Internal Constraints in Finite Elasticity: Manifolds or not

Maurizio Vianello

Received: 19 September 2012 / Published online: 16 February 2013

1 Introduction

A constraint is, broadly speaking, an *a priori* restriction placed on the set of configurations or motions which are deemed possible for a mechanical system. For N particles with cartesian coordinates (x_h, y_h, z_h), a set of *fixed holonomic* constraints is specified by s equations $f_i(x_h, y_h, z_h) = 0$, which are supposed to describe some $(3N - s)$-dimensional *submanifold* $\mathscr{M}$ of $\mathbb{R}^{3N}$. This assumption is not always made so explicit, but is frequently presented as the request that, for each permissible configuration, the Jacobian of functions $f_i(x_h, y_h, z_h)$ has maximum rank, so that, in view of Dini's Theorem, we may locally express the coordinates (x_h, y_h, z_h) through $n = 3N - s$ *free* variables q_j. Indeed, as explained in many textbooks on Differential Geometry (see, e.g., [11] and [1]), this is equivalent to the request that the set of points (x_h, y_h, z_h) which satisfy $f_i(x_h, y_h, z_h) = 0$ describes a smooth submanifold of $\mathbb{R}^{3N}$. There are good reasons for such an assumption; in particular, there is an obvious need for

M. Vianello (✉)
Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
e-mail: maurizio.vianello@polimi.it

avoiding constraints with a weaker mathematical structure, a topic I shall further comment later on.

We are interested here in *internal constraints in finite elasticity*, such as incompressibility or inextensibility in a given direction, which are just two among the many examples of restrictions placed on the set of deformations which are widely used for modelling specific elastic materials.

Many types of internal constraints have been discussed in the modern literature (see, e.g., [2, 4, 5, 8]). In order to provide a general framework, Gurtin and Podio-Guidugli proposed in [7] an axiomatic definition of what an internal constraint should be, at least for simple materials. Their proposal, which was further discussed in [9], is based on the idea that an internal constraint should be thought of as a smooth and connected submanifold $\mathscr{M}$ of Lin^+ (the open set of tensors with positive determinant) which, additionally, is required to include the identity tensor $\mathbf{I}$ and be invariant under the action of the rotation group. Since each tensor $\mathbf{F} \in \mathscr{M}$ represents the deformation gradient at a prescribed point, the requirement that $\mathbf{I}$ belongs to $\mathscr{M}$ is the natural assumption that the chosen reference configuration is permissible under the given constraint. Moreover, the condition that $\mathbf{QF} \in \mathscr{M}$ for each $\mathbf{F} \in \mathscr{M}$ and each rotation $\mathbf{Q}$ makes the constraint compatible with the principle of frame indifference.

A delicate but important aspect of this definition is the quite natural requirement that $\mathscr{M}$ be a differentiable manifold. Notice that the assumption that *virtual velocities* form a linear space, an ingredient of any well constructed mechanical theory, is strictly related to the manifold structure through the notion of *tangent space*. Such an assumption, however, has deeper consequences than expected.

The problem of characterizing *isotropic* internal constraints was thoroughly discussed by Podio-Guidugli and Vianello in [9]. The main and perhaps surprising conclusion is that, beside rigidity and conformality, the only possible isotropic constraints have dimension 8 in Lin^+. Thus, for example, one cannot have a seven-dimensional constraint manifold just adding some other isotropic restriction to the request of incompressibility.

In a fairly recent and interesting contribution to the literature, however, Carroll [3] introduced some isotropic constraints of dimension 7, suggesting that they might be possible counterexamples to the results proved in [9] (more precisely, in ref. [3] see the paragraph at the bottom of p. 1142, the top of p. 1143, Proposition 5, and the comments on p. 1147, just before Sect. 5.1). The main goal of the present research is to show that at least one such constraint is not described by a differentiable manifold and, moreover, to give reasonable arguments to convince the reader that the same happens with all other proposed examples, which, thus, do not qualify as "counterexamples".

The conclusion I anticipate is that all of the constraints proposed in [3] do not have a proper tangent space at $\mathbf{I} \in \mathscr{M}$. Is this important? Does it matter? This is not so obvious, and could be the subject of an interesting discussion, but certainly the lack of a proper manifold structure should be acknowledged and the related issues appropriately discussed.

For the sake of completeness in the final section of this research I propose an alternative proof of the main result contained in [9], which, in my opinion, is thus confirmed.

2 Constraint Manifolds

We use small and capital boldface letters for vectors and second-order tensors of the Euclidean three-dimensional space. Moreover, Lin denotes the space of such tensors, and $\mathrm{Lin}^+ \subset \mathrm{Lin}$ denotes the open subset of all $\mathbf{T} \in \mathrm{Lin}$ such that $\det \mathbf{T} > 0$. Symmetric tensors, for which $\mathbf{S} = \mathbf{S}^{\mathrm{T}}$ (a superscript T denotes the transpose), are the elements of $\mathrm{Sym} \subset \mathrm{Lin}$; the open set of all positive definite $\mathbf{S} \in \mathrm{Sym}$ is $\mathrm{Sym}^+ \subset \mathrm{Lin}^+$. The first, second and third invariant of a symmetric tensor $\mathbf{S}$ shall be written as $I_1(\mathbf{S})$, $I_2(\mathbf{S})$, $I_3(\mathbf{S})$ (the explicit dependence on $\mathbf{S}$ can be omitted when no confusion arises). Of course,

$$\begin{aligned} I_1(\mathbf{S}) &= \mathrm{tr}\,\mathbf{S} \quad \text{(trace)},\\ I_2(\mathbf{S}) &= \left[\mathrm{tr}(\mathbf{S})^2 - \mathrm{tr}\left(\mathbf{S}^2\right)\right]/2 \quad \text{(quadratic invariant)},\\ I_3(\mathbf{S}) &= \det \mathbf{S} \quad \text{(determinant)}. \end{aligned}$$

The subspaces of spherical (isotropic) and deviatoric (traceless) symmetric tensors are denoted by

$$\mathrm{Sph} := \{\mathbf{S} \in \mathrm{Sym}\colon\ \mathbf{S} = \alpha \mathbf{I}\}, \qquad \mathrm{Dev} := \left\{\mathbf{S} \in \mathrm{Sym}\colon\ \mathrm{tr}(\mathbf{S}) = 0\right\}.$$

Finally, Orth and Rot $\subset$ Orth are, respectively, the group of orthogonal tensors and the group of rotations.

An internal constraint at a given material point of an elastic body can be defined, loosely speaking, as a restriction placed on the set of permissible deformation gradients $\mathbf{F}$, such as, for example,

$$(\mathscr{M}_1) \quad \det \mathbf{F} = 1, \qquad (\mathscr{M}_2) \quad \mathrm{tr}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) = 3, \qquad (\mathscr{M}_3) \quad \mathbf{Fe} \cdot \mathbf{Fe} = 1, \tag{1}$$

(where $\mathbf{e}$ is a fixed unit vector). The first condition corresponds to the requirement of incompressibility, the second is known as "Bell's constraint", and the third one enforces inextensibility in the $\mathbf{e}$ direction. Of course, one can freely construct many other examples of internal constraints but, in general, we should remember that they are meant to describe material properties of interest and of some physical significance.

It seems useful to lay down an appropriate list of requirements that all internal constraints should fulfill, in order to be treated from a unified perspective. The approach chosen in [7] and [9] is to call a subset $\mathscr{M}$ of Lin^+ a *constraint manifold* if it has the following properties:

($\mathscr{P}_1$) $\mathscr{M}$ is a connected *submanifold* of Lin^+;
($\mathscr{P}_2$) $\mathbf{I} \in \mathscr{M}$;
($\mathscr{P}_3$) $\mathbf{Q}\mathscr{M} = \mathscr{M}$, for all $\mathbf{Q} \in \mathrm{Rot}$.

Condition ($\mathscr{P}_1$) implies that a tangent space is defined at each $\mathbf{F} \in \mathscr{M}$, so that the set of virtual velocities from a given configuration is a proper linear space, as is common in Mechanics. Condition ($\mathscr{P}_2$) is simply the requirement that the reference configuration itself is permitted, while ($\mathscr{P}_3$) means that for each admissible deformation gradient $\mathbf{F}$ the whole orbit $\mathbf{QF}$ is also admissible (this is the request needed to make the constraint compatible with material frame-indifference).

It is important to comment on why the word "submanifold" has been written in italics in the statement of property ($\mathscr{P}_1$). There are subtly different concepts which might go under this name, which is often preceded or followed by appropriate specifications.

A subset $\mathscr{S}$ of a Euclidean space $\mathscr{E}$ is a *submanifold* if, in the relative topology induced on $\mathscr{S}$ by $\mathscr{E}$, there is on $\mathscr{S}$ a (unique) C^∞ differentiable structure such that the inclusion map i of $\mathscr{S}$ into $\mathscr{E}$ is a C^∞ immersion. For a discussion of such definition and some related concepts, we refer to the detailed explanations provided in [11, Chap. 1]. Thus, at each point $p \in \mathscr{S}$, a submanifold has a tangent space $\mathscr{S}_p$ which can be identified with a subspace of $\mathscr{V}$, the translation space of $\mathscr{E}$, and, moreover, can be seen as the set of all derivatives $\dot{x}(0)$ for smooth curves $x(t) \in \mathscr{S}$ such that $x(0) = p$. It seems clear enough that this is the same approach taken in [7] and [9], even if, there, such definitions were not stated so explicitly.

Notice that a subset $\mathscr{S}$ contained in an open set $\mathscr{U} \subset \mathscr{E}$ is a submanifold of $\mathscr{E}$ if and only if it is a submanifold of $\mathscr{U}$. Thus, it will make no difference to declare a set contained in Lin^+ to be a submanifold of Lin^+ or Lin, and, analogously, for Sym^+ and Sym.

In order to avoid possible misunderstandings, we recall that an *algebraic variety* is, essentially, the set of solutions of a system of polynomial equations, for which the existence of a tangent space at each point is not guaranteed. Some (but not all) algebraic varieties are submanifolds and, in this case, they take the name of *algebraic manifolds*.

Our key request, thus, is that an internal constraint be a differentiable manifold which, by definition, is endowed with a tangent space of a constant dimension at each point.

One might ask the reason for such a choice and an interesting discussion could profitably be developed about this. Our motivation comes from Classical Mechanics, where virtual displacements from a given configuration are required to form a linear space. Difficulties which might otherwise arise are made clear with the help of a simple example.

The plane algebraic curve known as Bernoulli's Lemniscate, given by

$$\mathscr{C} := \big\{(x, y) \in \mathbb{R}^2 \colon x^4 - x^2 + y^2 = 0\big\}, \tag{2}$$

is "eight-shaped", similar to the symbol "∞". The set of virtual velocities (or displacements) for a point in the plane constrained to move on $\mathscr{C}$ does not form a linear space at the origin, where the double point is located. The assumption that the set of configurations for a constrained system should be described through a differentiable manifold is manifestly due to the need of avoiding such troublesome situations (indeed, $\mathscr{C}$ is an algebraic curve but not a submanifold of the plane) and is implicitly or explicitly made in all Classical Mechanics.

It is perhaps open to discussion if such considerations should be extended to internal constraints in Continuum Mechanics but, in my opinion, a cautious approach should retain the restriction that a constraint be a submanifold and not just an algebraic variety.

The main goal of the present article is to show that some internal constraints which might appear to be counterexamples to the results contained in [9] are indeed algebraic varieties but *not* manifolds. Since the proofs given in [9] are clearly based on the requirement that an internal constraint be a manifold, the contradiction between the "counterexamples" and the results found in [9] is shown to be mainly due to some misunderstanding about the terminology.

As a preliminary and reassuring observation, notice that the definition of an internal constraint given through properties ($\mathscr{P}_1$), ($\mathscr{P}_2$) and ($\mathscr{P}_3$) is satisfied by examples (1).

Connectivity of each $\mathscr{M}_i$ described in (1) can be deduced from the polar decomposition theorem and some known properties of the spectral decomposition of positive definite symmetric tensors. Here, we skip such details, and just verify the less trivial property ($\mathscr{P}_1$) for each $\mathscr{M}_i$.

For each of the smooth functions defined on Lin^+

$$\psi_1(\mathbf{F}) = \det \mathbf{F}, \quad \psi_2(\mathbf{F}) = \mathrm{tr}\big(\mathbf{F}\mathbf{F}^{\mathrm{T}}\big), \quad \psi_3(\mathbf{F}) = \mathbf{F}\mathbf{e} \cdot \mathbf{F}\mathbf{e},$$

we compute the differential at $\mathbf{F}$ in the direction $\mathbf{H} \in \mathrm{Lin}$ as

$$\begin{aligned}
\mathrm{D}\,\psi_1(\mathbf{F})[\mathbf{H}] &= (\det \mathbf{F})\,\mathrm{tr}\big(\mathbf{H}\mathbf{F}^{-1}\big),\\
\mathrm{D}\,\psi_2(\mathbf{F})[\mathbf{H}] &= 2\,\mathrm{tr}\big(\mathbf{F}^{\mathrm{T}}\mathbf{H}\big),\\
\mathrm{D}\,\psi_3(\mathbf{F})[\mathbf{H}] &= 2\mathbf{F}\mathbf{e} \cdot \mathbf{H}\mathbf{e}.
\end{aligned}$$

Constraints listed in (1) can be written, respectively, as

$$(\mathscr{M}_1) \quad \psi_1(\mathbf{F}) = 1, \qquad (\mathscr{M}_1) \quad \psi_2(\mathbf{F}) = 3, \qquad (\mathscr{M}_1) \quad \psi_3(\mathbf{F}) = 1$$

and, since for all $\mathbf{F}$ which belong to $\mathscr{M}_i$

$$\mathrm{D}\,\psi_i(\mathbf{F}) \neq 0,$$

we can apply the implicit function theorem in the context of differential geometry (see, e.g., [1, Chap. 1] or, more precisely, [11, Thm. 1.38]), and conclude that each $\mathscr{M}_i$, as defined in (1), is indeed a submanifold of Lin^+ with codimension one.

3 Isotropic Internal Constraints

A constraint is *isotropic* if

$$\mathbf{F} \in \mathscr{M}, \ \mathbf{Q} \in \mathrm{Rot} \quad \Rightarrow \quad \mathbf{FQ} \in \mathscr{M} \tag{3}$$

and only such constraints can be used to restrict the set of deformations of an isotropic material.

For $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ the left Cauchy-Green strain tensor and in view of the polar decomposition theorem constraints $\mathscr{M}_1$ and $\mathscr{M}_2$ in (1) can be written as

$$I_3(\mathbf{B}) = 1, \quad I_1(\mathbf{B}) = 3,$$

and their isotropicity is thus guaranteed by the well-known properties of the invariants. Notice, however, that while $\mathscr{M}_1$ and $\mathscr{M}_2$ satisfy requirement (3), $\mathscr{M}_3$ (inextensibility in a prescribed direction) does not, in agreement with physical intuition.

One might think that isotropic constraints could be easily generated as level sets of one or more polynomials in the invariants of the left Cauchy-Green strain tensor $\mathbf{B}$

$$\mathscr{S} := \left\{ \mathbf{B} \in \mathrm{Sym}^+ : \ g_i(I_1, I_2, I_3) = g_i(3, 3, 1) \quad i = 1, 2, \ldots, n \right\} \tag{4}$$

(the choice for the right-hand side is a guarantee that $\mathbf{I} \in \mathscr{S}$).

The are two problems, however. A first issue was discussed by Carroll in [3]: the set of tensors $\mathbf{B}$ satisfying conditions (4) should not be empty. Through a careful analysis, Carroll [3] was able to provide conditions that restrictions (4) should obey in order to be satisfied by some *non empty* set of strain tensors $\mathbf{B}$.

A second and very delicate issue, which does not seem to have been much discussed elsewhere, is the need to check that a constraint defined by (4) is not just an algebraic variety, but a differentiable manifold.

Since a polynomial function of the invariants I_h is polynomial in the components of $\mathbf{B}$, the set $\mathscr{S}$ defined in (4) is an algebraic variety in Sym^+. The constraint $\mathscr{M}$ is then

$$\mathscr{M} = \left\{ \mathbf{F} \in \mathrm{Lin}^+ : \ \mathbf{F}\mathbf{F}^{\mathrm{T}} \in \mathscr{S} \right\}, \tag{5}$$

and this should be a submanifold of Lin^+.

A result proved in [9, Theorem 5.1] can be re-stated as follows:

Proposition 1 *There is no* isotropic *constraint manifold* $\mathscr{M} \subset \mathrm{Lin}^+$ *of dimension* 5, 6 *or* 7.

Before discussing this result and the counterxamples which have been suggested, we introduce an alternative view of internal constraints. This is necessary, because some contributions to this subject take a slightly different approach. Indeed, it is reasonable to define an isotropic constraint to be a submanifold of Sym^+, and thus a restriction for the values of the left Cauchy-Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, rather than a submanifold of Lin^+, a restriction for the values of the deformation gradient $\mathbf{F}$.

According to this point of view, a subset $\mathscr{S} \subset \mathrm{Sym}^+$ is an isotropic constraint manifold if:

($\mathscr{U}_1$) $\mathscr{S}$ is a connected *submanifold* of Sym^+;
($\mathscr{U}_2$) $\mathbf{I} \in \mathscr{S}$;
($\mathscr{U}_3$) $\mathbf{Q}\mathscr{S}\mathbf{Q}^{\mathrm{T}} = \mathscr{S}$, for all $\mathbf{Q} \in \mathrm{Rot}$.

Proposition 1 has an analogue in the following statement.

Proposition 2 *There is no isotropic constraint manifold* $\mathscr{S} \subset \mathrm{Sym}^+$ *of dimension* 2, 3 *or* 4.

One may wonder if, by means of the map described by (5), an isotropic constraint manifold $\mathscr{M}$ in Lin^+ always corresponds to an isotropic constraint manifold $\mathscr{S}$ in Sym^+. The answer is positive, according to Lemma 5.2 of [9], and this makes Propositions 1 and 2 equivalent.

We shall make no appeal to such result, however, and, for simplicity, we confine our discussion to isotropic constraints $\mathscr{S}$ seen as subsets of Sym.

We show that an example proposed in [3] and first discussed by Saccomandi in [10] is *not* a manifold, in the sense that the algebraic variety defined by such constraint does not have a tangent space at the identity. The situation is thus quite similar to what is suggested by curve $\mathscr{C}$ of (2), where the role of the origin is now played by the identity tensor $\mathbf{I}$.

More explicitly, we consider the constraint

$$\mathscr{S} := \left\{\mathbf{B} \in \mathrm{Sym}^+ :\ I_1(\mathbf{B}) - I_2(\mathbf{B}) = 0,\ I_3(\mathbf{B}) = 1\right\}, \tag{6}$$

and prove that it is *not* a submanifold. Indeed, if $\mathscr{S}$ *were* a submanifold of Sym, then it would be a counterexample to Proposition 2: an isotropic constraint of dimension $d = 4\ (6 - 2)$.

It should be noted that Saccomandi in [10] writes that "this result [as stated here in Proposition 1] is true for constraints describing manifolds [...]; otherwise, a counterexample to this assertion is given by $I_1 - I_2 = 0$, $I_3 = 1$".[1]

Saccomandi's comment was not noticed by Carroll who, apparently, in [3] did not consider important to assess whether the algebraic variety (6) were a manifold or not.

On top of p. 1143 of [3] Carroll writes: "Saccomandi's counterexample is not the only one. Indeed, it is easy to construct an infinity of sets of constraints that imply one-parameter principal stretch states, such as uniaxial stretch $(\lambda, 1, 1)$, equibiaxial stretch $(\lambda, \lambda, 1)$ and symmetric isochoric stretch $(\lambda, \lambda, 1/\lambda^2)$, all of which correspond to 7-dimensional constraint manifolds."

[1] I entirely agree with this sentence, since it clearly states that the proposed example might lack the property of being a submanifold. Indeed, before writing his note, Saccomandi asked for my opinion, and I anticipated to him what the problem might be with the example he had in mind.

Of course, one might discuss in detail other hypothetical counterexamples proposed in [3], but this would be tedious and perhaps not very useful. My opinion is that all such examples have a common property: they are isotropic algebraic varieties but *not* submanifolds, since all of them lack a tangent space at the identity. I shall provide further motivations and clues which should convince the reader but I shall not go into detailed discussions on a case by case basis.

In order to make this presentation complete we conclude this work with a proof of the result stated in Proposition 2, as an alternative to what can be found in [9].

Thus, the main results of this research can be summarized as follows:

1. Constraints
$$I_1 - I_2 = 0 \quad \text{and} \quad I_3 = 1$$
define two isotropic manifolds.
2. Constraint
$$\begin{cases} I_1 - I_2 = 0 \\ I_3 = 1 \end{cases}$$
is shown *not* to be a manifold.
3. A new proof of Proposition 2 is given.

It is perhaps useful to anticipate the crucial detail which makes impossible for the proposed constraint to be a manifold. We shall prove that the set of curves lying in the constraint and which go through the identity $\mathbf{I}$ have, at that point, a set of tangent vectors which span a space with a higher dimension than the constraint itself. Thus, since the tangent spaces to a manifold must have the same dimension everywhere, equal to the dimension of the manifold itself, this observation will suffice to reach the desired conclusion.

Moreover, it is important to notice that away from the identity the constraint is indeed a manifold. Thus, it appears that a crucial role is played by the request that $\mathbf{I}$ be an admissible strain tensor for any given constraint. Could we do without this assumption? Perhaps, and this could be discussed. This article is less ambitious, however, and its goal is limited to pointing out where some difficulties might lie.

Finally, we point out that if

$$\mathscr{M} := \left\{ \mathbf{F} \in \mathrm{Lin}^+ : \ I_1\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) - I_2\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) = 0, \ I_3\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) = 1 \right\} \tag{7}$$

were a submanifold of Lin, then it would be a counterexample to Proposition 1: an isotropic constraint of dimension $d = 7$ $(9 - 2)$.

One can prove, however, that $\mathscr{M}$ is not a submanifold of Lin. Such proof is here omitted, for the sake of brevity, but this result is mentioned as a further confirmation that, as $\mathscr{S}$, defined in (6), is *not* a counterexample to Proposition 2, so $\mathscr{M}$, defined in (7), is *not* a counterexample to Proposition 1.

4 A Constraint Which Is not a Manifold

As a useful preliminary we list the differential of the invariants with respect to the tensor variable $\mathbf{B} \in \mathrm{Sym}^+$:

$$\mathrm{D}\, I_1(\mathbf{B})[\mathbf{H}] = \mathrm{tr}\,\mathbf{H} = \mathbf{I} \cdot \mathbf{H}, \tag{8}$$

$$\mathrm{D}\,I_2(\mathbf{B})[\mathbf{H}] = (\operatorname{tr}\mathbf{B})(\operatorname{tr}\mathbf{H}) - \operatorname{tr}(\mathbf{BH}) = \big[(\operatorname{tr}\mathbf{B})\mathbf{I} - \mathbf{B}\big]\cdot\mathbf{H}, \tag{9}$$

$$\mathrm{D}\,I_3(\mathbf{B})[\mathbf{H}] = (\det\mathbf{B})\mathbf{B}^{-1}\cdot\mathbf{H} = \operatorname{cof}\mathbf{B}\cdot\mathbf{H}. \tag{10}$$

The set $\mathscr{S}$ defined by (6) can be seen as the intersection of

$$\mathscr{S}_1 := \big\{\mathbf{B}\in\mathrm{Sym}^+ :\ I_1(\mathbf{B}) = I_2(\mathbf{B})\big\} \tag{11}$$

and

$$\mathscr{S}_2 := \big\{\mathbf{B}\in\mathrm{Sym}^+ :\ I_3(\mathbf{B}) = 1\big\}. \tag{12}$$

Theorem 1 *The sets* $\mathscr{S}_1$ *and* $\mathscr{S}_2$*, defined in* (11) *and* (12)*, are* 5*-dimensional isotropic submanifolds of* Sym.

Proof It is easy to verify that $\mathscr{S}_1$ and $\mathscr{S}_2$ are isotropic and we concentrate on the proof that each one is a submanifold. We begin our analysis from $\mathscr{S}_1$ and compute at each $\mathbf{B}\in\mathscr{S}_1$

$$\mathrm{D}(I_1 - I_2)(\mathbf{B}) = \mathbf{I} - (\operatorname{tr}\mathbf{B})\mathbf{I} + \mathbf{B}.$$

Thus, condition $\mathrm{D}(I_1 - I_2)(\mathbf{B}) = 0$ is equivalent with

$$(1 - \operatorname{tr}\mathbf{B})\mathbf{I} = -\mathbf{B}, \tag{13}$$

which in turn, after taking the trace of both sides, implies

$$3(1 - \operatorname{tr}\mathbf{B}) = -\operatorname{tr}\mathbf{B}$$

and

$$\operatorname{tr}\mathbf{B} = 3/2. \tag{14}$$

Upon substitution of (14) in (13) we conclude that the differential $\mathrm{D}(I_1 - I_2)$ is zero only for $\mathbf{B} = \mathbf{I}/2$. However, since

$$I_1(\mathbf{I}/2) = 3/2, \qquad I_2(\mathbf{I}/2) = 3/4,$$

we conclude that $\mathbf{I}/2$ does *not* belong to $\mathscr{S}_1$. Thus, the differential of $I_1 - I_2$ is not zero at each point $\mathbf{B}$ of the set defined by $I_1 - I_2 = 0$. In view of a standard theorem of differential geometry (see, e.g., [11, Theorem 1.38]) we conclude that each connected component of $\mathscr{S}_1$ is a submanifold of $\mathrm{Sym}^+\subset\mathrm{Sym}$ of dimension $5 = 6 - 1$.

We now turn our attention to $\mathscr{S}_2$. Since

$$\mathrm{D}\,I_3(\mathbf{B}) = (\det\mathbf{B})\mathbf{B}^{-1},$$

it is easy to conclude that at each $\mathbf{B}\in\mathscr{S}_2$ (for which $\det\mathbf{B} = 1$) we have

$$\mathrm{D}\,I_3(\mathbf{B}) \neq 0.$$

Again, this suffices to show that $\mathscr{S}_2$ is a submanifold of $\mathrm{Sym}^+\subset\mathrm{Sym}$ of dimension $5 = 6 - 1$, exactly as for $\mathscr{S}_1$. □

Our first conclusion, thus, is that both $\mathscr{S}_1$ and $\mathscr{S}_2$ are isotropic submanifolds of Sym.

We now turn our attention to $\mathscr{S} = \mathscr{S}_1 \bigcap \mathscr{S}_2$, with the aim of showing that they are *not* submanifolds of Sym and Lin, respectively.

The fact that $\mathscr{S}$ is not empty can be deduced from the observation that any tensor $\mathbf{B}$ which, with respect to some orthonormal basis, has components

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix}, \quad (\lambda > 0) \tag{15}$$

satisfies all conditions which define both $\mathscr{S}_1$ and $\mathscr{S}_2$ and thus belongs to $\mathscr{S} = \mathscr{S}_1 \bigcap \mathscr{S}_2$.

Indeed, it is useful to prove that *only* tensors described (with respect to some orthonormal basis) by one of the matrices listed in (15) belong to $\mathscr{S}$.

Theorem 2 *For each tensor* $\mathbf{B} \in \mathscr{S} \subset \mathrm{Sym}^+$ *there is at least one orthonormal basis* $\mathbf{e}_i$ *and a real number* $\lambda > 0$ *such that the matrix of the components takes one of the forms listed in* (15).

Proof For $I_3 = 1$ and $I_1 = I_2$ the characteristic polynomial for $\mathbf{B} \in \mathscr{S}$

$$-\gamma^3 + \gamma^2 I_1 - \gamma I_2 + I_3 = 0$$

takes the form

$$-\gamma^3 + \gamma^2 I_1 - \gamma I_1 + 1 = 0$$

and is factorized as

$$(\gamma - 1)\big[\gamma^2 + (1 - I_1)\gamma + 1\big] = 0.$$

One eigenvalue is thus equal to 1 and the product of the remaining pair is also 1.

The conclusion is that the diagonalized form of the matrix of components of $\mathbf{B} \in \mathscr{S} \subset \mathrm{Sym}^+$ is necessarily one among the three which appear in (15) (condition $\lambda > 0$ is forced by $\mathbf{B} \in \mathrm{Sym}^+$). □

We now show that, even if both $\mathscr{S}_1$ and $\mathscr{S}_2$ are submanifolds of Sym^+, their intersection $\mathscr{S} = \mathscr{S}_1 \bigcap \mathscr{S}_2$ is *not*.

First of all we show that *away* from $\mathbf{I}$ the set $\mathscr{S}$ is a 4-dimensional submanifold.

Theorem 3 *For each* $\mathbf{B} \in \mathscr{S} \subset \mathrm{Sym}$ *which is different from* $\mathbf{I}$ *there is neighborhood such that the restriction of* $\mathscr{S}$ *to that neighborhood is a* 4*-dimensional submanifold.*

Proof For two submanifolds, defined as zero sets of functions g_1 and g_2, a sufficient condition for their intersection to be itself a submanifold is that there the gradients (or differentials) $\mathrm{D}\, g_1$ and $\mathrm{D}\, g_2$ are linearly independent. Since, in our context,

$$g_1(\mathbf{B}) = I_1(\mathbf{B}) - I_2(\mathbf{B}), \qquad g_2(\mathbf{B}) = I_3(\mathbf{B}), \tag{16}$$

we need to verify if and where, for some α,

$$\mathrm{D}(I_1 - I_2)(\mathbf{B}) = \alpha\, \mathrm{D}\, I_3(\mathbf{B}), \quad \text{for } \mathbf{B} \in \mathscr{S}.$$

This condition is

$$(1 - \operatorname{tr}\mathbf{B})\mathbf{I} + \mathbf{B} = \alpha(\det\mathbf{B})\mathbf{B}^{-1}, \tag{17}$$

which, since $\det\mathbf{B} = 1$, after multiplication of both sides by $\mathbf{B}$ is equivalent with

$$(1 - \operatorname{tr}\mathbf{B})\mathbf{B} + \mathbf{B}^2 = \alpha\mathbf{I}. \tag{18}$$

In view of Theorem 2 we assume that, with respect to some orthonormal basis, the matrix of $\mathbf{B} \in \mathscr{S}$ has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix}$$

(the other matrices listed in (15) would give the same results, as can be readily checked). Thus, equation (18) becomes

$$\left(-\lambda - \frac{1}{\lambda}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & 1/\lambda^2 \end{bmatrix} = \alpha\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which leads to the system

$$\begin{cases} -\lambda - 1/\lambda + 1 = \alpha, \\ -\lambda^2 - 1 + \lambda^2 = \alpha, \\ -1 - 1/\lambda^2 + 1/\lambda^2 = \alpha. \end{cases}$$

The second and third equation give $\alpha = -1$ while, with such a value, the first equation becomes

$$\lambda + \frac{1}{\lambda} = 2,$$

which in turn yields $\lambda = 1$. This analysis keeps its validity for all other forms of the component matrices which appear in (15) and, in any case, we deduce that $\alpha = -1$, $\lambda = 1$.

The conclusion is that (17) is satisfied on $\mathscr{S}$ only for $\mathbf{B} = \mathbf{I}$. Thus, at all points which lie on $\mathscr{S}$, except for $\mathbf{B} = \mathbf{I}$, the differentials in (16) are linearly independent and this suffices to prove that, *away from point* $\mathbf{B} = \mathbf{I}$, $\mathscr{S}$ is a submanifold of dimension 4. □

What happens at $\mathbf{B} = \mathbf{I}$? Since the condition used is in Theorem 3 is sufficient but *not* necessary we cannot yet draw any conclusion about the geometric structure of $\mathscr{S}$ at $\mathbf{I}$.

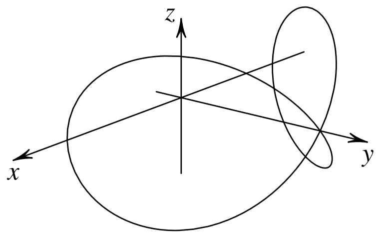
The differentials of $I_1 - I_2$ and I_3 at $\mathbf{B} = \mathbf{I}$ are, respectively,

$$\mathrm{D}(I_1 - I_2)(\mathbf{I}) = -\mathbf{I}, \qquad \mathrm{D}\, I_3(\mathbf{I}) = \mathbf{I},$$

and, because of this, the tangent spaces to $\mathscr{S}_1$ and $\mathscr{S}_2$ at $\mathbf{I}$ (and only there) coincide with Dev, the space of symmetric tensors which are orthogonal to the identity or, equivalently, traceless. We conclude that $\mathscr{S}_1$ and $\mathscr{S}_2$ have a common tangent space at $\mathbf{I}$, and this is the precise reason for which at that point the sufficient condition for their intersection to be a submanifold cannot be applied.

In the ordinary three-dimensional Euclidean space it is possible to provide examples of pairs of surfaces with an intersection which *is* a manifold, as it is possible to provide examples of pairs of surfaces with an intersection which *is not* a manifold, near a point

Fig. 1 The graph of Viviani's curve



where they share a common tangent plane. Thus, we need a more explicit analysis of the behavior of $\mathscr{S}$ at $\mathbf{I}$ in order to understand in which situation we are.

As an help to the reader's intuition we describe a classical algebraic curve which, as we shall see, has a behavior quite similar to what can be deduced for $\mathscr{S}$. We consider in $\mathbb{R}^3$ the intersection between the unit sphere $x^2 + y^2 + z^2 = 1$ and the circular cylinder $y^2 - y + z^2 = 0$, which has radius $1/2$ and is parallel to the x axis, with a basis centered at $(0, 1/2, 0)$. The intersection of such smooth surfaces, which have the set $y = 1$ as their common tangent plane at $(0, 1, 0)$, is the celebrated Viviani's curve, whose graph is shown in Fig. 1. Notice that at $(x, y, z) = (0, 1, 0)$ the curve does not have a tangent space but, rather, a tangent cone given by the union of two lines. As a consequence, this curve is an algebraic variety but not a submanifold of $\mathbb{R}^3$. Notice that Viviani's curve is defined through the intersection of two smooth submanifolds, a sphere and a cylinder, exactly as it happens for the constraint $\mathscr{S}$. As we shall see, there is a close similarity between the structure of $\mathscr{S}$ and what can be seen in the graph of Viviani's curve.

Let $\mathscr{S}'$ be the set of all tangent vectors to curves in $\mathscr{S}$ as they go through the point $\mathbf{I}$ (of course we should rather say "tangent tensors" but it is perhaps more convenient to abuse our terminology).

We now prove that the span of $\mathscr{S}'$ is Dev, and this result is sufficient to deduce that $\mathscr{S}$ itself is not a submanifold. Indeed, if $\mathscr{S}$ were a manifold the set $\mathscr{S}'$ would coincide with the tangent space, and at $\mathbf{B} = \mathbf{I}$ the tangent space to $\mathscr{S}$ would then be Dev, which has dimension 5. However, away from $\mathbf{I}$ the dimension of $\mathscr{S}$ is just $4 = 6 - 2$, a clear contradiction, since the dimension of the tangent space to a manifold is constant at each point.

Theorem 4 *The set* $\mathscr{S} \subset \mathrm{Sym}$ *is not a submanifold in any neighborhood of* $\mathbf{I}$.

Proof With respect to its eigenbasis a symmetric traceless tensor $\mathbf{A}_0$ has diagonal form

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\alpha - \beta \end{bmatrix}.$$

Let

$$\lambda_1(t) = \alpha t + 1, \qquad \lambda_2(t) = \beta t + 1,$$

for which

$$\lambda_1(0) = \lambda_2(0) = 1, \quad \dot{\lambda}_1(0) = \alpha, \quad \dot{\lambda}_2(0) = \beta.$$

The functions

$$\mathbf{A}_1(t) = \begin{bmatrix} \lambda_1(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda_1(t) \end{bmatrix}, \qquad \mathbf{A}_2(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & 1/\lambda_2(t) \end{bmatrix},$$

for t close to zero define curves on $\mathscr{S}$ which go through $\mathbf{I}$ for $t=0$ and

$$\dot{\mathbf{A}}_1(0) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha \end{bmatrix}, \qquad \dot{\mathbf{A}}_2(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\beta \end{bmatrix}.$$

Thus, while both $\dot{\mathbf{A}}_1(0)$ and $\dot{\mathbf{A}}_2(0)$ belong to $\mathscr{S}'$, their sum is

$$\dot{\mathbf{A}}_1(0) + \dot{\mathbf{A}}_2(0) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\beta \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\alpha-\beta \end{bmatrix} = \mathbf{A}_0.$$

This shows that the span of $\mathscr{S}'$ (set of tangent vectors at $\mathbf{I}$ to curves in $\mathscr{S}$) is Dev, the (5-dimensional) space of traceless symmetric tensors. As stated before this shows that $\mathscr{S}$ cannot be a submanifold of Sym^+. (A straightforward conjecture is that $\mathscr{S}_c = \{\mathbf{H} \in \mathrm{Dev}: \det \mathbf{H} = 0\}$ is the tangent *cone* to $\mathscr{S}$ at $\mathbf{I}$, but we shall not expand on this.) □

It is interesting to show the existence of a strong clue suggesting that any intersection of a pair of isotropic constraint manifolds in Sym^+ might not be a manifold near the identity. Let $\hat{g}_h(I_1, I_2, I_3)$ $(h = 1, 2)$ be polynomial functions of the invariants $I_i(\mathbf{B})$ and, for

$$g_h(\mathbf{B}) := \hat{g}_h(I_1, I_2, I_3),$$

assume that each one of the constraints $g_h(\mathbf{B}) = g_h(\mathbf{I})$ defines a submanifold of Sym^+. Notice that such constraints can be written as

$$\hat{g}_1(I_1, I_2, I_3) = \hat{g}_1(3, 3, 1),$$
$$\hat{g}_2(I_1, I_2, I_3) = \hat{g}_2(3, 3, 1).$$

From (8), (9) and (10) we have

$$\mathrm{D}\, I_1(\mathbf{I})[\mathbf{H}] = \mathrm{tr}\,\mathbf{H} = \mathbf{I} \cdot \mathbf{H}, \tag{19}$$

$$\mathrm{D}\, I_2(\mathbf{I})[\mathbf{H}] = (\mathrm{tr}\,\mathbf{I})(\mathrm{tr}\,\mathbf{H}) - \mathrm{tr}(\mathbf{IH}) = \big[(\mathrm{tr}\,\mathbf{I})\mathbf{I} - \mathbf{I}\big] \cdot \mathbf{H} = 2\mathbf{I} \cdot \mathbf{H}, \tag{20}$$

$$\mathrm{D}\, I_3(\mathbf{I})[\mathbf{H}] = (\det \mathbf{I})\mathbf{I}^{-1} \cdot \mathbf{H} = \mathrm{cof}\,\mathbf{I} \cdot \mathbf{H} = \mathbf{I} \cdot \mathbf{H}. \tag{21}$$

The differential of g_h with respect to $\mathbf{B}$, in view of (19), (20), (21), yields

$$\mathrm{D}\, g_h(\mathbf{I})[\mathbf{H}] = \left(\frac{\partial \hat{g}_h}{\partial I_1} + 2\frac{\partial \hat{g}_h}{\partial I_2} + \frac{\partial \hat{g}_h}{\partial I_3}\right)\mathbf{I} \cdot \mathbf{H},$$

which shows that the tangent space to the level sets of g_1 and g_2 at $\mathbf{I}$ is Dev, the space of traceless symmetric tensors.

Thus, even if $g_1(\mathbf{B}) = g_1(\mathbf{I})$ and $g_2(\mathbf{B}) = g_2(\mathbf{I})$ define two submanifolds of dimension 5 which go through $\mathbf{I}$, the condition of linear independence of the differentials of g_1 and g_2 at $\mathbf{I}$, which is sufficient for the intersection to be locally a submanifold of dimension 4, is *not* satisfied. Of course, this is not really a proof that such an intersection is not a manifold but it is a strong clue about what the problem might be.

Acknowledgements The Author is grateful to the anonymous Referee who took great care to give useful suggestions for improving this work.

Appendix

We give a new proof of Proposition 2, which can be stated as follows: any (proper) isotropic constraint manifold $\mathscr{S} \subset \mathrm{Sym}$ has either dimension 1 or dimension 5.

Let $\mathscr{S}_0 \subset \mathrm{Dev}$ be the tangent space to $\mathscr{S}$ at the identity $\mathbf{I}$. If $\dot{\mathbf{B}}(0) \in \mathscr{S}_0$ is the derivative of a curve $\mathbf{B}(t)$ in $\mathscr{S}$ which goes through $\mathbf{I}$ for $t = 0$ it follows that, for any rotation $\mathbf{Q}$, the curve

$$\hat{\mathbf{B}}(t) := \mathbf{Q}\mathbf{B}(t)\mathbf{Q}^{\mathrm{T}} \in \mathscr{S}$$

satisfies $\hat{\mathbf{B}}(0) = \mathbf{Q}\mathbf{B}(0)\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$ and

$$\frac{d}{dt}\hat{\mathbf{B}}(t)|_{t=0} = \mathbf{Q}\dot{\mathbf{B}}(0)\mathbf{Q}^{\mathrm{T}} \in \mathscr{S}_0.$$

Thus,

$$\dot{\mathbf{B}}(0) \in \mathscr{S}_0, \quad \Rightarrow \quad \mathbf{Q}\dot{\mathbf{B}}(0)\mathbf{Q}^{\mathrm{T}} \in \mathscr{S}_0,$$

and from this it follows that the tangent space $\mathscr{S}_0 \subset \mathrm{Sym}$ is invariant under the group of rotations, in the sense that

$$\mathbf{Q}\mathscr{S}_0\mathbf{Q}^{\mathrm{T}} = \mathscr{S}_0, \tag{22}$$

for all rotations $\mathbf{Q}$.

The problem now is: which subspaces of Sym satisfy condition (22)? Only two: Sph and Dev.

The proof will be given later. For the moment just notice that, since the dimension of the submanifold $\mathscr{S}$ is equal to the dimension of the tangent space $\mathscr{S}_0$, the only possibilities are

$$\dim \mathscr{S} = \dim \mathrm{Sph} = 1, \qquad \dim \mathscr{S} = \dim \mathrm{Dev} = 5,$$

and this is equivalent with the statement of Proposition 2.

One should notice that the crucial steps are: (1) $\mathscr{S}$ is a submanifold and, thus, has a tangent space at each point; (2) the identity tensor belongs to $\mathscr{S}$. Only if we neglect such requirements it is possible to construct examples of isotropic constraints (as algebraic varieties) which seemingly violate the results proved in [9].

Finally, how can we prove that Sph and Dev are the only proper subspaces of Sym which satisfy property (22) for all rotations?

The barehanded proof given in [9, Lemma 5.3] is perhaps a bit cumbersome and not very elegant. Indeed, the whole topic could be investigated within the framework of the theory of group actions on polynomials, for which one might refer to the (application-oriented) introduction given in [6, Chaps. XII and XIII]. Of course, this approach to the proof of Proposition 2 would not be self-contained with the additional disadvantage of requiring references to mathematical tools not so widely known in this context.

Here, however, I propose a new proof which, in my opinion, has the advantage of being reasonably simple and self-contained. Additionally, this approach is based on techniques which are commonly used in the literature about finite elasticity.

A preliminary technical lemma is useful.

Lemma 1 *Let* $\mathbf{B}$ *and* $\mathbf{A} \notin \mathrm{Sph}$ *be a pair of symmetric tensors such that, for all rotations* $\mathbf{Q}$,

$$\mathbf{B} \cdot \mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}} = 0. \tag{23}$$

Then $\mathbf{B} \in \mathrm{Sph}$.

Proof We show that condition $\mathbf{B} \notin \mathrm{Sph}$ would lead to a contradiction, and this will suffice to prove the lemma.

Under the assumptions, and in view of the spectral decomposition theorem for symmetric tensors, we know that $\mathbf{A}$ has (at least) one characteristic space of dimension precisely 1. Let $\mathbf{a}$ be an eigenvector spanning such space. Since $\mathbf{B}$ is also assumed to be not spherical we can easily find at least one rotation $\bar{\mathbf{Q}}$ such that $\bar{\mathbf{a}} := \bar{\mathbf{Q}}\mathbf{a}$ is *not* among its eigenvectors. Let $\bar{\mathbf{A}} := \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}}^{\mathrm{T}}$, so that $\bar{\mathbf{a}}$ spans a one-dimensional characteristic space $\bar{\mathscr{L}}$ for $\bar{\mathbf{A}}$. We now notice that $\bar{\mathbf{A}}$ and $\mathbf{B}$ *can not* commute since, otherwise, $\mathbf{B}$ would map $\bar{\mathscr{L}}$ into itself, and, in particular, this would imply that $\mathbf{B}\bar{\mathbf{a}} = \mu\bar{\mathbf{a}}$, for some μ, forcing $\bar{\mathbf{a}}$ to be an eigenvector of $\mathbf{B}$. Thus,

$$\mathbf{B}\bar{\mathbf{A}} - \bar{\mathbf{A}}\mathbf{B} \neq \mathbf{0}.$$

Next, define a curve on Rot as

$$\mathbf{Q}(t) = \exp(t\mathbf{W})\bar{\mathbf{Q}},$$

where $\mathbf{W}$ is an arbitrary skew-symmetric tensor, so that

$$\mathbf{Q}(0) = \bar{\mathbf{Q}}, \qquad \dot{\mathbf{Q}}(0) = \mathbf{W}\bar{\mathbf{Q}}. \tag{24}$$

The function

$$\sigma(t) := \mathbf{B} \cdot \mathbf{Q}(t)\mathbf{A}\mathbf{Q}(t)^{\mathrm{T}}$$

and its derivative are forced by (23) to be identically equal to zero. However, in view of (24), the symmetry of $\mathbf{A}$ and $\mathbf{B}$ and the definition of $\bar{\mathbf{A}}$,

$$\begin{aligned}\dot{\sigma}(0) &= \mathbf{B} \cdot \left[\dot{\mathbf{Q}}(0)\mathbf{A}\mathbf{Q}^{\mathrm{T}}(0) + \mathbf{Q}(0)\mathbf{A}\dot{\mathbf{Q}}^{\mathrm{T}}(0)\right] = \mathbf{B} \cdot \left[\mathbf{W}\bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}}^{\mathrm{T}} + \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}}^{\mathrm{T}}\mathbf{W}^{\mathrm{T}}\right] \\ &= \mathbf{B} \cdot \left[\mathbf{W}\bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}}^{\mathrm{T}} - \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}}^{\mathrm{T}}\mathbf{W}\right] = \mathbf{B} \cdot [\mathbf{W}\bar{\mathbf{A}} - \bar{\mathbf{A}}\mathbf{W}] = [\mathbf{B}\bar{\mathbf{A}} - \bar{\mathbf{A}}\mathbf{B}] \cdot \mathbf{W}.\end{aligned}$$

Now, let $\mathbf{W} := \mathbf{B}\bar{\mathbf{A}} - \bar{\mathbf{A}}\mathbf{B}$, which is skew-symmetric and not zero. With this choice, we have

$$\dot{\sigma}(0) = [\mathbf{B}\bar{\mathbf{A}} - \bar{\mathbf{A}}\mathbf{B}] \cdot [\mathbf{B}\bar{\mathbf{A}} - \bar{\mathbf{A}}\mathbf{B}] > 0,$$

a clear contradiction. □

Now we can easily prove our main and final result.

Theorem 5 *Let* $\mathscr{S}_0$ *be a proper subspace of* Sym *which satisfies condition* (22) *for all rotations* $\mathbf{Q}$. *Then, either* $\mathscr{S}_0 = \mathrm{Sph}$ *or* $\mathscr{S}_0 = \mathrm{Dev}$.

Proof If $\mathscr{S}_0 \neq \mathrm{Sph}$ there is a tensor $\mathbf{A}$ in $\mathscr{S}_0$ such that $\mathbf{A} \notin \mathrm{Sph}$. In view of property (22) any tensor $\mathbf{B}$ in the orthogonal complement $\mathscr{S}_0^{\perp}$ would satisfy (23), for all rotations $\mathbf{Q}$. Then, Lemma 1 implies that $\mathbf{B} \in \mathrm{Sph}$, so that and $\mathscr{S}_0^{\perp} \subset \mathrm{Sph}$. Since $\mathscr{S}_0$ is supposed to be proper, we conclude that $\mathscr{S}_0^{\perp} = \mathrm{Sph}$, forcing $\mathscr{S}_0$ to be Dev, the orthogonal complement of Sph. □

References

1. Bishop, R.L., Goldberg, S.I.: Tensor Analysis on Manifolds. Dover, New York (1980)
2. Boulos, P., Achcar, N.: Reduced constraint manifolds. Appl. Mech. Rev. **46**(11), S105–S109 (1993)
3. Carroll, M.M.: On isotropic constraints. Int. J. Eng. Sci. **47**, 1142–1148 (2009)
4. Cohen, H., Wang, C.C.: On the response and symmetry of elastic materials with internal constraints. Arch. Ration. Mech. Anal. **99**, 1–36 (1987)
5. Destrade, M., Scott, N.H.: Surface waves in a deformed isotropic hyperelastic material subject to an isotropic internal constraint. Wave Motion **40**, 347–357 (2004)
6. Golubitsky, M., Stewart, I., Schaeffer, D.G.: Singularities and Groups in Bifurcation Theory, vol. 2. Springer, New York (1985)
7. Gurtin, M.E., Podio-Guidugli, P.: The thermodynamics of constrained materials. Arch. Ration. Mech. Anal. **51**(3), 192–208 (1973)
8. Negahban, M.: Single and multiple material constraints in thermoelasticity. Math. Mech. Solids **12**, 623–664 (2007)
9. Podio-Guidugli, P., Vianello, M.: Constraint manifolds for isotropic solids. Arch. Ration. Mech. Anal. **105**(2), 105–121 (1989)
10. Saccomandi, G.: Universal results in finite elasticity. In: Fu, Y.B., Ogden, R.W. (eds.) Nonlinear Elasticity: Theory and Applications, pp. 97–134. Cambridge University Press, Cambridge (2001)
11. Warner, F.W.: Foundations of Differentiable Manifolds and Lie Groups. Scott, Foresman and Company, Glenview (1971)