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On the Choice of the Reference Frame for Beam Section Stiffness Properties

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Abstract

This work discusses the choice of a reference frame for beam section stiffness properties. Established concepts as the center of elasticity, the center of stiffness and the center of compliance are discussed and contextualized. An interpretation of univocally defined generalized strain transformations is given in terms of minimization of appropriate norms of the stiffness and compliance matrices of the beam section that univocally define special reference points. Transformations of generalized strain perturbations that preserve the angular strain are sought. They are subsequently constrained to represent a change of reference point, and further restricted to lie in the plane of the section. Each transformation is univocally defined and given a clear mathematical and geometrical interpretation. It is recognized that transformations that decouple forces and linear strains from moments and angular strains cannot be described as a mere change of reference point.

Keywords: Beam Model, Stiffness Matrix, Cross Section, Center of Stiffness

1. Introduction

The notion of 'elastic center' is well present in mechanics. In the second half of the nineteenth century, Karl Culmann developed graphical methods for the design of pile foundations for railroad bridges which involved the notion of elastic center (Culmann, 1866). In 1939, Vetter presents a method based on earlier works of other authors that involves the reduction of forces to an equivalent force applied in the elastic center, which causes a pure translation without rotation, and an equivalent moment which causes a pure rotation about the elastic center (Vetter, 1939). Such problems are extremely simple; they address two-dimensional systems with few rod elements acting along fixed axes; however, they indicate an attention to noteworthy definitions and the choice of points with special properties to find ingenious solutions to engineering problems (Kardestuncer, 1974).

The notions of 'center of stiffness' (CoS) and 'center of compliance' (CoC) have been introduced by Lončarić on solid mathematical foundations for compliant structures using screw theory (Lončarić, 1987), addressing compliant robotic applications. Lipkin et al., based on earlier work (Dimentberg, 1968), discussed the properties of the CoS and CoC, and introduced the 'center of elasticity' (CoE) as the center of the reciprocal three-systems that represent the wrench- and twist-compliant axes of a compliant system (Lipkin and Patterson, 1992; Ciblak and Lipkin, 1994, 1999). Such notions have been extensively used, and are still used nowadays, in several applications ranging from robotics (Roberts, 2002) to biomechanics (Enea et al., 2013). By referring the stiffness of a compliant system to the CoS, forces opposing rotations and moments opposing displacements are maximally decoupled.

In beam theory, the notions of ‘shear centroid’ (or ‘shear center’, ‘center of twist’, ‘flexural center’, namely the point that must lie along the line of action of a shear force for the section not to twist) and ‘axial strain centroid’ (or ‘tension center’, namely the point in a beam section where the neutral axes cross, and where an applied axial load does not produce any bending) are well understood. Nowinski in 1961 discussed an ‘axis of twist’ and ‘center of flexure’ for certain classes of anisotropic beams (Nowinski, 1961). Reissner and Tsai discussed the problem for cylindrical shell beams (Reissner and Tsai, 1972). In the seminal work (Giavotto et al., 1983), a simple transformation was proposed to identify the location of the shear and axial strain centroids of the beam section in terms of decoupling linear and angular generalized stresses and strains. However, such procedure cannot be described in terms of a change of reference system. In (Rehfield and Atilgan, 1989; Kosmatka, 1994; Yu et al., 2002) it is noted that some commonly accepted definitions of characteristic points like the shear center may depend on the spanwise location along the beam, e.g. when bending-torsion coupling is present. In (Andreaus and Ruta, 1998), a detailed review of the shear center problem is presented. Ecsedi discussed the centre of twist and the centre of shear for straight isotropic nonhomogeneous beams (Ecsedi, 2000). Bottasso et al. discussed invariance issues associated with the application of numerical methods, also addressing the case of referring beam sections to arbitrary points (Bottasso et al., 2002). Sapountzakis and Mokos presented an original Boundary Element Method (BEM) solution to transverse shear loading of beams (Sapountzakis and Mokos, 2005) in which transverse loads are applied in the shear center to avoid the induction of twisting moment. The discussion about twist and shear centers is active, as testified by very recent literature on the topic (Barretta, 2012; Ecsedi and Baksa, 2012).

In recent times, the so-called Absolute Nodal Coordinate Formulation (ANCF) became popular also for the analysis of deformable continua, including beams. Apparently, such an approach does not need to care about such issues as the definition of special centroids, since the absolute coordinates of the points that define the geometry of the beam represent the degrees of freedom of the problem, much like for solid nonlinear finite elements.; The comparison of ANCF with so-called Geometrically Exact Beam Formulations (GEBF) is an active topic of research (Romero, 2008).

This work presents an interpretation of the CoS concept in relation with beam section characterization. Univocally defined generalized strain transformations are interpreted in terms of minimization of appropriate norms of the stiffness matrix of the beam section. To the author's knowledge, such interpretation has never been pointed out before. The beam model is briefly presented in Section 2, focusing on referring linear constitutive properties to an arbitrary reference. The choice of the reference frame for beam section stiffness properties is discussed in Section 3, with a newly proposed definition that specializes Lončarić's CoS to beam stiffness properties. Examples are proposed in Section 4.

2. Beam Model

The beam model is formulated using generalized coordinates, namely the position of an arbitrary reference point and the orientation of an arbitrary triad that define the 'pose' of the beam section as a one dimensional Cosserat continuum. See for example the so-called geometrically exact beam formulation named after Reissner-Simo in (Ritto-Corrêa and Camotim, 2002; Merlini and Morandini, 2013).

The main focus of this work is on the definition of a possibly advantageous frame of reference to express the elastic properties of the beam section, so the choice of a specific approach is deemed inessential, and only the strain energy per unit span of the beam, $\mathcal{W}_{\text{sec}}$, is actually considered.

2.1. Constitutive Model

Consider the strain energy per unit span of a beam, $\mathcal{W}_{\text{sec}} = \mathcal{W}_{\text{sec}}(\boldsymbol{\psi})$, where $\boldsymbol{\psi} = \{\boldsymbol{\nu}; \boldsymbol{\kappa}\}$ represents a suitable measure of the generalized strains, namely the linear strain, $\boldsymbol{\nu}$, and the angular strain $\boldsymbol{\kappa}$, as defined, for example, in (Ritto-Corrêa and Camotim, 2002) and (Merlini and Morandini, 2013).

The generalized internal forces, namely the internal force, $\mathbf{f}$, and the internal moment, $\mathbf{m}$, are defined as the partial derivatives of the strain energy with respect to the generalized strains, namely

$$\mathbf{f} = \frac{\partial \mathcal{W}_{\text{sec}}}{\partial \boldsymbol{\nu}} \tag{1a}$$

$$\mathbf{m} = \frac{\partial \mathcal{W}_{\text{sec}}}{\partial \boldsymbol{\kappa}} \tag{1b}$$

As a consequence, the internal force and moment are intrinsically expressed with respect to the reference point and orientation of the section, as much as the generalized strains are. In this sense, the stiffness matrix can be seen as the Hessian matrix of the strain energy with respect to the generalized strains; thus,

$$\left\{ \begin{array}{c} \partial \mathbf{f} \\ \partial \mathbf{m} \end{array} \right\} = \mathbf{K} \left\{ \begin{array}{c} \partial \boldsymbol{\nu} \\ \partial \boldsymbol{\kappa} \end{array} \right\}, \tag{2}$$

in which $\partial(\cdot)$ indicates a perturbation, following the notation used in (Merlini and Morandini, 2013). In fact, the constitutive relationship of Eq. (2) must be interpreted as the tangent map that expresses the generalized force increments as functions of the generalized strain increments when beam sections made of hypereleastic material are considered. It applies to generalized finite forces and strains when $\mathbf{K}$ is constant, i.e. when the strains are small (although not necessarily infinitesimal), despite the overall displacements and rotations being arbitrary.

The object of this work is the determination of special reference points for the tangent map between generalized strains and generalized forces. It is worth anticipating that when such map is not constant, those reference points depend on the straining of the beam section, and thus lose their practical appeal, although they preserve a strong mathematical and physical significance. For the sake of simplicity, in the following a stiffness matrix representing a constant tangent map is considered; this fact is taken axiomatically.

In simple models, e.g. those analogous to Conventional Laminate Theory (CLT), the actual inplane straining of the section is implicitly dealt with considering constitutive properties for axial stress state. More sophisticated models, like the one proposed in (Giavotto et al., 1983) and subsequent developments (the interested reader may refer to Hodges' book (Hodges, 2006) for more details, and the recent works (Ghiringhelli et al., 2008; Morandini et al., 2010)), explicitly (although often approximately, either axiomatically or in a finite element sense) account for inplane and out-of-plane warping.

The matrix can be partitioned as

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}; \tag{3}$$

submatrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are 3×3, with $\mathbf{A}^T = \mathbf{A} > 0$, $\mathbf{C}^T = \mathbf{C} > 0$. The positive definiteness of $\mathbf{K}$, $\mathbf{A}$, and $\mathbf{C}$ can be lost only in degenerate cases that in practice do not need to be considered in this context.

Consider now the corresponding compliance matrix,

$$\mathbf{F} = \mathbf{K}^{-1} = \begin{bmatrix} \overline{\mathbf{A}} & \overline{\mathbf{B}} \\ \overline{\mathbf{B}}^T & \overline{\mathbf{C}} \end{bmatrix}, \tag{4}$$

with

$$\begin{aligned} \overline{\mathbf{A}} &= \left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1} \\ &= \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{A}^{-1} \end{aligned} \tag{5a}$$

$$\begin{aligned} \overline{\mathbf{B}} &= -\left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1}\mathbf{B}\mathbf{C}^{-1} \\ &= -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \end{aligned} \tag{5b}$$

$$\begin{aligned} \overline{\mathbf{C}} &= \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}^T\left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1}\mathbf{B}\mathbf{C}^{-1} \\ &= \left(\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^{-1}. \end{aligned} \tag{5c}$$

Later on, it will be used to discuss the reference frame transformation in more detail.

2.2. Change of Reference Frame

The internal force $\mathbf{f}$ and moment $\mathbf{m}$ can be expressed as functions of the internal force $\mathbf{f}'$ and $\mathbf{m}'$ referred to a different pole, offset by $\mathbf{p}$ from the original reference, and with respect to a different orientation $\mathbf{R}$, both expressed in the reference frame of the section, namely

$$\left\{ \begin{array}{c} \mathbf{f} \\ \mathbf{m} \end{array} \right\} = \left[\begin{array}{cc} \mathbf{R} & \mathbf{0} \\ \mathbf{p} \times \mathbf{R} & \mathbf{R} \end{array} \right] \left\{ \begin{array}{c} \mathbf{f}' \\ \mathbf{m}' \end{array} \right\}, \tag{6}$$

where the symbol $(\cdot)\times$ denotes the skew-symmetric linear operator that applied to a vector $\mathbf{a}$ and acting on any vector $\mathbf{b}$ results into the cross product $\mathbf{a} \times \mathbf{b}$. The inverse[1] of Eq. (6) yields

$$\left\{ \begin{array}{c} \mathbf{f}' \\ \mathbf{m}' \end{array} \right\} = \left[\begin{array}{cc} \mathbf{R} & \mathbf{p} \times \mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{array} \right]^T \left\{ \begin{array}{c} \mathbf{f} \\ \mathbf{m} \end{array} \right\} = \mathbf{H}^T_{\mathbf{R},\mathbf{p}\times} \left\{ \begin{array}{c} \mathbf{f} \\ \mathbf{m} \end{array} \right\}, \tag{9}$$

where $\mathbf{H}_{(\cdot)}$ indicates a transformation matrix characterized by the subscript as appropriate.

One may legitimately ask how the generalized strains are affected by such transformation. Regardless of the formulation used to determine the stiffness matrix, the virtual complementary work of the generalized internal forces $\delta\mathbf{f}$

[1] The inverse of matrix $\mathbf{H}_{\mathbf{R},\mathbf{p}\times}$ can be easily computed using the generic formulas for block matrix inversion; however, it is illustrative to show that

$$\left[\begin{array}{cc} \mathbf{R} & \mathbf{0} \\ \mathbf{p} \times \mathbf{R} & \mathbf{R} \end{array} \right]^{-1} = \left[\begin{array}{cc} \mathbf{R} & \mathbf{p} \times \mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{array} \right]^T = \left[\begin{array}{cc} \mathbf{R}^T & \mathbf{0} \\ \mathbf{R}^T \mathbf{p}\times^T & \mathbf{R}^T \end{array} \right]. \tag{7}$$

In fact,

$$\left[\begin{array}{cc} \mathbf{R} & \mathbf{0} \\ \mathbf{p} \times \mathbf{R} & \mathbf{R} \end{array} \right] \left[\begin{array}{cc} \mathbf{R}^T & \mathbf{0} \\ \mathbf{R}^T \mathbf{p}\times^T & \mathbf{R}^T \end{array} \right] = \left[\begin{array}{cc} \mathbf{R}\mathbf{R}^T & \mathbf{0} \\ \mathbf{p} \times \mathbf{R}\mathbf{R}^T + \mathbf{R}\mathbf{R}^T \mathbf{p}\times^T & \mathbf{R}\mathbf{R}^T \end{array} \right], \tag{8}$$

which corresponds to the identity matrix considering the orthogonality of rotation matrices, $\mathbf{R}\mathbf{R}^T = \mathbf{I}$, and the skew-symmetry of operator $(\cdot)\times$, which implies $\mathbf{p}\times^T = -\mathbf{p}\times$.

and $\delta\mathbf{m}$ conjugated with the generalized strains, $\boldsymbol{\nu}$, $\boldsymbol{\kappa}$, does not change when the new reference frame is considered, i.e.

$$\delta\mathbf{f}' \cdot \boldsymbol{\nu}' + \delta\mathbf{m}' \cdot \boldsymbol{\kappa}' = \delta\mathbf{f} \cdot \boldsymbol{\nu} + \delta\mathbf{m} \cdot \boldsymbol{\kappa}. \tag{10}$$

According to the previously defined transformation,

$$\left\{ \begin{array}{c} \delta\mathbf{f} \\ \delta\mathbf{m} \end{array} \right\}^T \left[\begin{array}{cc} \mathbf{R} & \mathbf{p} \times \mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{\nu}' \\ \boldsymbol{\kappa}' \end{array} \right\} = \left\{ \begin{array}{c} \delta\mathbf{f} \\ \delta\mathbf{m} \end{array} \right\}^T \left\{ \begin{array}{c} \boldsymbol{\nu} \\ \boldsymbol{\kappa} \end{array} \right\}, \tag{11}$$

i.e.

$$\left\{ \begin{array}{c} \boldsymbol{\nu} \\ \boldsymbol{\kappa} \end{array} \right\} = \left[\begin{array}{cc} \mathbf{R} & \mathbf{p} \times \mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{\nu}' \\ \boldsymbol{\kappa}' \end{array} \right\} = \mathbf{H}_{\mathbf{R},\mathbf{p}\times} \left\{ \begin{array}{c} \boldsymbol{\nu}' \\ \boldsymbol{\kappa}' \end{array} \right\}. \tag{12}$$

Apart from the re-orientation represented by $\mathbf{R}$, which changes the physical interpretation of the components of the angular strain vector, the norm of the generalized angular strain is not altered by the change of reference (in fact, $\boldsymbol{\kappa}' \cdot \boldsymbol{\kappa}' = \boldsymbol{\kappa} \cdot \boldsymbol{\kappa}$), whereas that of the generalized linear strain changes (in fact, $\boldsymbol{\nu}' \cdot \boldsymbol{\nu}' = \boldsymbol{\nu} \cdot \boldsymbol{\nu} - 2\boldsymbol{\nu} \cdot (\mathbf{p} \times \boldsymbol{\kappa}) + (\mathbf{p} \times \boldsymbol{\kappa}) \cdot (\mathbf{p} \times \boldsymbol{\kappa}) \neq \boldsymbol{\nu} \cdot \boldsymbol{\nu}$ as long as $\mathbf{p} \neq \mathbf{0}$ and $\mathbf{p} \times \boldsymbol{\kappa} \neq \mathbf{0}$). The transformed stiffness matrix is thus

$$\mathbf{K}' = \mathbf{H}_{\mathbf{R},\mathbf{p}\times}^T \mathbf{K} \mathbf{H}_{\mathbf{R},\mathbf{p}\times}. \tag{13}$$

The Ky Fan n-norm (or nuclear norm, or trace norm; see Johnson and Horn, 1991) of the stiffness matrix is by definition equal to the sum of the singular values of the matrix and thus, being the matrix symmetric positive-definite, to the sum of its eigenvalues. It can be considered a measure of the "specific strain energy" of the section. The term "specific" is used in the sense of "per unit strain", thus dependent on the definition of the strain measure.

Since the elements of the matrix are not dimensionally homogeneous, submatrix $\mathbf{A}$ could be normalized using an arbitrary reference length, ρ, which may be interpreted as a radius of gyration, the characteristic measure that is used to scale shear and bending stiffness parameters to define the slenderness of a beam. This approach would make the matrix eigenvalues depend on the selected length. However, in the following, it is shown that the choice of such length is inessential, since the portion of the trace of $\mathbf{K}$ that contains the trace of submatrix $\mathbf{A}$ is not affected by the transformations that will be considered. As a consequence, the trace of $\mathbf{K}$ can be conveniently limited to the trace of submatrix $\mathbf{C}$, which is dimensionally homogeneous. For this purpose, a specific trace operator is defined, $\mathrm{tr}_{\mathbf{C}}(\cdot)$, which, applied to a section stiffness matrix $\mathbf{K}$ as defined in Eq. (3), produces $\mathrm{tr}_{\mathbf{C}}(\mathbf{K}) = \mathrm{tr}(\mathbf{C})$.

This specific strain energy definition is introduced with the objective of isolating information about how the stiffness properties of the section store strain energy in a manner that is intrinsic and cannot be modified by a redefinition of the strains that, apart from a change of reference orientation, preserves the definition of the angular strains.

Apart from the reorientation operated by $\mathbf{R}$, which has been considered for completeness, but does not affect the matrix singular values, the trace of the transformed matrix is

$$\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{p}\times}^{T}\mathbf{K}\mathbf{H}_{\mathbf{p}\times}\right) = \mathrm{tr}\left(\mathbf{p}\times^{T}\mathbf{A}\mathbf{p}\times + \mathbf{B}^{T}\mathbf{p}\times + \mathbf{p}\times^{T}\mathbf{B} + \mathbf{C}\right), \tag{14}$$

which clearly depends on the position of the reference point $\mathbf{p}$. One may legitimately ask whether a special reference point exists, which corresponds to some invariant property of the matrix. This question will be answered in a later section.

3. Beam Section Stiffness Matrix

3.1. Homogeneous Isotropic Beam Section

The stiffness matrix of a homogeneous isotropic beam section has a specific layout. It is discussed here to introduce the topic of this work using notions that are familiar to engineers. Assuming that the beam axis is along direction 1, indicated by the unit vector $\mathbf{e}_1$, the stiffness matrix takes the general form

$$\mathbf{K} = \begin{bmatrix} a_{11} & 0 & 0 & 0 & a_{15} & a_{16} \\ & s_{22} & s_{23} & s_{24} & 0 & 0 \\ & & s_{33} & s_{34} & 0 & 0 \\ & & & s_{44} & 0 & 0 \\ & & & & a_{55} & a_{56} \\ & \text{sym.} & & & & a_{66} \end{bmatrix}, \tag{15}$$

where a_{ij} and s_{ij} respectively indicate elements associated with axial and shear strain. Specifically, adopting the terminology in use in normal engineering practice,

$$a_{11} = EA \tag{16a}$$

$$a_{15} = z_{\text{as}} EA \tag{16b}$$

$$a_{16} = -y_{\text{as}} EA \tag{16c}$$

$$a_{55} = EJ_y \cos^2 \alpha + EJ_z \sin^2 \alpha + z_{\text{as}}^2 EA \tag{16d}$$

$$a_{56} = (EJ_y - EJ_z) \cos \alpha \sin \alpha - y_{\text{as}} z_{\text{as}} EA \tag{16e}$$

$$a_{66} = EJ_y \sin^2 \alpha + EJ_z \cos^2 \alpha + y_{\text{as}}^2 EA \tag{16f}$$

$$s_{22} = GA_y \cos^2 \beta + GA_z \sin^2 \beta \tag{16g}$$

$$s_{23} = (GA_z - GA_y) \cos \beta \sin \beta \tag{16h}$$

$$s_{24} = y_{\text{sc}} GA_z \sin \beta - z_{\text{sc}} GA_y \cos \beta \tag{16i}$$

$$s_{33} = GA_y \sin^2 \beta + GA_z \cos^2 \beta \tag{16j}$$

$$s_{34} = y_{\text{sc}} GA_z \cos \beta + z_{\text{sc}} GA_y \sin \beta \tag{16k}$$

$$s_{44} = GJ + z_{\text{sc}}^2 GA_y + y_{\text{sc}}^2 GA_z, \tag{16l}$$

where the meaning of the symbols is summarized in Table 1, and illustrated in Fig. 1. The relations of Eqs. (16) can be inverted to compute the stiffness parameters of Table 1 from the elements of the matrix.

According to the structure of submatrix $\mathbf{B}$, a change of reference system origin $\mathbf{p}$ within the plane of the section (i.e. with $\mathbf{p} = \{0; y; z\}$),

$$\mathbf{H}_{\mathbf{p}\times} = \begin{bmatrix} \mathbf{I} & \mathbf{p}\times \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{17}$$

Table 1: Engineering beam section characterization symbols.

Symbol	**Description**	**Units**
EA	axial stiffness	force
EJ_y	bending stiffness about principal axis y	force·length^2
EJ_z	bending stiffness about principal axis z	force·length^2
y_{as}	component along y of axial strain centroid	length
z_{as}	component along z of axial strain centroid	length
α	orientation of bending principal axes[a]	angle
GA_y	shear stiffness along principal axis y	force
GA_z	shear stiffness along principal axis z	force
GJ	torsional stiffness about shear centroid	force·length^2
y_{sc}	component along y of shear centroid	length
z_{sc}	component along z of shear centroid	length
β	orientation of shear principal axes[a]	angle

[a] counter-clockwise rotation about axis x

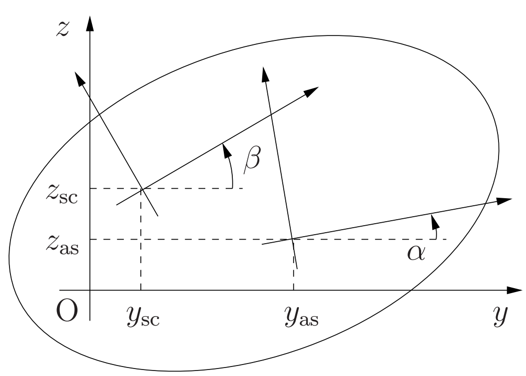

Figure 1: Engineering beam section geometric parameters.

results in

$$\mathbf{H}_{\mathbf{p}\times}^T \mathbf{K} \mathbf{H}_{\mathbf{p}\times} = \begin{bmatrix} \mathbf{A} & \mathbf{A}\mathbf{p}\times + \mathbf{B} \\ \mathbf{p}\times^T \mathbf{A} + \mathbf{B}^T & \mathbf{p}\times^T \mathbf{A}\mathbf{p}\times + \mathbf{B}\mathbf{p}\times + \mathbf{p}\times^T \mathbf{B} + \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}' \\ (\mathbf{B}')^T & \mathbf{C}' \end{bmatrix} \tag{18}$$

which, in the present case, corresponds to

$$\mathbf{B}' = \begin{bmatrix} 0 & a_{15} - z a_{11} & a_{16} + y a_{11} \\ s_{24} + z s_{22} - y s_{23} & 0 & 0 \\ s_{34} + z s_{23} - y s_{33} & 0 & 0 \end{bmatrix}. \tag{19}$$

Note that the structure of matrix $\mathbf{B}'$ remains the same of matrix $\mathbf{B}$.

Consider now a reorientation of the stiffness properties, consisting of separate rotations of forces and moments and of the conjugated generalized strains about axis 1,

$$\mathbf{H}_{\mathbf{R}_\alpha,\mathbf{R}_\beta} = \begin{bmatrix} \mathbf{R}_\beta & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_\alpha \end{bmatrix}, \tag{20}$$

with

$$\mathbf{R}_{(\cdot)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\cdot) & -\sin(\cdot) \\ 0 & \sin(\cdot) & \cos(\cdot) \end{bmatrix} = \exp((\cdot)\mathbf{e}_1\times) \tag{21}$$

such that

$$\mathbf{H}_{\mathbf{R}_\alpha,\mathbf{R}_\beta}^T \mathbf{K} \mathbf{H}_{\mathbf{R}_\alpha,\mathbf{R}_\beta} = \begin{bmatrix} \mathbf{R}_\beta^T \mathbf{A} \mathbf{R}_\beta & \mathbf{R}_\beta^T \mathbf{B} \mathbf{R}_\alpha \\ \mathbf{R}_\alpha^T \mathbf{B}^T \mathbf{R}_\beta & \mathbf{R}_\alpha^T \mathbf{C} \mathbf{R}_\alpha \end{bmatrix} = \begin{bmatrix} \mathbf{A}' & \mathbf{B}' \\ (\mathbf{B}')^T & \mathbf{C}' \end{bmatrix}. \tag{22}$$

In the present case, the reorientation yields

$$\mathbf{B}' = \begin{bmatrix} 0 & a_{15}\cos\alpha + a_{16}\sin\alpha & -a_{15}\sin\alpha + a_{16}\cos\alpha \\ s_{24}\cos\beta + s_{34}\sin\beta & 0 & 0 \\ -s_{24}\sin\beta + s_{34}\cos\beta & 0 & 0 \end{bmatrix}. \tag{23}$$

Also in this case the structure of matrix $\mathbf{B}'$ remains identical to that of matrix $\mathbf{B}$.

This analysis indicates that by choosing either the shear centroid or the axial strain centroid as the reference point, matrix $\mathbf{K}$ can take a simpler form (i.e. either the a_{15}, a_{16} or the s_{24}, s_{34} elements of the matrix can be eliminated); however, unless the centroids are coincident, matrix $\mathbf{B}$ cannot vanish, and thus no decoupling is possible using a change of reference frame.

Non-homogeneous sections and sections containing anisotropic materials may fully populate matrix $\mathbf{K}$ and significantly submatrix $\mathbf{B}$; in those cases, a redefinition of the origin in the y, z plane and of the reference frame of the section through a rotation about axis 1 might not produce analogous simplifications.

3.2. Center of Elasticity Transformation

A transformation of the stiffness matrix $\mathbf{K}$ turns submatrix $\mathbf{A}$ into $\mathbf{A}' = \mathbf{R}^T\mathbf{A}\mathbf{R}$. A transformation of the compliance matrix $\mathbf{F}$ turns submatrix $\overline{\mathbf{C}}$ into $\overline{\mathbf{C}}' = \overline{\mathbf{R}}^T\overline{\mathbf{C}}\overline{\mathbf{R}}$. This is consistent with the intuitive consideration that a pole change does not change forces nor rotations.

Matrices $\mathbf{A}'$ and $\overline{\mathbf{C}}'$ can be chosen to be diagonal, $\mathbf{A}' = \mathbf{\Lambda}$ and $\overline{\mathbf{C}}' = \overline{\mathbf{\Lambda}}$, consisting of the eigenvalues of the corresponding matrices prior to transformation. The corresponding rotation matrices $\mathbf{R}$ and $\overline{\mathbf{R}}$ are constructed from the eigenvectors of the corresponding matrices.

The transformed constitutive relationships are thus

$$
\begin{aligned}
\left\{ \begin{array}{c} \mathbf{R}^T\mathbf{f} \\ \overline{\mathbf{R}}^T\mathbf{m} \end{array} \right\} &= \left[\begin{array}{cc} \boldsymbol{\Lambda} & \mathbf{R}^T\mathbf{B}\overline{\mathbf{R}} \\ -\overline{\boldsymbol{\Lambda}}^{-1}\overline{\mathbf{R}}^T\overline{\mathbf{B}}^T\mathbf{R}\boldsymbol{\Lambda} & \overline{\boldsymbol{\Lambda}}^{-1}\overline{\mathbf{R}}^T\left(\mathbf{I} - \overline{\mathbf{B}}^T\mathbf{B}\right)\overline{\mathbf{R}} \end{array} \right] \left\{ \begin{array}{c} \mathbf{R}^T\boldsymbol{\nu} \\ \overline{\mathbf{R}}^T\boldsymbol{\kappa} \end{array} \right\} \\
&= \left[\begin{array}{cc} \boldsymbol{\Lambda} & \mathbf{R}^T\mathbf{B}\overline{\mathbf{R}} \\ \overline{\mathbf{R}}^T\mathbf{B}^T\mathbf{R} & \overline{\boldsymbol{\Lambda}}^{-1} + \overline{\boldsymbol{\Lambda}}^{-1}\overline{\mathbf{R}}^T\overline{\mathbf{B}}^T\left(\overline{\mathbf{A}} - \overline{\mathbf{B}\mathbf{C}}^{-1}\overline{\mathbf{B}}^T\right)^{-1}\overline{\mathbf{B}\mathbf{R}\boldsymbol{\Lambda}}^{-1} \end{array} \right] \left\{ \begin{array}{c} \mathbf{R}^T\boldsymbol{\nu} \\ \overline{\mathbf{R}}^T\boldsymbol{\kappa} \end{array} \right\}
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
\left\{ \begin{array}{c} \mathbf{R}^T\boldsymbol{\nu} \\ \overline{\mathbf{R}}^T\boldsymbol{\kappa} \end{array} \right\} &= \left[\begin{array}{cc} \boldsymbol{\Lambda}^{-1}\mathbf{R}^T\left(\mathbf{I} - \mathbf{B}\overline{\mathbf{B}}^T\right)\mathbf{R} & -\boldsymbol{\Lambda}^{-1}\mathbf{R}^T\mathbf{B}\overline{\mathbf{R}\boldsymbol{\Lambda}} \\ \overline{\mathbf{R}}^T\overline{\mathbf{B}}^T\mathbf{R} & \overline{\boldsymbol{\Lambda}} \end{array} \right] \left\{ \begin{array}{c} \mathbf{R}^T\mathbf{f} \\ \overline{\mathbf{R}}^T\mathbf{m} \end{array} \right\} \\
&= \left[\begin{array}{cc} \left(\boldsymbol{\Lambda} - \mathbf{R}^T\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\mathbf{R}\right)^{-1} & \mathbf{R}^T\overline{\mathbf{B}\mathbf{R}} \\ \overline{\mathbf{R}}^T\overline{\mathbf{B}}^T\mathbf{R} & \overline{\boldsymbol{\Lambda}} \end{array} \right] \left\{ \begin{array}{c} \mathbf{R}^T\mathbf{f} \\ \overline{\mathbf{R}}^T\mathbf{m} \end{array} \right\}.
\end{aligned}
\tag{25}
$$

Such transformation, as discussed in (Lipkin and Patterson, 1992; Ciblak and Lipkin, 1994, 1999), independently determines the principal directions for forces and curvatures, but does not act on the cross-couplings. Actually, no attempt is made to change the origin of the section reference frame to reduce the cross-coupling. The center of elasticity or compliant axes are recognized, if they exist, as noteworthy loci, for example when some of the couplings vanish.

3.3. Minimum Strain Energy Transformation

The opportunity of decoupling forces and moments was noticed in (Giavotto et al., 1983), where a transformation $\mathbf{Y}$ was suggested such that the moments can be expressed as $\mathbf{m}' = \mathbf{m} + \mathbf{Y}^T\mathbf{f}$, namely

$$\left\{ \begin{array}{c} \mathbf{f}' \\ \mathbf{m}' \end{array} \right\} = \left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{Y}^T & \mathbf{I} \end{array} \right] \left\{ \begin{array}{c} \mathbf{f} \\ \mathbf{m} \end{array} \right\} = \mathbf{H}_{\mathbf{Y}}^T \left\{ \begin{array}{c} \mathbf{f} \\ \mathbf{m} \end{array} \right\}, \tag{26}$$

with no assumption on the structure of $\mathbf{Y}$. The corresponding generalized strain transformation is

$$\left\{ \begin{array}{c} \boldsymbol{\nu} \\ \boldsymbol{\kappa} \end{array} \right\} = \left[\begin{array}{cc} \mathbf{I} & \mathbf{Y} \\ \mathbf{0} & \mathbf{I} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{\nu}' \\ \boldsymbol{\kappa}' \end{array} \right\}. \tag{27}$$

Then the stiffness matrix transforms as

$$\mathbf{H}_{\mathbf{Y}}^T \mathbf{K} \mathbf{H}_{\mathbf{Y}} = \left[\begin{array}{cc} \mathbf{A} & \mathbf{A}\mathbf{Y} + \mathbf{B} \\ \mathbf{Y}^T\mathbf{A} + \mathbf{B}^T & \mathbf{Y}^T\mathbf{A}\mathbf{Y} + \mathbf{B}^T\mathbf{Y} + \mathbf{Y}^T\mathbf{B} + \mathbf{C} \end{array} \right]. \tag{28}$$

By choosing $\mathbf{Y} = -\mathbf{A}^{-1}\mathbf{B}$, the transformed matrix is

$$\mathbf{H}_{\mathbf{Y}}^T \mathbf{K} \mathbf{H}_{\mathbf{Y}} = \left[\begin{array}{cc} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \end{array} \right]. \tag{29}$$

Finally, it is suggested to independently diagonalize the remaining submatrices along the diagonal of the transformed matrix using the spectral decompositions[2] $\mathbf{A} = \mathbf{U}_{\mathbf{A}}\boldsymbol{\Lambda}_{\mathbf{A}}\mathbf{U}_{\mathbf{A}}^T$ and $\mathbf{C}' = \mathbf{U}_{\mathbf{C}'}\boldsymbol{\Lambda}_{\mathbf{C}'}\mathbf{U}_{\mathbf{C}'}^T$, where $\mathbf{C}'$ indicates $\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$, to obtain the principal shear and bending axes. From $\mathbf{Y}$ they identify[3] the 'shear centroid' coordinates as $y_{\text{sc}} = -y_{23}$ and $z_{\text{sc}} = y_{13}$, and the 'normal stresses centroid' coordinates as $y_{\text{as}} = -y_{32}$ and $z_{\text{as}} = y_{31}$.

[2] Matrix $\mathbf{U}$ must have $\det(\mathbf{U}) = +1$ to belong to the $SO(3)$ group, and thus represent a rotation in a 3-dimensional Euclidean space.

[3] In (Giavotto et al., 1983), the axis of the beam was labeled 3, whereas in this work it is labeled 1; the subscripts used for y and z are given in the original notation.

For example, in the case of a beam section made of isotropic material,

$$\mathbf{Y} = -\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 0 & -\dfrac{a_{15}}{a_{11}} & -\dfrac{a_{16}}{a_{11}} \\ -\dfrac{s_{33}s_{24} - s_{23}s_{34}}{s_{22}s_{33} - s_{23}^2} & 0 & 0 \\ \dfrac{s_{23}s_{24} - s_{22}s_{34}}{s_{22}s_{33} - s_{23}^2} & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -z_{as} & y_{as} \\ -y_{sc}\sin\beta + z_{sc}\cos\beta & 0 & 0 \\ -y_{sc}\cos\beta - z_{sc}\sin\beta & 0 & 0 \end{bmatrix}, \qquad (30)$$

where $y_{13} = y_{as}$ and $y_{12} = -z_{as}$ are the coordinates of the axial strain centroid, whereas $y_{31} = -y_{sc}$ and $y_{21} = z_{sc}$ are the coordinates of the shear centroid in a reference frame rotated by angle $-\beta$ about axis 1.

Apparently, in (Giavotto et al., 1983) the authors failed to recognize that matrix $\mathbf{Y}$ takes such form only when the stiffness matrix has the structure of Eq. (15), i.e. the beam is made of isotropic, homogeneous material. In any case such transformation cannot be interpreted as a change of reference pole, since in general $y_{21} \neq -y_{12}$ and $y_{31} \neq -y_{13}$.

One may legitimately ask whether there exists a special transformation that minimizes some norm related to the specific strain energy. The minimization of the Ky Fan n-norm of the transformed matrix, $\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{Y}}^T\mathbf{K}\mathbf{H}_{\mathbf{Y}}\right)$, with respect to $\mathbf{Y}$ yields again $\mathbf{Y} = -\mathbf{A}^{-1}\mathbf{B}$, i.e. the transformation proposed in (Giavotto et al., 1983) minimizes the trace of the stiffness matrix without any constraint on the structure of the transformation itself. A detailed proof is given in Appendix A.

Consider now a transformation of the section compliance matrix,

$$\left(\mathbf{H}_{\mathbf{Y}}^{T}\mathbf{K}\mathbf{H}_{\mathbf{Y}}\right)^{-1} = \mathbf{H}_{\mathbf{Y}}^{-1}\mathbf{F}\mathbf{H}_{\mathbf{Y}}^{-T}, \tag{31}$$

with

$$\mathbf{H}_{\mathbf{Y}}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{Y} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{32}$$

Clearly, when $\mathbf{Y} = -\mathbf{A}^{-1}\mathbf{B}$, the coupling term of the transformed compliance matrix vanishes as well, since

$$\left(\mathbf{H}_{\mathbf{Y}}^{T}\mathbf{K}\mathbf{H}_{\mathbf{Y}}\right)^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} - \mathbf{B}^{T}\mathbf{A}^{-1}\mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{C} - \mathbf{B}^{T}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \end{bmatrix}. \tag{33}$$

The compliance matrix of Eq. (4) can be reduced to block diagonal form using a transformation $\mathbf{H}_{\mathbf{Y}}^{-1}\mathbf{F}\mathbf{H}_{\mathbf{Y}}^{-T}$ by setting $\mathbf{Y} = \overline{\mathbf{B}}\overline{\mathbf{C}}^{-1} = -\mathbf{A}^{-1}\mathbf{B}$, as one may easily check using Eqs. (5). This implies that when the minimum norm transformation is used, $\overline{\mathbf{C}} = (\mathbf{C} - \mathbf{B}^{T}\mathbf{A}^{-1}\mathbf{B})^{-1}$.

3.4. Center of Stiffness Transformation

Transform the reference frame in which the stiffness properties of a beam are expressed using the notion of center of stiffness presented by Lončarić (Lončarić, 1987), considering the previously described transformation

$$\mathbf{H}_{\mathbf{R},\mathbf{p}\times} = \begin{bmatrix} \mathbf{R} & \mathbf{p}\times\mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}. \tag{34}$$

Then

$$\mathbf{H}_{\mathbf{R},\mathbf{p}\times}^{T}\mathbf{K}\mathbf{H}_{\mathbf{R},\mathbf{p}\times} = \begin{bmatrix} \mathbf{R}^{T}\mathbf{A}\mathbf{R} & \mathbf{R}^{T}\left(\mathbf{A}\mathbf{p}\times+\mathbf{B}\right)\mathbf{R} \\ \mathbf{R}^{T}\left(\mathbf{p}\times^{T}\mathbf{A}+\mathbf{B}^{T}\right)\mathbf{R} & \mathbf{R}^{T}\left(\mathbf{p}\times^{T}\mathbf{A}\mathbf{p}\times+\mathbf{B}^{T}\mathbf{p}\times+\mathbf{p}\times^{T}\mathbf{B}+\mathbf{C}\right)\mathbf{R} \end{bmatrix}. \tag{35}$$

Lončarić's normal form is obtained by finding the transformation $\mathbf{H}_{\mathbf{R},\mathbf{p}\times}$ that diagonalizes the block of the transformed matrix corresponding to $\mathbf{B}$. This corresponds to first determining $\mathbf{p}$ such that matrix $\mathbf{S} = \mathbf{A}\mathbf{p} \times + \mathbf{B}$ is symmetric, and then diagonalizing it, i.e. determining the rotation matrix $\mathbf{R}$ that diagonalizes the resulting matrix.

Matrix $\mathbf{S}$ is symmetric when $\mathbf{S} = \mathbf{S}^T$, i.e. when $\mathbf{S} - \mathbf{S}^T = \mathbf{0}$, namely

$$\mathbf{A}\mathbf{p} \times + \mathbf{p} \times \mathbf{A} + \mathbf{B} - \mathbf{B}^T = \mathbf{0}. \tag{36}$$

An explicit solution of this skew-symmetric Sylvester equation is[4]

$$\mathbf{p} = \left(\mathbf{A} - \operatorname{tr}\left(\mathbf{A}\right)\mathbf{I}\right)^{-1} \operatorname{ax}\left(\mathbf{B} - \mathbf{B}^T\right) \tag{37}$$

(a proof is given in Appendix B). Finally, matrix $\mathbf{R}$ is obtained from the spectral decomposition (see note 2) of $\mathbf{S}$, $\mathbf{S} = \mathbf{R}\boldsymbol{\Gamma}\mathbf{R}^T$.

When $p_1 \neq 0$, the CoS lies outside the beam section plane. Since nothing prevents this occurrence in an arbitrary stiffness matrix, a beam model suitable for making use of such description of the sectional stiffness must be able to handle this circumstance.

Consider the trace of the stiffness matrix, which corresponds to the sum of the eigenvalues of the matrix, after the generic transformation $\mathbf{H}_{\mathbf{R},\mathbf{p}\times}$ of Eq. (34), consisting in a displacement and a rotation, is applied, namely

$$\operatorname{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{R},\mathbf{p}\times}^T \mathbf{K} \mathbf{H}_{\mathbf{R},\mathbf{p}\times}\right) = \operatorname{tr}\left(\mathbf{p} \times^T \mathbf{A}\mathbf{p} \times + \mathbf{B}^T\mathbf{p} \times + \mathbf{p} \times^T \mathbf{B} + \mathbf{C}\right). \tag{38}$$

[4]Operator $\operatorname{ax}(\cdot)$ extracts the vector that characterizes the skew-symmetric part of a generic matrix; it acts as the inverse of operator $(\cdot)\times$, namely, given a vector $\mathbf{v} \in \mathbb{R}^3$, $\operatorname{ax}(\mathbf{v}\times) = \mathbf{v}$, and given a matrix $\mathbf{M} \in \mathbb{R}^{3\times3}$, $(\operatorname{ax}(\mathbf{M}))\times = \operatorname{skw}(\mathbf{M})$.

The minimization of $\mathrm{tr}_{\mathbf{C}}(\mathbf{H}_{\mathbf{R},\mathbf{p}\times}^T \mathbf{K} \mathbf{H}_{\mathbf{R},\mathbf{p}\times})$ with respect to $\mathbf{p}$ yields the same expression of $\mathbf{p}$ given in Eq. (37), i.e. the CoS as defined in (Lončarić, 1987) is the transformation with the structure of a change of reference frame that minimizes the trace of the stiffness matrix and maximally decouples forces from angular strains and moments from linear strains. A detailed proof is given in Appendix C.

In a similar manner a center of compliance can be defined. The procedure is analogous; reported here without proof, it yields

$$\overline{\mathbf{p}} = \left(\overline{\mathbf{C}} - \mathrm{tr}(\overline{\mathbf{C}})\mathbf{I}\right)^{-1} \mathrm{ax}\left(\overline{\mathbf{B}}^T - \overline{\mathbf{B}}\right). \tag{39}$$

The resulting point $\overline{\mathbf{p}}$ in general differs from the CoS $\mathbf{p}$ of Eq. (37).

3.5. Beam Section Specific Center of Stiffness Transformation

A reference frame transformation for beam section stiffness properties that is physically meaningful must be expressible as a change of reference pole that lies within the plane of the section. In analogy with the interpretation of the CoS in terms of constrained minimization, consider a cost function f consisting of the trace norm of the transformed stiffness matrix of Eq. (38) augmented by the constraint that the component of $\mathbf{p}$ along the beam axis be zero, namely

$$f = \frac{1}{2}\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{p}\times}^T \mathbf{K} \mathbf{H}_{\mathbf{p}\times}\right) + \lambda \mathbf{e}_1^T \mathbf{p} \tag{40}$$

and minimize it with respect to $\mathbf{p}$ and the scalar Lagrange multiplier λ,

$$\frac{\partial f}{\partial \mathbf{p}} = \left(\mathbf{A} - \mathrm{tr}\left(\mathbf{A}\right)\mathbf{I}\right)\mathbf{p} + \mathbf{e}_1\lambda - \mathrm{ax}\left(\mathbf{B} - \mathbf{B}^T\right) = \mathbf{0} \tag{41a}$$

$$\frac{\partial f}{\partial \lambda} = \mathbf{e}_1^T \mathbf{p} = 0 \tag{41b}$$

(see Appendix C for details on computing Eq. (41a)). The solution of Eqs. (41) can be explicitly written by eliminating the Lagrange multiplier,

$$\mathbf{p} = \left(\mathbf{I} - \mathbf{Z}^{-1}\mathbf{e}_1 \left(\mathbf{e}_1^T\mathbf{Z}^{-1}\mathbf{e}_1\right)^{-1} \mathbf{e}_1^T\right) \mathbf{Z}^{-1}\mathrm{ax}\left(\mathbf{B} - \mathbf{B}^T\right), \tag{42}$$

with $\mathbf{Z} = \mathbf{A} - \mathrm{tr}(\mathbf{A})\mathbf{I}$. Equation (42) tells that the position of the beam CoS corresponds to Lončarić's CoS projected in the plane of the beam section by the non-orthogonal projector

$$\mathbf{P} = \mathbf{I} - \mathbf{Z}^{-1}\mathbf{e}_1 \left(\mathbf{e}_1^T\mathbf{Z}^{-1}\mathbf{e}_1\right)^{-1} \mathbf{e}_1^T \tag{43}$$

that accounts for the axial and shear stiffness properties of the beam.

The solution exists, is unique and uniquely defined as the displacement $\mathbf{p}$ that lies in the plane of the section and minimizes the trace norm of the section stiffness matrix.

3.6. Discussion

The stiffness properties of the beam section depend on the reference frame they are formulated in. A change of reference frame consisting in a change of reference orientation is intuitively expected to not alter intrinsic properties of the section like its eigenvalues; however, Eq. (14) shows that a change of reference pole may change the eigenvalues, since it may change their sum (which is equal to the trace of the matrix). At a first glance, this may sound counterintuitive, since a change of reference frame does not change the strain energy; on second thoughts, however, a change of reference frame is not a unitary transformation, and thus can modify the eigenvalues of the matrix. In fact, the strain energy does not change, because the change in the stiffness matrix is accompanied by a corresponding redefinition of strain measure, but the dependence of the stiffness matrix on the choice of the reference frame obfuscates intrinsic properties of the section stiffness.

A transformation like the one discussed in (Giavotto et al., 1983) completely decouples internal forces and angular strains, as well as internal moments and linear strains. We have proved that it minimizes the trace norm of the stiffness matrix. However, such transformation cannot be expressed in terms of a change of reference pole so it must change the specific strain energy of the system. Indeed, the strain energy change is hidden in the redefinition of the generalized strains operated by such transformation: it preserves the meaning of the curvature, whereas it redefines the linear strain as $\boldsymbol{\nu}' = \boldsymbol{\nu} - \mathbf{Y}\boldsymbol{\kappa}$, with $\mathbf{Y} + \mathbf{Y}^T \neq \mathbf{0}$ and thus not expressible as a change of reference pole.

Using the notion of center of elasticity, one may easily find that

$$\begin{aligned} \mathrm{tr}_{\mathbf{C}}\left(\mathbf{K}\right) &= \mathrm{tr}\left(\mathbf{C}\right) = \mathrm{tr}\left(\overline{\mathbf{C}}^{-1} + \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right) \\ &> \mathrm{tr}\left(\overline{\mathbf{C}}^{-1}\right) = \mathrm{tr}\left(\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right) = \mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{Y}}^T\mathbf{K}\mathbf{H}_{\mathbf{Y}}\right). \end{aligned} \quad (44)$$

This shows that the minimum strain energy is associated with the force-linear strain diagonal block of the stiffness matrix, $\mathbf{A}$, and the inverse of the angular strain-moment diagonal block of the compliance matrix, $\overline{\mathbf{C}}$. The term in excess in the trace of the untransformed matrix, $\mathrm{tr}(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})$, is removed by the transformation, and stored in the redefinition of the linear strains.

Using the notion of CoS of (Lončarić, 1987), the transformed coupling matrix $\mathbf{B}$ is made symmetric, $\mathbf{S} = \mathbf{A}\mathbf{p}\times + \mathbf{B}$. The transformed stiffness matrix can be rearranged as

$$\mathbf{H}_{\mathbf{p}\times}^T \mathbf{K} \mathbf{H}_{\mathbf{p}\times} = \begin{bmatrix} \mathbf{A} & \mathbf{S} \\ \mathbf{S} & \mathbf{C} - \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B} + \mathbf{S} \mathbf{A}^{-1} \mathbf{S} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} - \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{S} \\ \mathbf{S} & \mathbf{S} \mathbf{A}^{-1} \mathbf{S} \end{bmatrix}. \tag{45}$$

Further reduction is only possible considering the transformation of (Giavotto et al., 1983). After setting $\mathbf{Y}' = -\mathbf{A}^{-1}\mathbf{S}$, the completely decoupled form is obtained. Notice that

$$\mathbf{Y}' = -\mathbf{A}^{-1} (\mathbf{A} \mathbf{p} \times + \mathbf{B}) = -\mathbf{p} \times - \mathbf{A}^{-1} \mathbf{B} = -\mathbf{p} \times + \mathbf{Y}, \tag{46}$$

or

$$\mathbf{Y} = \mathbf{p} \times + \mathbf{Y}'. \tag{47}$$

Equation (47) confirms that the minimum energy transformation of (Giavotto et al., 1983), $\mathbf{Y}$, corresponds to the change of reference pole of (Lončarić, 1987), $\mathbf{p}\times$, plus a remainder $\mathbf{Y}'$ that, although not necessarily symmetric, cannot be expressed in terms of a change of reference pole. The minimum term in excess after the optimal change of reference pole is exactly $\mathrm{tr}(\mathbf{S}\mathbf{A}^{-1}\mathbf{S})$. Similar considerations apply to the beam section specific change of reference pole transformation discussed in Section 3.5.

In conclusion, as regards the reference point, there appears to be no ideal choice (the same holds for the reference orientation, or the separate reference orientations for linear and angular strains and thus for internal forces and moments). An attempt to completely decouple linear and angular entities, as the center of mass and principal axes do for rigid body inertia, yields a transformation that is not purely kinematic but changes the specific strain energy, so it sounds a bit like papering over the cracks. Transformations that can be represented as changes of reference pole, thus being kinematically meaningful, and minimize some clearly defined measure of the strain energy, like the proposed trace norm of the stiffness matrix, provide some form of normalization of the section reference frame. The use of such transformations is not mandatory, as any beam formulation that can handle arbitrary reference point choice can implicitly take them into account. Uniquely defined transformations that can be expressed as changes of reference pole provide a means to compare stiffness matrices on a common ground.

4. Examples

This section presents the analytical and numerical computation of the proposed stiffness matrix transformations applied to problems of increasing complexity.

4.1. Offset Axial Strain Centroid

Consider the stiffness matrix of a beam referred to the shear centroid, with coincident principal shear and axial strain orientations but axial strain centroid distinct from the shear centroid, namely

$$\mathbf{K} = \begin{bmatrix} EA & 0 & 0 & 0 & zEA & -yEA \\ 0 & GA_y & 0 & 0 & 0 & 0 \\ 0 & 0 & GA_z & 0 & 0 & 0 \\ 0 & 0 & 0 & GJ & 0 & 0 \\ zEA & 0 & 0 & 0 & EJ_y + z^2EA & -yzEA \\ -yEA & 0 & 0 & 0 & -yzEA & EJ_z + y^2EA \end{bmatrix}, \tag{48}$$

where EA is the axial stiffness, GA_y and GA_z are the shear stiffnesses along the principal shear axes, GJ is the torsional stiffness, and EJ_y and EJ_z are the bending stiffnesses about the principal bending axes.

In this case $\mathbf{A} = \mathrm{diag}(\{EA, GA_y, GA_z\})$ and $\mathrm{ax}(\mathbf{B}-\mathbf{B}^T) = \{0; -yEA; -zEA\}$. Then

$$\mathbf{p} = \begin{Bmatrix} 0 \\ p_y \\ p_z \end{Bmatrix}, \tag{49}$$

after defining

$$p_y = \frac{y}{1 + GA_z/EA} \qquad p_z = \frac{z}{1 + GA_y/EA}; \tag{50}$$

thus

$$\mathbf{S} = \begin{bmatrix} 0 & p_zGA_y & -p_yGA_z \\ p_zGA_y & 0 & 0 \\ -p_yGA_z & 0 & 0 \end{bmatrix}, \tag{51}$$

whose eigenvalues are

$$\Gamma_1 = 0 \tag{52}$$

$$\Gamma_{2|3} = \pm\sqrt{(p_zGA_y)^2 + (p_yGA_z)^2}. \tag{53}$$

The corresponding unit-norm eigenvectors are

$$\mathbf{r}_1 = \left\{ \begin{array}{c} 0 \\ \dfrac{p_y GA_z}{\sqrt{(p_z GA_y)^2 + (p_y GA_z)^2}} \\ \dfrac{p_z GA_y}{\sqrt{(p_z GA_y)^2 + (p_y GA_z)^2}} \end{array} \right\} \tag{54}$$

$$\mathbf{r}_{2|3} = \frac{1}{\sqrt{2}} \left\{ \begin{array}{c} 1 \\ \dfrac{\pm\, p_z GA_y}{\sqrt{(p_z GA_y)^2 + (p_y GA_z)^2}} \\ \dfrac{\mp\, p_y GA_z}{\sqrt{(p_z GA_y)^2 + (p_y GA_z)^2}} \end{array} \right\}, \tag{55}$$

with $\mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3]$. So a beam section referred to the shear centroid, with an offset axial strain centroid, yields an optimal reference position $\mathbf{p}$ consisting in a point in between the two centroids, whose location is weighted by the ratio between the shear and axial stiffnesses, and an optimal orientation consisting of vector $\mathbf{r}_1$ that lies in the plane of the section, and two other vectors orthogonal to $\mathbf{r}_1$ and mutually orthogonal, none of which is along axis $\mathbf{e}_1$.

4.2. Offset Shear Centroid

Consider the stiffness matrix of a beam referred to the axial strain centroid, with coincident principal shear and axial strain orientations, but shear centroid distinct from the axial strain centroid, namely

$$\mathbf{K} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & GA_y & 0 & -zGA_y & 0 & 0 \\ 0 & 0 & GA_z & yGA_z & 0 & 0 \\ 0 & -zGA_y & yGA_z & GJ + z^2GA_y + y^2GA_z & 0 & 0 \\ 0 & 0 & 0 & 0 & EJ_y & 0 \\ 0 & 0 & 0 & 0 & 0 & EJ_z \end{bmatrix} \tag{56}$$

In this case $\mathbf{A} = \mathrm{diag}(\{EA,\, GA_y,\, GA_z\})$, and $\mathrm{ax}(\mathbf{B} - \mathbf{B}^T) = \{0;\, -yGA_z;\, -zGA_y\}$. Then

$$\mathbf{p} = \begin{Bmatrix} 0 \\ \dfrac{y}{1 + EA/GA_z} \\ \dfrac{z}{1 + EA/GA_y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{GA_z}{EA} p_y \\ \dfrac{GA_y}{EA} p_z \end{Bmatrix}, \tag{57}$$

thus

$$\mathbf{S} = \begin{bmatrix} 0 & -p_zGA_y & p_yGA_z \\ -p_zGA_y & 0 & 0 \\ p_yGA_z & 0 & 0 \end{bmatrix} \tag{58}$$

i.e. the opposite of the matrix obtained in the previous case, which has exactly the same eigenvalues and the same eigenvector for the null-valued eigenvalue, and eigenvectors

$$\mathbf{r}_{2|3} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ \dfrac{\mp\, p_zGA_y}{\sqrt{(p_zGA_y)^2 + (p_yGA_z)^2}} \\ \dfrac{\pm\, p_yGA_z}{\sqrt{(p_zGA_y)^2 + (p_yGA_z)^2}} \end{Bmatrix} \tag{59}$$

for the other two eigenvalues (i.e. the sign of the components in the plane of the beam section is reversed), and thus the two problems have similar normal form. Indeed, matrices $\mathbf{A}$ and $\mathbf{B}$ of the two problems, the only portions of $\mathbf{K}$ that are involved in the determination of the CoS, only differ by a change of reference point.

4.3. Arbitrary Reference Point

The stiffness matrix of the section is referred to an arbitrary point by the transformation

$$\mathbf{K}' = \mathbf{H}_{\mathbf{p}\times}^T \mathbf{K} \mathbf{H}_{\mathbf{p}\times}, \tag{60}$$

with

$$\mathbf{H}_{\mathbf{p}'\times} = \begin{bmatrix} \mathbf{I} & \mathbf{p}'\times \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{61}$$

yielding

$$\mathbf{K}' = \begin{bmatrix} \mathbf{A} & \mathbf{A}\mathbf{p}' \times + \mathbf{B} \\ \mathbf{p}' \times^T \mathbf{A} + \mathbf{B}^T & \mathbf{p}' \times^T \mathbf{A}\mathbf{p}' \times + \mathbf{B}^T \mathbf{p}' \times + \mathbf{p}' \times^T \mathbf{B} + \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}' \\ (\mathbf{B}')^T & \mathbf{C}' \end{bmatrix} \tag{62}$$

It is obvious that if $\mathbf{B} \equiv \mathbf{0}$ the transformation $\mathbf{p}$ that makes matrix $\mathbf{A}\mathbf{p}\times + \mathbf{B}'$ symmetric (actually, null) is $\mathbf{p} = -\mathbf{p}'$.

Consider the problem proposed in (Bottasso et al., 2002) as Example 3.5; the corresponding baseline stiffness matrix is $\mathbf{K} = \mathrm{diag}(EA,\, GA_y,\, GA_z,\, GJ,\, EJ_y,\, EJ_z)$, with $\mathbf{B} \equiv \mathbf{0}$. In that reference, a simple problem is repeatedly analyzed after referring the stiffness matrix to a set of points arbitrarily offset from the baseline one.

Considering the CoS proposed by Lončarić, the beam section would always be referred to the centroid, thus overcoming the errors introduced by numerical methods because of their lack of invariance, as discussed in (Bottasso et al., 2002). Note that Lončarić's CoS in this case intrinsically complies with the constraint $\mathbf{p} \cdot \mathbf{e}_1 = 0$, thus being equivalent to the proposed constrained CoS.

4.4. Fully Coupled Smart Helicopter Blade Section

Consider the fully populated stiffness matrix of the smart composite helicopter rotor blade section shown in Fig. 2 and described in (Ghiringhelli et al., 2008),

$$\mathbf{K} = \begin{bmatrix} 8.187\text{e}{+7} & 1.718\text{e}{+6} & 6.110\text{e}{+4} & -2.241\text{e}{+4} & 3.689\text{e}{+5} & 6.067\text{e}{+6} \\ 1.718\text{e}{+6} & 8.548\text{e}{+6} & 7.321\text{e}{+4} & -5.483\text{e}{+4} & 1.086\text{e}{+4} & 2.732\text{e}{+5} \\ 6.110\text{e}{+4} & 7.321\text{e}{+4} & 1.203\text{e}{+6} & -8.306\text{e}{+4} & 3.250\text{e}{+2} & 1.233\text{e}{+4} \\ -2.241\text{e}{+4} & -5.483\text{e}{+4} & -8.306\text{e}{+4} & 1.356\text{e}{+4} & -7.500\text{e}{+2} & -3.801\text{e}{+3} \\ 3.689\text{e}{+5} & 1.086\text{e}{+4} & 3.250\text{e}{+2} & -7.500\text{e}{+2} & 1.085\text{e}{+4} & 2.245\text{e}{+4} \\ 6.067\text{e}{+6} & 2.732\text{e}{+5} & 1.233\text{e}{+4} & -3.802\text{e}{+3} & 2.245\text{e}{+4} & 8.513\text{e}{+5} \end{bmatrix}. \tag{63}$$

Numerical data are in SI, i.e. data in submatrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ respectively are in N, N·m, and N·m^2, whereas data in vector $\mathbf{p}$ and matrix $\mathbf{Y}$ are in m.

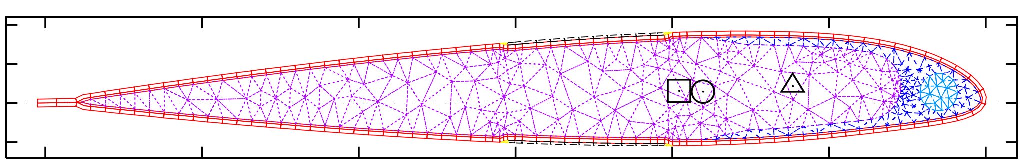

Figure 2: Smart composite helicopter blade section (from (Ghiringhelli et al., 2008)); ○: center of mass; △: shear center; □: normal stress center.

Using the notation of Appendix B, one obtains

$$\mathbf{p} = \begin{Bmatrix} 1.503\text{e–}2 \\ -7.372\text{e–}2 \\ 4.637\text{e–}3 \end{Bmatrix} \tag{64}$$

$$\mathbf{R} = \begin{bmatrix} -6.624\text{e–}1 & -7.454\text{e–}1 & 7.539\text{e–}2 \\ -5.710\text{e–}1 & 4.371\text{e–}1 & -6.949\text{e–}1 \\ 4.850\text{e–}1 & -5.033\text{e–}1 & -7.151\text{e–}1 \end{bmatrix} \tag{65}$$

$$\mathbf{H}_{\mathbf{p}\times}^T \mathbf{K} \mathbf{H}_{\mathbf{p}\times} = \begin{bmatrix} 4.021\text{e+}7 & 3.826\text{e+}7 & -3.603\text{e+}5 & -2.277\text{e+}4 & 0.0 & 0.0 \\ 3.826\text{e+}7 & 4.632\text{e+}7 & -5.784\text{e+}6 & 0.0 & -1.561\text{e+}2 & 0.0 \\ -3.603\text{e+}5 & -5.784\text{e+}6 & 5.095\text{e+}6 & 0.0 & 0.0 & 2.371\text{e+}4 \\ -2.277\text{e+}4 & 0.0 & 0.0 & 1.031\text{e+}5 & -9.855\text{e+}4 & -1.361\text{e+}5 \\ 0.0 & -1.561\text{e+}2 & 0.0 & -9.855\text{e+}4 & 1.091\text{e+}5 & 1.396\text{e+}5 \\ 0.0 & 0.0 & 2.371\text{e+}4 & -1.361\text{e+}5 & 1.396\text{e+}5 & 2.041\text{e+}5 \end{bmatrix} \tag{66}$$

where exact zeros were actually zeros to machine precision. Matrix $\mathbf{Y}$ as proposed in (Giavotto et al., 1983) is

$$\mathbf{Y} = \begin{bmatrix} 1.006\text{e–}4 & -4.498\text{e–}3 & -7.375\text{e–}2 \\ 5.806\text{e–}3 & -3.668\text{e–}4 & -1.710\text{e–}2 \\ 6.868\text{e–}2 & -1.938\text{e–}5 & -5.466\text{e–}3 \end{bmatrix}, \tag{67}$$

which does not take the form of Eq. (30). Its skew-symmetric part,

$$\mathrm{ax}\left(\mathbf{Y}\right) = \begin{Bmatrix} -1.938\text{e–}5 \\ -7.375\text{e–}2 \\ 5.806\text{e–}3 \end{Bmatrix}, \tag{68}$$

clearly differs from $\mathbf{p}$ of Eq. (64). The change of reference pole transformation of Eq. (42) is

$$\mathbf{p} = \left\{ \begin{array}{c} 0.0 \\ -7.403\text{e–}2 \\ 4.626\text{e–}3 \end{array} \right\}. \tag{69}$$

Note that its p_2, p_3 coefficients, although similar, differ from the corresponding ones of the change of reference pole transformation of Eq. (64), not constrained to remain on the plane of the section.

5. Conclusions

Beam section stiffness properties can be referred to an arbitrary point. Beam formulations should be able to handle such arbitrariness. Nonetheless, it may be desirable to be able to uniquely define a reference point. It is noted that a transformation that completely decouples internal force from angular strain and internal moment from linear strain minimizes the trace norm of the stiffness matrix; however, such transformation cannot be expressed only in terms of a change of reference frame. It is also noted that a transformation that maximally decouples internal force from angular strain and internal moment from linear strain also minimizes the trace norm of the stiffness matrix subjected to the constraint of being representable as a change of reference frame. Such transformation takes as reference point the center of stiffness; however, such point may lie outside the plane of the beam section. A novel unique definition of reference point is proposed, which minimizes the trace norm of the stiffness matrix subjected to the constraint of being representable as a change of reference frame within the plane of the section.

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Appendix

Appendix A. Proof of Optimality of Complete Decoupling (Matrix Y)

Consider the transformation

$$\mathbf{H}_{\mathbf{Y}} = \begin{bmatrix} \mathbf{I} & \mathbf{Y} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{A.1}$$

The trace of the transformed stiffness matrix is

$$\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{Y}}^T\mathbf{K}\mathbf{H}_{\mathbf{Y}}\right) = \mathrm{tr}\left(\mathbf{Y}^T\mathbf{A}\mathbf{Y} + \mathbf{B}^T\mathbf{Y} + \mathbf{Y}^T\mathbf{B} + \mathbf{C}\right) \tag{A.2}$$

Exploiting the properties of the trace operator, one obtains

$$\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{Y}}^T\mathbf{K}\mathbf{H}_{\mathbf{Y}}\right) = \mathrm{tr}\left(\mathbf{A}\mathbf{Y}\mathbf{Y}^T + 2\mathbf{B}\mathbf{Y}^T + \mathbf{C}\right) \tag{A.3}$$

and

$$\frac{\partial}{\partial \mathbf{Y}}\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{Y}}^T\mathbf{K}\mathbf{H}_{\mathbf{Y}}\right) = 2\mathbf{A}\mathbf{Y} + 2\mathbf{B}. \tag{A.4}$$

By requiring that Eq. (A.4) be equal to zero, one computes the transformation $\mathbf{Y}$ that minimizes $\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{Y}}^{T}\mathbf{K}\mathbf{H}_{\mathbf{Y}}\right)$, since the latter is a positive definite form in $\mathbf{Y}$, yielding

$$\mathbf{Y} = -\mathbf{A}^{-1}\mathbf{B} \tag{A.5}$$

■.

Appendix B. Direct Computation of Center of Stiffness (Vector p)

An explicit solution of the skew-symmetric Sylvester equation

$$\mathbf{A}\mathbf{p}\times + \mathbf{p}\times\mathbf{A} + \mathbf{B} - \mathbf{B}^{T} = \mathbf{0} \tag{B.1}$$

can be found by first decomposing matrix $\mathbf{A}$, which is symmetric positive definite, in spectral form, $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{T}$, and then transforming the problem in

$$\mathbf{\Lambda}\hat{\mathbf{p}}\times - \left(\mathbf{\Lambda}\hat{\mathbf{p}}\times\right)^{T} + \hat{\mathbf{b}}\times = \mathbf{0}, \tag{B.2}$$

with $\hat{\mathbf{p}} = \mathbf{U}^{T}\mathbf{p}$ and $\hat{\mathbf{b}} = \mathrm{ax}(\mathbf{U}^{T}(\mathbf{B} - \mathbf{B}^{T})\mathbf{U})$. The solution is then

$$\hat{p}_i = -\frac{\hat{b}_i}{\mathrm{tr}(\mathbf{\Lambda}) - \Lambda_i}, \qquad i \in [1,3] \tag{B.3}$$

and $\mathbf{p} = \mathbf{U}\hat{\mathbf{p}}$. Eq. (B.3) can be rearranged as

$$\mathbf{U}^{T}\mathbf{p} = -\left(\mathrm{tr}(\mathbf{\Lambda})\mathbf{I} - \mathbf{\Lambda}\right)^{-1}\mathbf{U}^{T}\mathbf{b} \tag{B.4}$$

which, after premultiplication by $\mathbf{U}$, yields

$$\begin{aligned}
\mathbf{p} &= -\mathbf{U}\left(\mathrm{tr}(\boldsymbol{\Lambda})\mathbf{I} - \boldsymbol{\Lambda}\right)^{-1}\mathbf{U}^T\mathbf{b} \\
&= -\mathbf{U}^{-T}\left(\mathrm{tr}(\boldsymbol{\Lambda})\mathbf{I} - \boldsymbol{\Lambda}\right)^{-1}\mathbf{U}^{-1}\mathbf{b} \\
&= -\left(\mathbf{U}\left(\mathrm{tr}(\boldsymbol{\Lambda})\mathbf{I} - \boldsymbol{\Lambda}\right)\mathbf{U}^T\right)^{-1}\mathbf{b} \\
&= -\left(\mathbf{U}\mathrm{tr}(\boldsymbol{\Lambda})\mathbf{U}^T - \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T\right)^{-1}\mathbf{b} \\
&= \left(\mathbf{A} - \mathrm{tr}(\mathbf{A})\mathbf{I}\right)^{-1}\mathbf{b},
\end{aligned} \tag{B.5}$$

since $\mathrm{tr}(\boldsymbol{\Lambda}) = \mathrm{tr}(\mathbf{A})$ $\blacksquare$.

Equation (B.3) requires that $\Lambda_i \neq \mathrm{tr}(\boldsymbol{\Lambda})$. In the present context, it implies that the sum of any pair of eigenvalues of matrix $\mathbf{A}$ be non zero. Since matrix $\mathbf{A}$ expresses the force portion of the beam section stiffness, it may be safely assumed to be positive definite, so all its eigenvalues are positive and the requirement is always met. A discussion on special cases where matrix $\mathbf{A}$, from a different context, is not positive definite is presented in (Roberts, 2002).

Appendix C. Proof of Suboptimality of Center of Stiffness (Vector p)

Consider the transformation

$$\mathbf{H}_{\mathbf{R},\mathbf{p}\times} = \begin{bmatrix} \mathbf{R} & \mathbf{p}\times\mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \tag{C.1}$$

After some manipulation exploiting the properties of the trace operator, the trace of the transformed stiffness matrix is

$$\begin{aligned}
\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{R},\mathbf{p}\times}^T\mathbf{K}\mathbf{H}_{\mathbf{R},\mathbf{p}\times}\right) &= \mathrm{tr}_{\mathbf{C}}\left(\begin{bmatrix} \mathbf{R}^T\mathbf{A}\mathbf{R} & \mathbf{R}^T\left(\mathbf{A}\mathbf{p}\times+\mathbf{B}\right)\mathbf{R} \\ \text{sym.} & \mathbf{R}^T\left(\mathbf{p}\times^T\mathbf{A}\mathbf{p}\times+\mathbf{B}^T\mathbf{p}\times+\mathbf{p}\times^T\mathbf{B}+\mathbf{C}\right)\mathbf{R}\end{bmatrix}\right) \\
&= \mathrm{tr}\left(\mathbf{R}^T\left(\mathbf{p}\times^T\mathbf{A}\mathbf{p}\times+\mathbf{B}^T\mathbf{p}\times+\mathbf{p}\times^T\mathbf{B}+\mathbf{C}\right)\mathbf{R}\right) \\
&= \mathrm{tr}\left(\mathbf{p}\times^T\mathbf{A}\mathbf{p}\times+\mathbf{B}^T\mathbf{p}\times+\mathbf{p}\times^T\mathbf{B}+\mathbf{C}\right) \\
&= \mathrm{tr}\left(\mathbf{A}\mathbf{p}\times\mathbf{p}\times^T+2\mathbf{B}\mathbf{p}\times^T+\mathbf{C}\right) \qquad \text{(C.2)}
\end{aligned}$$

The derivative with respect to $\mathbf{p}$ is not as straightforward as that of Appendix A with respect to $\mathbf{Y}$. Consider that

$$\frac{\partial}{\partial g_{ij}}\mathrm{tr}\left(\mathbf{M}\mathbf{G}^T\right) = \mathrm{tr}\left(\mathbf{M}\boldsymbol{\Delta}_{(ji)}\right) = m_{ij} \qquad \text{(C.3)}$$

where $\boldsymbol{\Delta}_{(ji)}$ indicates a matrix of all zeros with the exception of coefficient ji, which is equal to 1.

The derivative of $\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{p}\times}^T\mathbf{K}\mathbf{H}_{\mathbf{p}\times}\right)$ with respect to $\mathbf{p}$, given the structure of $\mathbf{p}\times$, namely

$$\mathbf{p}\times = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \qquad \text{(C.4)}$$

yields

$$\frac{\partial}{\partial p_i}\mathrm{tr}\left(\mathbf{M}\mathbf{p}\times^T\right) = \mathrm{tr}\left(\mathbf{M}\left(\boldsymbol{\Delta}_{(kj)}-\boldsymbol{\Delta}_{(jk)}\right)\right) = m_{jk}-m_{kj} \qquad \text{(C.5)}$$

with $i = 1,2,3,\ j = 3,1,2,\ k = 2,3,1$. As a consequence,

$$\frac{\partial}{\partial \mathbf{p}}\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{p}\times}^T\mathbf{K}\mathbf{H}_{\mathbf{p}\times}\right) = 2\mathrm{ax}\left(\mathbf{A}\mathbf{p}\times+\mathbf{B}\right) \qquad \text{(C.6)}$$

By requiring that Eq. (C.6) be equal to zero, which is equivalent to solving the Sylvester equation

$$\mathbf{A}\mathbf{p}\times + \mathbf{p}\times\mathbf{A} + \mathbf{B} - \mathbf{B}^T = \mathbf{0}, \tag{C.7}$$

one computes the displacement $\mathbf{p}$ of the reference frame that minimizes $\mathrm{tr}_{\mathbf{C}}\left(\mathbf{H}_{\mathbf{p}\times}^T\mathbf{K}\mathbf{H}_{\mathbf{p}\times}\right)$, since the latter is a positive definite form in $\mathbf{p}\times$ which, in accordance with Appendix B, yields

$$\mathbf{p} = \left(\mathbf{A} - \mathrm{tr}(\mathbf{A})\mathbf{I}\right)^{-1}\mathrm{ax}(\mathbf{B} - \mathbf{B}^T) \tag{C.8}$$

■.

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