Application of the dynamics of variable mass systems to the Pelton turbine

Michele Mantegna[1,2] and Luigi P. M. Colombo[1,3]

[1] *Politecnico di Milano, Department of Energy, Via Lambruschini 4, 20156 Milan, Italy*

[2] *Assistant Professor of Applied Thermodynamics and Fluid Machines*

[3] *Assistant Professor of Heat and Mass Transfer, corresponding author,*

Nomenclature

Symbol	Definition	Unit
a	acceleration	m s^{-2}
m	mass	kg
n	normal vector	-
p	linear momentum	kg m s^{-1}
r	bucket radius	m
t	time	s
v	velocity	m s^{-1}
x	spatial coordinate	m
A	surface area	m^2
E	power	W
F	force	N
L	length	m
P	pressure	Pa
R	volume	m^3
S	surface	m^2
T	torque	N m
W	weight	N
λ	unit vector	-
μ	dynamic viscosity	kg m^{-1} s^{-1}
ρ	density	kg m^{-3}

Subscript

i	i-th component
m	mechanical
n	n-th component
r	radial
x	x-coordinate
rel	relative
tot	total
B	body
E	external
J	jet
R	rotor
S	surface

Introduction

Most treatises follow the standard approach to the Pelton turbine, based upon velocity triangles (1–3). This approach, introduced by the well-known Swiss mathematician L. Euler (1707–83) (4), is founded on the angular momentum theorem. The notion of mechanic torque, essential for rotating machines, appeared later on in a treatise on Statics written by L. Poinsot (5). Three years later, the American inventor L. A. Pelton deposited the patent request for the turbine bearing his name. The standard approach provides a straightforward expression of the torque; there is no need to examine closely the bucket-flow interaction. Nevertheless, the same results are achieved by an alternative approach based on the dynamics of variable mass systems (6). Although a bit harder, it provides a deeper analysis of different contributions to the torque. Furthermore, the presented approach could become an effective design tool for improved buckets by removing the usual simplifying hypotheses on which the basic theory of Pelton turbine is based. Actually, the structure and setup of this machine are well established nowadays in the high power range. However, a new interest on improved performance has grown very recently, due to the development of small hydropower plants (7).

Fundamentals on the dynamics of variable mass systems

Particle dynamics rests on the so-called Newton's second law (6) establishing a relationship between the resultant of the external forces $\vec{F}$ [kg], the linear momentum $\vec{p}$ [kg m s^{-1}], and time t [s], valid for inertial reference frames, as follows:

$$\vec{F} = \frac{d\,\vec{p}}{dt} \tag{1}$$

where the linear momentum is given by the product of the particle mass m [kg] and its velocity $\vec{v}$ [m s^{-1}] i.e.

$$\vec{p} = m\,\vec{v} \tag{2}$$

Equations are easily extended to rigid bodies by the notion of barycentre. Whether the reference frame is not inertial, fictitious forces (e.g. the centrifugal force or the Coriolis force) can be added to the left-hand side of eqn (1) to keep its validity.

If the mass of the body is constant, replacing eqn (2) in eqn (1), we obtain

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} \tag{3}$$

where $\vec{a}$ [m s^{-2}] is the acceleration.

Notice that eqn (1) is invariant with respect to Galileo's transformations only if the mass of the body is constant. Let us consider a body with variable mass $m(t)$ and two distinct inertial reference frames, which are indicated in the following with the subscripts 1 and 2, respectively. The (constant) relative velocity is then

$$\vec{v}_{rel} = \vec{v}_1 - \vec{v}_2. \tag{4}$$

Newton's second law, eqn (1), becomes

$$\vec{F}_1 = \vec{v}_1\frac{dm}{dt} + m\frac{d\vec{v}_1}{dt} \tag{5}$$

in the first reference frame, and

$$\vec{F}_2 = \vec{v}_2\frac{dm}{dt} + m\frac{d\vec{v}_2}{dt} = \left(\vec{v}_1 - \vec{v}_{rel}\right)\frac{dm}{dt} + m\frac{d\vec{v}_1}{dt} \tag{6}$$

in the second one.

Subtracting eqn (6) from eqn (5), one gets

$$\vec{F}_1 - \vec{F}_2 = \vec{v}_{rel}\frac{dm}{dt}. \tag{7}$$

Invariance with respect to Galileo's transformations is verified if $\vec{F}_1 = \vec{F}_2$; hence, one of the following conditions must be met:

(1) $\vec{v}_{rel} = 0$, which means that the two reference frames are identical or they are shifted by the same constant vector;
(2) $\frac{dm}{dt} = 0$, which means that the mass of the system is constant.

The principle of mass conservation is useful to generalise Newton's second law to the dynamics of a mass variable system. Let us consider a system composed of:

(1) a body of mass $m_B(t)$ and velocity $\vec{v}_B(t)$;
(2) a particle of infinitesimal mass dm and velocity $\vec{v}_E(t)$

in a given inertial reference frame.

The linear momentum of the whole system at the time t is given by

$$\vec{p}\,(t) = m_B \vec{v}_B + dm \cdot \vec{v}_E. \tag{8}$$

During the infinitesimal time interval dt, an anelastic collision between the body and the particle takes place so that, at time $t + dt$, the system is composed of a unique body of mass

$$m = m_B + dm \tag{9}$$

velocity

$$\vec{v} = \vec{v}_B + d\vec{v}_B = \vec{v}_B + \vec{a}_B \cdot dt \tag{10}$$

and linear momentum

$$\vec{p}\,(t + dt) = (m_B + dm) \cdot \left(\vec{v}_B + \vec{a}_B \cdot dt\right). \tag{11}$$

Subtracting eqn (8) from eqn (11), and neglecting second-order infinitesimals, we obtain

$$d\,\vec{p} = m_B \vec{a}_B \cdot dt - dm \cdot \left(\vec{v}_E - \vec{v}_B\right) \tag{12}$$

where $\vec{v}_{rel} = \vec{v}_E - \vec{v}_B$ is the relative velocity of the external particle with respect to the body.

Equation (12) is easily rewritten as

$$\frac{d\,\vec{p}}{dt} = m_B \vec{a}_B - \frac{dm}{dt} \cdot \vec{v}_{rel}. \tag{13}$$

Since the total system mass (body and particle) remains constant, eqn (1) applies, so that

$$\vec{F} = m_B \vec{a}_B - \frac{dm}{dt} \cdot \vec{v}_{rel}. \tag{14}$$

is the generalised form of Newton's second law for variable mass systems.

Another useful form of eqn (14) is obtained as

$$\vec{F} + \frac{dm}{dt} \cdot \vec{v}_{rel} = m_B \vec{a}_B. \tag{15}$$

Here, the second term on the left-hand side is interpreted as an additional force, which acts on the body according to its rate of mass variation. Usually it is called thrust

$$\vec{T} = \frac{dm}{dt} \cdot \vec{v}_{rel}. \tag{16}$$

Hence, for a variable mass system, Newton's second law is generalised as

$$\frac{d\vec{p}}{dt} = \vec{F} + \vec{T}. \tag{17}$$

Typical examples of application of eqn (17) are conveyor belts, jet engines and rocket propulsion (6). In the following, the same equation will be used to describe the interaction between a fluid flow and the buckets of a Pelton turbine. To this purpose, the integral form of the well-known Navier-Stokes (N-S) equation describing incompressible flow fields (8) is interpreted according to the theory of variable mass systems. Let us consider at first the local formulation of the N-S equation for the generic i-th component of the linear momentum in conservative form:

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{n=1}^{3} \frac{\partial(\rho v_i v_n)}{\partial x_n} = \sum_{n=1}^{3} \frac{\partial}{\partial x_n}\left(\mu \frac{\partial v_i}{\partial x_n}\right) - \frac{\partial P}{\partial x_i} + \rho g_i \tag{18}$$

where x [m] is the spatial coordinate, t [s] the time, v [m s^{-1}] the velocity component, g_i [m s^{-2}] the i-th component of the acceleration due to gravity, ρ [kg m^3] the density, P [Pa] the pressure and μ [kg m^{-1} s^{-1}] the dynamic viscosity. In case of turbulent flow, eqn (18) still applies, provided that velocity components and pressure are replaced by their time averages and the sum of molecular and eddy viscosity occurs inside the spatial derivative (9).

We integrate eqn (18) on a finite region of volume *R (t)*, delimited by a boundary of area *S (t)* generally moving with a velocity $\vec{v}_S$. The details of the integration are omitted since they can be found in Ref. (10). The resulting expression is

$$\frac{d}{dt}\left[\iiint_{R(t)} \rho v_i \, d^3\vec{x}\right] = \oiint_{S(t)} \left[\rho v_i\left(\vec{v}_S - \vec{v}\right) + \mu \vec{\nabla} v_i - P\vec{\lambda}_i\right] \cdot \vec{n}_S \, d^2\vec{x} + W_i \tag{19}$$

where $\vec{x}$ [m] is the position vector, $\vec{\lambda}_i$ the unit vector along the i-axis, $\vec{n}_S$ the outer-pointing normal to the boundary and W_i [N] the i-th component of the weight.

Equation (19) may be regarded as a special case of eqn (17), where

- the volume integral in the left-hand side is the i-th component of $\vec{p}$;
- $\oiint_{S(t)} \left[\rho v_i\left(\vec{v}_S - \vec{v}\right)\right] \cdot \vec{n}_S d^2\vec{x}$ is the i-th component of the thrust $\vec{T}$;
- the other terms are the i-th components of external forces.

This equation proves useful to study the flow through a Pelton turbine by computing the forces exerted by the fluid on the buckets, hence the torque, as it is shown in the following section.

Analysis of the Pelton turbine

For a detailed discussion of the Pelton turbine, as depicted in Fig. 1a, the reader is referred to any textbook on hydraulic machines (1–3). Only aspects relevant to the present discussion will be mentioned here, namely:

(1) The water jet is discharged from the distributor directly into the atmosphere.
(2) The water jet runs freely from the distributor to the buckets.
(3) The mechanical work produced by the rotor only stems from the conversion of the jet kinetic energy.
(4) The buckets, shown in detail in Fig. 1b, exhibit the typical double-spoon shape.
(5) The water jet undergoes a deviation due to the interaction with a bucket, and, ideally, it leaves the bucket in the opposite direction after a U-turn.
(6) Water is treated as an ideal liquid (i.e. incompressible and viscosity free).
(7) One-dimensional flow.
(8) Steady-state regime.

Figure 2 shows schematically the distributor, the free jet and the U-shaped bend (elbow) representing the bucket in the inertial reference frame of the plant, where the distributor is at rest, the jet velocity is v_J and the U-shaped bend translates at a velocity v_R. For sake of simplicity, the cross-sectional area A_J is assumed to be uniform throughout the system.

The resistance torque applied to the rotor determines the force F_R equal and opposed to the motive force that the water jet exerts on the bucket. The rotor reference frame is not inertial, but it can be replaced by an inertial one translating along the x-axis with velocity v_R. In the latter, the bucket impinged by the jet is temporarily fixed, as depicted in Fig. 3a, whereas in the former it moves with velocity v_R, as shown in Fig. 3b, where it is seen that the jet velocities at the inlet and at the outlet of the bucket are v_J and $2v_R - v_J$, respectively. Hence, in the inertial

a b

Fig. 1 *The Pelton wheel (a) and a detail of the buckets (b).*

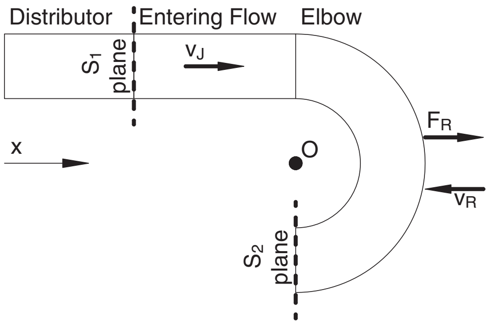

Fig. 2 *Schematic of the idealized water jet.*

reference frame integral with the distributor, the control volume bounded by the distributor outlet and the bucket outlet is a variable mass system, as the mass flow rate $\rho A_J v_J$ [kg s^{-1}] leaving the distributor is different from the one leaving the bucket, equal to $\rho A_J(v_J - 2v_R)$. The mass variation is due to the motion of the bucket, as shown in Fig. 4.

Due to the hypotheses 6 and 7, eqn (19) for the grey-shaded control volume writes

$$\frac{d}{dt}\left[\iiint_{R(t)} \rho v_x \; d^3\vec{x}\right] = \oint\!\!\!\oint_{S(t)} \left[\rho v_x\left(\vec{v}_S - \vec{v}\right) - p\vec{\lambda}_x\right]\cdot\vec{n}_S \; d^2\vec{x} \tag{20}$$

where the x-component of the weight is zero as the jet is horizontal.

To evaluate the left-hand term, it is convenient to separate two subsystems, namely the fluid within the free jet and the fluid within the bucket (see Fig. 4).

(1) The momentum of the fluid within the free jet is given by

$$p_J = \rho A_J[L_J(0) + v_R t]v_J \tag{21}$$

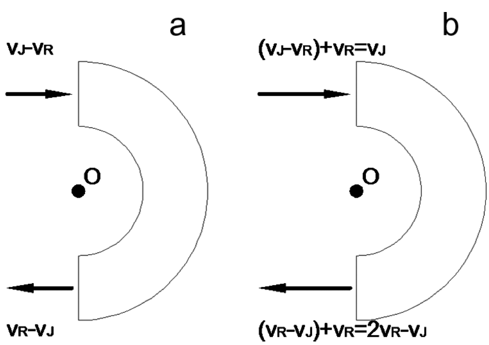

Fig. 3 *The elbow in a reference frame integral with the jet (a) and with the distributor (b).*

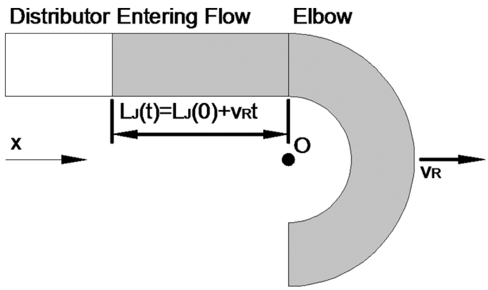

Fig. 4 *Control volume (in grey) used to evaluate the force* $\vec{F}_R$.

Hence, its time derivative is

$$\frac{dp_J}{dt} = \rho A_J v_R v_J \tag{22}$$

(2) The momentum of the fluid within the bucket is constant in time, since this subsystem has constant mass and the flow regime is steady (see Fig. 3a). Hence

$$\frac{dp_R}{dt} = 0. \tag{23}$$

Consequently

$$\frac{d}{dt}\left[\iiint\limits_{R(t)} \rho v_x \, d^3\vec{x}\right] = \frac{dp_J}{dt} + \frac{dp_R}{dt} = \rho A_J v_R v_J. \tag{24}$$

To evaluate the right hand term, it is convenient to split it into four surface integrals by considering four different portions of the boundary surface, namely, referring to Fig. 4, the distributor outlet section S_1, the jet lateral surface S_l, the inner surface of the bucket S_R, and the jet discharge surface S_2. Furthermore, as the Pelton turbine operates at atmospheric pressure, relative pressure will be considered to avoid constant terms vanishing in the final computation.

(1) To integrate over the surface S_1, we mind that:

$v_x = v_J$;
$\vec{v}_S = 0$;
$\vec{n}_S = -\vec{\lambda}_x$;
$P = 0$.

Hence

$$\oiint_{S_1} \left[\rho v_x\left(\vec{v}_S - \vec{v}\right) - P\vec{\lambda}_x\right]\cdot\vec{n}_S \; d^2\,\vec{x} = \oiint_{S_1} \rho v_J^2 \; d^2\,\vec{x} = \rho A_J v_J^2. \tag{25}$$

(2) To integrate over the surface S_l, we mind that:

$v_x = v_J$;
$\vec{v} = v_J\,\vec{\lambda}_x$ and $\vec{n}_S = \vec{\lambda}_r \perp \vec{\lambda}_x$, so that $\vec{v}\cdot\vec{n}_S = 0$;
$\vec{v}_S = 0$;
$P = 0$.

Hence

$$\oiint_{S_l} \left[\rho v_x\left(\vec{v}_S - \vec{v}\right) - P\vec{\lambda}_x\right]\cdot\vec{n}_S \; d^2\,x = 0. \tag{26}$$

(3) To integrate over the surface S_R, we mind that:

$\vec{v}_S - \vec{v}$ is directed circumferentially whereas $\vec{n}_S$ is directed radially, so that $\left(\vec{v}_S - \vec{v}\right)\cdot\vec{n}_S = 0$; the relative pressure is zero at the free surface of the jet, whereas at the jet-bucket interface the force $\vec{F}_R = -F_R\,\vec{\lambda}_x$ is exerted by the bucket on the jet. Hence

$$\oiint_{S_R} \left[\rho v_x\left(\vec{v}_S - \vec{v}\right) - P\vec{\lambda}_x\right]\cdot\vec{n}_S \; d^2\,\vec{x} = -F_R. \tag{27}$$

(4) To integrate over the surface S_2, we mind that:

$v_x = 2v_R - v_J$;
$\vec{v} = v_x\vec{\lambda}_x$;
$\vec{v}_S = v_R\vec{\lambda}_x$;
$\vec{n}_S = -\vec{\lambda}_x$;
$P = 0$.

Hence

$$\oiint_{S_R} \left[\rho v_x\left(\vec{v}_S - \vec{v}\right) - P\vec{\lambda}_x\right]\cdot\vec{n}_S \; d^2\,\vec{x} = \rho(2v_R - v_J)(v_R - v_J)A_J. \tag{28}$$

Finally, eqn (20) becomes

$$\rho A_J v_R v_J = \rho A_J v_J^2 - F_R + \rho(2v_R - v_J)(v_R - v_J)A_J. \tag{29}$$

Rearranging, the force directly exerted on the bucket by the jet is

$$F_R = 2\,\rho A_J (v_J - v_R)^2. \tag{30}$$

However, eqn (30) does not represent the only force acting on the rotor. Two further contributions arise from the intake and the discharge of the jet in the bucket. They can be evaluated referring to Fig. 5, where, in the reference frame integral with the distributor, two auxiliary variable mass systems are highlighted, namely $R_1(t)$ and $R_2(t)$. Notice that both systems have the same mass

$$m = \rho R_1(t) = \rho R_2(t) = \rho A_J (v_J - v_R) t. \tag{31}$$

During the time interval dt, the mass variation is then $dm = \rho A_J(v_J - v_R)dt$. The corresponding thrust can be evaluated according to eqn 16.

As the jet velocity relative to the distributor at the bucket intake is v_J, the thrust acting on system R_1 is

$$F_1 = \rho A_J (v_J - v_R) v_J. \tag{32}$$

On the other hand, as the jet velocity relative to the distributor at the discharge section is $2v_R - v_J$, the thrust acting on system R_2 is

$$F_2 = \rho A_J (v_J - v_R)(2v_R - v_J). \tag{33}$$

The total force acting on the bucket is then

$$F_{tot} = F_R + F_1 + F_2 = 2\rho A_J v_J (v_J - v_R) = \dot{m}(v_J - v_R) \tag{34}$$

where $\dot{m} = 2\rho A_J v_J$ [kg s^{-1}] is the water mass flow rate.

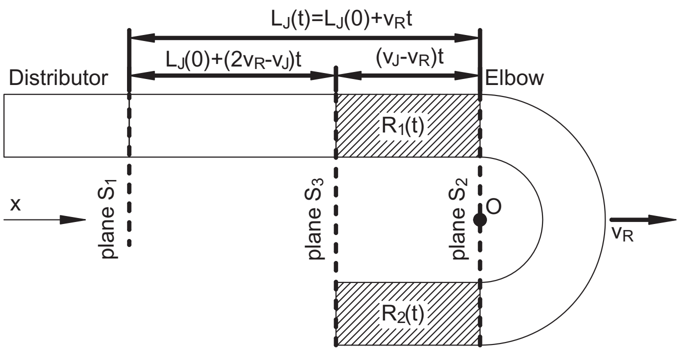

Fig. 5 *Control volumes (shaded) used to evaluate forces* $\vec{F}_1$ and $\vec{F}_2$.

It immediately follows the mechanical torque T_m [N m]

$$T_m = F_{tot} r \tag{35}$$

where r [m] is the mean bucket radius, and the mechanical power E_m [W]

$$E_m = m\, v_R(v_J - v_R). \tag{36}$$

Conclusions

Although the main results of the basic theory on the Pelton turbine, summarized in eqns (34), (35) and (36), are deduced more easily by applying the angular momentum theorem, the argument presented in this paper can be viewed as an alternative teaching approach at the advanced graduate level. This is particularly useful to highlight the different contributions of the total force acting on a bucket, namely:

(1) the force exerted through the liquid column between the distributor and the bucket inlet i.e. F_1;
(2) the force arising from the direct interaction between the flow and the bucket i.e. F_R;
(3) the force exerted through the liquid column between the bucket outlet and the discharge channel i.e. F_2.

From a technical point of view, the performance decay in a real machine is a consequence of the reduction of all these forces compared with the idealized case. In particular, F_R is lowered by the viscous stresses, whereas F_1 and F_2 are no longer effective as the jet integrity is compromised. This is due, for example, to the interference between consecutive buckets that reduces F_1: in actual Pelton turbines, the jet does not undergo exactly a U-turn; its deviation is slightly less than 180 degrees. On the other hand, fluid-dynamic instabilities determine the disintegration of the jet at the bucket outlet, thus considerably reducing F_2. Numerical methods are then required to solve eqn (18) and simulate the complex jet-bucket interaction (11).

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