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The real Schur decomposition estimates Lyapunov characteristic exponents with multiplicity greater than one

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Abstract

Lyapunov Characteristic Exponents are indicators of the nature and of the stability properties of solutions of differential equations. The estimation of Lyapunov exponents of algebraic multiplicity greater than 1 is troublesome. In this work, we intuitively derive an interpretation of higher multiplicity Lyapunov exponents in forms that occur in simple linear time invariant problems of engineering relevance. We propose a method to determine them from the real Schur decomposition of the state transition matrix of the linear, non-autonomous problem associated with the fiducial trajectory. So far, no practical way has been found to formulate the method as an algorithm capable of mitigating over- or underflow in the numerical computation of the state transition matrix. However, this interesting approach in some practical cases is shown to provide quicker convergence than usual methods like the discrete QR and the continuous QR and SVD methods when Lyapunov exponents with multiplicity greater than one are present.

Keywords

Lyapunov Characteristic Exponents, Multiplicity, Real Schur Decomposition, Stability, Dynamics

Introduction

Lyapunov Characteristic Exponents (LCE) or, in short, Lyapunov Exponents are indicators of the nature and of the stability properties of solutions of differential equations (see[1;2] and references therein). Their definition stems from the seminal work on stability published in 1892 by Aleksandr M. Lyapunov[3], but only in 1968, thanks to the work by Oseledec, and specifically by his general Noncommutative Ergodic Theorem[4], their theory was "adapted to the needs of the theory of dynamical system"[2]. In 1980, Benettin et al.[2] showed that all LCEs could be estimated under certain assumptions; before their contribution, only methods capable of estimating the largest LCE were available. In the same work, Benettin et al. laid the foundations of modern methods for LCE estimation. Most of them, noticeably the discrete QR method and the continuous QR and SVD methods, were formulated by several authors around 1985 (e.g. Eckmann and Ruelle[5]). A robust method for LCE estimation from time series was proposed by Wolf et al.[6]. A detailed review of the topic, including computational methods, can be found for example in the work of Geist et al.[7] and Dieci and Van Vleck[8].

It is known that the estimation of non-simple LCEs, of algebraic multiplicity greater than 1, is troublesome. All the literature known to the authors insists on the importance of LCEs being distinct[8]. Being distinct may not be enough of LCEs; only the so-called integral separation property guarantees the stability of their numerical estimation (see for example[9;10]). The topic of multiple LCEs is the subject of active research efforts[11].

In this work, we intuitively derive an interpretation of higher multiplicity LCEs in forms that occur in simple linear time invariant problems. We propose a method to determine them from a robust (with respect to multiplicity) decomposition of the state transition matrix (STM) of the linear, non-autonomous problem associated with the fiducial trajectory, based on unitary similarity transformations.

Unfortunately, so far no practical way has been found to formulate the method as an algorithm capable of mitigating over- or underflow in the numerical computation of the STM, so LCE estimation is not yet robust per se. However, it is the authors' opinion that the definition of the method represents itself an interesting improvement for this type of analysis.

Furthermore, in some practical cases it is shown to provide quicker convergence than usual methods like the discrete QR, and definitely overcomes the limitation of the continuous QR and SVD methods when LCEs with multiplicity greater than one are present[8].

Lyapunov Characteristic Exponents

Consider a Cauchy problem, defined without loss of generality in terms of a system of ordinary differential equations in explicit form,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}, t\right), \tag{1}$$

with $\boldsymbol{f} : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$, which can be nonlinear and non-autonomous, i.e. explicitly dependent on time, and a set of initial conditions

$$\boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2}$$

where we arbitrarily consider 0 as the origin of the time t. We assume that the corresponding solution, $\boldsymbol{x}(t)$, called fiducial trajectory, is known for $t \geq 0$, either analytically or numerically.

We define a corresponding linear, time variant problem

$$\dot{\boldsymbol{y}} = \mathbf{J}\left(\boldsymbol{x}(t), t\right)\, \boldsymbol{y}, \tag{3}$$

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where $\mathbf{J}(\boldsymbol{x}, t)$ (or, in short, $\mathbf{J}(t)$) is the partial derivative of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$, evaluated on the fiducial trajectory $\boldsymbol{x}(t)$. The solution of this problem with initial conditions corresponding to the identity matrix yields the STM $\mathbf{Y}(t) = \mathbf{\Phi}(t, 0)$ from 0 to an arbitrary time t,

$$\dot{\mathbf{Y}} = \mathbf{J}\mathbf{Y} \qquad \mathbf{Y}(0) = \mathbf{I}. \tag{4}$$

According to the Ostrogradskiĭ-Jacobi-Liouville formula[1], the determinant of $\mathbf{Y}(t)$ (the *Wronskian determinant* of the independent solutions that constitute $\mathbf{Y}(t)$) is

$$\det\left(\mathbf{Y}(t)\right) = \det\left(\mathbf{Y}(0)\right) \mathrm{e}^{\int_0^t \mathrm{tr}(\mathbf{J}(\tau))\, \mathrm{d}\tau}, \tag{5}$$

where $\mathrm{tr}(\cdot)$ is the trace operator. Thus, the Wronskian never vanishes when $\mathbf{J}(t)$ is regular in $[0, t]$, since $\mathbf{Y}(0) \equiv \mathbf{I}$. The Wronskian geometrically represents the evolution in time of the N-dimensional volume of an infinitesimal portion of the state space.

The LCEs are usually defined as the limit for $t \to \infty$ of the logarithm of the singular values of the STM, divided by the time itself (see for example[7]),

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log\left(\mathrm{svd}\left(\mathbf{Y}(t)\right)\right). \tag{6}$$

Geometrically, one may interpret them as the exponents that determine the rate of either expansion or contraction of an infinitesimal volume of the state space along principal directions. Such volume envelopes the evolution of a perturbation of the fiducial trajectory. If the volume contracts along all principal directions, the fiducial trajectory is asymptotically stable; if it expands along at least one direction, it is unstable or describes a chaotic motion. An n-dimensional LCO is expected when the largest n LCEs are exactly zero.

Alternative definitions involve the real part of the eigenvalues of the STM,

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \mathrm{Re}\left(\log\left(\mathrm{eig}\left(\mathbf{Y}(t)\right)\right)\right), \tag{7}$$

or the diagonal coefficients of the upper-triangular matrix $\mathbf{R}$ that results from the QR decomposition of the STM,

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log\left(\mathrm{diag}\left(\mathrm{qr}\left(\mathbf{Y}(t)\right)\right)\right). \tag{8}$$

Notice that all definitions converge to the same LCEs for $t \to \infty$; in general, the estimates differ for a finite value of t.

The definition based on the QR decomposition of the STM is at the roots of the so-called discrete and continuous QR algorithms for the estimation of LCEs[7;8], whereas the definition based on the SVD is at the roots of the continuous SVD algorithm[7;8]. The interested reader is directed to those references for more details on the algorithms themselves.

The definition based on the eigenvalues of the STM has little practical use; it is worth noticing its resemblance with the definition of the stability indicators provided by the Floquet-Lyapunov method for linear, time periodic (LTP) systems,

$$\lambda_i = \frac{1}{T} \mathrm{Re}\left(\log\left(\mathrm{eig}\left(\mathbf{Y}(T)\right)\right)\right), \tag{9}$$

where $\mathbf{Y}(T)$, the STM over one period T, is the so-called monodromy matrix.

The definition of the LCEs requires the evaluation of a limit for $t \to \infty$. In practice, their numerical computation requires one to continue the estimation until convergence. The algorithms proposed in the literature do not make direct use of the STM, since it is destined to either over- or underflow at a pace that depends on the value of the LCEs themselves (the larger the LCEs in modulus, the sooner the matrix over- or underflows). The discrete QR method operates on the incremental STM, i.e. the STM across a limited time interval. The continuous SVD and QR methods operate on the time derivative of the decomposition of the STM, and take measures to mitigate over- and underflow.

The Problem of Multiplicity

LCEs, like eigenvalues, can be algebraically simple, i.e. occur only once, or multiple. When an eigenvalue χ of matrix $\mathbf{M}$ is algebraically multiple, i.e. it is a multiple root of the characteristic polynomial of $\mathbf{M}$ with multiplicity m, its geometric multiplicity is the size n of the nullspace of matrix $(\chi\mathbf{I} - \mathbf{M})$. When $n < m$, the matrix cannot be diagonalized.

From the previous discussion, when the multiplicity of an LCE is greater than one, its rate of growth is the same along multiple principal directions. Some of the previously mentioned LCE definitions yield slow convergence rates when the multiplicity of some LCEs is greater than 1. The algorithms available in the literature either fail or converge slowly, according to the convergence properties of the above mentioned definitions.

In order to exemplify the issues related to LCE estimation in presence of multiplicity greater than one, consider the simple linear, time invariant problem of a damped oscillator of mass m, damping characteristic c and stiffness k,

$$\ddot{q} + 2\xi\omega\dot{q} + \omega^2 q = 0, \tag{10}$$

with $\omega = \sqrt{k/m}$ and $\xi = c/(2\sqrt{km})$. The LCEs correspond to the real part of the roots χ of the characteristic polynomial,

$$\chi = \omega\left(-\xi \pm \sqrt{\xi^2 - 1}\right) = \omega\left(-\xi \pm i\sqrt{1 - \xi^2}\right) \tag{11}$$

When the absolute value of the damping factor is below its critical value, $-1 < \xi < 1$, the roots are distinct, but the LCE $\lambda = -\xi\omega$, which corresponds to the real part of the roots, has multiplicity 2. When the damping is critical, $|\xi| = 1$, the roots are real coincident, i.e. the root $\chi = -\omega$ (respectively $\chi = \omega$ for $\xi = -1$) has multiplicity 2 and directly corresponds to the LCE, which also has multiplicity 2.

In state space form, the damped oscillator problem is

$$\dot{\boldsymbol{x}} = \left\{\begin{array}{c} \dot{q} \\ \ddot{q} \end{array}\right\} = \left[\begin{array}{cc} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{array}\right] \left\{\begin{array}{c} q \\ \dot{q} \end{array}\right\} = \mathbf{A}\boldsymbol{x}. \tag{12}$$

Its STM is

$$\mathbf{Y}(t) = \mathrm{e}^{\mathbf{A}t}. \tag{13}$$

Subcritical Damping

When damping is subcritical, the coefficients of the STM are

$$y_{11} = \left(C(t) + \frac{\xi}{\sqrt{1-\xi^2}} S(t)\right) \mathrm{e}^{-\xi\omega t} \tag{14a}$$

$$y_{12} = \frac{1}{\sqrt{1-\xi^2}\omega} S(t) \mathrm{e}^{-\xi\omega t} \tag{14b}$$

$$y_{21} = -\frac{\omega}{\sqrt{1-\xi^2}} S(t) \mathrm{e}^{-\xi\omega t} \tag{14c}$$

$$y_{22} = \left(C(t) - \frac{\xi}{\sqrt{1-\xi^2}} S(t)\right) \mathrm{e}^{-\xi\omega t}, \tag{14d}$$

with

$$C(t) = \cos\left(\sqrt{1-\xi^2}\omega t\right) \tag{15a}$$

$$S(t) = \sin\left(\sqrt{1-\xi^2}\omega t\right). \tag{15b}$$

Eigenvalues The eigenvalues of the STM are the roots of the characteristic polynomial

$$\chi^2 - \chi \cdot \operatorname{tr}(\mathbf{Y}) + \det(\mathbf{Y}) = 0, \tag{16}$$

which can be written as

$$\chi_{1,2} = \mathrm{e}^{\pm i\sqrt{1-\xi^2}\omega t} \mathrm{e}^{-\xi\omega t}. \tag{17}$$

Thus, according to Eq. (7),

$$\lambda_{1,2} = \lim_{t\to\infty} \frac{1}{t} \operatorname{Re}\left(\log\left(\mathrm{e}^{\pm i\sqrt{1-\xi^2}\omega t} \mathrm{e}^{-\xi\omega t}\right)\right) = -\xi\omega. \tag{18}$$

The definition based on the eigenvalues directly yields the expected value, regardless of the value of t.

SVD The singular values of the STM are

$$\sigma_{1,2} = \sqrt{\frac{a_1(t)}{2} \pm \sqrt{\left(\frac{a_1(t)}{2}\right)^2 - a_2(t)}}, \tag{19}$$

with

$$a_1(t) = \operatorname{tr}\left(\mathbf{Y}\mathbf{Y}^T\right) \tag{20a}$$

$$a_2(t) = \det(\mathbf{Y})^2, \tag{20b}$$

which can be rewritten as

$$\sigma_{1,2} = \hat{\sigma}(t)^{\pm 1} \mathrm{e}^{-\xi\omega t}, \tag{21}$$

with $\hat{\sigma}(t)$ periodic, limited and positive. Thus, according to Eq. (6),

$$\begin{aligned} \lambda_{1,2} &= \lim_{t\to\infty} \frac{1}{t} \log\left(\hat{\sigma}(t)^{\pm 1} \mathrm{e}^{-\xi\omega t}\right) \\ &= \lim_{t\to\infty} \pm \frac{\log(\hat{\sigma}(t))}{t} - \xi\omega \\ &= -\xi\omega. \end{aligned} \tag{22}$$

The definition based on the SVD converges to the expected value.

QR The QR decomposition of the STM is

$$\mathbf{Y} = \mathbf{Q}\mathbf{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \tag{23}$$

The diagonal coefficients are

$$r_{11} = \sqrt{y_{11}^2 + y_{21}^2} \tag{24a}$$

$$r_{22} = \frac{\det(\mathbf{Y})}{r_{11}}; \tag{24b}$$

r_{12} and α are inessential. The diagonal coefficients can be rewritten as

$$r_{11,22} = \hat{r}(t)^{\pm 1} \mathrm{e}^{-\xi\omega t}, \tag{25}$$

with $\hat{r}(t)$ periodic, limited and positive. Thus, according to Eq. (8),

$$\begin{aligned} \lambda_{1,2} &= \lim_{t\to\infty} \frac{1}{t} \log\left(\hat{r}(t)^{\pm 1} \mathrm{e}^{-\xi\omega t}\right) \\ &= \lim_{t\to\infty} \pm \frac{\log(\hat{r}(t))}{t} - \xi\omega \\ &= -\xi\omega. \end{aligned} \tag{26}$$

The definition based on the QR decomposition converges to the expected value.

Critical Damping

When damping is critical, the coefficients of the STM are

$$y_{11} = (1 + \omega t)\, \mathrm{e}^{-\omega t} \tag{27a}$$

$$y_{12} = t\mathrm{e}^{-\omega t} \tag{27b}$$

$$y_{21} = -\omega^2 t\mathrm{e}^{-\omega t} \tag{27c}$$

$$y_{22} = (1 - \omega t)\, \mathrm{e}^{-\omega t}, \tag{27d}$$

where it is understood that a negative value is used for ω in case $\xi = -1$.

Eigenvalues The eigenvalues of the STM are the roots of its characteristic polynomial, which in this case yields

$$\chi_{1,2} = \mathrm{e}^{-\omega t}. \tag{28}$$

Thus, according to Eq. (7),

$$\lambda_{1,2} = \lim_{t\to\infty} \frac{1}{t} \operatorname{Re}\left(\log\left(\mathrm{e}^{i(1\pm 1)\pi/2} \mathrm{e}^{-\omega t}\right)\right) = -\omega. \tag{29}$$

The definition based on the eigenvalues directly yields the expected value, regardless of the value of t. It is worth noticing that in this case the STM is not diagonalizable.

SVD The function $\hat{\sigma}(t)$ the diagonal coefficients of the SVD of the STM depend on is now

$$\hat{\sigma}(t) = \sqrt{1 + \frac{t^2}{2}\left(1+\omega^2\right)^2 + \frac{t\left(1+\omega^2\right)}{2}\sqrt{4 + t^2\left(1+\omega^2\right)^2}}, \tag{30}$$

which is no longer periodic, but still positive, and asymptotically growing as $(1 + \omega^2)t$. Thus, according to

Eq. (6), Eq. (22) still holds, namely

$$\begin{aligned}\lambda_{1,2} &= \lim_{t\to\infty} \frac{1}{t} \log\left(\hat{\sigma}(t)^{\pm 1} \mathrm{e}^{-\omega t}\right) \\ &= \lim_{t\to\infty} \pm \frac{\log\left(\hat{\sigma}(t)\right)}{t} - \omega \\ &= -\omega. \end{aligned} \tag{31}$$

As a consequence, the definition based on the SVD again converges to the expected value.

QR The function $\hat{r}(t)$ the diagonal coefficients of the QR decomposition of the STM depend on is now

$$\hat{r}(t) = \sqrt{1 + 2\omega t + \omega^2(1+\omega^2)t^2}, \tag{32}$$

which is no longer periodic but still positive, and asymptotically growing as $\omega\sqrt{1+\omega^2}t$. Thus, according to Eq. (8), Eq. (26) still holds, namely

$$\begin{aligned}\lambda_{1,2} &= \lim_{t\to\infty} \frac{1}{t} \log\left(\hat{r}(t)^{\pm 1} \mathrm{e}^{-\omega t}\right) \\ &= \lim_{t\to\infty} \pm \frac{\log\left(\hat{r}(t)\right)}{t} - \omega \\ &= -\omega. \end{aligned} \tag{33}$$

As a consequence, the definition based on the QR decomposition again converges to the expected value.

Discussion of Results

It is clear from the example in the previous section that both the definitions based on the SVD and the QR decomposition only converge to the exact value for very large values of t, although the desired exact solution is directly and immediately contained in the eigenvalues of the STM.

It is worth noticing that the sum of the LCE estimates always yields the correct value regardless of the value of t. Unfortunately, one could easily prove that the latter is true also in case of real, distinct eigenvalues of matrix $\mathbf{J}$. As a consequence, strategies based on considering the average of two close enough LCE estimates may not be satisfactory.

The problem of LCE multiplicity is twofold:

a) assuming that we know the multiplicity of an LCE is greater than 1, we do not know how to tell the algorithm how to exploit this information; this aspect is discussed below;
b) in a generic problem, we also need some criterion to understand that the multiplicity is (going to be) greater than 1; this is discussed in the subsequent Section.

Let us consider these problems one at a time. The previously mentioned interpretation of single LCEs given in[2] as the rate of expansion along principal directions of the state space, and of clusters of LCEs as the rate of expansion of a sub-volume of the state space, suggests that when the multiplicity of an LCE is greater than 1, the rate of expansion along the principal directions related to the cluster of LCEs should be determined simultaneously for all of them.

Consider, for example, the definition of the LCEs in terms of the eigenvalues of the STM: the evolution volume is given by the determinant of the STM, which can be expressed as the product of its eigenvalues ($\chi_1 \cdot \chi_2 = \mathrm{e}^{-2\xi\omega t}$ for the damped oscillator case). The LCEs are the real part of the logarithm of the volume, divided by t and by the multiplicity (2 in the case at hand) and taken to the limit, yielding

$$\begin{aligned}\lambda_{1,2} &= \lim_{t\to\infty} \frac{1}{2t} \mathrm{Re}\left(\log\left(\chi_1 \cdot \chi_2\right)\right) \\ &= \lim_{t\to\infty} \frac{1}{2t} \log\left(\mathrm{e}^{-2\xi\omega t}\right) \\ &= -\xi\omega. \end{aligned} \tag{34}$$

Similarly, considering the definition of the LCEs in terms of the SVD of the STM, the evolution volume is represented by the product of the singular values,

$$\begin{aligned}\lambda_{1,2} &= \lim_{t\to\infty} \frac{1}{2t} \log\left(\sigma_1 \cdot \sigma_2\right) \\ &= \lim_{t\to\infty} \frac{1}{2t} \log\left(\mathrm{e}^{-2\xi\omega t}\right) \\ &= -\xi\omega, \end{aligned} \tag{35}$$

whereas in the case of the QR decomposition of the STM, the evolution volume is represented by the product of the diagonal elements of matrix $\mathbf{R}$,

$$\begin{aligned}\lambda_{1,2} &= \lim_{t\to\infty} \frac{1}{2t} \log\left(r_{11} \cdot r_{22}\right) \\ &= \lim_{t\to\infty} \frac{1}{2t} \log\left(\mathrm{e}^{-2\xi\omega t}\right) \\ &= -\xi\omega. \end{aligned} \tag{36}$$

So, assuming that we can detect when two or more LCEs end up being identical, i.e. the multiplicity of an LCE is going to be greater than 1, we can improve the convergence of their estimation by computing it through the sub-volume of the related principal directions.

We obtain exactly the same result, with $\xi = \pm 1$, when considering the critical damping case.

The Real Schur Decomposition

It is now time to address the second point, about how to detect whether two or more LCEs are going to converge to the same value. To this end, the so-called real Schur decomposition can provide useful insight.

We recall the real Schur decomposition as the real orthogonal transformation $\mathbf{U}$, with $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, that reduces a real STM in real, quasi-triangular form,

$$\mathbf{U}^T\mathbf{Y}\mathbf{U} = \mathbf{S}. \tag{37}$$

Matrix $\mathbf{S}$ is block upper triangular. Since the transformation operated by $\mathbf{U}$ is a similarity, the eigenvalues of the diagonal blocks and of matrix $\mathbf{Y}$ are the same. Further details can be found in[12].

Diagonal 1×1 blocks (i.e. scalars) correspond to real eigenvalues of $\mathbf{Y}$ with multiplicity equal to 1. Diagonal 2×2 blocks with equal diagonal values correspond to complex conjugated eigenvalues of matrix $\mathbf{Y}$. Upper triangular blocks of size greater than 1 with identical diagonal entries correspond to eigenvalues of $\mathbf{Y}$ with multiplicity greater than 1. The subspaces spanned by the block columns of $\mathbf{U}$ can be ordered arbitrarily; as a consequence, the eigenvalues of $\mathbf{Y}$ can be sorted at leisure. LCEs are thus estimated from the eigenvalues of matrix $\mathbf{Y}$ through its real Schur decomposition, matrix $\mathbf{S}$.

The complex Schur decomposition of the STM could be used instead; it yields an upper triangular matrix whose diagonal entries are the complex eigenvalues. LCEs would then be estimated using Eq. (7). Incidentally, this is a popular algorithm for the computation of eigenvalues, used for example by Matlab, Octave, and other widespread mathematical environments.

Consider now the real Schur decomposition of the STM in the previous example with subcritical damping; after defining

$$\mathbf{U} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} C_\phi & -S_\phi \\ S_\phi & C_\phi \end{bmatrix}, \tag{38}$$

it yields

$$s_{11} = C_\phi^2 y_{11} + C_\phi S_\phi (y_{21} + y_{12}) + S_\phi^2 y_{22} \tag{39a}$$
$$s_{12} = C_\phi^2 y_{12} - C_\phi S_\phi (y_{11} - y_{22}) - S_\phi^2 y_{21} \tag{39b}$$
$$s_{21} = C_\phi^2 y_{21} - C_\phi S_\phi (y_{11} - y_{22}) - S_\phi^2 y_{12} \tag{39c}$$
$$s_{22} = C_\phi^2 y_{22} - C_\phi S_\phi (y_{21} + y_{12}) + S_\phi^2 y_{11}. \tag{39d}$$

By imposing $s_{11} = s_{22}$, to solve the indetermination in the orthogonal vectors that constitute the subspace of the eigenvalues with equal evolution rate, one obtains

$$\phi = -\frac{1}{2}\tan^{-1}\left(\frac{y_{11} - y_{22}}{y_{21} + y_{12}}\right). \tag{40}$$

In the subcritical damping case, this yields

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{2\xi\omega}{\omega^2 - 1}\right) \tag{41}$$

and

$$s_{11} = s_{22} = \mathrm{e}^{-\xi\omega t}\cos\left(\sqrt{1-\xi^2}\omega t\right) \tag{42a}$$

$$s_{12} = \mathrm{e}^{-\xi\omega t}\left(1 - \frac{2\xi^2\sqrt{4\omega^2\xi^2 + (\omega^2-1)^2}}{(\omega^2-1)^2 + 4\omega^2\xi^2 - (\omega^2-1)\sqrt{4\omega^2\xi^2 + (\omega^2-1)^2}}\right)\frac{\omega}{\sqrt{1-\xi^2}}\sin\left(\sqrt{1-\xi^2}\omega t\right) \tag{42b}$$

$$s_{21} = -\mathrm{e}^{-\xi\omega t}\left(1 + \frac{2\omega^2\xi^2\sqrt{4\omega^2\xi^2 + (\omega^2-1)^2}}{(\omega^2-1)^2 + 4\omega^2\xi^2 - (\omega^2-1)\sqrt{4\omega^2\xi^2 + (\omega^2-1)^2}}\right)\frac{1}{\omega\sqrt{1-\xi^2}}\sin\left(\sqrt{1-\xi^2}\omega t\right) \tag{42c}$$

whereas, in the critical damping case,

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{2\omega}{\omega^2 - 1}\right) \tag{43}$$

(namely, Eq. (41) with $\xi = 1$), and

$$s_{11} = s_{22} = \mathrm{e}^{-\omega t} \tag{44a}$$
$$s_{12} = \left(1 + \omega^2\right) t\mathrm{e}^{-\omega t} \tag{44b}$$
$$s_{21} = 0. \tag{44c}$$

Of course, in both cases

$$\det(\mathbf{S}) = \det(\mathbf{Y}) = \mathrm{e}^{-2\xi\omega t} \tag{45}$$

(with $\xi = \pm 1$ in the critical damping case.)

By giving up the unitarity of the transformation matrices $\mathbf{U}$, the decomposition can be further improved, bringing matrix $\mathbf{S}$ to take block-diagonal form (i.e. as close as possible to diagonal form). In this case, the blocks can either be of unit size (independent real eigenvalues), full 2×2 (complex conjugated eigenvalues), upper triangular (eigenvalues with higher multiplicity), upper block-triangular (complex conjugated eigenvalues with higher multiplicity)[12].

The key point is that only when a pair of complex conjugated eigenvalues is expected is the s_{21} element of matrix $\mathbf{S}$ nonzero. When reduced to block diagonal form and the s_{21} element is zero, the s_{12} element is nonzero only when identical eigenvalues with multiplicity 2 are expected. Thus, the real Schur decomposition provides an indication that multiplicity can (and should) be exploited to improve the convergence of LCE estimates.

To summarize, the real Schur decomposition:

- operates on real numbers;
- is a similarity transformation, thus the resulting matrix has the same eigenvalues of the STM;
- robustly detects the existence of, and computes, real eigenvalues with multiplicity greater than 1
- directly provides the square sub-blocks whose determinant is required for the computation of LCE estimates with multiplicity equal to 2 because the related eigenvalues are complex conjugated.

A major drawback, so far, is that no practical manner has been found to implement it in a form that prevents over- or underflow of the transformed matrix. However, one should consider that the proposed method appears to provide a quicker convergence of LCE estimates than conventional ones. This is particularly true for problems characterized by "small" damping and thus potentially subject to higher multiplicity of the LCEs. As a consequence, it requires the integration of the STM for shorter times, which may contribute to mitigating the risk of over- or underflow.

Numerical Examples

In this Section we present three numerical examples aimed at showing the convergence properties of the proposed approach. The first example is the linear time invariant damped oscillator which was discussed earlier. The second one is a nonlinear, excited oscillator obtained by adding a nonlinear spring to the previous problem. The third one describes the flapping of a helicopter blade in forward flight, which yields a linear time periodic problem.

These are first integrated in time using the implicit, second-order accurate, A/L stable multistep integration scheme recently discussed in [13].

Subsequently, the incremental STM across each time step is computed using a 'leapfrog' variant of the second-order accurate, A-stable Crank-Nicolson scheme: considering

$$\boldsymbol{x}_{k+1/2} = \boldsymbol{x}_{k-1/2} + h\dot{\boldsymbol{x}}_k, \tag{46}$$

where h is the time step,

$$\dot{\boldsymbol{x}}_k = \mathbf{J}(t_k)\boldsymbol{x}_k, \tag{47}$$

and

$$\boldsymbol{x}_k = \frac{\boldsymbol{x}_{k+1/2} + \boldsymbol{x}_{k-1/2}}{2}, \tag{48}$$

in short, one obtains

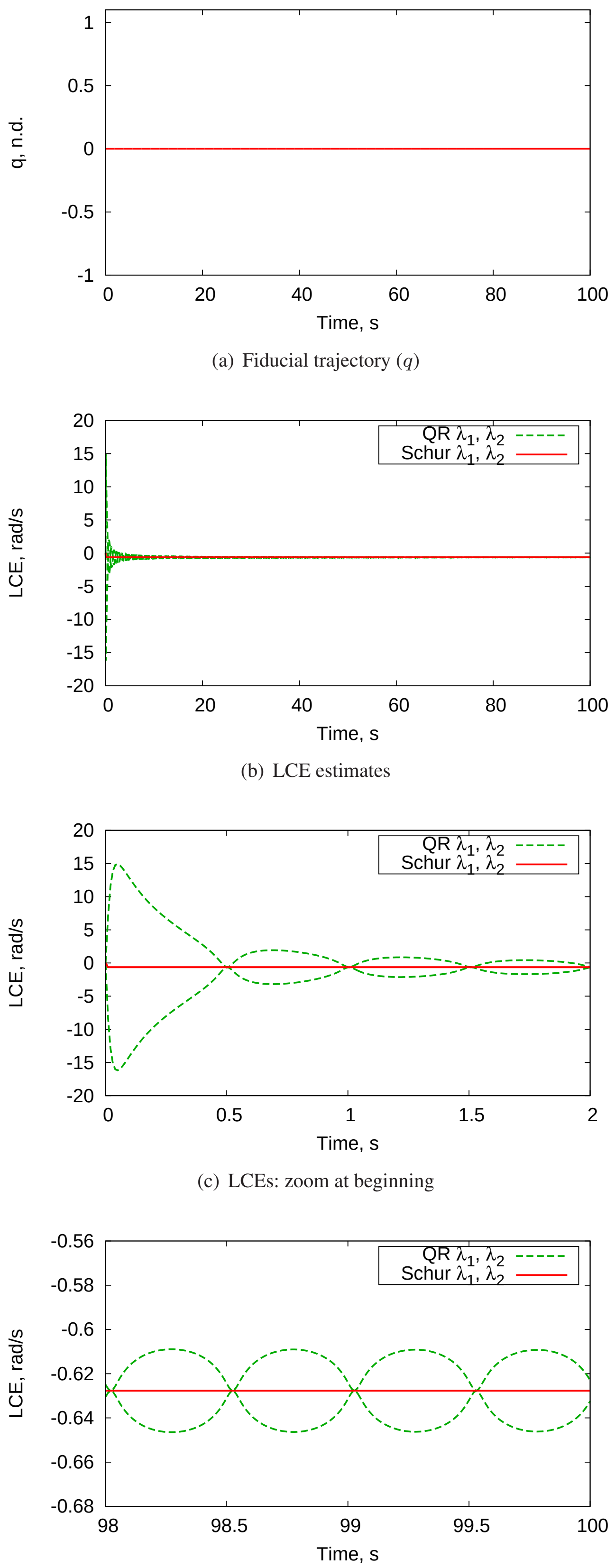
$$\boldsymbol{x}_{k+1/2} = \underbrace{\left(\mathbf{I} - \frac{h}{2}\mathbf{J}(t_k)\right)^{-1}\left(\mathbf{I} + \frac{h}{2}\mathbf{J}(t_k)\right)}_{\tilde{\mathbf{Y}}(t_k+h/2,t_k-h/2)}\boldsymbol{x}_{k-1/2}, \tag{49}$$

where $\tilde{\mathbf{Y}}(t_k + h/2, t_k - h/2)$ is a second-order accurate approximation of the STM from $t_{k-1/2}$ to $t_{k+1/2}$ of a linear, time invariant problem, as discussed in [14]. Of course, other, more accurate schemes can be used.

The LCEs are first estimated using the discrete QR algorithm, directly exploiting the incremental STM. Subsequently, the overall STM is computed and analyzed. Its QR decomposition is first evaluated, to check whether it over- or underflows; then, its real Schur decomposition is computed. In all the problems reported in the following, the STM rarely approached underflow; it occurs only for the largest modulus negative LCEs in the time periodic blade flapping example.

Linear, Time Invariant Problem: Mass-Spring-Damper

The problem presented in Section is analyzed numerically, using $\omega = 2\pi$ rad/s and $\xi = 0.1$ in Eq. (10). Figure 1 shows the LCEs estimated using the QR and the real Schur decomposition. It is clear from Fig. 1(d) that the LCEs estimated by the QR decomposition slowly converge to the exact value, which could be easily obtained by considering their average value. Conversely, the real Schur decomposition always produces the exact value, right from the first estimate, as illustrated in Fig. 1(c).



(a) Fiducial trajectory (q)

(b) LCE estimates

(c) LCEs: zoom at beginning

(d) LCEs: zoom at end

Figure 1. Linear mass-spring-damper problem

It is clear from the plots in Fig. 1 that the mean of the estimates computed with the QR decomposition also provides the exact value directly from the beginning, regardless of the value of t, according to Eq. (36).

The LCEs estimated using the QR decomposition, Eq. (26), are identical to those computed numerically and shown in the plots of Fig. 1. The estimates obtained using the SVD, Eq. (22), differ from those obtained using the QR decomposition only in the first 0.1 s (not shown in the plots).

Nonlinear, Time Invariant Problem: Mass-Nonlinear Spring-Damper

The problem of the previous section is modified by introducing a cubic spring and a harmonic forcing term that turns Eq. (10) into

$$\ddot{q} + 2\xi\omega\dot{q} + \omega^2\left(1 + \alpha q^2\right) q = A\cos(\Omega t), \tag{50}$$

with $\Omega = 1$. The results are identical to those of the previous case when $A = 0$ and $q(t) = 0$ is considered as fiducial trajectory, as one would expect, since matrix $\mathbf{J}$ is constant and the same of that case. When non-zero initial conditions are considered (Fig. 2), also the real Schur decomposition yields a sequence of 'bubbles' (i.e. distinct LCE estimates) alternating with coincident values, as shown in Fig. 2. It is worth noticing that coincident estimates are still obtained right from the beginning, although intermittently.

When forcing is introduced (Fig. 3), producing a periodic motion of maximum amplitude of the order of 0.7, the 'bubbles' in the LCEs estimated using the real Schur decomposition nearly vanish, producing very accurate estimates right from the beginning (Fig. 3(c)). A markedly oscillatory trend about the exact value persists in the estimates obtained with the QR method (Fig. 3(d)).

Linear, Time Periodic Problem: Flapping of Helicopter Blade

The problem of the flapping of a helicopter blade is analyzed. This classical problem has been recently discussed in[15] to exemplify stability analysis of LTP problems based on Floquet-Lyapunov theory. The equation

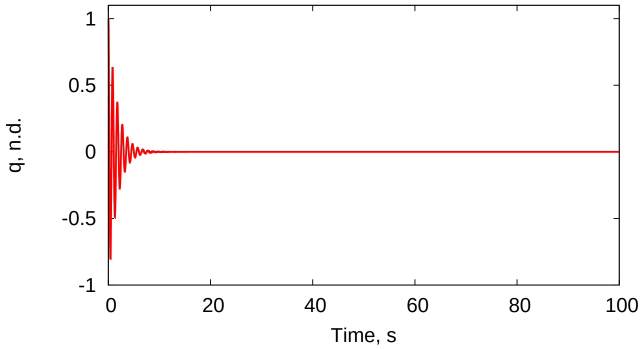
$$\ddot{\beta} + c(t)\dot{\beta} + k(t)\beta = 0, \tag{51}$$

with

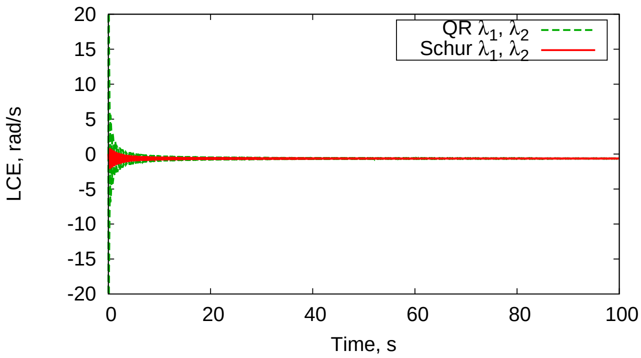
$$c(t) = \frac{\gamma}{8}\left(1 + \frac{4}{3}\mu\sin(t)\right) \tag{52a}$$

$$k(t) = \left(\nu_\beta^2 + \frac{\gamma}{8}\left(\frac{4}{3}\mu\cos(t) + \mu^2\sin(2t)\right)\right) \tag{52b}$$

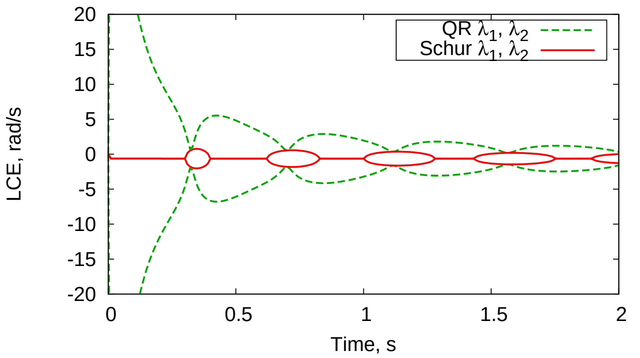
represents the flapping of a helicopter blade; β is the blade flap angle, γ is the Lock number (the non-dimensional ratio between aerodynamic and inertial flapping loads), μ is the advance ratio (the ratio between the helicopter forward velocity and the blade tip velocity), ν_β is the non-dimensional flapping frequency, and t is the azimuth angle; the dots represent derivation with respect to t. Periodicity in the aerodynamic loads originates from the interaction between the rotation of the blades and the uniform flow caused by forward motion of the helicopter. The dynamics of the blade have been oversimplified by linearizing the kinematics, considering two-dimensional steady aerodynamics, and neglecting reverse flow conditions; at values of μ larger than 0.3 it merely represents an example of LTP problem.



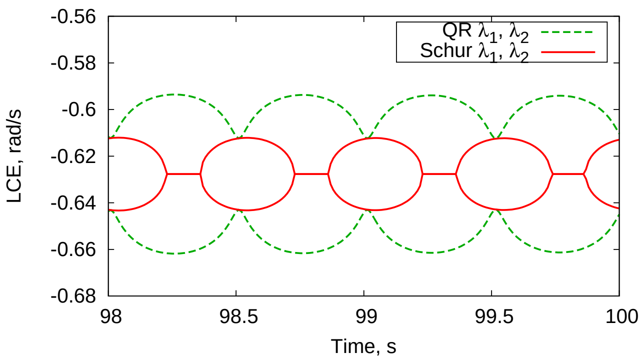
(a) Fiducial trajectory (q)



(b) LCE estimates



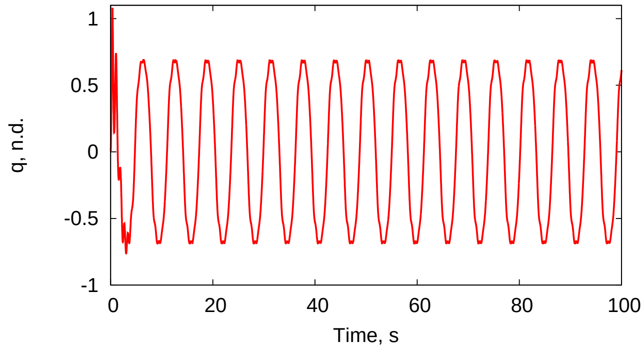
(c) LCEs: zoom at beginning



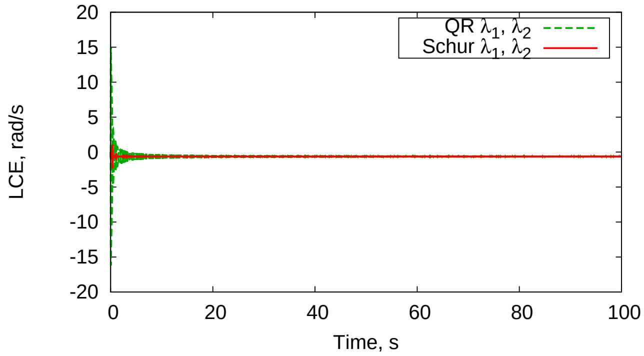
(d) LCEs: zoom at end

Figure 2. Nonlinear mass-spring-damper problem with non-null initial conditions and without excitation ($\alpha = 1$, $q(0) = 1$, $A = 0$)

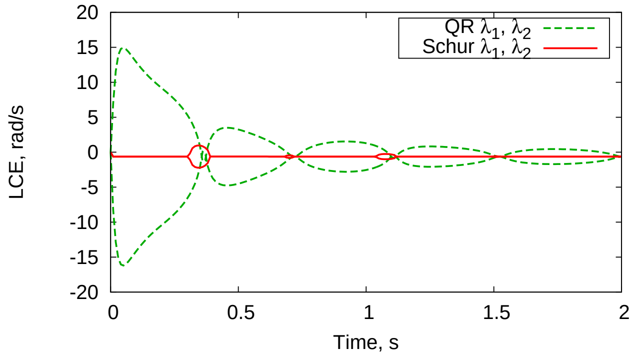
Clearly, a fiducial trajectory is $\beta(t) = 0$, which is obtained for $\beta(0) = 0$ and $\dot{\beta}(0) = 0$. Other trajectories can be obtained starting from arbitrary initial conditions; when



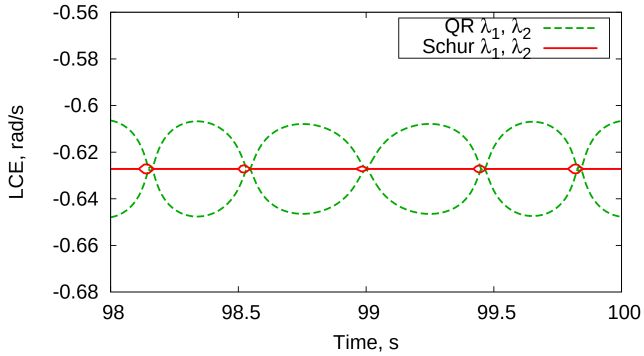
(a) Fiducial trajectory (q)



(b) LCE estimates



(c) LCEs: zoom at beginning



(d) LCEs: zoom at end

Figure 3. Nonlinear mass-spring-damper problem with null initial conditions and with excitation ($\alpha = 1$, $q(0) = 0$, $A = w^2$)

$\beta(t) = 0$ is asymptotically stable, the solution converges on it.

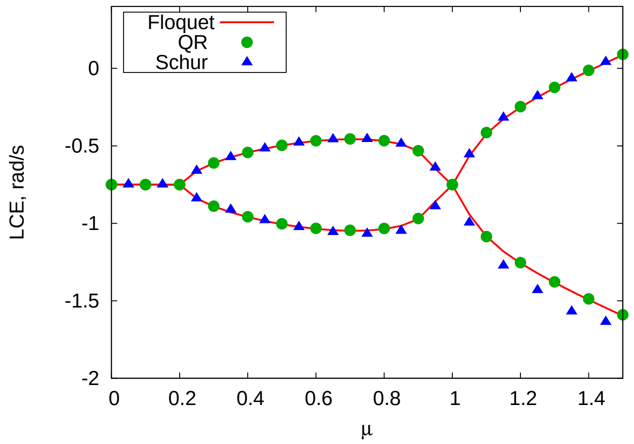


Figure 4. Blade flapping, $\gamma = 12$

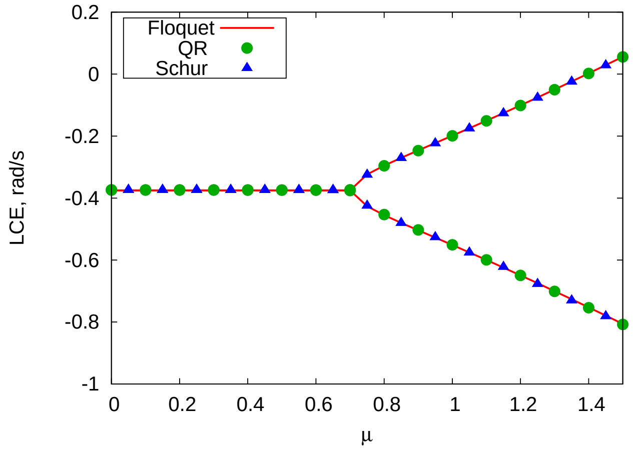


Figure 5. Blade flapping, $\gamma = 6$

Consider $\nu_\beta = 1$ and $\gamma = 12$, as in[15]. The LCE estimates as functions of μ are reported in Fig. 4, compared with the corresponding results obtained using the Floquet-Lyapunov method for LTP problems (Eq. (9), see[16–20] for a detailed discussion). Results obtained using the discrete QR method are equivalent to those obtained using Floquet-Lyapunov. The corresponding results obtained using the real Schur decomposition differ slightly, mostly because with $\gamma = 12$ (indeed an unusually large value for a helicopter rotor; γ is about 16 times the damping factor of a damped oscillator) the most negative LCE is very large in modulus. Thus, the STM quickly underflows, and the estimation had to be stopped relatively early.

Figure 5 instead refers to $\gamma = 6$, with LCEs correspondingly closer to zero. In that case, the estimate using the real Schur decomposition is fairly accurate.

In all cases, time integration has been performed for about 200 cycles, with an angular azimuth increment $h = 0.04\pi$ rad, corresponding to 50 steps per period. In many cases, the complete STM underflew; consequently, LCE estimation using the real Schur decomposition was performed on a small data set. LCE estimation converges slowly, owing to the periodicity of the problem. The results presented in Figs. 4 and 5 are averaged over the final 4 periods for estimates computed using both the QR and the real Schur decomposition.

Conclusions

This work showed how the real Schur decomposition can be used to estimate the Lyapunov Characteristic Exponents associated with a given fiducial trajectory of a nonlinear, non-autonomous problem, when their multiplicity is greater than one. Multiplicity either makes the convergence of usual Lyapunov exponents estimation algorithms very slow, or causes their failure. Multiplicity characterizes problems with complex conjugated eigenvalues of the state transition matrix, which are often characteristic of lightly damped mechanical systems. The proposed approach based on the real Schur decomposition can detect such occurrences of multiplicity, and correctly estimates those Lyapunov exponents. The properties of the proposed decomposition have been illustrated analytically with reference to a simple linear, time invariant problem, and numerically for nonlinear, time invariant problems with respect to arbitrary fiducial trajectories. Linear, time periodic problems have been considered as well, comparing the results with equivalent ones obtained using the Floquet-Lyapunov approach, namely the real part of the logarithm of the eigenvalues of the monodromy matrix. A major limitation of the proposed decomposition is that no feasible way has been found yet to formulate it in either incremental or differential manner. Consequently, it requires the accumulation of the state transition matrix, which can easily under- or overflow in case of large modulus exponents. Attempts to overcome this limitation are the object of actively ongoing research.

Nomenclature

$\boldsymbol{f}$	differential problem vector
m	geometric multiplicity of a root
n	algebraic multiplicity of a root
q	generic coordinate
r_{ij}	element ij of matrix $\mathbf{R}$
s_{ij}	element ij of matrix $\mathbf{S}$
t	the time
$\boldsymbol{x}$	state vector
$\boldsymbol{x}_0$	initial state vector
$\boldsymbol{y}$	state vector of auxiliary problem
y_{ij}	element ij of matrix $\mathbf{Y}$
$\mathbf{A}$	state matrix of linear, time invariant problem
$\mathbf{I}$	identity matrix
$\mathbf{J}$	derivative matrix of problem $\boldsymbol{f}$ with respect to state $\boldsymbol{x}$
$\mathbf{M}$	generic matrix
N	number of states
$\mathbf{Q}$	orthogonal matrix of QR decomposition
$\mathbf{R}$	upper-triangular matrix of QR decomposition
$\mathbf{S}$	block upper-triangular matrix of Schur decomposition
T	period of time-periodic problems
$\mathbf{U}$	orthogonal matrix of Schur decomposition
$\mathbf{Y}$	state transition matrix
λ_i	ith Lyapunov Characteristic Exponent
ξ	damping coefficient
σ_i	ith singular value
χ	characteristic polynomial roots
ω	characteristic frequency

Symbols that are specific of examples are defined in place.

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