COMPARISON RESULTS FOR GARCH PROCESSES

FABIO BELLINI,* *Universitá di Milano Bicocca*
FRANCO PELLEREY,** *Politecnico di Torino*
CARLO SGARRA,*** *Politecnico di Milano*
SALIMEH YASAEI SEKEH,**** *University of Bojnord*

1. Introduction

An extensive literature is available on applications of stochastic orders to finance and insurance markets. The implications of stochastic orders for derivative pricing and risk management have been used to provide bounds for option prices or related quantities in incomplete markets; the literature focused on this subject is quite extensive and we just mention the papers by El Karoui *et al.* [5], Bellamy and Jeanblanc [1], and Gushchin and Mordecki [9]. The increasing dependence of European option prices on the riskiness of the underlying asset is a well-known property for basic models such as the Black–Merton–Scholes model, in which riskiness is expressed in terms of the logreturns distribution variance: the uncertainty is quantified there through the dispersion around the expected value and the distribution functions can be ordered according to their 'peakedness'; the larger the dispersion, the higher the option prices. This very elementary and intuitive observation for simple models becomes more involved when turning attention to more complex models, where a more rigorous approach is necessary in order to avoid wrong conclusions.

The Black–Merton–Scholes model is nowadays considered fairly inadequate to describe the asset price dynamics; several empirical facts cannot be explained on the basis of this

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* Postal address: Dipartimento di Metodi Quantitativi, Universitá di Milano Bicocca, Via Bicocca degli Arcimboldi 8, 20126 Milano, Italy. Email address: fabio.bellini@unimib.it
** Postal address: Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca Degli Abruzzi 24, 10129 Torino, Italy. Email address: franco.pellerey@polito.it
*** Postal address: Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo Da Vinci 32, 20133 Milano, Italy. Email address: carlo.sgarra@polimi.it
**** Postal address: Department of Mathematics, University of Bojnord, Bojnord, Iran.
Email address: sa_yasaei@yahoo.com

model: some statistical features exhibited by logreturns, such as fat tails, volatility clustering, aggregational Gaussianity, and the so-called leverage effect, are completely outside of the prevision properties of the Black–Merton–Scholes model. Moreover, a very relevant phenomenon exhibited by option prices, the 'volatility smile' (and its term structure) cannot be explained on the basis of this model. In order to provide a more satisfactory description, several different models have been introduced. Some of these models introduce a stochastic dependence in volatility and/or jumps in asset logreturn (and/or in volatility) dynamics both in continuous- and discrete- time settings.

Among discrete-time models introduced in order to remove some of the drawbacks of the Black–Merton–Scholes model, the class of autoregressive conditioned heteroschedastic (ARCH) models introduced by Engle [6] and their general extension (GARCH models) proposed by Bollerslev [4] have aroused considerable interest.

Several results related to stochastic orders are available for the continuous-time models class: in [2], a systematic investigation on semimartingale models is performed; the models considered there include the Heston, and the Barndorff-Nielsen and Shephard models. Møller [12] provided some results on stochastic orders in a dynamic reinsurance market where the traded risk process is driven by a compound Poisson process and the claim amount is unbounded.

The purpose of the present paper is to present a systematic investigation of stochastic order propagation in a GARCH context.

Comparison with stochastic orders in incomplete market models can give rise to different classes of problems: one in which the comparison of models under the same probability measure but with different parametric specification is considered, and another in which the problem of comparing the same model under different probability measures is examined; as a matter of fact, when markets are incomplete, there are several probability measures equivalent to the historical one, under which the dynamics of prices can be given. In this paper we shall focus on the first class of mentioned problems: we shall provide a systematic comparison of logreturns and then of prices when the model parameters change, but the dynamics are specified under the same probability measure. In a GARCH context the parameters entering into play are three parameters assuming a numerical value and the innovations, which are random independent and identically distributed variables for which the density function is assigned.

We present a numerical experiment in order to motivate more strongly our investigation. Considering a standard GARCH(1, 1) model as in (3) Section 2, we simulate 1 000 000 trajectories of length $n = 50$, and, for each trajectory, we compute the total logreturn $S_{50} = \sum_{k=0}^{50} X_k$. The chosen parameters are $\alpha_0 = 1.3 \times 10^{-6}$, $\alpha_1 = 0.08$, $\beta_1 = 0.8, 0.85, 0.9$; the initial variance σ_0 is initialized in its unconditional value.

In the left-hand diagram of Figure 1 we compare the kernel-smoothed densities of the logreturn sums S_{50} for the three values 0.8, 0.85, and 0.9 of the parameter β_1; it is evident that these densities are increasing in the convex order (whose definition is recalled in Section 2) when the parameter β_1 increases. It is then natural to ask if this simple remark can be made rigorous, and if this conclusion can be cast into a more general framework that includes different kinds of stochastic orders and enables a comparison between stochastic order propagation from innovations to logreturns and from innovations to logreturn sums.

Let us consider the sequence of 'stock prices' defined by $Q_{k+1} = Q_0 \exp(S_k),\ k = 0, 1, \ldots$. Assuming an initial value of the underlying $Q_0 = 10\$$ and a zero interest rate, we then compute the Monte Carlo prices C_i of 21 call options with equally spaced strikes K_i ranging from 9\$ to 11\$, given by

$$C_i = \hat{\mathbb{E}}[\max(Q_{51} - K_i, 0)] = \hat{\mathbb{E}}[\max(Q_0 \exp(S_{50}) - K_i, 0)],$$

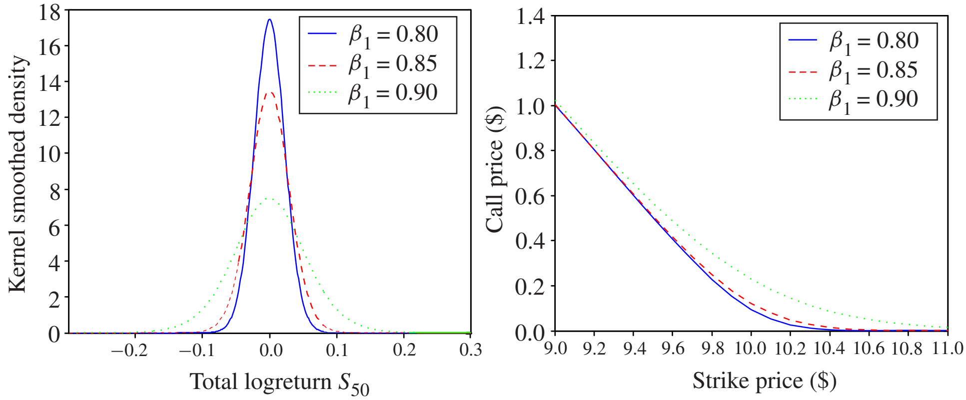

Figure 1: Comparison of the densities of the total logreturns in a GARCH(1, 1) model (*left*) and of the corresponding call option prices (*right*) for different values of the parameter β_1 in (3).

where the average $\hat{\mathbb{E}}$ is computed for each value of the parameter β_1 over the set of 1 000 000 simulated values of S_{50}.

The results are reported in the right-hand side of Figure 1; it is evident that call option prices are increasing in the parameter β_1. This is actually a consequence of the convex ordering of the total logreturns S_{50}, since the payoff of the call option is a convex function of the total logreturns S_{50}. The same argument would apply to every financial derivative with increasing and convex payoff; thus, convex ordering of total logreturns gives a sufficient condition for the comparison of prices of increasing and convex payoff across different models.

The paper is structured as follows. In the following section we briefly present the GARCH models and an auxiliary lemma. In Section 3 we present the univariate stochastic comparisons for logreturns in a GARCH setting. In Section 4 we study several stochastic orders that are naturally propagated from the innovations to the logreturns. In Section 5 we focus on convex ordering of the logreturn sums, while their multivariate convex ordering is the subject of Section 6. In the last section we focus on ordering with respect to the parameters in the GARCH(1, 1) case.

The main results provided in this paper require the assumption of a symmetric probability density for innovations. The extension of the stochastic order propagation analysis presented here to the case of nonsymmetric innovation densities would be of great interest: both the convex order propagation result and the comparison with respect to parameter variations will be the subject of future investigation in this more general setting. Moreover, the identification of some convex multivariate order which naturally propagate from innovations to logreturns is another target on which our research interest will be focused. These will be the subject of our future work.

2. General GARCH models

We consider GARCH models of two different very general forms; in both cases the innovations ε_n are assumed to be independent and identically distributed (i.i.d.). The first model (M1) is

$$\begin{gathered} X_n = \sigma_n \varepsilon_n, \quad n = 0, 1, \dots, \\ \varepsilon_n \perp \sigma_n, \qquad \mathbb{E}[\varepsilon_n] = 0, \qquad \sigma_{n+1} = f^I(|\varepsilon_n|, \sigma_n) \end{gathered} \tag{1}$$

with $f^I \colon \mathbb{R}_+^2 \to \mathbb{R}_+$ increasing and componentwise convex (iccx for brevity). Here, and everywhere throughout the paper, the symbol '$\perp$' denotes the usual stochastic independence between random variables, while the term increasing (decreasing) should be read in the nonstrict sense. Also, recall that a real function defined on $\mathbb{R}^n$ is said to be increasing whenever it is componentwise increasing.

The second model (M2) is

$$X_n = \sigma_n \varepsilon_n, \quad n = 0, 1, \ldots, \qquad \varepsilon_n \perp \sigma_n, \qquad \mathbb{E}[\varepsilon_n] = 0, \qquad \sigma_{n+1}^2 = f^{II}(\varepsilon_n^2, \sigma_n^2) \tag{2}$$

with $f^{II} \colon \mathbb{R}_+^2 \to \mathbb{R}_+$ iccx.

The difference between model M1 and model M2 is that in the first case the recursive dynamics are defined in terms of the volatility σ_n, while in the second they are defined in terms of the variance σ_n^2.

The usual GARCH(1, 1) model is a particular case of both M1 and M2, and is defined as

$$\begin{gathered} X_n = \sigma_n \varepsilon_n, \quad n = 0, 1, \ldots, \\ \varepsilon_n \perp \sigma_n, \qquad \mathbb{E}[\varepsilon_n] = 0, \qquad \sigma_{n+1}^2 = \alpha_0 + \alpha_1 X_n^2 + \beta_1 \sigma_n^2 \end{gathered} \tag{3}$$

with $\alpha_0, \alpha_1, \beta_1 > 0$ and $\alpha_1 + \beta_1 < 1$, in order to guarantee covariance stationarity. Both models start with a possibly random $\sigma_0 > 0$, by drawing a random ε_0.

The recursive equations for σ_{n+1} and σ_{n+1}^2 are examples of 'stochastic recurrences' in the sense of Chapter 4 of [11]. Thus, the volatilities in (1) and the variances in (2) can be also expressed as

$$\begin{aligned} \sigma_{n+1} &= f^I(|\varepsilon_n|, \sigma_n) \\ &= f^I(|\varepsilon_n|, f^I(|\varepsilon_{n-1}|, \sigma_n)) \\ &= f^I(|\varepsilon_n|, f^I(|\varepsilon_{n-1}|, f^I(|\varepsilon_{n-2}|, \sigma_{n-2}))) \\ &= \ldots \\ &= f^I(|\varepsilon_n|, f^I(|\varepsilon_{n-1}|, f^I(\ldots, f^I(|\varepsilon_0|, \sigma_0)))) \\ &:= g_{n+1}^I(\sigma_0, \varepsilon_0, \ldots, \varepsilon_n), \end{aligned} \tag{4}$$

and, similarly,

$$\sigma_{n+1}^2 := g_{n+1}^{II}(\sigma_0^2, \varepsilon_0^2, \ldots, \varepsilon_n^2), \tag{5}$$

for suitable functions g_{n+1}^I and g_{n+1}^{II}. Observing that compositions of iccx functions are still iccx (see [10] for this and further properties of iccx functions), as in [11], the following property holds.

Lemma 1. *Let $g_{n+1}^I, g_{n+1}^{II} \colon \mathbb{R}_+^{n+2} \to \mathbb{R}_+$ be defined as in (4) and (5). If f^I and f^{II} are* iccx*, then g_{n+1}^I and g_{n+1}^{II} are also* iccx.

3. Univariate comparisons of logreturns

The aim of this section is to establish comparison results for X_n when the distributions of the innovations are changed from ε_k to $\widetilde{\varepsilon}_k$ for $k \leq n$. In order to establish these results, the assumption that the innovations are identically distributed is not necessary (while the independency assumption is essential). In the following theorems only the distribution of a single innovation ε_k will be changed, and the impact of this change on X_n will be investigated.

We recall the definitions of the basic stochastic orderings.

Definition 1. The random variable X is said to be smaller than Y in the usual stochastic order (convex order, increasing convex order), denoted by $X \leq_{\mathrm{st}} Y$ $[X \leq_{\mathrm{cx}} Y,\ X \leq_{\mathrm{icx}} Y]$, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for all increasing (convex, increasing convex) functions $\phi\colon \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

We will see that in the general context of models M1 and M2 the orderings that are naturally propagated from the innovations ε_k to X_n are the '$\leq_{\mathrm{st}}$' and the '$\leq_{\mathrm{icx}}$' ordering between *absolute values or squared variables*. This clearly completely modifies their interpretation; in particular, in the next section we will see that the '$\leq_{\mathrm{st}}$' ordering between absolute values or squares can be interpreted as a *variability ordering*, while the '$\leq_{\mathrm{icx}}$' ordering between absolute values or squares can be interpreted as a *kurtosis* ordering.

In order to establish these results, we proceed in two steps: first we consider the volatilities σ_n and then the variables X_n. In the following, the variables with a tilde denote the variables obtained by substituting the innovations ε_k with $\widetilde{\varepsilon}_k$. The first step is an immediate consequence of Lemma 1.

Theorem 1. (Comparisons of σ_n and σ_n^2.)

(a) *Let σ_{n+1} be as in (1) and $|\varepsilon_k| \leq_{\mathrm{st}} |\widetilde{\varepsilon}_k|$. It follows that $\sigma_{n+1} \leq_{\mathrm{st}} \widetilde{\sigma}_{n+1}$.*

(b) *Let σ_{n+1} be as in (1) and $|\varepsilon_k| \leq_{\mathrm{icx}} |\widetilde{\varepsilon}_k|$. It follows that $\sigma_{n+1} \leq_{\mathrm{icx}} \widetilde{\sigma}_{n+1}$.*

(c) *Let σ_{n+1}^2 be as in (2) and $\varepsilon_k^2 \leq_{\mathrm{st}} \widetilde{\varepsilon}_k^2$. It follows that $\sigma_{n+1}^2 \leq_{\mathrm{st}} \widetilde{\sigma}_{n+1}^2$.*

(d) *Let σ_{n+1}^2 be as in (2) and $\varepsilon_k^2 \leq_{\mathrm{icx}} \widetilde{\varepsilon}_k^2$. It follows that $\sigma_{n+1}^2 \leq_{\mathrm{icx}} \widetilde{\sigma}_{n+1}^2$.*

Proof. Since, from Lemma 1 for model M1, it holds that $\sigma_{n+1} = g_{n+1}^{I}(\sigma_0, |\varepsilon_0|, \ldots, |\varepsilon_n|)$ with g_{n+1}^{I} iccx, (a) and (b) respectively follow from Theorem 1.A.3 of [14] and Theorem 4.A.15 of [14]. Similarly, since from Lemma 1 for model M2, it holds that $\sigma_{n+1}^2 = g_{n+1}^{II}(\sigma_0^2, \varepsilon_0^2, \ldots, \varepsilon_n^2)$ with g_{n+1}^{II} iccx, from the same theorems we obtain (c) and (d).

The comparison results for σ_n and σ_n^2 lead to the following comparisons of the variables X_n.

Theorem 2. (Comparisons of X_n.)

(a) *Let X_n be as in (1) and $|\varepsilon_k| \leq_{\mathrm{st}} |\widetilde{\varepsilon}_k|$. It follows that $|X_n| \leq_{\mathrm{st}} |\widetilde{X}_n|$.*

(b) *Let X_n be as in (1) and $|\varepsilon_k| \leq_{\mathrm{icx}} |\widetilde{\varepsilon}_k|$. It follows that $|X_n| \leq_{\mathrm{icx}} |\widetilde{X}_n|$.*

(c) *Let X_n be as in (2) and $\varepsilon_k^2 \leq_{\mathrm{st}} \widetilde{\varepsilon}_k^2$. It follows that $X_n^2 \leq_{\mathrm{st}} \widetilde{X}_n^2$.*

(d) *Let X_n be as in (2) and $\varepsilon_k^2 \leq_{\mathrm{icx}} \widetilde{\varepsilon}_k^2$. It follows that $X_n^2 \leq_{\mathrm{icx}} \widetilde{X}_n^2$.*

Proof. Since $|X_n| = \sigma_n |\varepsilon_n|$ and $X_n^2 = \sigma_n^2 \varepsilon_n^2$, with σ_n independent from ε_n, (a) and (c) follow from Theorem 1.A.3(b) of [14]. Similarly, (b) and (d) follow from Theorem 4.A.15 of [14].

A natural question that arises at this point is if the convex order is also propagated, that is, if $\varepsilon_k \leq_{\mathrm{cx}} \widetilde{\varepsilon}_k$ implies that $X_n \leq_{\mathrm{cx}} \widetilde{X}_n$. We prove that this is indeed the case for model M1. We start with a simple preliminary result.

Lemma 2. *Let σ and $\tilde{\sigma}$ be nonnegative, with $\sigma \leq_{\mathrm{st}} \tilde{\sigma}$. Let ε be independent from σ and $\tilde{\sigma}$, with $\mathbb{E}[\varepsilon] = 0$. Then $\sigma\varepsilon \leq_{\mathrm{cx}} \tilde{\sigma}\varepsilon$.*

Proof. Let $0 \leq \alpha_1 < \alpha_2$. It is a well-known fact that, for ε such that $\mathbb{E}[\varepsilon] = 0$, it holds that $\alpha_1\varepsilon \leq_{\mathrm{cx}} \alpha_2\varepsilon$ (see, e.g. Theorem 1.5.18 of [11]). Now let φ be any convex function, and consider the function $h(\alpha) := \mathbb{E}[\varphi(\alpha\varepsilon)]$. By $\alpha_1\varepsilon \leq_{\mathrm{cx}} \alpha_2\varepsilon$ and the definition of the convex order, it immediately follows that $h(\alpha_1) := \mathbb{E}[\varphi(\alpha_1\varepsilon)] \leq \mathbb{E}[\varphi(\alpha_2\varepsilon)] = h(\alpha_2)$, i.e. the function $h(\alpha)$ is increasing in $\alpha \geq 0$. Thus, by the inequality $\sigma \leq_{\mathrm{st}} \tilde{\sigma}$, it holds that $\mathbb{E}[\varphi(\sigma\varepsilon)] = \mathbb{E}[\mathbb{E}[\varphi(\sigma\varepsilon) \mid \sigma]] = \mathbb{E}[h(\sigma)] \leq \mathbb{E}[h(\tilde{\sigma})] = \mathbb{E}[\mathbb{E}[\varphi(\tilde{\sigma}\varepsilon) \mid \tilde{\sigma}]] = \mathbb{E}[\varphi(\tilde{\sigma}\varepsilon)]$ and the thesis follows.

Theorem 3. (Propagation of the convex order.) *Let X_n be as in (1) and $\varepsilon_k \leq_{\mathrm{cx}} \widetilde{\varepsilon}_k$; it follows that $X_n \leq_{\mathrm{cx}} \widetilde{X}_n$.*

Proof. We first note that, since $\varepsilon_k \leq_{\mathrm{cx}} \widetilde{\varepsilon}_k$, it follows that $|\varepsilon_k| \leq_{\mathrm{icx}} |\widetilde{\varepsilon}_k|$. Indeed, for each ϕ increasing and convex, the composition $\phi(|\cdot|)$ is convex; this implies that $\mathbb{E}[\phi(|\varepsilon_k|)] \leq \mathbb{E}[\phi(|\widetilde{\varepsilon}_k|)]$, that is, $|\varepsilon_k| \leq_{\mathrm{icx}} |\widetilde{\varepsilon}_k|$. From Proposition 1(b), it then follows that $\sigma_{n+1} \leq_{\mathrm{icx}} \widetilde{\sigma}_{n+1}$. From Theorem 4.A.6 of [14], there exists a random variable $\overline{\sigma}_{n+1}$ such that $\sigma_{n+1} \leq_{\mathrm{st}} \overline{\sigma}_{n+1} \leq_{\mathrm{cx}} \widetilde{\sigma}_{n+1}$. By Lemma 2, $\sigma_{n+1} \leq_{\mathrm{st}} \overline{\sigma}_{n+1}$ implies that $\sigma_{n+1}\varepsilon_{n+1} \leq_{\mathrm{cx}} \overline{\sigma}_{n+1}\varepsilon_{n+1}$; on the other hand, $\overline{\sigma}_{n+1} \leq_{\mathrm{cx}} \widetilde{\sigma}_{n+1}$ implies that $\overline{\sigma}_{n+1}\varepsilon_{n+1} \leq_{\mathrm{cx}} \widetilde{\sigma}_{n+1}\varepsilon_{n+1}$. By transitivity we obtain $\sigma_{n+1}\varepsilon_{n+1} \leq_{\mathrm{cx}} \widetilde{\sigma}_{n+1}\varepsilon_{n+1}$.

4. The relevant orderings

In the preceding section the orderings defined by $|X| \leq_{\mathrm{st}} |Y|$, $X^2 \leq_{\mathrm{st}} Y^2$, $|X| \leq_{\mathrm{icx}} |Y|$, and $X^2 \leq_{\mathrm{icx}} Y^2$ have arisen naturally. In order to better understand their meaning, in the following lemmas we identify some necessary and sufficient conditions in the symmetric case. Before stating our result, we need to define the *number of sign changes* of a function defined on the real line and the notion of the *sign sequence*.

Definition 2. Given a function f defined on the real line $\mathbb{R}$, the *number of sign changes $S^-(f)$* of f in $\mathbb{R}$ is equal to n, $S^-(f) = n$, if and only if there exists a partition $\mathcal{P}$ of $\mathbb{R}$ into $n+1$ subintervals I_k, $k = 1, \dots, n+1 < \infty$, such that

- f has constant sign in each subinterval,
- f does not vanish in any subinterval,
- f changes sign from each interval I_k to the next.

Definition 3. The *sign sequence* is the alternating sequence of signs of f on the partition $\mathcal{P} = (I_1, I_2, \dots, I_{n+1})$ defined above.

We have the following statement.

Lemma 3. *Let X and Y be symmetric with distributions F and G, respectively. The following conditions are equivalent.*

(a) $X^2 \leq_{\mathrm{st}} Y^2$.

(b) $|X| \leq_{\mathrm{st}} |Y|$.

(c) $X \leq_{\mathrm{peak}} Y$*, where '*$\leq_{\mathrm{peak}}$*' is the peakedness ordering (see [3]).*

(d) $S^-(G-F) = 1$ *with sign sequence* $+, -$, *where* $S^-(G-F)$ *is the number of intersections between G and F as defined above.*

Proof. The equivalence of (a) and (b) is an immediate consequence of Theorem 1.A.3 of [14]. The equivalence of (b) and (c) is the definition of the peakedness ordering, while the equivalence of (c) and (d) follows from Theorem 3.D.1 in [14].

Lemma 4. *Let X and Y be symmetric with distributions F and G. The following conditions are equivalent.*

(a) $X^2 \leq_{\mathrm{icx}} Y^2$.

(b) $\int_x^{+\infty} u\overline{F}(u)\,\mathrm{d}u \leq \int_x^{+\infty} u\overline{G}(u)\,\mathrm{d}u$ *for each* $x \geq 0$, *where* $\overline{F}(u) = 1 - F(u)$ *and* $\overline{G}(u) = 1 - G(u)$.

(c) $\mathbb{E}[(X^2 - k)^+] \leq \mathbb{E}[(Y^2 - k)^+]$ *for each* $k \geq 0$.

Proof. Under our hypothesis, $F_{X^2}(t) = 2F(\sqrt{t}) - 1$ and $\overline{F}_{X^2}(t) = 2 - 2F(\sqrt{t})$ for $t \geq 0$. The equivalence of (a) and (b) then follows from Theorem 4.A.2 of [14] with a simple change of variable. The equivalence of (a) and (c) is also a consequence of Theorem 1.5.7 of [11].

The first lemma shows that, for symmetric variables, the orderings $|X| \leq_{\mathrm{st}} |Y|$ and $X^2 \leq_{\mathrm{st}} Y^2$ are variability comparisons equivalent to the peakedness ordering, which in this case boils down to (d), that is, the validity of a single crossing condition between the distribution functions. In the typical econometric applications these orderings are however not very relevant since the innovations satisfy $\mathbb{E}[\varepsilon_k^2] = 1$, and, hence, $\varepsilon_k^2 \leq_{\mathrm{st}} \widetilde{\varepsilon}_k^2$ would imply that $\varepsilon_k^2 =_{\mathrm{st}} \widetilde{\varepsilon}_k^2$.

In the normalized case the ordering $X^2 \leq_{\mathrm{icx}} Y^2$ becomes equivalent to $X^2 \leq_{\mathrm{cx}} Y^2$; we prove a sufficient and a necessary condition for this.

Lemma 5. *Let X and Y be symmetric with absolutely continuous distributions F and G and densities f and g, respectively, with* $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$.

(a) *If the densities f and g cross four times, with the density f being lower in the tails and in the center, and higher in the intermediate region, then* $X^2 \leq_{\mathrm{icx}} Y^2$.

(b) *If $X^2 \leq_{\mathrm{icx}} Y^2$ and X and Y have finite fourth moments, then $\beta_2(X) < \beta_2(Y)$, where β_2 is Pearson's kurtosis coefficient.*

Proof. (a) Under our hypothesis, $f_{X^2}(t) = f(\sqrt{t})/\sqrt{t}$ for $t > 0$. Since X and Y are symmetric, the four intersection points between the densities f and g are symmetrical with respect to the origin. Hence, the densities of X^2 and Y^2 cross in two points and, since $\mathbb{E}[X^2] = \mathbb{E}[Y^2]$ from Theorem 3.A.44 of [14] we have $X^2 \leq_{\mathrm{cx}} Y^2$.

(b) In our case $\beta_2(X) = \mathbb{E}[X^4]$ and, hence, the thesis follows from the definition of the convex order.

This lemma shows that the comparison $X^2 \leq_{\mathrm{icx}} Y^2$ can be interpreted as a classical kurtosis ordering; the crossing condition is usually referred to in the kurtosis ordering literature as a Dyson–Finucan condition (see, for example, [8]).

5. Convex comparisons for total logreturns

In financial applications the variables X_n typically represent logreturns, which are additive quantities. The over-the-period total logreturn is given by $S_n = \sum_{k=0}^{n} X_k$. It is therefore natural to ask if some of the comparison results of Section 2 extend to the variables S_n. In this section we consider the case of the convex order, that is, whether, for $k \leq n$, $\varepsilon_k \leq_{\mathrm{cx}} \widetilde{\varepsilon}_k$ implies that $S_n \leq_{\mathrm{cx}} \widetilde{S}_n$, where $\widetilde{S}_n$ is the sum of logreturns when the innovations ε_k are substituted by $\widetilde{\varepsilon}_k$. The problem is not trivial since S_n cannot be expressed as a sum of independent variables, so standard results about convex ordering of sums cannot be applied; we are able to prove a positive result in the case of model M1 and for symmetric innovations. We start with a basic lemma.

Lemma 6. *Let $\phi \in C^2(\mathbb{R})$ be convex, and let $g_i \in C^2(\mathbb{R})$, $i = 1, \dots, m$, be convex and nonnegative. Let $a, b \in \mathbb{R}$ and $\Pi_m := \{-1, 1\}^m = \{\underline{p} = (p_1, \dots, p_m),\ p_i \in \{-1, 1\},\ i = 1, \dots, m\}$. It follows that*

$$h(u) = \sum_{\underline{p} \in \Pi_m} \phi\Big(a + bu + \sum_{i=1}^{m} p_i g_i(u)\Big)$$

is convex.

Proof. We can compute

$$\begin{aligned} h'(u) &= \sum_{\underline{p} \in \Pi_m} \phi'\Big(a + bu + \sum_{i=1}^{m} p_i g_i(u)\Big)\Big(b + \sum_{i=1}^{m} p_i g_i'(u)\Big), \\ h''(u) &= \sum_{\underline{p} \in \Pi_m} \phi''\Big(a + bu + \sum_{i=1}^{m} p_i g_i(u)\Big)\Big(b + \sum_{i=1}^{m} p_i g_i'(u)\Big)^2 \\ &\quad + \sum_{\underline{p} \in \Pi_m} \phi'\Big(a + bu + \sum_{i=1}^{m} p_i g_i(u)\Big)\Big(\sum_{i=1}^{m} p_i g_i''(u)\Big). \end{aligned}$$

The first term is positive; the second is given by

$$A_m = \sum_{\underline{p} \in \Pi_m} \phi'\Big(a + bu + \sum_{i=1}^{m} p_i g_i(u)\Big)\Big(\sum_{i=1}^{m} p_i g_i''(u)\Big).$$

Let us denote by $\underline{P} = (P_1, P_2, \dots, P_m)$ a random vector with a discrete uniform distribution on Π_m; clearly, $\mathbb{E}[\underline{P}] = \underline{0}$, the components of $\underline{P}$ are independent, and

$$A_m = 2^m \mathbb{E}[\phi'(a + bu + \underline{g}(u) \cdot \underline{P})(\underline{g}''(u) \cdot \underline{P})],$$

where $\underline{g}(u) = (g_1(u), \dots, g_m(u))$ and $\underline{g}''(u) = (g_1''(u), \dots, g_m''(u))$. Since the functions $\phi'(a + bu + \underline{g}(u) \cdot \underline{p})$ and $\underline{g}''(u) \cdot \underline{p}$ are increasing in $\underline{p} \in \Pi_m$, from the covariance inequality, it follows that

$$\begin{aligned} A_m &= 2^m \mathbb{E}[\phi'(a + bu + \underline{g}(u) \cdot \underline{P})(\underline{g}''(u) \cdot \underline{P})] \\ &\ge 2^m \mathbb{E}[\phi'(a + bu + \underline{g}(u) \cdot \underline{P})]\mathbb{E}[(\underline{g}''(u) \cdot \underline{P})] \\ &= 0. \end{aligned}$$

This completes the proof.

Remark 1. As is well known, a random vector whose components are independent is associated, in the sense of Esary *et al.* [7]. We recall that the random vector $\underline{X} = (X_1, X_2, \dots, X_n)$ is said to be *associated* if

$$\operatorname{cov}(h_1(X_1, X_2, \dots, X_n), h_2(X_1, X_2, \dots, X_n)) \ge 0$$

for all increasing functions h_1 and h_2 for which the above covariance is defined. We want to emphasize the role played by this property in the proof of Lemma 6, and later in the proofs of Lemma 8 and Theorem 6.

We also note that in the previous lemma the smoothness requirements on ϕ and on the g_i can be dropped; we preferred this formulation in order to simplify the proof. Since in this section we consider only model M1, we define

$$g_n(\sigma_0, \varepsilon_0, \ldots, \varepsilon_{n-1}) := g_n^I(\sigma_0, |\varepsilon_0|, \ldots, |\varepsilon_{n-1}|);$$

from Lemma 1, it is clear that g_n is even and ccx. We have

$$\begin{aligned} S_n &= X_0 + X_1 + \cdots + X_n \\ &= \sigma_0\varepsilon_0 + \sigma_1\varepsilon_1 + \cdots + \sigma_n\varepsilon_n \\ &= \sigma_0\varepsilon_0 + g_1(\sigma_0, \varepsilon_0)\varepsilon_1 + \cdots + g_n(\sigma_0, \varepsilon_0, \ldots, \varepsilon_{n-1})\varepsilon_n \\ &= S_n(\sigma_0, \varepsilon_0, \ldots, \varepsilon_n). \end{aligned} \tag{6}$$

The main problem in proving the propagation of convexity to the sums is that S_n is not a convex function of the innovations ε_k; indeed, each g_k in (6) is multiplied by a possibly negative innovation ε_k. This prevents the application of standard results and requires the development of a specific technique based on Lemma 6. The basic idea is that in the case of symmetric innovations it is possible to restore the convexity by averaging over all the possible sign changes, as in Lemma 6. This will be done in a recursive way, using the following preliminary statement, where $\mathbb{E}_{\varepsilon_{k+1},\ldots,\varepsilon_n}$ denotes the expectation with respect to the random variables $\varepsilon_{k+1}, \ldots, \varepsilon_n$, i.e. for every function f for which the following integral exists:

$$\begin{aligned} &\mathbb{E}_{\varepsilon_{k+1},\ldots,\varepsilon_n}[f(S_n(\sigma_0, \varepsilon_0, \ldots, \varepsilon_n))] \\ &\quad = \int_{R^{n-k}} f(S_n(\sigma_0, \varepsilon_0, \ldots, \varepsilon_k, x_{k+1}, \ldots, x_n))F_{\varepsilon_{k+1},\ldots,\varepsilon_n}(\mathrm{d}x_{k+1}, \ldots, \mathrm{d}x_n). \end{aligned}$$

Here $F_{u_{k+1},\ldots,u_n}$ denotes the joint distribution function of the random variables $u_{k+1}, \ldots, u_n$.

Lemma 7. *Let X_n and S_n be as in (1) and (6). Let ϕ be convex, and let the innovations ε_i be symmetric. Then the function*

$$h(\sigma_0, \varepsilon_0, \ldots, \varepsilon_k) := \mathbb{E}_{\varepsilon_{k+1},\ldots,\varepsilon_n}[\phi(S_n(\sigma_0, \varepsilon_0, \ldots, \varepsilon_n))] \tag{7}$$

is convex in ε_k for each fixed value of $\sigma_0, \varepsilon_0, \ldots, \varepsilon_{k-1}$.

Proof. To avoid cumbersome notation, we drop the arguments of the functions g_i. Since the innovations are symmetric and g_i is even, we can write

$$\begin{aligned} &\mathbb{E}_{\varepsilon_{k+1},\ldots,\varepsilon_n}[\phi(S_n(\sigma_0, \varepsilon_0, \ldots, \varepsilon_n))] \\ &\quad = \mathbb{E}_{\varepsilon_{k+1},\ldots,\varepsilon_n}[\phi(\sigma_0\varepsilon_0 + \cdots + g_n\varepsilon_n)] \\ &\quad = \mathbb{E}_{\varepsilon_{k+1},\ldots,\varepsilon_n}\Bigg[\sum_{\underline{p}\in\Pi_{n-k}} \phi(\sigma_0\varepsilon_0 + \cdots + p_1 g_{k+1}\varepsilon_{k+1} + \cdots + p_{n-k}g_n\varepsilon_n)\,\mathbf{1}_{\{\varepsilon_{k+1}\geq 0,\ldots,\varepsilon_n\geq 0\}}\Bigg]. \end{aligned}$$

Defining

$$\overline{h}(\sigma_0, \varepsilon_0, \ldots, \varepsilon_k, \ldots, \varepsilon_n) = \sum_{\underline{p}\in\Pi_{n-k}} \phi(\sigma_0\varepsilon_0 + g_1\varepsilon_1 + \cdots + g_k\varepsilon_k + p_1 g_{k+1}\varepsilon_{k+1} + \cdots + p_{n-k}g_n\varepsilon_n),$$

it follows that

$$h(\sigma_0, \varepsilon_0, \ldots, \varepsilon_{k-1}, \varepsilon_k) = \mathbb{E}_{\varepsilon_{k+1},\ldots,\varepsilon_n}[\mathbf{1}_{\{\varepsilon_{k+1}\geq 0,\ldots,\varepsilon_n\geq 0\}}\,\overline{h}(\sigma_0, \varepsilon_0, \ldots, \varepsilon_k, \ldots, \varepsilon_n)]$$

and $\overline{h}$ is convex in ε_k from Lemma 6. It also follows that $h(\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}, \varepsilon_k)$ is convex in ε_k for each value of $\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}$.

We can finally state the main result on the propagation of the convex order to S_n.

Theorem 4. *Let X_n and S_n be as in (1) and (6). Let the innovations ε_i be symmetric. If $\widetilde{\varepsilon}_k$ is also symmetric and $\widetilde{\varepsilon}_k \geq_{\mathrm{cx}} \varepsilon_k$, then $\widetilde{S}_n := S_n(\sigma_0, \varepsilon_0, \dots, \widetilde{\varepsilon}_k, \dots, \varepsilon_n) \geq_{\mathrm{cx}} S_n(\sigma_0, \varepsilon_0, \dots, \varepsilon_k, \dots, \varepsilon_n) = S_n$.*

Proof. Let ϕ be convex. From the independence of the ε_i we can write

$$\begin{aligned}\mathbb{E}[\phi(\widetilde{S}_n)] &= \mathbb{E}_{\varepsilon_0,\dots,\varepsilon_{k-1}}[\mathbb{E}_{\widetilde{\varepsilon}_k}[\mathbb{E}_{\varepsilon_{k+1},\dots,\varepsilon_n}[\phi(S_n(\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}, \widetilde{\varepsilon}_k, \varepsilon_{k+1}, \dots, \varepsilon_n))]]] \\ &= \mathbb{E}_{\varepsilon_0,\dots,\varepsilon_{k-1}}[\mathbb{E}_{\widetilde{\varepsilon}_k}[h(\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}, \widetilde{\varepsilon}_k)]],\end{aligned}$$

where, as in (7),

$$h(\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}, \widetilde{\varepsilon}_k) := \mathbb{E}_{\varepsilon_{k+1},\dots,\varepsilon_n}[\phi(S_n(\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}, \widetilde{\varepsilon}_k, \varepsilon_{k+1}, \dots, \varepsilon_n))],$$

which is a convex function of $\widetilde{\varepsilon}_k$ for each value of $\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}$ from Lemma 7.

It follows that

$$\mathbb{E}_{\widetilde{\varepsilon}_k}[h(\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}, \widetilde{\varepsilon}_k)] \geq \mathbb{E}_{\varepsilon_k}[h(\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}, \varepsilon_k)],$$

which gives

$$\begin{aligned}\mathbb{E}[\phi(\widetilde{S}_n)] &= \mathbb{E}_{\varepsilon_0,\dots,\varepsilon_{k-1}}[\mathbb{E}_{\widetilde{\varepsilon}_k}[h(\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}, \widetilde{\varepsilon}_k)]] \\ &\geq \mathbb{E}_{\varepsilon_0,\dots,\varepsilon_{k-1}}[\mathbb{E}_{\varepsilon_k}[h(\sigma_0, \varepsilon_0, \dots, \varepsilon_{k-1}, \varepsilon_k)]] \\ &= \mathbb{E}[\phi(S_n)],\end{aligned}$$

that is, $\widetilde{S}_n \geq_{\mathrm{cx}} S_n$.

Remark 2. We note that, since the price of the underlying asset Q at maturity $t = n + 1$ is given by

$$Q_{n+1} = Q_0 \exp(S_n),$$

which is an increasing and convex function of the total logreturn S_n, from a well-known lemma (see, for example, [14, Theorem 4.A.15]), it follows that the convex ordering of S_n implies *increasing convex ordering* of Q_n; thus, as a corollary, the price of any increasing convex payoff is higher with innovations $\widetilde{\varepsilon}_k$.

6. Multivariate comparisons of logreturns

In the literature there are several multivariate generalizations of the convex order (see, for example, [11], [14], and the references therein). Until now we have proved only univariate comparison results for the *scalar* variables X_n and S_n; in this section we wonder whether the more general multivariate comparisons for the *vector* $(X_1, \dots, X_n)$ also hold. Before stating a positive result, we recall two basic definitions.

Definition 4. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is directionally convex if, for any $\boldsymbol{x}_1, \dots, \boldsymbol{x}_4 \in \mathbb{R}^n$ such that $\boldsymbol{x}_1 \leq \boldsymbol{x}_2$, $\boldsymbol{x}_3 \leq \boldsymbol{x}_4$, and $\boldsymbol{x}_1 + \boldsymbol{x}_4 = \boldsymbol{x}_2 + \boldsymbol{x}_3$, it holds that

$$\varphi(\boldsymbol{x}_2) + \varphi(\boldsymbol{x}_3) \leq \varphi(\boldsymbol{x}_1) + \varphi(\boldsymbol{x}_4).$$

Definition 5. A function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is supermodular if, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, it holds that

$$\varphi(\boldsymbol{x}) + \varphi(\boldsymbol{y}) \le \varphi(\boldsymbol{x} \wedge \boldsymbol{y}) + \varphi(\boldsymbol{x} \vee \boldsymbol{y}),$$

where the operators '$\wedge$' and '$\vee$' respectively denote the coordinatewise minimum and maximum (see Section 7.A.8 of [14]).

In the univariate case directional convexity is equivalent to convexity, while in the multivariate case there are no implications between the two concepts. Directional convexity implies supermodularity and it is equivalent to supermodularity plus componentwise convexity. For smooth functions, directional convexity is equivalent to the nonnegativity of all entries in the Hessian matrix, while supermodularity is equivalent to the nonnegativity of all entries out of the principal diagonal. Clearly, no implications exist between this concept and the usual convexity of φ, which corresponds to the positive semidefiniteness of the Hessian matrix. However, φ is directionally convex, and convex if and only if it is supermodular and convex. Finally, in the smooth case, φ is directionally convex if and only if its gradient is increasing, i.e. if all the partial derivatives are increasing functions.

In order to establish multivariate comparison results, we need a generalization of Lemma 6.

Lemma 8. *Let $\phi \in C^2(\mathbb{R}^m)$ be convex and supermodular, and let $g_i \in C^2(\mathbb{R})$ be convex and nonnegative. Let $\Pi_m := \{-1, 1\}^m$. It follows that*

$$h(u) = \sum_{\underline{p} \in \Pi_m} \phi(p_1 g_1(u), \ldots, p_m g_m(u))$$

is convex.

Proof. If we denote by y_i the arguments of the function ϕ, we can write

$$\begin{aligned}
h'(u) &= \sum_{\underline{p} \in \Pi_m} \sum_{i=1}^{m} p_i g_i'(u) \frac{\partial \phi}{\partial y_i}(p_1 g_1(u), \ldots, p_m g_m(u)),\\
h''(u) &= \sum_{\underline{p} \in \Pi_m} \Bigg[\sum_{i=1}^{m} p_i g_i''(u) \frac{\partial \phi}{\partial y_i}(p_1 g_1(u), \ldots, p_m g_m(u))\\
&\qquad + \sum_{i=1}^{m} \sum_{j=1}^{m} p_i p_j g_i'(u) g_j'(u) \frac{\partial^2 \phi}{\partial y_i \partial y_j}(p_1 g_1(u), \ldots, p_m g_m(u)) \Bigg].
\end{aligned}$$

Clearly,

$$\sum_{i=1}^{m} \sum_{j=1}^{m} p_i p_j g_i'(u) g_j'(u) \frac{\partial^2 \phi}{\partial y_i \partial y_j}(p_1 g_1(u), \ldots, p_m g_m(u)) \ge 0,$$

since the Hessian of ϕ is positive semidefinite. Moreover, if we define $Z_i := p_i g_i''(u)$, the first term can be written as

$$\sum_{\underline{p} \in \Pi_m} \sum_{i=1}^{m} p_i g_i''(u) \frac{\partial \phi}{\partial y_i}(p_1 g_1(u), \ldots, p_m g_m(u)) = 2^m \mathbb{E}[\nabla \phi(p_1 g_1(u), \ldots, p_m g_m(u)) \cdot \underline{Z}],$$

where $\underline{Z} = (Z_1, \ldots, Z_m)$. Since ϕ is convex and supermodular, and g_i is nonnegative, all the components of $\nabla\phi$ are increasing in p_i. The same holds for the components of $\underline{Z}$, since, by hypothesis, g_i is convex. From the multivariate covariance inequality, it follows that

$$\mathbb{E}[\nabla\phi(p_1 g_1(u), \ldots, p_m g_m(u)) \cdot \underline{Z}] \geq \mathbb{E}[\nabla\phi(p_1 g_1(u), \ldots, p_m g_m(u))] \cdot \mathbb{E}[\underline{Z}] = 0.$$

The assertion follows.

As in Lemma 6, the smoothness requirements on ϕ and g_i can be dropped, but we added them in order to simplify the proof. The multivariate analogue of Lemma 7 is the following statement.

Lemma 9. *Let X_n and S_n be as in (1) and (6). Let $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be supermodular and convex, and let the innovations ε_i be symmetric. Then the function*

$$h_k(x) := \mathbb{E}[\phi(X_0, \ldots, X_n) \mid \varepsilon_k = x]$$

is convex.

Proof. From the symmetry of the innovations we can write

$$\begin{aligned} h_k(x) &= \mathbb{E}_{\varepsilon_0,\ldots,\varepsilon_{k-1},\varepsilon_{k+1},\ldots,\varepsilon_n}[\phi(\sigma_0\varepsilon_0, g_1\varepsilon_1, \ldots, g_k x, \ldots, g_n\varepsilon_n)] \\ &= \mathbb{E}_{\varepsilon_0,\ldots,\varepsilon_{k-1},\varepsilon_{k+1},\ldots,\varepsilon_n}\left[\mathbf{1}_{\mathcal{E}} \sum_{\underline{p}\in\Pi_{n+1}} \phi(\sigma_0 p_0\varepsilon_0, g_1 p_1\varepsilon_1, \ldots, g_k p_k x, \ldots, g_n p_n\varepsilon_n)\right], \end{aligned}$$

where $\mathcal{E} = \{(\varepsilon_0, \ldots, \varepsilon_{k-1}, \varepsilon_{k+1}, \ldots, \varepsilon_n)\colon \varepsilon_i \geq 0,\ i \neq k\}$. Since each g_i is convex in ε_k, from Lemma 8, it follows that, for each $\sigma_0 > 0$ and $\varepsilon_i \geq 0,\ i \neq k$, the function

$$\overline{h}_k(x) = \sum_{\underline{p}\in\Pi_{n+1}} \phi(\sigma_0 p_0\varepsilon_0, g_1 p_1\varepsilon_1, \ldots, g_k p_k x, \ldots, g_n p_n\varepsilon_n)$$

is convex. Averaging with respect to ε_i, with $i \neq k$, it also follows that $h_k(x)$ is convex.

This enables us to state our main multivariate comparison result.

Theorem 5. *Let X_n and S_n be as in (1) and (6). Let the innovations ε_i be symmetric. If $\widetilde{\varepsilon}_k$ is also symmetric and $\widetilde{\varepsilon}_k \geq_{\mathrm{cx}} \varepsilon_k$, then*

$$\mathbb{E}[\phi(X_0, \ldots, X_k, \ldots, X_n)] \leq \mathbb{E}[\phi(X_0, \ldots, \tilde{X}_k, \ldots, \tilde{X}_n)]$$

for every function $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}$ which is supermodular and convex.

Proof. From the previous lemma we have

$$\mathbb{E}[\phi(X_0, \ldots, X_k, \ldots, X_n)] = \mathbb{E}_{\varepsilon_k}[h_k(\varepsilon_k)] \leq \mathbb{E}_{\widetilde{\varepsilon}_k}[h_k(\widetilde{\varepsilon}_k)] = \mathbb{E}[\phi(X_0, \ldots, \tilde{X}_k, \ldots, \tilde{X}_n)].$$

We note that we are not able to prove supermodular or componentwise convex orderings for $(X_0, \ldots, X_n)$; at the moment both hypotheses on ϕ seem to be necessary for Lemma 8.

7. The GARCH(1, 1) case

We now focus on the GARCH(1, 1) model specified by

$$X_n = \sigma_n \varepsilon_n, \qquad \varepsilon_n \perp \sigma_n, \qquad E[\varepsilon_n] = 0, \qquad \sigma_{n+1}^2 = \alpha_0 + \alpha_1 X_n^2 + \beta_1 \sigma_n^2, \tag{8}$$

with $\alpha_0, \alpha_1, \beta_1 > 0$ and $\alpha_1 + \beta_1 < 1$. For this model, the recursive dynamics of the volatility or of the variance (4) can easily be made explicit as follows (see [13]):

$$\sigma_{n+1}^2 = \sigma_0^2 \prod_{i=1}^{n+1} (\beta_1 + \alpha_1 \varepsilon_{n-i+1}^2) + \alpha_0 \left[1 + \sum_{k=1}^{n} \prod_{i=1}^{k} (\beta_1 + \alpha_1 \varepsilon_{n-i+1}^2) \right].$$

From this expression, it is immediate that σ_{n+1}^2 and σ_{n+1} are increasing functions of the parameters α_0, α_1, and β_1. We already noted that this model is a special case of both M1 and M2, so all the comparison results for varying innovations of the preceding sections hold. In this section we are interested in establishing comparison results for varying parameters α_0, α_1, and β_1. As mentioned in the introduction, intuition suggests that an increase in α_0, α_1, and β_1 should correspond to an increase in the variability of X_n and S_n; in this section we prove it rigorously. As in the previous sections, here the tilde denotes variables obtained by substituting α_0, α_1, or β_1 with $\widetilde{\alpha}_0$, $\widetilde{\alpha}_1$, or $\widetilde{\beta}_1$.

Proposition 1. *Let X_n be as in (8). If $\alpha_0 \le \widetilde{\alpha}_0$, $\alpha_1 \le \widetilde{\alpha}_1$, and $\beta_1 \le \widetilde{\beta}_1$, then $|X_n| \le_{\mathrm{st}} |\widetilde{X}_n|$, $X_n^2 \le_{\mathrm{st}} \widetilde{X}_n^2$, and $X_n \le_{\mathrm{cx}} \widetilde{X}_n$.*

Proof. Since σ_n and σ_n^2 are increasing functions of the parameters, if $\alpha_0 \le \widetilde{\alpha}_0$, $\alpha_1 \le \widetilde{\alpha}_1$, and $\beta_1 \le \widetilde{\beta}_1$, it follows that $\sigma_n \le_{\mathrm{st}} \widetilde{\sigma}_n$ and $\sigma_n^2 \le_{\mathrm{st}} \widetilde{\sigma}_n^2$. As in the proof of Theorem 2, it follows that $|X_n| \le_{\mathrm{st}} |\widetilde{X}_n|$ and $X_n^2 \le_{\mathrm{st}} \widetilde{X}_n^2$. From Lemma 2, $\sigma_n \le_{\mathrm{st}} \widetilde{\sigma}_n$ implies that $X_n \le_{\mathrm{cx}} \widetilde{X}_n$.

The last point is to prove the convex comparison of the sums S_n; again, this is nontrivial since the X_n are not independent; we provide a proof in the case of symmetric innovations.

Theorem 6. *Let X_n be as in (8) and S_n as in (6). Let all the innovations ε_i be symmetric. If $\alpha_0 \le \widetilde{\alpha}_0$, $\alpha_1 \le \widetilde{\alpha}_1$, and $\beta_1 \le \widetilde{\beta}_1$, then $S_n \le_{\mathrm{cx}} \widetilde{S}_n$.*

Proof. As before, we write

$$S_n = \sigma_0 \varepsilon_0 + g_1(\varepsilon_0, \alpha_0, \alpha_1, \beta_1)\varepsilon_1 + \cdots + g_n(\varepsilon_0, \ldots, \varepsilon_{n-1}, \alpha_0, \alpha_1, \beta_1)\varepsilon_n,$$

where the functions g_i are increasing in the parameters α_0, α_1, and β_1. Let ϕ be any convex function. We first prove that $\mathbb{E}[\phi(S_n)]$ is increasing in the parameters α_0, α_1, and β_1. From the symmetry of the innovations ε_i we can write

$$\begin{aligned}\mathbb{E}[\phi(S_n)] &= \mathbb{E}_{\varepsilon_0,\ldots,\varepsilon_n}[\phi(\sigma_0\varepsilon_0 + \cdots + g_n\varepsilon_n)] \\ &= \mathbb{E}_{\varepsilon_0,\ldots,\varepsilon_n}\left[\sum_{\underline{p} \in \Pi_{n+1}} \phi(\sigma_0 p_0 \varepsilon_0 + \cdots + p_n g_n \varepsilon_n)\, \mathbf{1}_{\{\varepsilon_0 \ge 0, \ldots, \varepsilon_n \ge 0\}} \right].\end{aligned}$$

For each nonnegative value of $\varepsilon_0, \ldots, \varepsilon_n$, we define

$$\overline{h}(\sigma_0, \alpha_0, \alpha_1, \beta_1) = \sum_{\underline{p} \in \Pi_{n+1}} \phi(\sigma_0 p_0 \varepsilon_0 + \cdots + p_n g_n \varepsilon_n).$$

We see that $\overline{h}$ is increasing in α_0, α_1, and β_1; indeed, we can compute

$$\frac{\partial \overline{h}}{\partial \alpha_0} = \sum_{\underline{p} \in \Pi_{n+1}} \phi'(\sigma_0 p_0 \varepsilon_0 + \cdots + p_n g_n \varepsilon_n) \cdot (p_1 \varepsilon_1 g_1' + \cdots + p_n \varepsilon_n g_n') \geq 0$$

from the multivariate covariance inequality, as in the proof of Lemma 6. The same reasoning shows that $\partial \overline{h}/\partial \alpha_1 \geq 0$ and $\partial \overline{h}/\partial \beta_1 \geq 0$.

It follows that $\mathbb{E}[\phi(S_n)]$ is increasing in α_0, α_1, and β_1; but then, if $\alpha_0 \leq \widetilde{\alpha}_0$, $\alpha_1 \leq \widetilde{\alpha}_1$, and $\beta_1 \leq \widetilde{\beta}_1$,

$$\mathbb{E}[\phi(S_n(\alpha_0, \alpha_1, \beta_1)] \leq \mathbb{E}[\phi(S_n(\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\beta}_1)],$$

that is, $S_n \leq_{\mathrm{cx}} \tilde{S}_n$.

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