\right|$$
$$=\mathbb{E}^{t,x}\left|\int_t^T\int_K [v(s,y)-v(s,X_s)-v^n(s,y)+v^n(s,X_s)]\,\nu(s,X_s,dy)\,ds\right|$$
$$=\mathbb{E}^{t,x}\left|\int_t^T\int_K (Z_s^{t,x}-Z_s^{t,x,n})\,\nu(s,X_s,dy)\,ds\right|$$
$$\le (T-t)^{1/2}\sup_{t,x}\nu(t,x,K)^{1/2}\left(\mathbb{E}^{t,x}\int_t^T\int_K |Z_s^{t,x}-Z_s^{t,x,n}|^2\nu(s,X_s,dy)\,ds\right)^{1/2}$$

which tends to zero, by (4.9). So for a subsequence (still denoted v^n) we have $\int_t^T \mathcal{L}v^n(s,X_s)\,ds\to\int_t^T \mathcal{L}v(s,X_s)\,ds$ $\mathbb{P}^{t,x}$-a.s. Note that, according to (2.3), the first jump time T_1^t has exponential law, so the set $\{\omega\in\Omega\,:\,T_1^t(\omega)>T\}$ has positive $\mathbb{P}^{t,x}$ probability, and on this set we have $X_s(\omega)=x$. Taking such an ω we conclude that $\int_t^T \mathcal{L}_s v^n(s,x)\,ds\to\int_t^T \mathcal{L}_s v(s,x)\,ds$.

To prove (4.10) we compute

$$\mathbb{E}^{t,x}\bigg|\int_t^T f\Big(s,X_s,v(s,X_s),v(s,\cdot)-v(s,X_s)\Big)\,ds$$
$$-\int_t^T f^n\Big(s,X_s,v^n(s,X_s),v^n(s,\cdot)-v^n(s,X_s)\Big)\,ds\bigg|$$
$$=\mathbb{E}^{t,x}\left|\int_t^T [f(s,X_s,Y_s^{t,x},Z_s^{t,x})-f^n(s,X_s,Y_s^{t,x,n},Z_s^{t,x,n})]\,ds\right|$$
$$\le\mathbb{E}^{t,x}\int_t^T |f(s,X_s,Y_s^{t,x},Z_s^{t,x})-f^n(s,X_s,Y_s^{t,x},Z_s^{t,x})|\,ds$$
$$+\mathbb{E}^{t,x}\int_t^T |f^n(s,X_s,Y_s^{t,x},Z_s^{t,x})-f^n(s,X_s,Y_s^{t,x,n},Z_s^{t,x,n})|\,ds.$$

In the right-hand side, the first integral tends to zero by monotone convergence. Since f^n is a truncation of f, it satisfies the Lipschitz condition (3.3) with the same constants L, L'

independent of n; therefore the second integral can be estimated by

$$
\begin{aligned}
& L' \, \mathbb{E}^{t,x} \int_t^T |Y_s^{t,x} - Y_s^{t,x,n}| \, ds \\
& \quad + L \, \mathbb{E}^{t,x} \int_t^T \left(\int_K |Z_s^{t,x}(y) - Z_s^{t,x,n}(y)|^2 \, \nu(s, X_s, dy) \right)^{1/2} ds \\
& \le L' \left((T-t) \, \mathbb{E}^{t,x} \int_t^T |Y_s^{t,x} - Y_s^{t,x,n}|^2 \, ds \right)^{1/2} \\
& \quad + L \left((T-t) \, \mathbb{E}^{t,x} \int_t^T \int_K |Z_s^{t,x}(y) - Z_s^{t,x,n}(y)|^2 \, \nu(s, X_s, dy) \, ds \right)^{1/2},
\end{aligned}
$$

which tends to zero, again by (4.9). So for a subsequence (still denoted v^n) we have

$$
\begin{aligned}
& \int_t^T f^n\Big(s, X_s, v^n(s, X_s), v^n(s, \cdot) - v^n(s, X_s)\Big) \, ds \\
& \quad \to \int_t^T f\Big(s, X_s, v(s, X_s), v(s, \cdot) - v(s, X_s)\Big) \, ds
\end{aligned}
$$

$\mathbb{P}^{t,x}$-a.s. Picking an ω in the set $\{\omega \in \Omega \, : \, T_1^t(\omega) > T\}$ as before we conclude that (4.10) holds, and the proof is finished. □

5. Optimal control

5.1. Formulation of the problem

In this section we start again with a measurable space $(K, \mathcal{K})$ and a transition measure ν on K, satisfying (2.1). The process X is constructed as described in Section 2.

The data specifying the optimal control problem that we will address are a measurable space $(U, \mathcal{U})$, called the action (or decision) space, a running cost function l, a (deterministic, finite) time horizon $T > 0$, a terminal cost function g, and another function r specifying the effect of the control process.

For every $t \in [0, T]$ we define an admissible control process, or simply a control, as an $\mathbb{F}^t$-predictable process $(u_s)_{s \in [t,T]}$ with values in U. The set of admissible control processes is denoted by $\mathcal{A}^t$.

We will make the following assumptions.

Hypothesis 5.1. 1. $(U, \mathcal{U})$ is a measurable space.

2. $r : [0, T] \times K \times K \times U \to \mathbb{R}$ is $\mathcal{B}([0, T]) \otimes \mathcal{K} \otimes \mathcal{K} \otimes \mathcal{U}$-measurable and there exist a constant $C_r > 0$ such that

$$
0 \le r(t, x, y, u) \le C_r, \quad t \in [0, T], \; x, y \in K, u \in U. \tag{5.1}
$$

3. $g : K \to \mathbb{R}$ is $\mathcal{K}$-measurable and

$$
\mathbb{E}^{t,x} |g(X_T)|^2 < \infty, \quad t \in [0, T], \; x \in K. \tag{5.2}
$$

4. $l : [0, T] \times K \times U \to \mathbb{R}$ is $\mathcal{B}([0, T]) \otimes \mathcal{K} \otimes \mathcal{U}$-measurable, and there exists $\alpha > 1$ such that for every $t \in [0, T]$, $x \in K$ and $u(\cdot) \in \mathcal{A}^t$ we have

$$
\inf_{u \in U} l(t, x, u) > -\infty, \qquad \mathbb{E}^{t,x} \int_t^T \left| \inf_{u \in U} l(s, X_s, u) \right|^2 ds < \infty, \tag{5.3}
$$

$$\mathbb{E}^{t,x}\left(\int_t^T |l(s, X_s, u_s)|\, ds\right)^{\alpha} < \infty. \tag{5.4}$$

Remark 5.2. We note that the cost functions g and l need not be bounded. Clearly, (5.4) follows from the other assumptions if we assume for instance that $\mathbb{E}^{t,x} \int_t^T |\sup_{u\in U} l(s, X_s, u)|\, ds < \infty$ for all $t \in [0, T]$ and $x \in K$.

To any $(t, x) \in [0, T] \times K$ and any control $u(\cdot) \in \mathcal{A}^t$ we associate a probability measure $\mathbb{P}_u^{t,x}$ on $(\Omega, \mathcal{F})$ by a change of measure of Girsanov type, as we now describe. Recalling the definition of the jump times T_n^t in (2.5) we define, for $s \in [t, T]$,

$$L_s^t = \exp\left(\int_t^s \int_K (1 - r(z, X_z, y, u_z))\, \nu(z, X_z, dy)\, dz\right) \times \prod_{n\geq 1\,:\, T_n^t \leq s} r(T_n^t, X_{T_{n-}^t}, X_{T_n^t}, u_{T_n^t}),$$

with the convention that the last product equals 1 if there are no indices $n \geq 1$ satisfying $T_n^t \leq s$. It is a well-known result that L^t is a nonnegative supermartingale relative to $\mathbb{P}^{t,x}$ and $\mathbb{F}^t$ (see [16] Proposition 4.3, or [4]), solution to the equation

$$L_s^t = 1 + \int_t^s \int_K L_{z-}^t\, (r(z, X_{z-}, y, u_z) - 1)\, q^t(dz\, dy), \quad s \in [t, T].$$

As a consequence of the boundedness assumption in (2.1) it can be proved, using for instance Lemma 4.2 in [8], or [5] Chapter VIII Theorem T11, that for every $\gamma > 1$ we have

$$\mathbb{E}^{t,x}[|L_T^t|^{\gamma}] < \infty, \qquad \mathbb{E}^{t,x} L_T^t = 1, \tag{5.5}$$

and therefore the process L^t is a martingale (relative to $\mathbb{P}^{t,x}$ and $\mathbb{F}^t$). Defining a probability $\mathbb{P}_u^{t,x}$ by $\mathbb{P}_u^{t,x}(d\omega) = L_T^t(\omega)\mathbb{P}^{t,x}(d\omega)$, we introduce the cost functional corresponding to $u(\cdot) \in \mathcal{A}^t$ as

$$J(t, x, u(\cdot)) = \mathbb{E}_u^{t,x}\left[\int_t^T l(s, X_s, u_s)\, ds + g(X_T)\right],$$

where $\mathbb{E}_u^{t,x}$ denotes the expectation under $\mathbb{P}_u^{t,x}$. Taking into account (5.2), (5.4), (5.5) and using the Hölder inequality it is easily seen that the cost is finite for every admissible control. The control problem starting at (t, x) consists in minimizing $J(t, x, \cdot)$ over $\mathcal{A}^t$.

We finally introduce the value function

$$v(t, x) = \inf_{u(\cdot)\in\mathcal{A}^t} J(t, x, u(\cdot)), \quad t \in [0, T],\ x \in K.$$

The previous formulation of the optimal control problem by means of a change of probability measure is classical (see e.g. [11,12,5]). Some comments may be useful at this point.

Remark 5.3. 1. We recall (see e.g. [5], Appendix A2, Theorem T34) that a process u is $\mathbb{F}^t$-predictable if and only if it admits the representation

$$u_s(\omega) = \sum_{n\geq 0} u_s^{(n)}(\omega)\, 1_{T_n^t(\omega) < s \leq T_{n+1}^t(\omega)} \tag{5.6}$$

where for each $n \geq 0$ the mapping $(\omega, s) \mapsto u_s^{(n)}(\omega)$ is $\mathcal{F}_{[t,T_n^t]} \otimes \mathcal{B}([t, \infty))$-measurable. Moreover, $\mathcal{F}_{[t,T_n^t]} = \sigma(T_i^t, X_{T_i^t}, 0 \leq i \leq n)$ (see e.g. [5], Appendix A2, Theorem T30). Thus the fact that controls are predictable processes admits the following interpretation: at each time T_n^t (i.e., immediately after a jump) the controller, having observed the random variables T_i^t, $X_{T_i^t}$ $(0 \le i \le n)$, chooses his current control action, and updates her/his decisions only at time T_{n+1}^t.

2. It can be proved (see [16] Theorem 4.5) that the compensator of $p^t(ds\,dy)$ under $\mathbb{P}_u^{t,x}$ is

$$r(s, X_{s-}, y, u_s)\,\nu(s, X_{s-}, dy)\,ds,$$

whereas the compensator of $p^t(ds\,dy)$ under $\mathbb{P}^{t,x}$ was $\nu(s, X_{s-}, dy)\,ds$. This explains that the choice of a given control $u(\cdot)$ affects the stochastic system multiplying its compensator by $r(s, x, y, u_s)$.

3. We call control law an arbitrary measurable function $\underline{u} : [0, T] \times K \to U$. Given a control law one can define an admissible control u setting $u_s = \underline{u}(s, X_{s-})$. Controls of this form are called feedback controls. For a feedback control the compensator of $p^t(ds\,dy)$ is $r(s, X_{s-}, y, \underline{u}(s, X_{s-}))\,\nu(s, X_{s-}, dy)\,ds$ under $\mathbb{P}_u^{t,x}$. Thus, in this case the controlled system is a Markov process corresponding to the transition measure

$$r(s, x, y, \underline{u}(s, x))\,\nu(s, x, dy) \tag{5.7}$$

instead of $\nu(s, x, dy)$.

We will see later that an optimal control can often be found in feedback form. In this case, even if the original process was time-homogeneous (i.e. ν did not depend on time) the optimal process is not, in general, since the control law may depend on time.

Remark 5.4. Our formulation of the optimal control problem should be compared with another classical approach (see e.g. [14,9]) that we describe informally. One may start with the same running and terminal cost functions l, g as before, but with a jump rate function $\lambda^u(t, x)$ and a jump measure $\pi^u(t, x, A)$ which also depend on the control parameter $u \in U$ as well as on $t \in [0, T]$, $x \in K$, $A \in \mathcal{K}$. Controls only consist in feedback laws, i.e. functions $\underline{u} : [0, T] \times K \to U$. Given any such $\underline{u}(\cdot, \cdot)$ one constructs a jump Markov process, on some probability space, with jump rate function and jump measure given, respectively, by

$$\lambda^{\underline{u}(t,x)}(t, x), \qquad \pi^{\underline{u}(t,x)}(t, x, A),$$

or, equivalently, with rate measure $\lambda^{\underline{u}(t,x)}(t, x)\,\pi^{\underline{u}(t,x)}(t, x, A)$. Thus, together with the initial state and starting time, the choice of a control law $\underline{u}(\cdot, \cdot)$ determines the law of the process and consequently the corresponding cost that we now denote $J(\underline{u})$ (the cost functional being defined in terms of l and g similarly as before).

Under appropriate conditions this optimal control problem can be reduced to our setting. For instance suppose that there exist (fixed) jump rate function and jump measure $\lambda(t, x)$, $\pi(t, x, A)$ (equivalently, a rate measure $\nu(t, x, A) = \lambda(t, x)\,\pi(t, x, A)$) as in Section 2 and that we have the implications

$$\pi(t, x, A) = 0 \;\Rightarrow\; \pi^u(t, x, A) = 0, \qquad \lambda(t, x) = 0 \;\Rightarrow\; \lambda^u(t, x) = 0, \tag{5.8}$$

for every t, x, A, u. Then denoting $y \mapsto r_0(x, t, y, u)$ the Radon–Nikodym derivative of $\pi^u(t, x, \cdot)$ with respect to $\pi(t, x, \cdot)$, whose existence is granted by (5.8), we can define

$$r(t, x, y, u) = r_0(x, t, y, u)\,\frac{\lambda^u(t, x)}{\lambda(t, x)},$$

with the convention that $0/0 = 1$. Suppose also that r is measurable and bounded, so that it satisfies Hypothesis 5.1-2. Then we have the identity

$$r(t,x,y,u)\,\nu(t,x,A) = r(t,x,y,u)\,\lambda(t,x)\,\pi(t,x,A) = \lambda^u(t,x)\,\pi^u(t,x,A),$$

whence it follows that the choice of any control law $\underline{u}(\cdot,\cdot)$, giving rise to the rate measure (5.7), will correspond to a cost equal to $J(\underline{u})$. Therefore the required reduction is in fact possible.

We mention however that, unless some condition like (5.8) is verified, the class of control problems specified by the initial data $\lambda^u(t,x)$ and $\pi^u(t,x,A)$ is in general larger than the one we address in this paper. This can be seen noting that in our framework all the controlled processes have laws which are absolutely continuous with respect to a single uncontrolled process (the one corresponding to $r \equiv 1$) whereas this might not be the case for the rate measures $\lambda^{\underline{u}(t,x)}(t,x)\,\pi^{\underline{u}(t,x)}(t,x,A)$ when $\underline{u}(\cdot,\cdot)$ ranges in the set of all possible control laws: a precise verification might be based on the results of Section 4 in [16] where absolute continuity of the laws of marked point processes is characterized in terms of their compensators.

5.2. *The Hamilton–Jacobi–Bellman equation and the solution to the control problem*

The Hamilton–Jacobi–Bellman (HJB) equation is the following nonlinear Kolmogorov equation: for every $t \in [0,T]$, $x \in K$,

$$v(t,x) = g(x) + \int_t^T \mathcal{L}_s v(s,x)\,ds + \int_t^T f\Big(s,x,v(s,\cdot) - v(s,x)\Big)\,ds, \tag{5.9}$$

where $\mathcal{L}_s$ denotes the generator of the Markov process X as before, and f is the Hamiltonian function defined by

$$f(s,x,z(\cdot)) = \inf_{u\in U}\left\{ l(s,x,u) + \int_K z(y)\,(r(s,x,y,u)-1)\,\nu(s,x,dy)\right\}, \tag{5.10}$$

for $s \in [0,T]$, $x \in K$, $z \in L^2(K,\mathcal{K},\nu(s,x,dy))$. The (possibly empty) set of minimizers will be denoted by

$$\begin{aligned}\Gamma(s,x,z(\cdot)) = \Big\{ u \in U \;:\; f(s,x,z(\cdot)) &= l(s,x,u) \\ &+ \int_K z(y)\,(r(s,x,y,u)-1)\,\nu(s,x,dy)\Big\}.\end{aligned} \tag{5.11}$$

We note that the HJB equation can be written in the alternative form:

$$\begin{cases} \partial_t v(t,x) + \inf_{u\in U}\left\{ l(t,x,u) + \int_K (v(t,y)-v(t,x))\,r(t,x,y,u)\,\nu(t,x,dy)\right\} = 0, \\ v(T,x) = g(x), \quad t\in[0,T],\ x \in K, \end{cases}$$

but we will rather use (5.9) in order to make a connection with previous results. To study the HJB equation we use the notion of solution presented in Definition 4.1. We have the following preliminary result:

Lemma 5.5. *Under Hypothesis 5.1 the assumptions of Hypothesis 3.1 hold true for every $(t,x) \in [0,T]\times K$ and consequently the HJB equation has a unique solution according to Theorem 4.4.*

Proof. Hypothesis 3.1-1 and 4 coincide with (5.2) and (5.3) respectively. The only non trivial verification is the Lipschitz condition (3.3): this follows from the boundedness assumption (5.1) which implies that, for every $s \in [0, T]$, $x \in K$, $z, z' \in L^2(K, \mathcal{K}, \nu(s, x, dy))$, $u \in U$,

$$\begin{aligned}
&\int_K z(y)\,(r(s, x, y, u) - 1)\,\nu(s, x, dy) \\
&\quad \le \int_K |z(y) - z'(y)|\,(r(s, x, y, u) - 1)\,\nu(s, x, dy) \\
&\qquad + \int_K z'(y)\,(r(s, x, y, u) - 1)\,\nu(s, x, dy) \\
&\quad \le (C_r + 1)\,\nu(s, x, K)^{1/2} \left(\int_K |z(y) - z'(y)|^2\,\nu(s, x, dy) \right)^{1/2} \\
&\qquad + \int_K z'(y)\,(r(s, x, y, u) - 1)\,\nu(s, x, dy),
\end{aligned}$$

so that adding $l(s, x, u)$ to both sides and taking the infimum over $u \in U$ it follows that

$$f(s, x, z(\cdot)) \le L \left(\int_K |z(y) - z'(y)|^2\,\nu(s, x, dy) \right)^{1/2} + f(s, x, z'(\cdot))$$

where $L = (C_r + 1)\ \sup_{t,x} \nu(t, x, K)^{1/2} < \infty$; exchanging z and z' we obtain (3.3). $\square$

We can now state our main result.

Theorem 5.6. *Suppose that Hypothesis 5.1 holds.*

Then there exists a unique solution v to the HJB equation. Moreover, for any $t \in [0, T]$, $x \in K$ and any admissible control $u(\cdot) \in \mathcal{A}^t$ we have $v(t, x) \le J(t, x, u(\cdot))$.

Suppose in addition that the sets Γ introduced in (5.11) are non empty and for every $t \in [0, T]$, $x \in K$ one can find an $\mathbb{F}^t$-predictable process $u^{*,t,x}(\cdot)$ in U satisfying

$$u_s^{*,t,x} \in \Gamma(s, X_{s-}, v(s, \cdot) - v(s, X_{s-})), \tag{5.12}$$

$\mathbb{P}^{t,x}$-a.s. for almost all $s \in [t, T]$.

Then $u^{*,t,x}(\cdot) \in \mathcal{A}^t$, it is an optimal control, and $v(t, x)$ coincides with the value function, i.e. $v(t, x) = J(t, x, u^{*,t,x}(\cdot))$.

Remark 5.7. 1. The existence of a process $u^{*,t,x}$ satisfying (5.12) is crucial in order to apply the theorem and solve the optimal control problem in a satisfactory way. It is possible to formulate general sufficient conditions for the existence of $u^{*,t,x}$: see Proposition 5.9 below. The proof of this proposition makes it clear that in general the process $u^{*,t,x}$ may depend on t, x.

2. Suppose that there exists a measurable function $\underline{u} : [0, T] \times K \to U$ such that

$$\begin{aligned}
&l(s, x, \underline{u}(s, x)) + \int_K \Big(v(s, y) - v(s, x)\Big) \Big(r(s, x, y, \underline{u}(s, x)) - 1\Big)\,\nu(s, x, dy) \\
&\quad = \inf_{u \in U} \left\{ l(s, x, u) + \int_K \Big(v(s, y) - v(s, x)\Big) \Big(r(s, x, y, u) - 1\Big)\,\nu(s, x, dy) \right\},
\end{aligned} \tag{5.13}$$

for all $s \in [0, T]$, $x \in K$, where v denotes the solution of the HJB equation. We note that in specific situations it is possible to compute explicitly the function $\underline{u}$. Then the process

$$u_s^{*,t,x} = \underline{u}(s, X_{s-})$$

satisfies (5.12) and is therefore optimal. Note that in this case the optimal control is in feedback form and the feedback law $\underline{u}$ is the same for every starting point (t, x).

Proof. Existence and uniqueness of a solution to the HJB equation, in the sense of Definition 4.1, is a consequence of Lemma 5.5 and Theorem 4.4. All the other statements of the theorem are immediately deduced from the following identity, sometimes called the fundamental relation: for any $t \in [0, T]$, $x \in K$ and any admissible control $u(\cdot) \in \mathcal{A}^t$,

$$\begin{aligned} v(t, x) = J(t, x, u(\cdot)) + \mathbb{E}_u^{t,x} \int_t^T \Big\{ f\Big(s, X_s, v(s, \cdot) - v(s, X_s)\Big) - l(s, X_s, u_s) \\ - \int_K (v(s, y) - v(s, X_s))\, (r(s, X_s, y, u_s) - 1)\, \nu(s, X_s, dy) \Big\}\, ds. \end{aligned} \tag{5.14}$$

Indeed, the term in curly brackets $\{\dots\}$ is non positive by the definition of the Hamiltonian function (5.10), and it equals zero when $u(\cdot)$ coincides with $u^{*,t,x}(\cdot)$ by (5.11).

To finish the proof we show that (5.14) holds. Applying the Ito formula (2.9) to the process $v(s, X_s)$, $s \in [t, T]$, and proceeding as in the proof of Theorem 4.4 we arrive at equality (4.7), that we write for $s = T$: recalling that $v(T, x) = g(x)$ for all $x \in K$ we obtain

$$\begin{aligned} v(t, x) = g(X_T) + \int_t^T f\Big(s, X_s, v(s, X_s), v(s, \cdot) - v(s, X_s)\Big)\, ds \\ - \int_t^T \int_K \Big(v(s, y) - v(s, X_{s-})\Big)\, q^t(ds\, dy). \end{aligned}$$

Since $q^t(ds\, dy) = p^t(ds\, dy) - \nu(s, X_{s-}, dy)\, ds$, we have, adding and subtracting some terms,

$$\begin{aligned} v(t, x) &= g(X_T) + \int_t^T l(s, X_s, u_s)\, ds \\ &+ \int_t^T \Big\{ f\Big(s, X_s, v(s, X_s), v(s, \cdot) - v(s, X_s)\Big) - l(s, X_s, u_s) \\ &- \int_K (v(s, y) - v(s, X_s))\, (r(s, X_s, y, u_s) - 1) \nu(s, X_s, dy) \Big\}\, ds \\ &- \int_t^T \int_K \Big(v(s, y) - v(s, X_{s-})\Big) \Big(p^t(ds\, dy) - r(s, X_s, y, u_s) \nu(s, X_{s-}, dy)\, ds\Big). \end{aligned}$$

Now (5.14) follows by taking the expectation with respect to $\mathbb{P}_u^{t,x}$, provided we can show that the last term (the stochastic integral) has mean zero with respect to $\mathbb{P}_u^{t,x}$. Since the $\mathbb{P}_u^{t,x}$-compensator of $p^t(ds\, dy)$ is $r(s, X_{s-}, y, u_s) \nu(s, X_{s-}, dy)\, ds$, it is enough to verify that the integrand $v(s, y) - v(s, X_{s-})$ belongs to $\mathcal{L}^1(p^t)$ (with respect to $\mathbb{P}_u^{t,x}$), i.e. that the following integral, denoted I, is finite:

$$I = \mathbb{E}_u^{t,x} \int_t^T \int_K |v(s, y) - v(s, X_{s-})|\, r(s, X_s, y, u_s) \nu(s, X_s, dy)\, ds.$$

We have, by (5.1) and the Hölder inequality,

$$I \le C_r\, \mathbb{E}_u^{t,x} \int_t^T \int_K |v(s, y) - v(s, X_s)|\, \nu(s, X_s, dy)\, ds$$

$$
\begin{aligned}
&= C_r\, \mathbb{E}^{t,x}\left[L_T^t \int_t^T \int_K |v(s,y) - v(s,X_s)|\, \nu(s,X_s,dy)\, ds\right] \\
&\le C_r \left(\mathbb{E}^{t,x}|L_T^t|^2\right)^{\frac{1}{2}} \left(\mathbb{E}^{t,x}\left|\int_t^T \int_K |v(s,y) - v(s,X_s)|\, \nu(s,X_s,dy)\, ds\right|^2\right)^{\frac{1}{2}} \\
&\le C_r \left(\mathbb{E}^{t,x}|L_T^t|^2\right)^{\frac{1}{2}} \\
&\quad \times \left((T-t)\sup_{t,x} \nu(t,x,K) \mathbb{E}^{t,x} \int_t^T \int_K |v(s,y) - v(s,X_s)|^2\, \nu(s,X_s,dy)\, ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

Recalling (5.5) and the integrability condition in Definition 4.1 we conclude that $I < \infty$ and this finishes the proof. □

As a consequence of Theorem 4.4 we can also conclude that the value function and the optimal control law can also be represented by means of the solution $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$ of the following BSDE: $\mathbb{P}^{t,x}$-a.s.

$$
Y_s^{t,x} + \int_s^T \int_K Z_r^{t,x}(y)\, q^t(dr\, dy) = g(X_T) + \int_s^T f(r, X_r, Z_r^{t,x}(\cdot))\, dr, \quad s \in [t,T],
$$

with fixed $(t,x) \in [0,T] \times K$ and the generator equal to the Hamiltonian function f. As before, equalities (5.15) below are understood as explained in Remark 4.5.

Corollary 5.8. *Under the assumptions of Theorem 5.6, for every $(t,x) \in [0,T] \times K$ we have*

$$
Y_s^{t,x} = v(s, X_s), \qquad Z_s^{t,x}(y) = v(s,y) - v(s, X_{s-}). \tag{5.15}
$$

In particular, the value function and an optimal control are given by the formulae

$$
v(t,x) = Y_t^{t,x}, \qquad u_s^{*,t,x} = \underline{u}(s, X_{s-}, Z_s^{t,x}(\cdot)).
$$

As mentioned before, general conditions can be formulated for the existence of a process $u^{*,t,x}$ satisfying (5.12), hence of an optimal control. This is done in the following proposition, by means of an appropriate selection theorem.

Proposition 5.9. *In addition to the assumptions in Hypothesis 5.1, suppose that U is a compact metric space with its Borel σ-algebra $\mathcal{U}$ and that the functions $r(s,x,\cdot), l(s,x,\cdot) : U \to \mathbb{R}$ are continuous for every $s \in [0,T]$, $x \in K$. Then a process $u^{*,t,x}$ satisfying (5.12) exists and all the conclusions of Theorem 5.6 hold true.*

Proof. We fix t, x and consider the measure $\mu(d\omega\, ds) = \mathbb{P}^{t,x}(d\omega)\, ds$ on the product σ-algebra $\mathcal{G} := \mathcal{P}^t \otimes \mathcal{B}([t,T])$. Let $\bar{\mathcal{G}}$ denote its μ-completion and consider the complete measure space $(\Omega \times [t,T], \bar{\mathcal{G}}, \mu)$. Let v denote the solution of the HJB equation. Define a map $F : \Omega \times [0,T] \times U \to \mathbb{R}$ setting

$$
\begin{aligned}
F(\omega, s, u) &= l(s, X_{s-}(\omega), u) \\
&\qquad + \int_K \Big(v(s,y) - v(s, X_{s-}(\omega))\Big) \Big(r(s, X_{s-}, y, u) - 1\Big) \nu(s, X_{s-}(\omega), dy).
\end{aligned}
$$

Then $F(\cdot,\cdot,u)$ is $\bar{\mathcal{G}}$-measurable for every $u \in U$, and it is easily verified that $F(\omega,s,\cdot)$ is continuous for every $(\omega,s) \in \Omega \times [0,T]$. By a classical selection theorem (see [1], Theorems

8.1.3 and 8.2.11) there exists a function $u^{*,t,x} : \Omega \times [t, T] \to U$, measurable with respect to $\bar{\mathcal{G}}$ and $\mathcal{U}$, such that $F(\omega, s, u^{*,t,x}(\omega, s)) = \min_{u \in U} F(\omega, s, u)$ for every $(\omega, s) \in \Omega \times [t, T]$, so that (5.12) holds true for every (ω, s). Note that $u^{*,t,x}$ may depend on t, x because μ does. After modification on a set of μ-measure zero, the function $u^{*,t,x}$ can be made measurable with respect to $\mathcal{P}^t \otimes \mathcal{B}([t, T])$ and $\mathcal{U}$, and (5.12) still holds, as it is understood as an equality for μ-almost all (ω, s). □

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