On Dynamic State-Space models for fatigue-induced structural degradation

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Article history:
Received 10 June 2013
Received in revised form 7 November 2013
Accepted 11 November 2013
Available online 20 November 2013

1. Introduction

Structural damage identification and lifetime prediction has raised increasing interest over the last years. The continuous rise of maintenance costs of complex systems highlights the necessity to maximise the operating life of the structures without any performance drop. Moreover, deterioration monitoring stands in accordance with the strict regulations on safety of dangerous systems such as bridges [1] nuclear power plants [2] or civil and military aircrafts [3]. For these reasons, the paper proposes an in-depth study of Fatigue Crack Growth (FCG) models and their combination with statistical tools for dynamic processes, both non-linear and non-Gaussian. A typical *model-based* approach for dynamic systems in which a dynamic model describes the observed phenomenon is used. In order to produce an effective model-based algorithm, at least two different fields have to be studied in-depth: (i) Damage evolution over structures, and (ii) theory of random processes. The first one is required to understand all the possible time-evolution of the fatigue-damage according to the phenomenological aspects of damage propagation, while the second one is essential to account for all the uncertainties that can be encountered in a real environment. Usually the algorithms able to produce a lifetime prediction are based on the mathematical description of the degradation (achieved by point (i)) altered by a random process (ii), which is heuristically selected. Thanks to this alteration of the deterministic deterioration process, a statistical lifetime prediction of the structure is provided. The approach can be considered as a particular branch of structural reliability and maintenance technologies having damage tolerance, condition based or predictive maintenance as the final purpose.

The necessity of a stochastic process for the efficient (and possibly on-line) lifetime prediction is justified by the wide scatter of data about fatigue failure of structures. This scatter is due to the intrinsic uncertainty about the atomic arrangement, the impurity and the statistical presence of flaws altering the crystal lattice of materials. Moreover, the uncertainties about the loads and the measurement of the damage markedly affect the Residual Useful Life (RUL) estimation of the monitored structure. The scatter in fatigue and fracture data appears also in the case of constant-amplitude loading [4–6] and it markedly grows if the applied load is not well defined or, in the worst case, has a random spectrum such as in the case of aeronautical and civil structures [7–10]. Yokobori in [11,12] highlights the necessity to consider crack nucleation and crack-induced failure as a stochastic process since 1953. Therefore, the entire literature about prognostic systems and fatigue life

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prediction of structures involves some degree of stochastic processes, in particular if the loads are unknown [13–18]. Uncertainty on crack propagation parameters has been studied by several authors [19–22] on the basis of the Virkler's database [4] which contains the results of 68 fatigue crack propagations on Al2024-T3 specimens in which the variability of crack growth velocity has been highlighted. Wu et al. have considered the dependence of the specimen orientations on FCG rate [23] and the uncertainties of crack propagation near the threshold limit [24]. Ray and Tangirala [25,26] described in-depth the variability of the FCG phenomenon to provide prognostic tools. They have used models for the FCG rate with additional random noise to produce an estimation of the probability of failure through Gaussian–Markov process filters or using the principle of Karhunen–Loève expansion of the crack length covariance [27]. Instead, approaches based on State-Space models can be found in [28,29]. Yang and Manning [30] have statistically analysed the FCG by a stochastic model composed of the deterministic crack propagation law multiplied by a lognormal random process $x(t)$ producing a Stochastic Differential Equation (SDE). The integration of the equation provides the median crack growth curve and the probability of the critical crack length with respect to time. Wu et al. [31] have presented stochastic FCG models built by experimental data. Xiang et al. [32] have proposed a life prediction algorithm of specimens based on equivalent initial flaw size concept combined with asymptotic Stress Intensity Factor (SIF) solution for notch cracks. Farhangdoost et al. [33] propose a methodology similar to those in [30] using stochastic Markov process and taking into account the closure effect. Scafetta et al. [34] have analysed the correlation of random phenomena inside the FCG problem on ductile alloys. They have shown that some features of the crack dynamics have to be modelled by correlated stochastic processes, while some other characteristics can be considered as a random noise. Gangloff et al. [35] have made predictions about FCG on cracked specimen studying the closure and environmental effects, load frequency and relative interactions. Newman et al. [36] have studied the FCG on a cracked helicopter component under variable-amplitude loading by using FASTRAN software. This complex problem emphasizes the scatter of data and the uncertainty about FCG and the lifetime prediction. The threshold SIF variability and the closure effect is affected by many parameters, since a lot of work has been dedicated to the study of this particular region of the FCG curve ([24,37–42]). As a consequence, a real component subjected to fatigue loading is also subjected to high fluctuations in fatigue crack propagation.

The need for a stochastic process in order to design an effective lifetime predictor stands in agreement with analytical and numerical algorithms based on Stochastic Differential Equation or Dynamic State-Space (DSS) models for random processes. The literature about stochastic system evolution and stochastic filtering is at least as extensive as the FCG literature. A series of numerical algorithms and filters have been developed to combine the model evolution equation with the statistics of the observed phenomenon. This combination merges the information about the probable evolution of the degradation (in this case, the fatigue crack propagation) with the direct or indirect measure of the crack length on the structure. The basic idea is to screen the most probable crack evolution according to the measured crack length to produce an accurate time to failure or RUL distribution. The main techniques of interest are based on Bayesian statistics and Monte Carlo Sampling (MCS), for example, the Markov chain Monte Carlo (MCMC) or the Bayesian filters. Both of these methods can be used to produce a reliable failure time distribution given a mathematical model of the phenomenon exists and sequential measures on the system are available. One of the most widespread MCMC method is the algorithm proposed by Metropolis [43] and generalised by Hastings in 1970 [44], or the Metropolis Adjusted Langevin Algorithm (Roberts and Rosenthal [45–47] and references therein). These MCMC techniques are used to identify the actual parameters of the evolution equations that are roughly known at the beginning of the process. Inside the field of Bayesian filters, a series of techniques for nonlinear problems with non-Gaussian noise have been developed starting from the Kalman Filter (KF) ([48] and [49]) up to the most advanced as Sequential Importance Sampling often called Particle Filter. Haug [50], Arulampalam et al. [51] and Chen [52] reported a useful explanation of these different techniques. Moreover, recent studies on Bayesian filtering focus on techniques that evaluate the most probable system evolution by means of a model-degradation equation built by random variable parameters [53–56]. Interesting and useful applications of Sequential Importance Sampling on FCG problems can be found in [57–59].

As clearly visible from the quoted works, the scientific community studied the problem of Fatigue Crack Growth and stochastic lifetime prediction for many years. Several methodologies address the variability of crack growth phenomenon trying to evaluate the RUL of the structure. One of these methods is the statistical modelling of FCG parameters. Several papers show the application of this technique using Paris' equation or more advanced models. However, an overall dissertation on FCG modelling combined with Markov chain Monte Carlo methods is missing. Many papers dedicated to crack monitoring focus on the mathematical implementation of the problem, sometimes overlooking the applicability of the methodologies to real systems. Moreover, some features such as the artificial noise of simulated models or the parameters defined in statistical terms are arbitrarily selected in some cases. Again, one of the main drawbacks of the common Metropolis–Hastings algorithm, which is the proper selection of the proposal distribution, is not faced for the problem of fatigue crack propagation. It is a crucial step for the algorithm development. At the beginning of the process, the spread of the measures is unknown, so previous experience and practical considerations drive the selection of the proposal variance. Given these premises, the present work provides a full description of the FCG modelling using statistical parameters and the combination of this stochastic modelling within a Bayesian framework. It tries to make a comprehensive treatment of the matter underlying advantage and drawbacks of the statistical definition for each model parameter. The different sections describe all the peculiar features mandatory to develop a prognostic tool based on statistical parameters and MCS techniques. First, the variability of the crack propagation curves is expressed as mean and variance of the FCG rate models, defining the output variability as a function of different uncertainties on the input variables. Several models with different parameters are analysed based on this concept, providing brief discussions on the applicability of the analytical solutions. The approach is suitable for Damage Tolerant design introducing a statistical definition of the crack growth velocity, rather than the standard deterministic curves implemented nowadays. Moreover, it is suitable for the integration of the FCG rate equation and provides useful information to build a Stochastic DSS (SDSS) model for damage propagation, as presented in this work. The development of the Dynamic State-Space model for FCG constitutes the second part of the work. It takes into account the variability of FCG models previously studied to solve the problem of residual life prediction. The description of the method and the peculiar variables of interest have been presented, accounting for the threshold SIF range as random variable, which is crucial at least in the first part of the damage propagation. The prognostic tool is based on Metropolis–Hastings algorithm with a particular *adaptive proposal distribution*, which is detailed during the description of the DSS model updating. The application of this adaptive proposal to fatigue crack propagation monitoring constitutes an additional novelty of the paper. The method is applied to simulated fatigue crack propagation, with a preliminary analysis of the threshold SIF range updating, for the purposes of RUL prediction. The final step forward of the work is represented by the application of the prognostic tool for the real-time estimation of the residual life of real portions of helicopter fuselages, subjected to fatigue load, however in a laboratory environment. The algorithm is tested on four different crack propagations, highlighting the effectiveness and the robustness of the method for real structures.

The paper organises as follows: Section 2 shows a statistical description of several FCG rate models, starting from the simplest equation defined by Paris and Erdogan [60] up to the most advanced NASGRO equation [61]. Focusing on the NASGRO model, different kind of uncertainties are studied separately, in order to underline some statistical features of the sigmoid curve describing the crack velocity as a function of the SIF range. Section 3 provides the implementation of the stochastic FCG equations into a Dynamic State-Space model suitable for the numerical resolution and the real-time updating of the residual lifetime prediction. Section 4 firstly shows the application of the methodology to simulated crack propagations with a discussion about the updating of the FCG parameters; then the algorithm is tested on a real aluminium helicopter panel subjected to fatigue load in a laboratory environment. The SDSS model of the crack evolution is updated by means of a Metropolis–Hastings algorithm with a particular adaptive proposal distribution used to improve the performance into the framework of crack propagation monitoring. The lifetime prediction is compared with the actual residual useful life of the structures. The conclusive section (Section 5) summarises the main novelties introduced by this work, the lifetime prediction performances of the proposed technique and finally the advantages and drawbacks of the method for real-time crack propagation monitoring. An appendix is also provided at the end of the paper to show the analytical formulations used to evaluate the statistics of the crack propagation velocity related to different FCG rate models.

2. Stochastic modelling of Fatigue Crack Growth

This section provides a detailed statistical insight to the fatigue crack propagation models. The mathematical quantities within these models are statistically defined after a very brief introduction about FCG. The treatment is made as general as possible in the beginning, with a final application to the most advanced crack growth equation, which is analysed distinguishing among different types of uncertainty. The dependence of the fatigue crack propagation on the stress affecting the crack tip is a concept generally accepted by the scientific community [62–64]. Several models describe the phenomenological aspects of crack propagation. Within these models, the SIF range near the crack tip drives the crack growth; nevertheless a direct dependence on the crack length also exists. Consequently, the crack propagation velocity with respect to the applied fatigue load is usually described by ordinary differential Eq. (1) that inherently includes the formulation of SIF range.

$$\frac{dy}{dt} = \phi(y) \tag{1}$$

where y is the crack length, t denotes the increasing time and ϕ is the function used to describe the FCG process. The number of necessary parameters and variables is allied with the model complexity. The analytical or numerical resolution of (1) produces the expected time to reach selected certain crack length. The most common and easy way is to use the Paris' equation to describe ϕ and integrate the Eq. (1) using the method of variable separation to calculate the failure cycle N_f (2). Considering constant-amplitude loading.

$$N_f = \int_{y_0}^{y_f} \frac{dy}{\phi(y)} \tag{2}$$

where y_0 and y_f are the starting and the final crack length, respectively. Unfortunately, the estimations provided by deterministic models are not sufficient to produce a reliable residual life assessment for any real component. This concept is evidenced by Fig. 1, which shows some results of the fatigue crack propagation on helicopter aluminium panels, a similar configuration is analysed in [65,66]. The results of the crack evolutions are compared with the theoretical crack propagation obtained by computer simulations with the NASGRO equation. The large differences between experimental data and simulation are clearly visible. Thus a probabilistic assessment of the possible crack evolutions is mandatory for structural maintenance strategies, safety of structures and advanced real-time prognostic tools. Further information about these experimental tests is provided in Section 4, in which the experimental data is used to validate the proposed methodology.

Two main techniques are typically implemented to predict the residual life of cracked structures. The first one is based on the FCG equation in (1), multiplied by stochastic processes taken into account during the integration procedure [27,28,30] or used in dynamic filter techniques [25,26]. These random processes are generated according to previously obtained experimental data, user's experience or heuristic considerations. The other method is based on the statistical description of the FCG rate parameters as just presented in [18,20–22,67–71]. The statistical considerations within these works are heavily interrelated with the implemented models. Nevertheless, a general treatment on the combination of FCG modelling within Bayesian frameworks is missing. Thus, the paragraph investigates a wide range of possible random variables within the field of FCG models, underlying some statistical features of the Eq. (1). The dissertation provides advantages, drawbacks and practical problems related to the statistical definition of the employed parameters. It provides a statistical definition of crack growth rate, useful for Damage Tolerance strategies, and certainly more reliable than the standard deterministic equations. Obviously, the result of probabilistic crack propagation analysis is dependent on the employed mathematical model. However, the proposed methodology is suitable for the application to any fatigue degradation process without restrictions. In all generality, the FCG rate models are constituted of:

(i) Empirical parameters of the FCG equation (model uncertainty).
(ii) Typical features of the material (material uncertainty).
(iii) Time-varying variables (environmental and physics uncertainty).

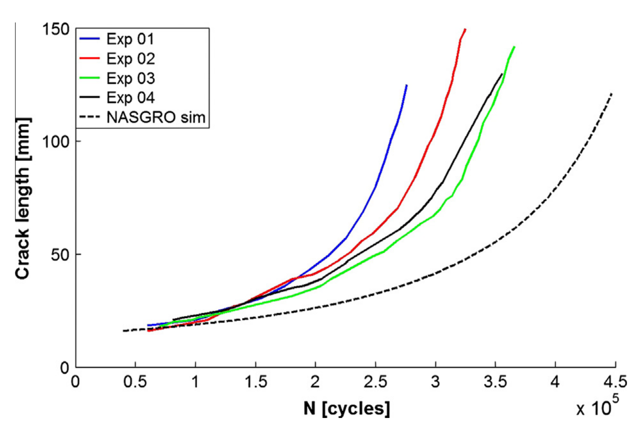

Fig. 1. Observed crack propagations on aluminium panels compared with the FCG simulation based on the NASGRO model.

The variability of the empirical parameters has been demonstrated by means of Virkler's data [4], at least for the two parameters describing the log-linear regime. Intuitively, the empirical parameters are estimated through regressions or other numerical techniques with an intrinsic uncertainty on the output related to the error of the model itself. This is the reason why it is defined as a model uncertainty, too. The features of the material represent the second source of uncertainty. The considerations about the material variability are similar of those made for the empirical parameters. However, they represent the variability of physical quantities, rather than the uncertainty related to the data-fitting practise. The uncertainties related to the time-dependent variables is mainly associated to the variability of the loads acting on the structure and the uncertainty related to the crack length, especially for automatic measurement systems[1]. Starting from these considerations, the ordinary differential Eq. (1) becomes an SDE. If the empirical parameters, the material properties and the time-varying variables are respectively collected in the vectors ϑ, η, ζ the SDE assumes the form in (3).

$$\frac{dY}{dt} = \phi(\Theta, \mathrm{H}, \mathrm{Z}(Y, t)) \tag{3}$$

where the capital letters state random variables. Y represents the crack length, Θ is the random vector of empirical parameters, H is the random vector of material properties and $\mathrm{Z}(Y,t)$ is the random vector of the time-dependent variables. Intuitively, it should be possible to evaluate the prediction intervals of the output dY/dt based on the statistical definition of the input variables. Nonetheless, one has to overcome the nonlinearity of the model in order to provide a statistical definition of the output. The analytical formulation of the output variability given the statistics of the input parameters is proposed hereafter. As explained above, the statistical evaluation of crack growth velocity can be applied for the optimisation of advanced maintenance strategies. In addition, this analysis highlights which parameters should be statistically defined, and checks whether a closed form solution of mean and variance exists. If the analytical solution does not exist, the FCG with statistical definition of parameters can be implemented in Dynamic State-Space models suitable for the numerical resolution via MCS.

Different FCG models are accounted for, highlighting the formulation of the variance, given different parameters and material properties are provided as input. In order to produce a closed form solution of the problem, three main hypotheses are considered: (i) the random variables contemplated in the dissertation are supposed normally distributed or log-normally distributed, (ii) the time-varying variables, that is SIF range and maximum applied SIF are supposed to be deterministic (zero variance), and (iii) the amplitude of the load is assumed constant, so that the crack velocity can be evaluated per load cycle dY/dN instead of the crack increment in time dY/dt. Thus the vector of time-varying variables becomes deterministic at determined time $\mathrm{Z}(Y,t) = \zeta(y,t)$. If the hypotheses cannot be satisfied or a closed form solution does not exist, the propagation of uncertainties within the FCG framework can be solved according with the Monte Carlo method presented in this paper (Section 4) or other numerical techniques.

2.1. Paris' model

Sometimes, the transformation of (1) in logarithmic form make easier the analytical evaluation of the variance associated to the crack propagation velocity. Suppose to start from the Paris–Erdogan law defined by the well-known parameters C, m and the SIF range acting on the crack tip [60]. The two distributions $N(\mu_{\log C}, \sigma^2_{\log C})$, $N(\mu_m, \sigma^2_m)$ define the statistics of $\log C$, m correlated by the coefficient $\rho_{\log C,m}$ [70]. The closed form solution of the logarithmic crack propagation velocity $\mu_{\log dY/dN}, \sigma^2_{\log dY/dN}$ are defined in (4) and (5), as presented in [72]. A more detailed explanation of the equation is available in the appendix.

$$\mu_{\log dY/dN} = \mu_{\log C} + \mu_M \log \Delta K \tag{4}$$

$$\sigma^2_{\log dY/dN} = \sigma^2_{\log C} + \log \Delta K \sigma_M (\log \Delta K \sigma_M + 2\rho_{\log C,M}\sigma_{\log C}) \tag{5}$$

2.2. Forman's model

The Paris' formulation is very simple and widely studied. On the other hand, it overrate the crack growth rate of the threshold region and underrate the crack rate in the unstable crack propagation zone. Moving towards more advanced models, Forman et al. [73] suggest a model capable of describing the region III of the sigmoid curve (6), namely the *critical region*. Similarly to the treatment made for Paris' law, Eqs. (7) and (8) shows the mean and variance associated to Forman's model.

$$\frac{dy}{dN} = C\Delta K^m \frac{K_{\max}}{K_C - K_{\max}} \tag{6}$$

$$\mu_{\log dY/dN} = \mu_{\log C} + \mu_M \log \Delta K + K_{\max} - \mu_{\log(K_C - K_{\max})} \tag{7}$$

$$\begin{aligned} \sigma^2_{\log dY/dN} = \sigma^2_{\log C} + \log \Delta K \sigma_M [&\log \Delta K \sigma_M + 2(\rho_{\log C,M}\sigma_{\log C} \\ &- \rho_{M,\log(K_C - K_{\max})}\sigma_{\log(K_C - K_{\max})})] + \\ +\sigma_{\log(K_C - K_{\max})}(\sigma_{\log(K_C - K_{\max})} &- 2\rho_{\log C,\log(K_C - K_{\max})}\sigma_{\log C}) \end{aligned} \tag{8}$$

Notwithstanding the logarithmic transformation of (6), the evaluation of the variance associated to the model is not easy. In fact, the critical SIF K_C and the maximum applied SIF $K_{\max}$ remain in the same logarithm, thus the evaluation of the correct variance $\sigma^2_{\log dY/dN}$ requires the knowledge of the terms $\sigma^2_{\log(K_C - K_{\max})}$ and the correlation coefficients $\rho_{\log C,\log(K_C - K_{\max})}$ and $\rho_{M,\log(K_C - K_{\max})}$. Unfortunately, the calculation of the latter variance and correlation coefficients are not trivial even if the critical SIF is normally or log-normally distributed. As a consequence, an approximated formulation using *delta method* is presented hereafter.

The delta method (explained in the appendix) is a common technique to approximate nonlinear functions of several random variables. The mean and variance of (6) approximated with the delta method are shown in (9) and (10). See the appendix for more information about this method and for the full description of the analytical calculations.

$$\mu_{dY/dN} \approx \frac{\mu_C \Delta K^{\mu_M} K_{\max}}{\mu_{K_C} - \mu_{K_{\max}}} \tag{9}$$

$$\begin{aligned} \sigma^2_{dY/dN} \approx & \left(\frac{\Delta K^{\mu_M} K_{\max}}{\mu_{K_C} - K_{\max}}\right)^2 \left[\sigma^2_C + (\mu_C \log \Delta K)^2 \sigma^2_M + \frac{\mu^2_C}{(\mu_{K_C} - K_{\max})^2}\sigma^2_{K_C} \right. \\ & \left. + 2\rho_{C,M}(\mu_C \log \Delta K)\sigma_C\sigma_M \right] + \\ -2 & \left(\frac{\Delta K^{\mu_M} K_{\max}}{\mu_{K_C} - K_{\max}}\right)^2 \left[\frac{\rho_{C,K_C}\mu_C\sigma_C\sigma_{K_C}}{(\mu_{K_C} - K_{\max})} + \frac{\rho_{M,K_C}(\mu^2_C \log \Delta K)\sigma_M\sigma_{K_C}}{(\mu_{K_C} - K_{\max})}\right] \end{aligned} \tag{10}$$

2.3. McEvily's model

Forman's model does not take into account the zone I of the crack growth rate curve. The so-named *threshold region* is fundamental from a RUL viewpoint, because most of the life of a cracked structure is spent in the first and second region of the sigmoid curve and relatively few cycles in the unstable or critical region

[1] One can think to apply the prognostic methodology herein described to estimate the RUL of a component based on the statistical output provided by a real time Structural Health Monitoring system [84].

(III). The model proposed by McEvily, in its first formulation, allows accounting for the threshold effect on the Fatigue Crack Growth rate [74]. The expression is only useful for the lower portion of the curve as just highlighted in [75]. Eq. (11) shows The formulation for inert atmosphere, supposing the load ratio $R = 0$.

$$\frac{dy}{dN} = \frac{8}{\pi E^2}(\Delta K^2 - \Delta K_{th,0}^2) \tag{11}$$

The mean and variance of the model should be evaluated according to the two material properties statistically defined, that is the Elastic modulus and the threshold SIF range. However, the formulation of the exact two moments of the model is not easy because of the ratio between E and $\Delta K_{th,0}$ raised to the power of two. Furthermore, the same problems of Forman's law are found in the McEvily's model; the logarithmic transformation does not simplify the formulation because the variance and the correlations of the term $\log(\Delta K^2 - \Delta K_{th,0}^2)$ should be evaluated. Then, the approximation of the moments with the delta method is presented. The resulting mean (12) and variance (13) are shown afterwards.

$$\mu_{dY/dN} \approx \frac{8}{\pi \mu_E^2}(\Delta K^2 - \mu_{\Delta K_{th,0}}^2) \tag{12}$$

$$\sigma_{dY/dN}^2 \approx \left(\frac{-16}{\pi \mu_E^2}\right)^2 \left[\frac{(\Delta K^2 - \mu_{\Delta K_{th,0}}^2)^2}{\mu_E^2}\sigma_E^2 + \mu_{\Delta K_{th,0}}^2 \sigma_{\Delta K_{th,0}}^2 + \frac{2\mu_{\Delta K_{th,0}}(\Delta K^2 - \mu_{\Delta K_{th,0}}^2)\rho_{E,\Delta K_{th,0}}\sigma_E \sigma_{\Delta K_{th,0}}}{\mu_E}\right] \tag{13}$$

The application of (13) is feasible if the correlation among the Young modulus and the threshold SIF range is available, or it can be evaluated. Otherwise, an approximated variance neglecting the correlation term remains valid for the (approximated) estimation of the statistical crack growth velocity in the first region and part of the second region of the sigmoid.

2.4. NASGRO model

The more advanced NASGRO model [61] is able to cover the entire FCG rate from the threshold up to unstable crack propagation [7,36,61,65–67]. Moreover, it accounts for the closure effect studied by Newman [41]. Consequently, NASGRO formulation (14) represents the state of the art of the crack velocity equations and it is used in the following part of the paper as the reference FCG model.

$$\frac{dy}{dN} = C\left(\frac{1-f}{1-R}\Delta K\right)^m \frac{\left(1 - \frac{\Delta K_{th}}{\Delta K}\right)^p}{\left(1 - \frac{K_{max}}{K_C}\right)^q} \tag{14}$$

C, m, p and q are empirical parameters, f is the crack opening function, R is the load ratio, K_{max} is the maximum SIF during the load cycle, K_C is the critical SIF value, ΔK and ΔK_{th} are the SIF range and the threshold SIF range, respectively. The two additional parameters p and q, together with the threshold SIF range ΔK_{th} and the critical SIF K_C governs the first and the third region of the sigmoid curve [61]. The dependence of ΔK_{th} from the crack length and the threshold SIF range for null load ratio ($\Delta K_{th,0}$) and the dependence of K_C from the fracture toughness K_{IC} in plane strain condition is also highlighted within NASGRO formulation [61]. Eq. (15) shows the logarithmic form of NASGRO law.

$$\log\frac{dy}{dN} = \log C + m\log\left(\frac{1-f}{1-R}\Delta K\right) + p \times \log\left(1 - \frac{\Delta K_{th}(\Delta K_{th,0})}{\Delta K}\right) - q\log\left(1 - \frac{K_{max}}{K_C(K_{IC})}\right) \tag{15}$$

In the most general case, all the material properties and the empirical parameters are statistically defined. On the other hand, the exact analytical calculation of mean and variance is unfeasible because of the high nonlinearity of the model, also in its logarithmic form. Delta method was used in the previous models to provide the approximated mean and variance of the crack growth rate model that unfortunately cannot be easily solved accounting for all the random variables. Before applying delta method to NASGRO equation, some simplifying hypothesis are proposed hereafter. For the best author knowledge, statistical studies about the two parameters p and q are missing. Moreover, the variability of threshold SIF range $\Delta K_{th,0}$ and fracture toughness K_{IC} (two material properties) markedly affects the uncertainty of the first and third region of the FCG rate rather than the variability of p and q. Roughly speaking, the threshold SIF range and K_C are the vertical asymptotes (and consequently, the limits of the domain) of NASGRO. As a consequence, the parameters p and q are considered deterministic afterwards. According to the above considerations, the approximated formulations of NASGRO mean and variance are expressed in (16) and (17). See the appendix for the complete mathematical treatment of the variance.

$$\mu_{\log dY/dN} \approx \mu_{\log C} + \mu_M \log\left(\frac{1-f}{1-R}\Delta K\right) + p \times \log\left(1 - \frac{\Delta K_{th}\left(\mu_{\Delta K_{th,0}}\right)}{\Delta K}\right) - q \times \log\left(1 - \frac{K_{max}}{K_C\left(\mu_{K_{IC}}\right)}\right) \tag{16}$$

$$\begin{aligned}\sigma_{\log dY/dN}^2 \approx\ & \sigma_{\log C}^2 + \log\left(\frac{1-f}{1-R}\Delta K\right)^2 \sigma_M^2 \\ & + p^2\omega_0^2\left[\Delta K\left(\frac{\mu_{\Delta K_{th,0}}\omega_0}{\Delta K \omega_1} - 1\right)\omega_1\right]^{-2}\sigma_{\Delta K_{th,0}}^2 \\ & + q^2\left[\frac{K_{max}}{\mu_{K_{IC}}^2\alpha_0} + \frac{\alpha_2}{\mu_{K_{IC}}^6\alpha_1\alpha_0^2}\right]^2 \left.\frac{K_{max}}{\mu_{K_{IC}}\alpha_0} - 1\right)^{-2}\sigma_{K_{IC}}^2 \\ & + 2\log\left(\frac{1-f}{1-R}\Delta K\right)\rho_{\log C,M}\sigma_{\log C}\sigma_M \\ & + 2p\omega_0\left[\Delta K\left(\frac{\mu_{\Delta K_{th,0}}\omega_0}{\Delta K\omega_1} - 1\right)\omega_1\right]^{-1}\rho_{\log C,\Delta K_{th,0}}\sigma_{\log C}\sigma_{\Delta K_{th,0}} \\ & + 2q\left[\frac{K_{max}}{\mu_{K_{IC}}^2\alpha_0} + \frac{\alpha_2}{\mu_{K_{IC}}^6\alpha_1\alpha_0^2}\right]\left.\frac{K_{max}}{\mu_{K_{IC}}\alpha_0} - 1\right)^{-1}\rho_{\log C,K_{IC}}\sigma_{\log C}\sigma_{K_{IC}} \\ & + 2\log\left(\frac{1-f}{1-R}\Delta K\right)p\omega_0\left[\Delta K\left(\frac{\mu_{\Delta K_{th,0}}\omega_0}{\Delta K\omega_1} - 1\right)\omega_1\right]^{-1}\rho_{M,\Delta K_{th,0}}\sigma_M\sigma_{\Delta K_{th,0}} \\ & + 2\log\left(\frac{1-f}{1-R}\Delta K\right)q\left[\frac{K_{max}}{\mu_{K_{IC}}^2\alpha_0} + \frac{\alpha_2}{\mu_{K_{IC}}^6\alpha_1\alpha_0^2}\right] \\ & \times \left.\frac{K_{max}}{\mu_{K_{IC}}\alpha_0} - 1\right)^{-1}\rho_{M,K_{IC}}\sigma_M\sigma_{K_{IC}} \\ & + 2p\omega_0\left[\Delta K\omega_1\left(\frac{\mu_{\Delta K_{th,0}}\omega_0}{\Delta K\omega_1} - 1\right)\left.\frac{K_{max}}{\mu_{K_{IC}}\alpha_0} - 1\right)\right]^{-1} q\left[\frac{K_{max}}{\mu_{K_{IC}}^2\alpha_0} + \frac{\alpha_2}{\mu_{K_{IC}}^6\alpha_1\alpha_0^2}\right] \\ & \times \rho_{\Delta K_{th,0},K_{IC}}\sigma_{\Delta K_{th,0}}\sigma_{K_{IC}}\end{aligned} \tag{17}$$

The terms ω_0, ω_1, α_0, α_1, α_2 depend on some constants highlighted in the appendix. Although the initial hypothesis largely simplify the most general case, the application of Eq. (17) remains unfeasible. As a matter of fact, the terms related to the first derivative of $\log\phi$ with respect to the vertical asymptotes, that is $\partial\log\phi/\partial\Delta K_{th,0}$, $\partial\log\phi/\partial K_{IC}$ tends to infinity when the applied ΔK approaches ΔK_{th} or K_C, respectively. It means that the first order Taylor expansion is not sufficient to correctly describe the variability of NASGRO law close to the limits of its domain. Unfortunately, the second or higher orders of the Taylor expansion produce very complicated formulation of the variance, and the method becomes not viable for practical applications. For that reason, the definition of a statistical FCG model requires further simplifications to obtain a closed form solution of the equation, otherwise numerical techniques should be implemented. A simple statistical study of NASGRO is presented afterwards. A splitting of the variability associated to empirical parameters and material properties is provided. It permits evaluating the uncertainty of the model related to the two categories separately. Basically, the additional underlying hypothesis in the next sub Sections 2.5 and 2.6 is the neglecting of any correlation between material properties and empirical parameters.

2.5. Uncertainty of empirical parameters

Starting from the previous considerations about the FCG rate models, the mean and the variance are evaluable according to the statistics of the empirical parameters. Belonging to the vector Θ. The most common parameters defined in statistical terms are $\log C$ and m describing the linear zone of the crack propagation velocity. Sometimes, simplified methods with one statistical parameter are implemented in literature. For instance, Beretta and Villa [22] propose a simplified approach using a fixed value for m and $\log(C)$ defined as a random variable, or a fixed value for $\log(C)$ and m as a random variable. However, the two random variables can be easily taken into account in the FCG equations, as visible in the Eqs. (18) and (19) for mean and variance of NASGRO model, respectively.

$$\mu_{\log dY/dN} = \mu_{\log C} + \mu_M \log\left(\frac{1-f}{1-R}\Delta K\right) + p\left[\log\left(1-\frac{\Delta K_{th}(\Delta K_{th,0})}{\Delta K}\right)\right] - q\left[\log\left(1-\frac{K_{\max}}{K_C(K_{IC})}\right)\right] \quad (18)$$

$$\sigma^2_{\log dY/dN} = \sigma^2_{\log C} + \left(\log\left(\frac{1-f}{1-R}\Delta K\right)\right)^2 \sigma^2_M + 2\rho_{\log C,M}\sigma_{\log C}\sigma_M \times \log\left(\frac{1-f}{1-R}\Delta K\right) \quad (19)$$

The standard formula $\mu_{\log dY/dN} \pm z_\alpha \sigma_{\log dY/dN}$ yield to the prediction intervals associated to the FCG rate at different confidence levels. Fig. 2 shows the average NASGRO curve with the confidence bands related to the statistics of C and m for the Al2024-T6 shown in Table 1. The mean values come from the NASGRO manual [61] while the variances of $\log C$, m and the correlation coefficients are already defined in [4,20–22] for the Paris' equation. All the bands

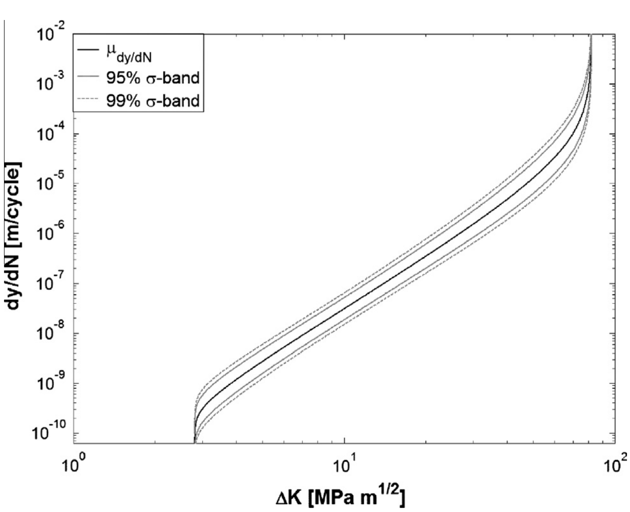

Fig. 2. NASGRO model with probabilistic definition of C and m parameters.

converge to the threshold SIF range from one side and the critical SIF from the other side, because the two material properties are considered deterministic within this subsection.

2.6. Uncertainty of material properties

The material properties involved in the FCG process can be handled in the same way as the other material characteristics used in stress analyses: ultimate tensile strength or fatigue stress limit, etc. All of these properties are statistically defined in the design phase of structures and reliability analyses, so the properties related to the FCG should be handled using the same approach. As it is mentioned above, the two main material features affecting the fatigue crack propagation are the threshold SIF range for $R = 0$ and the fracture toughness in plane strain condition. A statistical description of $\Delta K_{th,0}$ has been adopted by Beretta and Villa [22] analysing some data obtained from specimens of A4T steel. The same considerations can be made for the statistical definition of the plane strain toughness K_{IC} or whatever other property is considered in the FCG model. In this context, $\Delta K_{th,0}$ and K_{IC} are defined as the only two material properties statistically defined, so they belong to the vector H. If the $\Delta K_{th,0}$ and K_{IC} are normally distributed, the mean vector and covariance matrix in (20) provide the exact statistical description of H.

$$\mu_H = E(H) = \begin{bmatrix} \mu_{\Delta K_{th,0}} \\ \mu_{K_{IC}} \end{bmatrix}; \quad \Sigma_H = COV(H) = \begin{bmatrix} \sigma^2_{\Delta K_{th,0}} & \sigma_{\Delta K_{th,0},K_{IC}} \\ \sigma_{\Delta K_{th,0},K_{IC}} & \sigma^2_{K_{IC}} \end{bmatrix} = \begin{bmatrix} 20.808 & 0 \\ 0 & 3565 \end{bmatrix} \quad (20)$$

As shown above the derivatives of NASGRO model with respect to $\Delta K_{th,0}$ and K_{IC} tends to infinity, so it is difficult to evaluate the global variance of the model even if the other parameters are deterministic. However, under the hypothesis that the two material properties are uncorrelated, the boundaries of the crack velocity can be approximated in an easy way. In particular, the confidence bands of each parameters are combined to produce the confidence band of the crack growth rate. The upper and lower boundaries of the crack propagation velocity are shown in Fig. 3. The mean and variance of the threshold SIF range and fracture toughness are expressed in Table 1. The statistics of $\Delta K_{th,0}$ is extrapolated from the results of [37] for $R = 0.05$ (assuming negligible the difference of the variance between $R = 0$ and $R = 0.05$) while the variance of the fracture toughness is inferred by simple considerations of the range available in [76]. All the other empirical parameters of the equation are defined deterministically based on NASGRO database. This method does not take into account the correlation between the vertical asymptotes and the parameters describing the Paris' regime. However, it can be considered an acceptable approximation to solve the dY/dN near the threshold or the critical region if no other advanced methods are available.

After all these consideration, the definition of the SDE for the fatigue crack propagation can be formulated according to (3). Certainly, the simplified method where the different sources of uncertainties are split introduces some errors. First, the correlation among the material properties and the empirical constants is lost. Second, it prevents the formulation of a closed form solution of the whole variance of the model, and in any case the numerical integration is not trivial. It remains useful to predict the crack velocity interval given a series of parameters statistically defined, and it can be implemented into whatever maintenance approach based on crack propagation process. Therefore, a Dynamic State-Space model with the Stochastic definition of NASGRO equation is developed in section 3. It is solvable with numerical methods based on MCMC theory as presented in this paper, or other techniques based on Bayesian statistics. Both the methodologies can be used; the

Table 1
Parameters used for prior and proposal distributions for the simulated FCG.

Parameter		μ	σ^2	$\rho_{\log(C),m}$	$\rho_{\log(C),\Delta K_{th,0}}$	$\rho_{m,\Delta K_{th,0}}$
$\log(C)$	$(\log(\text{mm}/cycle \cdot (\text{MPa}\sqrt{\text{mm}})^{1/m}))$	−26.76	0.9966	0.9979	0	–
m	(–)	3.2	0.0346	0.9979	–	0
$\Delta K_{th,0}$	$(\text{MPa}\sqrt{\text{mm}})$	103	20.808	–	0	–
K_{IC}	$(\text{MPa}\sqrt{\text{mm}})$	1042	3565	–	–	–

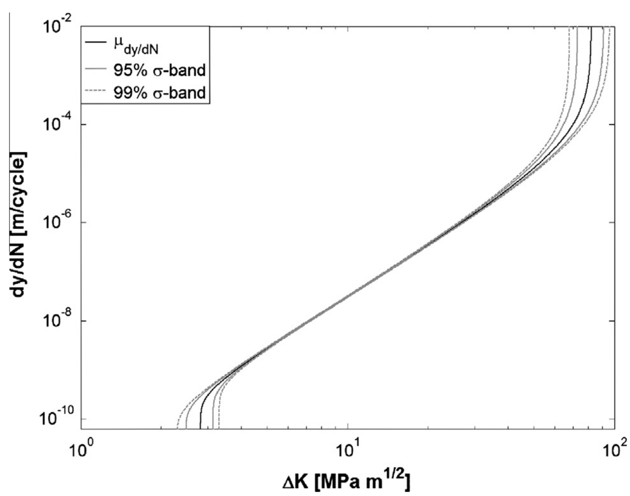

Fig. 3. NASGRO model with probabilistic definition of $\Delta K_{th,0}$ and K_{IC} parameters.

provided analytical formulation of mean and variance of FCG overlook some correlations and uncertainties that are present in the real phenomenon. It could provide useful information regarding the crack evolution in post-processing studies. The second method, based on a DSS definition and a well-known Monte Carlo algorithm, provides a full description of the possible crack propagation evolution, but requires higher computational costs.

3. Dynamic State-Space model for fatigue crack propagation

Many definitions of Dynamic State-Space models are present in the literature, mainly in the field of Bayesian filters and mathematical dissertations [50,52,57,77–78]. In this context, the definition of a discrete dynamic equation governing the crack propagation is considered, as already presented in [57–59] for crack monitoring via Bayesian filters. The purpose of the DSS model is to link the different states of the crack evolution in time to produce a swarm of possible crack lengths (the state of the system) according to the FCG model and its probabilities. Starting from the statistical definition of the FCG rate in Section 2, the next step is the definition of the possible crack evolutions according to the model namely from the statistical dy/dN to the probability distribution of the RUL. The following explanation is referred to the case of constant amplitude fatigue load, in which R and $S_{\max}$ lose the time dependence. However, the treatment can be extended to the variable-amplitude load condition with some complications [79].

3.1. Stochastic definition of DSS models

To stochastically define the Dynamic State-Space models the random variable Y describing the crack length in the discrete time domain has to be considered. At the general kth step, the Dynamic State-Space model linking the current crack length Y_k with the previous crack length y_{k-1} assumes the general formulation in Eq. (21).

$$Y_k = \psi(y_{k-1}, \Theta, \mathrm{H}, \zeta(y_{k-1}, \mathrm{H})) \tag{21}$$

Where the symbolism of the Eq. (21) is the same of the Section 2. It is however important to notice that the model ψ indicates the link between two sub-sequent crack lengths differently from ϕ in the Eq. (1) which indicates a general FCG rate model. According to the hypothesis of constant amplitude load, the crack length at the general kth step (22) depends on the previous crack length plus a linearisation of the damage growth defined by the term $\Delta N \cdot dY/dN$.

$$Y_k = y_{k-1} + \Delta N \frac{dY}{dN}\bigg|_{k-1} = y_{k-1} + \Delta N \phi_{k-1}(\Theta, \mathrm{H}, \zeta(y_{k-1}, \mathrm{H})) \tag{22}$$

Let assume to be able to evaluate the expected value and the variance of the model ϕ with the analytical formulation shown in Section 2 for different kind of models or other numerical methods. Again, according to the theory of random variables, the estimation of the crack length at the general kth step can be expressed in terms of mean (23) and variance (24). In this way, the uncertainty on the crack length after a relatively small number of loading cycle is available.

$$\mu_{Y_k} = y_{k-1} + \Delta N \mu_{dY/dN} \tag{23}$$

$$\sigma^2_{Y_K} = \Delta N^2 \sigma^2_{dY/dN} \tag{24}$$

The uncertainties rise if the crack length at the $k-1$th step is known in statistical terms due to possible noise in the measurement system. Thus, also the crack length becomes a random variable and the evaluation of $\sigma^2_{Y_K}$ becomes more complicated (25).

$$\sigma^2_{Y_K} = \sigma^2_{Y_{k-1}} + \Delta N^2 \sigma^2_{dY/dN} + 2\rho_{Y_{k-1},dY/dN} \sigma_{Y_{k-1}} \sigma_{dY/dN} \tag{25}$$

The evaluation of crack length distribution using the Eqs. (23)–(25) allows the estimation of the probability of failure at a certain time. Unfortunately, it requires the closed form solution of the variance of the FCG rate that is difficult to evaluate in many cases, as emphasized in Section 2.

Finally, given the statistical definition of the FCG rate equation with the uncertainty of empirical parameters and material properties, the numerical resolution of the general Stochastic Dynamic State-Space (SDSS) model in (22) by means of the Metropolis–Hastings algorithm is proposed afterwards. Given the Stochastic definition of the DSS model, the aim is to produce a lifetime prediction for the cracked structure overcoming the analytical difficulties. The presence of multiple uncertainties is faced by means of the Monte Carlo Sampling methods and recursively updated by Metropolis–Hastings algorithm with a particular adaptive proposal, as described afterwards. The quantity y_f is the crack length associated to the failure of the structure and $p_{\Theta,\mathrm{H},\mathrm{Z}}(\vartheta, \eta, \zeta(y, \eta))$ is the multidimensional Probability Density Function (PDF) of the FCG parameters and the time-varying variables. Starting from the initial crack length y_0 the crack propagation can be simulated by N_S samples drawn from $p_{\Theta,\mathrm{H},\mathrm{Z}}$. In particular, a simulation of the FCG up to the critical crack size is associated with each sample. On the other hand, if the starting crack is known in the PDF form $p_Y(y_0)$ by the use of the MCS method, N_S samples can be drawn from the two PDFs: the distribution of the crack length and the multidimensional distribution of the parameters. The simulation of these cracks for subsequent discrete time steps up to the critical crack length produces the density function of the time to failure, and the consequent RUL. The multidimensional PDF $p_{\Theta,\mathrm{H,Z}}(\vartheta,\eta,\zeta(y,\eta))$ depends on the previous crack length due to the presence of the variable vector $\zeta(y,\eta)$. Thus, a sequential sampling first from $p_Y(y_{k-1})$ and then from $p_{\Theta,\mathrm{H,Z|Y}}(\vartheta,\eta,\zeta|y)$ has to be made. The general formulation of the conditioned probability given the crack y_k, is expressed in (26).

$$p_{\Theta,\mathrm{H,Z|Y}}(\vartheta,\eta,\zeta|y_k)=\frac{p_{\Theta,\mathrm{H,Z,Y}}(\vartheta,\eta,\zeta,y_k)}{p_Y(y_k)} \tag{26}$$

Some simplifications occur if the empirical parameters, the material properties and the model variables are considered independent. In this case, the probabilities of the empirical parameters and material properties are separated from the other quantities and the multidimensional probability simplifies into Eq. (27).

$$p_{\Theta,\mathrm{H,Z|Y}}(\vartheta,\eta,\zeta|y_k)=p_\Theta(\vartheta)p_\mathrm{H}(\eta)p_{\mathrm{Z|Y}}(\zeta|y_k). \tag{27}$$

3.2. Sequential updating of the Dynamic State-Space model

Supposing that a measure of the crack length at the general kth step is available, intuitively, starting from a given measure y_0, this knowledge of the crack length at the step kth narrows the possible crack evolution down because the crack must "pass through" the given measure (under the hypothesis of a perfect measuring system). Therefore, if a numerical technique is able to filter the possible crack evolutions on the basis of the measures, it should increase the prognostic resolution. Starting from the statistical definition of the FCG equation, the filtering approach can be applied on the model parameters and material properties by means of the posterior distribution $p_{\Theta,\mathrm{H,Z|Y}}(\vartheta,\eta,\zeta|y_k)$. This is a crucial step because the parameters will affect the following evolution of the crack. If they are well-estimated from the previous measure, the subsequent simulation of the crack dynamics (i.e. the integration of the Stochastic Differential Equation in (3) or the dynamic simulation of (22)) will get close to the most probable evolutions (given the prior information on the crack growth). If a series of sequential measures becomes available, a sequential updating of the parameter PDFs can gradually enhance the prediction performance. Eq. (28) shows the updating of the multidimensional PDF of parameters and variables through the Bayes' rule.

$$p_{\Theta,\mathrm{H,Z}}(\vartheta,\eta,\zeta|y_k)=\frac{L(y_k|\vartheta,\eta,\zeta)f(\vartheta,\eta,\zeta)}{\iiint L(y_k|\vartheta,\eta,\zeta)f(\vartheta,\eta,\zeta)d\vartheta d\eta d\zeta} \tag{28}$$

In which $L(y_k|\vartheta,\eta,\zeta)$ is the likelihood of the measure y_k given the parameters ϑ,η,ζ while $f(\vartheta,\eta,\zeta)$ is the prior distribution of the parameters and the time-varying variable. The marginalized distribution $\iiint L(y_k|\vartheta,\eta,\zeta)f(\vartheta,\eta,\zeta)d\vartheta d\eta d\zeta$ is not calculable for the majority of complex problems. Hence, numerical methods like the MCMC are required to obtain the posterior distribution. Once the updated PDF is available, the MCS procedure to simulate the crack dynamics has to be repeated to produce a series of samples according to the new PDF of the Stochastic DSS model. There are different methods to merge the information of the measures inside the dynamic model; most of them are based on the MCS. Among these, the Metropolis–Hastings algorithm [43,44] and the SIS/SIR algorithms [50–52] are the most common techniques. In order to test the methodology, a particular Metropolis Hastings algorithm with an adaptive proposal distribution is described in sub-Section 3.3 and is subsequently used in the lifetime prediction problem described in Section 4.

3.3. Markov chain Monte Carlo method for parameter PDF updating

The statistical quantities employed in the FCG models has been described above. A methodology to update the prior knowledge of parameter PDFs and based on the MCMC theory is provided in this sub-section. Actually, any statistical method based on the Bayes' rule and able to evaluate the posterior probability of the parameters expressed in (24) or the posterior distribution of the crack length evolution can be used. In this work the Metropolis–Hastings algorithm [43,44] is used, being widespread algorithm founded on Monte Carlo Sampling [80]. The inputs of the algorithm are the initialized values of the parameters, the proposal distribution from which samples are generated and the (sequential) measures on the system (in this case, the crack lengths). Every time a new measure of the crack length becomes available, the MH algorithm is activated, producing a posterior PDF of the empirical parameters and material properties, given the prior knowledge on the parameters and the available measures up to the last step (the crack lengths $y_{1:k}$). One of the main benefits of the algorithm is its capability to estimate also the noise level affecting the measures. This becomes fundamental in the case of automatic and real-time measurement systems in which the uncertainty on the output quantity (crack length) is non-negligible [81]. The noise affecting the measures enters into the definition of the likelihood of the measures given a parameter vector.

The main drawback of this algorithm is the need for a proposal distribution to generate samples. In fact, this proposal distribution must have a properly tuned variance to avoid useless samples (i.e. too refused values or too slow convergence properties). In this work, a particular adaptive proposal developed by Haario et al. [82] is used. The Adaptive Metropolis Hastings algorithm implemented is able to automatically update the proposal distribution. This updating method is based on the residuals of the chain, and it depends on two parameters named *memory* and *frequency* parameter H and U, respectively. This method introduces a slight bias on the final distribution [82], however, it has been demonstrated that the chain of the algorithm remains ergodic [83] and the error has been declared as negligible in most cases. No mathematical dissertation of the algorithm is provided since it is not the purpose of the paper. The interested reader can refer to [43,44] for further information about the algorithm and [45–47,82,83] for adaptive proposal algorithms. In the context of real-time RUL estimation, the Adaptive MH algorithm constitutes a novelty and it is mandatory for the development of effective and robust lifetime predictor. In fact, if the proposal distribution has a mistaken variance with respect to the data will be provided by the measurement system during the crack evolution, the MH algorithm will not work and the resulting lifetime prediction will be wrong. The proposal distribution with self-adaptive variance is able to overcome the possible uncertainties and variability run into real crack propagation monitoring. In this way, the proposed SDSS model becomes able to correctly update the parameter distributions even if the initial variance of the proposal (usually empirically selected) is too wide or too low with respect to the variability of the measures.

Table 2 shows the pseudo-code of the Adaptive Metropolis algorithm used to update the distributions within the SDSS model. As proposed by Haario et al. [82] the standard MH algorithm is modified to update the covariance matrix of the proposal every U steps based on the residual of the last H samples. c_d is a heuristic scaling factor set equal to $2.4/\sqrt{d}$ as suggested in [47,82] for random Gaussian target variables. The factor d must to be equal to the number of parameters to estimate. The definitions of the quantities in Table 2 are the followings: x is the vector of the objective quantities containing the vector of the parameters $[\vartheta,\eta]$ and the variance of the noise associated to the measures σ_n^2, X_H is a matrix which contains the last H samples of the chain, $R(X_H)$ is the matrix

Table 2
MH algorithm with an adaptive proposal distribution.

i. Initialize the parameter vector $x_0 = [\vartheta_0, \eta_0, \sigma^2_{n,0}]$ according to the proposal PDF $p(x)$
ii. Initialize the likelihood of the measure y given the vector x_0, taking into account the prior PDF
iii. Initialize the matrix of the chain $X = [x_0]$
iv. Set the updating memory of the proposal H and the frequency parameter U
v. For $i = 1–N_S$
– If the remainder of $i/U = 0$
Store the last H residuals of the chain: $R(X_H)$
Generate the covariance matrix of the proposal: $COV(X_H) = \frac{c_d^2}{H-1} R(X_H) R(X_H)'$
End
– Draw sample x_i from $p(x)$ where the mean of the proposal is the last accepted sample vector x_{i-1} and the variance of the proposal is the covariance matrix $COV(X_H)$
– Calculate a fictitious crack length $y(x_i)$ according to the sample x_i.
– Evaluate the likelihood of the measure $L(z|x_i)$ and the prior $f(x_i)$ given the sample
– Accept the sample x_i with probability $\alpha = \min\left(1, \frac{L(z|x_i) f(x_i)}{L(z|x_{i-1}) f(x_{i-1})}\right)$.
End
*vi.*Erase the burn-in period and select far-between samples to avoid the possibility of correlated samples from the chain

of the residuals of X_H, $COV(X_H)$ is the covariance matrix to assign the proposal distribution $p(x)$. At the end of the Adaptive Proposal Metropolis–Hastings algorithm, N_S samples drawn from the updated parameter distributions are available. These samples are used to simulate the new crack evolution starting from the last measure y_k.

4. Numerical resolution of the Stochastic Dynamic State-Space model to crack growth problems

The implementation of the proposed lifetime predictor for damage monitoring of structures is the aim of the section. Some simplifications from the most general case are considered in this example, given the complexity of the problem. First, the method is tested on simulated data (Section 4.1) and then applied to real crack propagations (Section 4.2) measured during FCG tests on helicopter aluminium panels. Both cases are related to a constant-amplitude fatigue load. As mentioned above, the FCG rate model used is the NASGRO equation. According to the SDSS model presented in (22), the crack length at the general kth step assumes the form in (29). The constant value 2 is needed because the cracks have two tips (both in the simulated and in the real cases) [62].

$$Y_k = y_{k-1} + 2\Delta N C \left(\frac{1-f}{1-R}\Delta K\right)^M \frac{\left(1 - \frac{\Delta K_{th}(\Delta K_{th,0})}{\Delta K}\right)^p}{\left(1 - \frac{K_{\max}}{K_C(K_{IC})}\right)^q} \tag{29}$$

The empirical parameters and material properties of Al2024-T6 selected for the proposal distribution of the MH algorithm are visible in Table 1, where the fracture toughness K_{IC} is a deterministic value. The choice comes from the previous considerations of Section 2; the process of FCG spends few cycles in the third region of the sigmoid curve and the structure should be declared failed before reaching the critical SIF. It is a conservative hypothesis; besides, the statistics of the fracture toughness can be easily implemented in the algorithm.

4.1. Performance on simulated data

Let us consider a fictitious Al2024-T6 infinite thin plate with a 600 mm^2 resistant section and a through-thickness central crack. This plate is loaded by a constant-amplitude sinusoidal load with $P_{\max} = 40$ kN and $R = 0.1$. The simulation of the crack growth is made starting from a 3 mm length and a critical crack length equal to 120 mm. Fig. 4 shows the simulated plate with centre crack used to test the algorithm. The measure of the crack length is provided every 2000 load cycles with a simulated measurement system with a normally distributed error characterised by zero mean and a variance of 0.11 mm^2. The shape function inside the SIF is set as deterministic and equal to 1 according to the theory of fracture mechanics. The simulation of the crack starts from y_0 and it stops when all the fictitious propagations reach the limit crack length y_f. The number of samples for the Monte Carlo simulation of the crack propagation is 2000. The number of iteration for the MH algorithm is equal to 10,800 while the updating and frequency parameters are set equal to 1800 according to the advices in [83].

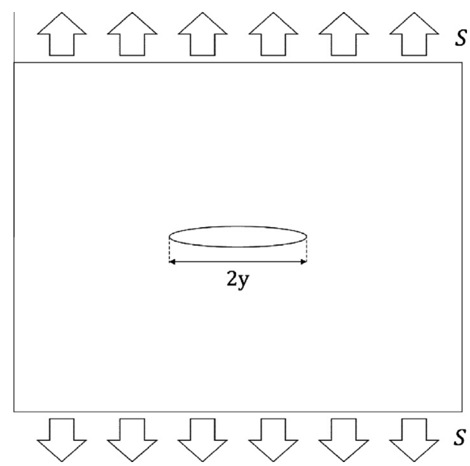

Fig. 4. Simulated plate with central crack.

4.1.1. Performance with the triplet log(C), m and ΔK_{th0} as random variables

The simulated FCG is built with values of $\log(C)$, m and $\Delta K_{th,0}$ slightly different from the average values found in the NASGRO manual (30), in order to highlight the performance of the proposed Stochastic DSS (26). They are randomly extracted by the multivariate distribution of the triplet $\log(C)$, m and $\Delta K_{th,0}$ built with the correlation matrix proposed by Beretta and Villa [22] for aluminium alloys. Parameters p and q are taken equal to 0.25 and 1, respectively. The starting values for prior and proposal distributions used during the algorithm operation are already shown in Table 1. The couples $(\log(C), \Delta K_{th,0})$ and $(m, \Delta K_{th,0})$ are considered as uncorrelated inside the prior knowledge and the proposal distribution. Even if they are actually correlated, the results in [67] show good performances in RUL prediction also considering the empirical parameters uncorrelated with respect to the material properties.

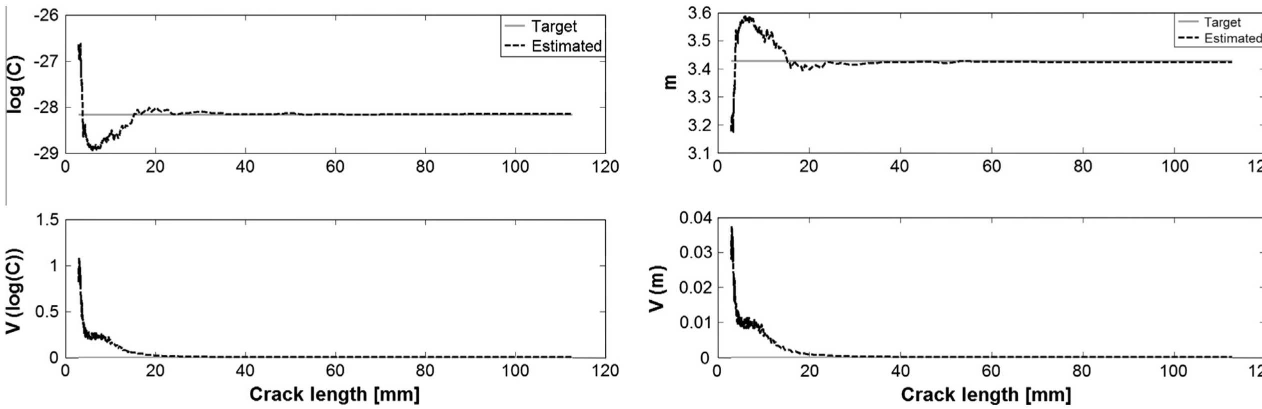

Fig. 5. Estimation of log(C) (left) and m (right) parameters as a function of the crack length.

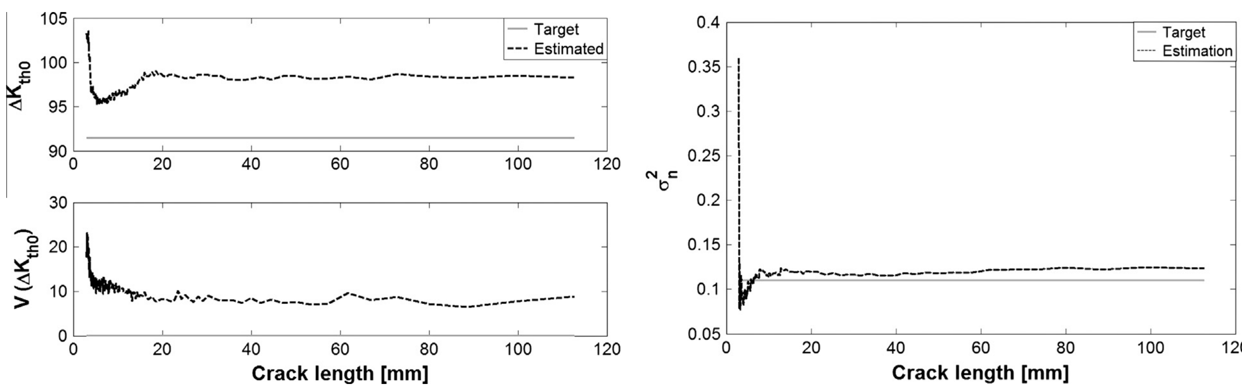

Fig. 6. Estimation of $\Delta K_{th,0}$ (left) and Noise variance (right) associated to the measures.

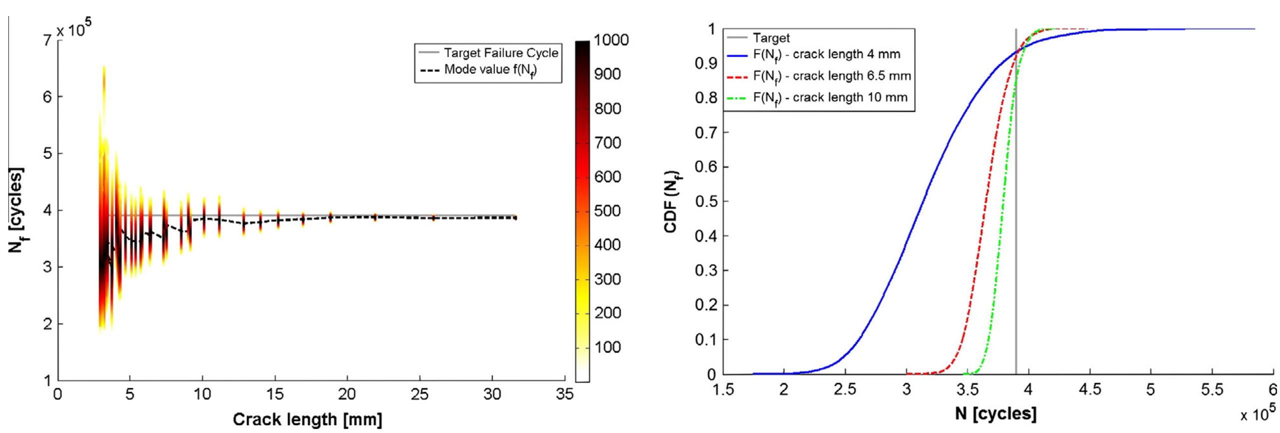

Fig. 7. Failure cycle PDF (left) and CDF (right) using three random variables (C, m and $\Delta K_{th,0}$).

$$C = 5.8162e - 013; \quad m = 3.4265;$$
$$\Delta K_{th,0} = 91.4159 \text{ MPa} \sqrt{\text{mm}} \tag{30}$$

During the simulated operation of the structure, the estimation of the parameters, the noise associated to measures and the RUL of the system are provided every time a measure becomes available. The updating of the multivariate normal PDFs is made through the Adaptive Proposal MH algorithm. Starting from the updated PDFs, the prediction of the RUL can be performed according to the MCS procedure. The successful updating of the log(C) and m parameters is visible in Fig. 5, while the $\Delta K_{th,0}$ and the noise estimation are shown in Fig. 6. Unfortunately, the MH algorithm is not able to estimate the correct value of the threshold SIF range because it affects the variability of the crack propagation velocity dy/dN in the first part of the sigmoid curve, when the uncertainty on the measure are very high with respect to the absolute value of the crack length. When the crack size increases, the uncertainty on the threshold affects the propagation less because the crack is inside the Paris' regime (so it is out from the first region where the threshold SIF range affects the crack velocity). Essentially, the prior knowledge of the $\Delta K_{th,0}$ is slightly updated by the MCMC algorithm, however the final value is overestimated and a wide variance remains at

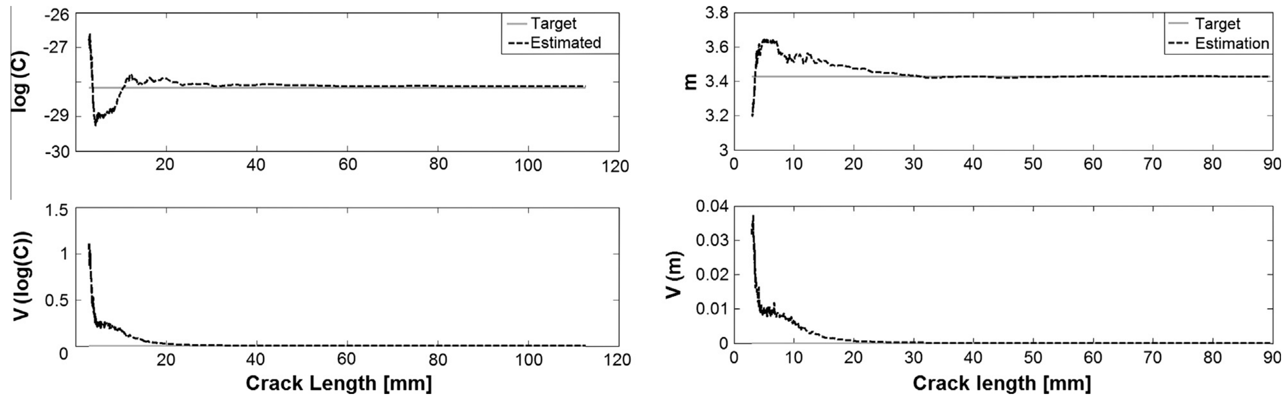

Fig. 8. Estimation of the log(C) (left) and m parameters (right) with respect to the crack length.

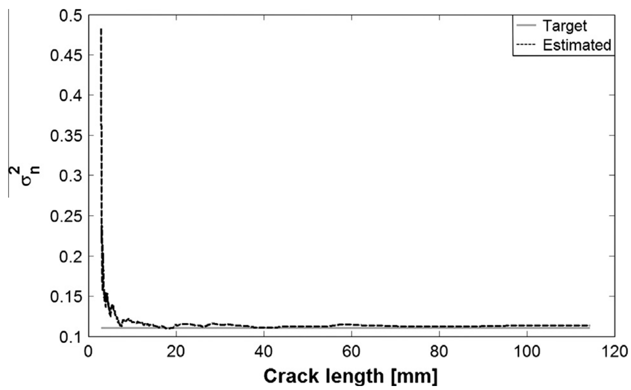

Fig. 9. Noise variance associated to the measures.

the end of the simulation. The results remain the same even if a correlation between the triplet (log(C), m, $\Delta K_{th,0}$) is put into the prior knowledge of the parameters, according to [67]. The error in the threshold SIF range falls back into a constant bias in the estimation of the noise variance visible in Fig. 6 (right). As a matter of fact, the inability of the algorithm to evaluate the correct value of $\Delta K_{th,0}$ produces higher noise estimation with respect to the target noise represented by the variance of the measures. Instead, the RUL prediction in Fig. 7 (left) tends to the correct value. The grey straight line is the target failure cycle, the blending lines are the normalised probability density functions (in order to have a peak value equal to 1000 as visible in the coloured bar[2]) of the failure cycles during the crack propagation and the dashed black line is the mode value of the normalised PDFs. The first blending line represents the time to failure distribution with the starting crack length as single information. Actually it is the lifetime distribution given the prior knowledge of C, m and $\Delta K_{th,0}$. The different lifetime distributions clearly tend to the correct value of the failure cycle during the operation of the algorithm. In Fig. 7 (right) the Cumulative Distribution Functions (CDFs) of the failure cycle at three different crack lengths is presented. The distribution evidently approaches the correct failure cycle when the crack length is just 10 mm.

4.1.2. Performance with log(C) and m as random variables

Since the algorithm seems unable to correctly estimate the three parameters and the artificial noise inserted in the measures of crack length, a simplified case is considered in this sub-section. The threshold SIF range is considered deterministic and centred on the correct value, while the two empirical parameters C and m remain identical to the previous case. The threshold SIF range for $R = 0$ is fixed according to the NASGRO database (103 MPa $\sqrt{mm}$) thus, the SDSS model is based on two random parameters (C and m) only. The distributions of the empirical parameters are the same of the previous case, while the threshold SIF range is deterministic (like the fracture toughness) and equal to the average value (Table 1). The deterministic parameters used to simulate the FCG are the same as in the sub-Section 4.1.1 (30). The estimation of the empirical parameters ($\log(C), m$), the measure noise and the RUL are shown in the Figs. 8–10. The algorithm is able to estimate the target parameters when the crack length is around 20 mm. As explained by the figures, the variances of the parameters tend to zero; in fact, the simulated crack propagation is built with deterministic log(C) and m values. The prior knowledge of the noise variance is arbitrarily set equal to 0.9, but the algorithm quickly reduces this value approaching the correct variance (Fig. 9). Since the $\Delta K_{th,0}$ is not estimated, the algorithm is able to associate the correct noise to the uncertainty of the measures. Fig. 10 shows the performance in terms of the failure cycle distribution. Also in this case the CDF of the failure cycle correctly approaches the target with a crack length of 10 mm. The simulations considering two (C, m) and three ($C, m, \Delta K_{th,0}$) random variables show very similar results in the RUL prediction, which constitutes the main purpose of the degradation model. Several analyses on simulation data have been carried out with similar results. The two empirical parameters are correctly estimated together with the noise variance, if the threshold SIF range is considered as deterministic. While if the $\Delta K_{th,0}$ is put into the algorithm as a random variable, the estimation of the noise is lightly altered, and the estimation of the threshold SIF range does not converge to the correct value. It means that an inability of the algorithm to correctly estimate (at least) one constant parameter produces an error in the estimation of the noise variance. Therefore, the algorithm is not able to associate the correct uncertainty to the measures in those cases. On the other hand, the lifetime predictions are similar if Figs. 7 and 10 are compared.

4.2. Analysis of simulated results

Paragraphs 4.1.1 and 4.1.2 show the application of the MH algorithm with and without the updating of the threshold SIF range, respectively. The effectiveness of the threshold SIF range updating is addressed here. Fifty different crack propagations are simulated, half of them with the updating of $\Delta K_{th,0}$ distribution and the remaining simulations without the updating of $\Delta K_{th,0}$ (thus set as deterministic parameter). The estimation of the failure cycle $\hat{N}_f$ is extracted at pre-determined number of cycles. The error on the failure cycle estimation ε follows the intuitive Eq. (31).

$$\varepsilon_i = N_{f,i} - \hat{N}_{f,i} \quad (31)$$

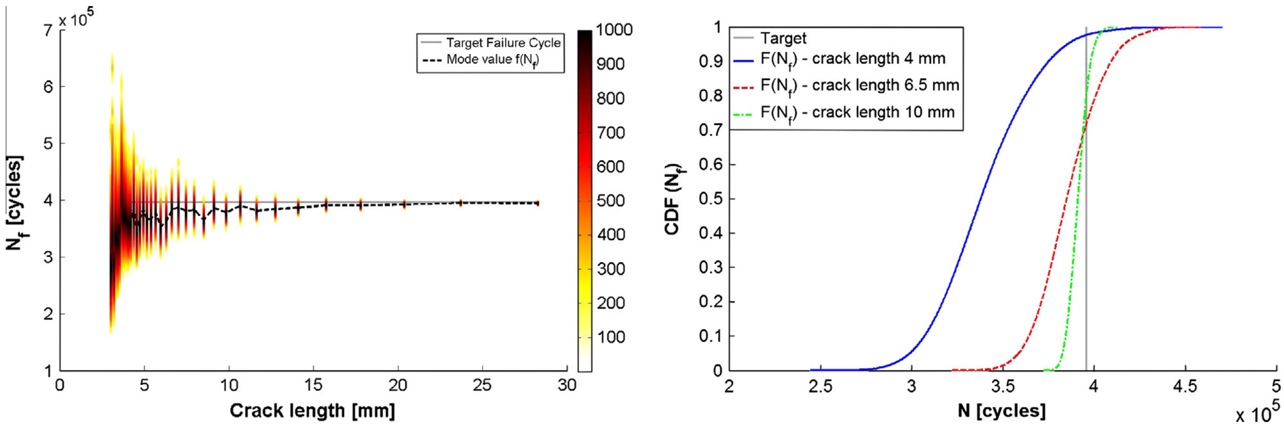

Fig. 10. Normalised PDF of failure cycle (left) and CDF of failure cycle (right).

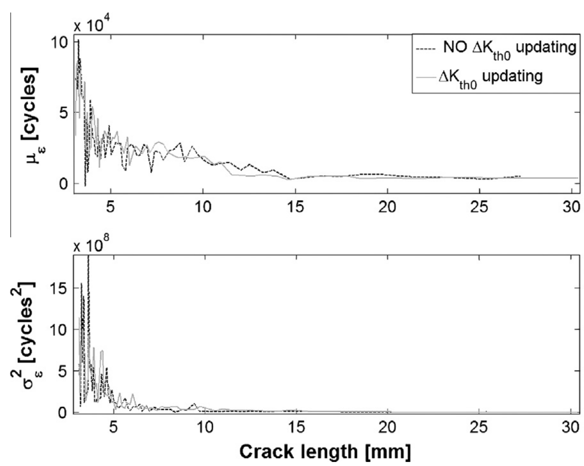

Fig. 11. Mean and variance of the error of the failure cycle estimation for different crack lengths.

where N_f represents the objective residual lifetime of the simulation, $\hat{N}_f$ is the estimated residual lifetime and the subscripts i indicates the general ith simulation. 25 errors are available at predetermined number of cycles for each category (with and without the threshold updating). Average and variance of the error are calculated assuming normal distribution (Fig. 11).

There is no a clear evidence of the difference between the two types of simulations made with and without the updating of the $\Delta K_{th,0}$ distribution. Further investigations are mandatories to prove the inefficiency of the $\Delta K_{th,0}$ updating in these types of crack propagations, for instance using the statistical hypothesis testing [84]. As a first approximation, the performances in RUL prediction are considered comparable, thus the tests on experimental data neglect the updating of the threshold SIF range distribution.

Table 3
Parameters relative to the crack propagation tests.

Item	Description
Load shape	Sinusoidal
Load frequency	12 Hz
Maximum amplitude load	35 kN
Load ratio (R)	0.1
Damage type	Skin crack
Damage position	Central on central bay
Damage initiation	Artificial, 16 mm
Material of the panel	Aluminum 2024 T6

4.3. Stochastic DSS model for experimental data

In this subsection the proposed SDSS model is tested on real crack propagation data obtained from aluminium panels representative of a helicopter rear fuselage. The complete test-rig is presented hereafter. The experimental test has been repeated four times with the same centre-crack. The number of samples of the simulated crack evolutions is again 2000 like in the simulated case. The number of iterations for the chain of the MH algorithm is

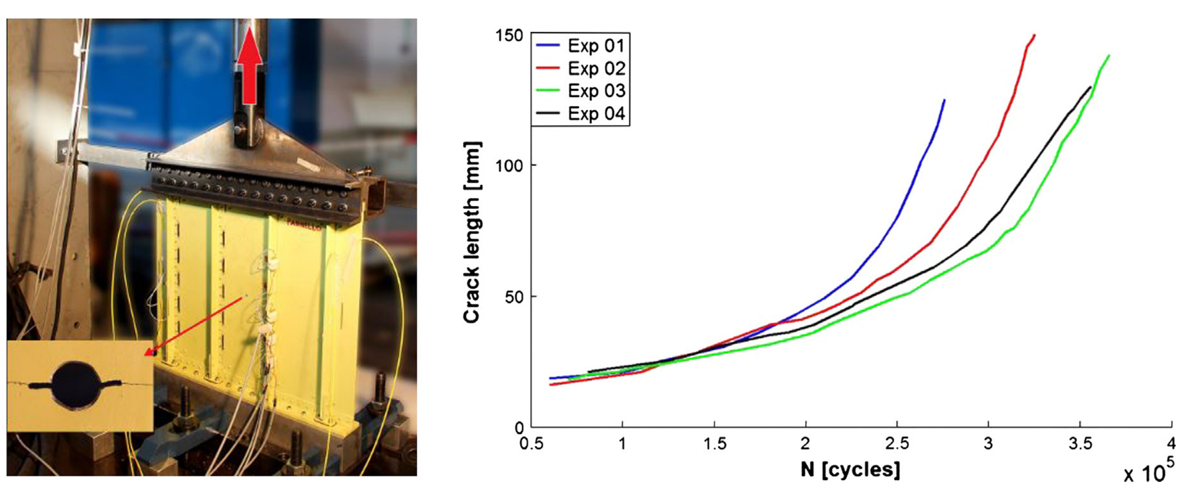

Fig. 12. Test rig for the experimental tests (left) and the resulting crack propagations (right).

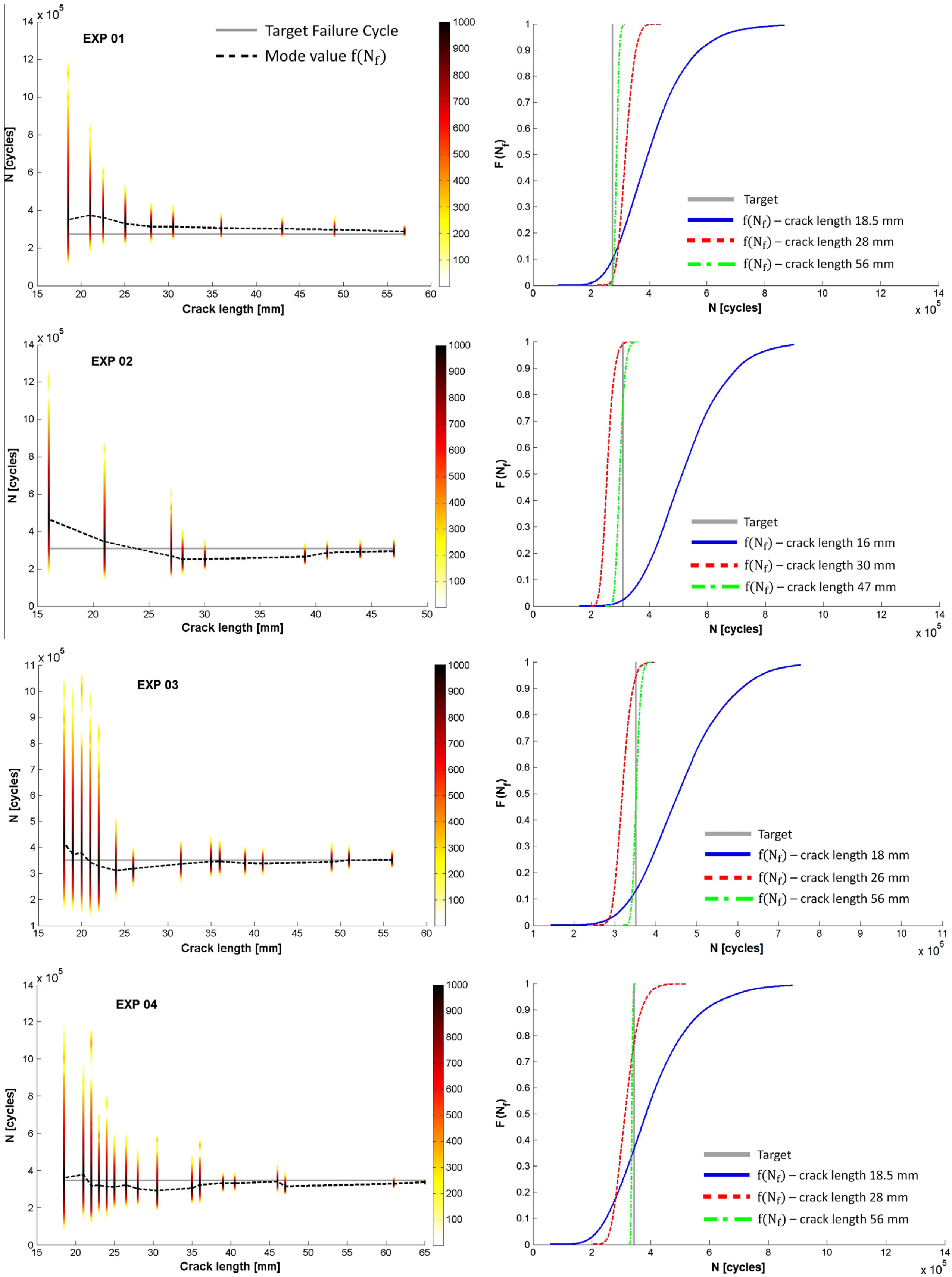

Fig. 13. Lifetime prediction for experimental data: failure cycle estimation (left) and corresponding CDF for the three different crack lengths (right).

14,400 and both the updating and frequency parameters are set equal to 1800.

4.3.1. Experimental activity

The helicopter panel presented in the left side of Fig. 12 has been rigidly grounded to its lower end and connected to the actuator through its upper end; more details about the experimental set-up are reported in [65,66,81]. Crack damage has been artificially initiated in the centre of the skin to guarantee the repeatability of the test. The initial length of the notched hole is 16 mm. A sinusoidal load with a constant amplitude and frequency has been applied in the vertical direction (indicated by the vertical arrow) to obtain damage propagation. The load reaches its maximum peak of 35 kN with a load ratio $R = 0.1$. The crack is manually measured with callipers during the system operation. The relevant features of the test are reported in Table 3. The experimental crack propagations obtained during the tests are shown in Fig. 12 (right).

4.3.2. Performances on real data

The component under discussion cannot be approximated as a simple aluminium plate because of the presence of the stringers. The estimation of the SIF can be made through a complex analytical model for plates with riveted stringers [85] or by means of numerical methods. In this case the estimation of ΔK is approximated by an Artificial Neural Network (ANN). In particular, the ANN receives the crack length and the crack position as input, and it provides the SIF range for that particular crack for a known fatigue load configuration. Examples for algorithm training have been provided by means of repeated FEM analyses with crack damages in different positions with different lengths.

Further dissertations about the ANN for the ΔK estimation and its relative inaccuracy are omitted for brevity, as it is not the central objective of the work. See [81] for additional information about the method.

It is however important to underline that, if a methodology (analytical or numerical) is available to estimate the ΔK with deterministic or statistical approaches, it can be inserted in the algorithm. According to this method, the SIF range becomes a function of the crack length and the crack position. However, all the experimental tests are made with a central crack, so the crack position is constant in all of the simulations. In this case, the SIF range value is deterministic if the crack length is given.

According to the presented methodology, N_S samples of the FCG parameters $(C, m, \Delta K_{th,0})$ are selected according to their PDFs defined in Table 1. Given the inability of the algorithm to correctly fit the threshold SIF range, the threshold in the experimental tests is considered as a random variable with a mean value and variance, without participating to the algorithm updating operation. The RUL of the structure is extracted as the difference between the time to failure distribution and the current load cycle. The obtained results in terms of C and m estimation are not shown in order to highlight the performance of the algorithm in terms of the RUL estimation only. As a matter of fact, the estimation of the parameters in these real cases produces results not comparable with any known quantities.

The results of the algorithm in terms of failure cycle PDFs are shown in Fig. 13 for the four tests. The failure distributions visible on the left side of the figure approach the real failure cycle in a relatively short time expressed as a crack measure on the abscissa. These good results are highlighted by the fact that in these real cases the number of available measures is much reduced than in the case of simulated data. Moreover, the discrepancies among the measures rise in the real cases, where the cracks do not follow the theoretical trend but they are affected by a lot of unpredictable environmental parameters. The Stochastic DSS presented is able to predict the Failure distribution of different panels having the same characteristics from a theoretical viewpoint but which can be affected by relevant discrepancies on real environment. It can be considered as a stable method to produce the lifetime prediction also thanks to the prior distributions of the parameters. As a matter of fact, the prior distributions work as inertia terms slowing down the updating of the parameter features based on the measures.

This approach is highly advantageous with respect to the case where the models of the phenomenon are not used such as data driven methods. In fact, a wrong measure (very probable in the case of automatic measurement systems) produces a completely different time to failure if a pure data-driven approach is adopted. This problem is attenuated with model-based approaches. During the early stage of the algorithm (after two or three measures only) the prior knowledge of the parameters prevails on the available measures and the estimated parameter PDFs are not so different with respect to the initial ones. After that, the measures start to have a relevant effect on the updating of the SDSS model moves the lifetime prediction towards the real failure cycle. In the presented results, the PDFs of the failure cycle tend to the correct value when the crack length is around 30 mm, when the number of available measures varies between five and eight. The failure cycle prediction becomes more precise around a crack length of 50–60 mm as underlined by the CDF graphs on the right side of Fig. 13. The CDFs are related to different crack lengths because the measurements during the tests have been taken at pre-determined load cycles, thus the crack lengths are slightly different for each test.

5. Conclusions

A Stochastic Dynamic State-Space model is proposed in this paper to statistically describe fatigue structural degradation problems. The SDSS is used to propagate the uncertainties on the model and material parameters into the estimation of the remaining life of a structure subjected to fatigue loads. Many studies are available on this subject, which is nowadays a very active research topic as it would represent the final linkage between any Structural Health Monitoring system and the Condition Based Maintenance strategies for the optimisation of structure operation and safety. It is the author intention to summarise in the following paragraphs the main contributes of this study to the state of art of residual useful life estimation.

Several FCG models have been statistically described, starting from the well-known Paris's equation up to the most advanced NASGRO model, all of them relating a damage condition and load/geometry configuration to crack propagation velocity. The most common sources of uncertainty in FCG problems have been categorised into model and material uncertainties (generally labelled as inter-specimen variability), while time-dependent variables are still deterministically treated. The influence of each parameter uncertainty to the output of the considered models has been analytically calculated, sometimes requiring the approximation of the first order statistical moments via the application of *delta method*; the advantages and drawbacks of these analytical solutions have been discussed. Considering some data are available to evaluate the statistical distributions of the different model and material parameters, one could implement the above FCG model equations into the DSS definition to obtain the probability of failure and the resulting RUL distribution. This statistical definition of model parameters can enhance the common Damage Tolerance strategies where the FCG rate is deterministically defined. Unfortunately, the procedure cannot be solved analytically unless simple equations and normal-distributed uncertainties are considered. The implementation of a Stochastic DSS model for crack growth becomes numerically viable. The stochastic implementation of degradation algorithms based on MCMC techniques (taking advantage from an adaptive proposal distribution) has been in-depth presented in the paper. Results in the simulated environment are used to verify the performances of the method. The MH algorithm is both able to update the prior knowledge on some selected parameters, thus estimating a posterior distribution conditional on measures, and to provide an accurate lifetime prediction. The results obtained in the simulated environment highlight as the updating of empirical parameters (C and m) has a very large benefit on the estimation of the target RUL distribution, while further investigations are required to prove the effectiveness of the updating of $\Delta K_{th,0}$ distribution; in practice the latter has a reduced influence on the crack propagation if the growing phenomenon is in zone II or III of the FCG NASGRO model.

The lifetime prediction capability of the defined Stochastic DSS model is verified on real structures in laboratory environment, subjected to crack propagation under fatigue load. The robustness of the methodology is demonstrated through repeated fatigue tests on the same structure. The uncertainty on FCG when real structure are addressed has been emphasized. Despite the FCG modelling with the statistical definition of model parameters is a problem already handled by many authors, the verification of the method on several real portions of aeronautical structures instead of simple specimens has been rarely addressed and it represents an important step forward in the panorama of prognostic system.

The application of *adaptive proposal distribution* (already developed in [82,83]) to MH algorithm in FCG problems constitutes the most important innovation inside this study. In fact, the adaptation of the variance associated to the prior parameter distribution is a crucial matter in such kind of mathematical tools. There is a very high probability that the MH algorithm alone will not be able to correctly predict the evolution of cracks, as the sequential updating of the parameter distributions could provide biased estimations, with a consequent drop in performance. Despite of the possibilities to implement different statistical techniques to monitor the crack, one of the most important statement inside the paper is the need for self-adaptive algorithms. In fact, there is a very high variability inside the crack propagation phenomenon, as it is highlighted in many papers (see for instance [25–28]) and in Fig. 1 of this work. Numerical methods taking into account the whole parameter variability at the beginning of the procedure and selecting the most probable parameters according to the measured crack evolution are mandatory. Standard MCMC algorithms without a self-adaptation of their features (like the variance of the adaptive proposal presented in this work) in most cases will not be enough to produce a robust lifetime prediction, especially when real structures are addressed.

As an alternative to the full MCMC method herein presented, a Sequential Importance Sampling/Resampling algorithm can be also implemented with a Stochastic Dynamic model to filter the most probable crack evolution. The authors recommend the last case for on-line applications, as proposed in [77]. As a matter of fact, the main drawback of the MH algorithm implemented in this work is the computational effort required to produce a real-time prognosis. Additionally, the computational time has a linear proportionality to the number of measures. Nevertheless, it can be drastically reduced if particular real-time algorithms (such as Particle Filter) are used. However, any MCS method can receive benefit if (i) a rigorous stochastic definition of degradation model parameters and (ii) a self-adaptation of the Dynamic State-Space model are implemented.

However, some simplifications with respect to reality are considered in these tests. (i) The function that relates crack length and geometry to the SIF is set as deterministic in both the simulated and the experimental case. (ii) The fatigue load has a constant amplitude (and load ratio). This condition is unrealistic if compared with real aerospace and civil structures; nevertheless, it is reasonable in mechanical rotating machineries. (iii) The estimation of the FCG parameters with the proposed methodology can be affected by constant bias due to factors not considered in the model. As an example, any error in the evaluation of the structure geometry can produce a slightly different crack propagation (given by a different geometric function). This is considered by the algorithm as different values of C and/or m. However, the purpose of the methodology is not the fitting of the actual parameter for the material, but is the proper fitting of the current data to produce better lifetime predictions. As a consequence, the obtained values of the empirical parameters are futile outside the context of RUL evaluations of the operating structure.

The necessity to have some *prior* knowledge of the statistical FCG model parameters, such as the material properties or the uncertainty of the measurement system, is an additional drawback of the method. In fact, if no prior information is available, more experimental measures are required to make inference about the unknown parameter distributions. This will result in a further increase of the computational costs, though the mathematical convergence is still guaranteed.

Appendix A. Analytical formulation of statistical FCG models

A.1. Paris' law

Let consider the Paris' law (A.1) and its logarithmic conversion (A.2); the mean and variance of the dependent variable dY/dN can be calculated according to the theory of random variables (Eqs. (A.3) and (A.4)) where the expected values and variances are indicated with the common notation μ, σ^2 respectively

$$\frac{dy}{dN} = C\Delta K^m \tag{A.1}$$

$$\log\frac{dy}{dN} = \log C + m\log\Delta K \tag{A.2}$$

$$\mu_{\log dY/dN} = \mu_{\log C} + \mu_M \log\Delta K \tag{A.3}$$

$$\sigma^2_{\log dY/dN} = \sigma^2_{\log C} + \log\Delta K\sigma_M(\log\Delta K\sigma_M + 2\rho_{\log C,M}\sigma_{\log C}) \tag{A.4}$$

A.2. Forman's law

The logarithmic form of Forman's model is expressed in (A.5). The additional terms with respect to Paris' law are the maximum SIF acting on the crack tip $K_{\max}$ and the critical SIF K_C. The latter depends on the fracture toughness in plane strain condition K_{IC} and the thickness of the structure under discussion.

$$\log\frac{dy}{dN} = \log C + m\log\Delta K + \log K_{\max} - \log(K_C - K_{\max}) \tag{A.5}$$

Suppose the critical SIF is statistically defined. The average and variance of the propagation velocity are expressed in (A.6) and (A.7).

$$\mu_{\log dY/dN} = \mu_{\log C} + \mu_M\log\Delta K + \log K_{\max} - \mu_{\log(K_C-K_{\max})} \tag{A.6}$$

$$\begin{aligned}\sigma^2_{\log dY/dN} &= \sigma^2_{\log C} + \log\Delta K^2\sigma^2_M + \sigma^2_{\log(K_C-K_{\max})} + 2\rho_{\log C,M} \\ &\quad\times \log\Delta K\sigma_{\log C}\sigma_M \\ &\quad - 2\rho_{\log C,\log(K_C-K_{\max})}\sigma_{\log C}\sigma^{+}_{\log(K_C-K_{\max})} \\ &\quad - 2\rho_{M,\log(K_C-K_{\max})}\log\Delta K\sigma_M\sigma_{\log(K_C-K_{\max})} \\ &= \sigma^2_{\log C} + \log\Delta K\sigma_M[\log\Delta K\sigma_M + 2(\rho_{\log C,M}\sigma_{\log C} \\ &\quad - \rho_{M,\log(K_C-K_{\max})}\sigma_{\log(K_C-K_{\max})})] \\ &\quad + \sigma_{\log(K_C-K_{\max})}(\sigma_{\log(K_C-K_{\max})} \\ &\quad - 2\rho_{\log C,\log(K_C-K_{\max})}\sigma_{\log C})\end{aligned} \tag{A.7}$$

The evaluation of (A.6) is the simple evaluation of the model with the mean values of the variables. However, the calculation of (A.7) becomes difficult for the presence of the variable $\log(K_C - K_{\max})$, as explained in Section 2. Then, an approximation of the exact solution is provided using the delta method. It consists in the first order Taylor expansion of the model, which can be used to approximate the first two moments of the dependent variable (that is the mean and the variance of the crack growth rate). Let assume to include all the random vectors of parameters and material properties defined in the Eq. (2) into a random vector $X = [\Theta, \mathrm{H}]$. According to delta method, Eqs. (A.8) and (A.9) shows the implicit approximation of the crack velocity (the hypothesis of constant-amplitude load changes the crack growth increment from dY/dt to dY/dN).

$$\mu_{dY/dN} \approx \phi(\mu_X) \tag{A.8}$$

$$\sigma^2_{dY/dt} \approx \sum_i \sum_j \left.\frac{\partial \phi(X)}{\partial X_i}\right|_{\mu_X} \left.\frac{\partial \phi(X)}{\partial X_j}\right|_{\mu_X} COV(X_i, X_j) \tag{A.9}$$

The term μ_X is the mean of vector X, i is the index that goes from 1 to the length of X, and $\partial\phi(X)/\partial X_i|_{\mu_X}$ is the derivative of the model ϕ with respect to the ith random variable of X evaluated at μ_X. $COV(X_i, X_j)$ is the covariance between the ith and jth random variables inside X, reminding that $COV(X_i, X_i) = \sigma^2_{X_i}$.

Obviously, the complexity of ϕ drives the complexity of (A.8) and (A.9). The application of the delta method to the Forman's model produces the subsequent formulation for mean (A.10) and variance (A.11), respectively.

$$\mu_{dY/dN} \approx \frac{\mu_C \Delta K^{\mu_M} K_{\max}}{\mu_{K_C} - \mu_{K_{\max}}} \tag{A.10}$$

$$\begin{aligned}
\sigma^2_{dY/dN} \approx{}& \left.\frac{\Delta K^{\mu_M} K_{\max}}{\mu_{K_c} - K_{\max}}\right)^2 \sigma^2_C + \left.\frac{\mu_C \Delta K^{\mu_M} K_{\max} \log \Delta K}{\mu_{K_c} - K_{\max}}\right)^2 \sigma^2_M + \left(-\frac{\mu_C \Delta K^{\mu_M} K_{\max}}{\left(\mu_{K_c} - K_{\max}\right)^2}\right)^2 \sigma^2_{K_c} \\
&+2 \left.\frac{\Delta K^{\mu_M} K_{\max}}{\mu_{K_c} - K_{\max}}\right) \left.\frac{\mu_C \Delta K^{\mu_M} K_{\max} \log \Delta K}{\mu_{K_c} - K_{\max}}\right) \sigma_{C,M} \\
&+2 \left.\frac{\Delta K^{\mu_M} K_{\max}}{\mu_{K_c} - K_{\max}}\right) \left(-\frac{\mu_C \Delta K^{\mu_M} K_{\max}}{\left(\mu_{K_c} - K_{\max}\right)^2}\right) \sigma_{C,K_c} \\
&+2 \left.\frac{\mu_C \Delta K^{\mu_M} K_{\max} \log \Delta K}{\mu_{K_c} - K_{\max}}\right) \left(-\frac{\mu_C \Delta K^{\mu_M} K_{\max}}{\left(\mu_{K_c} - K_{\max}\right)^2}\right) \sigma_{M,K_c} = \left.\frac{\Delta K^{\mu_M} K_{\max}}{\mu_{K_c} - K_{\max}}\right)^2 \\
&\times \left[\sigma^2_C + (\mu_C \log \Delta K)^2 \sigma^2_M + \frac{\mu^2_C}{\left(\mu_{K_c} - K_{\max}\right)^2} \sigma^2_{K_c} + 2\rho_{C,M} (\mu_C \log \Delta K) \sigma_C \sigma_M\right] \\
&+ -2 \left.\frac{\Delta K^{\mu_M} K_{\max}}{\mu_{K_c} - K_{\max}}\right)^2 \left[\frac{\rho_{C,K_c} \mu_C \sigma_C \sigma_{K_c}}{\left(\mu_{K_c} - K_{\max}\right)} + \frac{\rho_{M,K_c} (\mu^2_C \log \Delta K) \sigma_M \sigma_{K_c}}{\left(\mu_{K_c} - K_{\max}\right)}\right]
\end{aligned} \tag{A.11}$$

A.3. McEvily/s model

The statistical formulation of the McEvily's model presented in (12) is not trivial because of the ratio between two statistical material properties (Elastic modulus and threshold SIF range). Moreover, the two properties are raised to the power of two. The application of the delta method to the McEvily's model produce the following closed form solution (A.14) and (A.15).

$$\mu_{dY/dN} \approx \frac{8}{\pi \mu^2_E} \left(\Delta K^2 - \mu^2_{\Delta K_{th,0}}\right) \tag{A.14}$$

$$\begin{aligned}
\sigma^2_{dY/dN} \approx{}& \left(\frac{-16}{\pi \mu^3_E} \left(\Delta K^2 - \mu^2_{\Delta K_{th,0}}\right)\right)^2 \sigma^2_E + \left(\frac{-16}{\pi \mu^2_E} \mu_{\Delta K_{th,0}}\right)^2 \sigma^2_{\Delta K_{th,0}} \\
&+2 \left(\frac{-16}{\pi \mu^3_E} \left(\Delta K^2 - \mu^2_{\Delta K_{th,0}}\right)\right) \left(\frac{-16}{\pi \mu^2_E} \mu_{\Delta K_{th,0}}\right) \sigma_{E,\Delta K_{th,0}} \\
={}& \left(\frac{-16}{\pi \mu^3_E} \left(\Delta K^2 - \mu^2_{\Delta K_{th,0}}\right)\right)^2 \sigma^2_E + \left(\frac{-16}{\pi \mu^2_E} \mu_{\Delta K_{th,0}}\right)^2 \sigma^2_{\Delta K_{th,0}} \\
&+ \frac{512 \mu_{\Delta K_{th,0}}}{\pi^2 \mu^5_E} \left(\Delta K^2 - \mu^2_{\Delta K_{th,0}}\right) \sigma_{E,\Delta K_{th,0}} \\
={}& \left(\frac{-16}{\pi \mu^2_E}\right)^2 \left[\frac{\left(\Delta K^2 - \mu^2_{\Delta K_{th,0}}\right)^2}{\mu^2_E} \sigma^2_E + \mu^2_{\Delta K_{th,0}} \sigma^2_{\Delta K_{th,0}}\right. \\
&\left. + \frac{2 \mu_{\Delta K_{th,0}} \left(\Delta K^2 - \mu^2_{\Delta K_{th,0}}\right) \rho_{E,\Delta K_{th,0}} \sigma_E \sigma_{\Delta K_{th,0}}}{\mu_E}\right]
\end{aligned} \tag{A.15}$$

A.4. NASGRO equation

The logarithmic form of the model presented in (10) cannot be represented by the exact mean and variance formulas because of the high nonlinearity of the model. Then, if several random variables are considered in the equation, an approximation is needed.

The application of delta method to general NASGRO law produces an average value of $\log dY/dN$ that is the simple NASGRO equation evaluated with the average values of its random variables, defined in X. The implicit equation of the variance is expressed in (A.16).

$$\begin{aligned}
\sigma^2_{\log dY/dN} \approx{}& \left.\frac{\partial \log \phi}{\partial \log C}\right|_{\mu_X}\Bigg) \left[\sum_{X_i = [\log C, M, P, Q, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log \phi}{\partial X_i}\right|_{\mu_X} \sigma_{\log C, X_i}\right] \\
&+ \left.\frac{\partial \log \phi}{\partial M}\right|_{\mu_X}\Bigg) \left[\sum_{X_i = [\log C, M, P, Q, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log \phi}{\partial X_i}\right|_{\mu_X} \sigma_{M, X_i}\right] \\
&+ \left.\frac{\partial \log \phi}{\partial P}\right|_{\mu_X}\Bigg) \left[\sum_{X_i = [\log C, M, P, Q, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log \phi}{\partial X_i}\right|_{\mu_X} \sigma_{P, X_i}\right] \\
&+ \left.\frac{\partial \log \phi}{\partial Q}\right|_{\mu_X}\Bigg) \left[\sum_{X_i = [\log C, M, P, Q, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log \phi}{\partial X_i}\right|_{\mu_X} \sigma_{Q, X_i}\right] \\
&+ \left.\frac{\partial \log \phi}{\partial \Delta K_{th,0}}\right|_{\mu_X}\Bigg) \left[\sum_{X_i = [\log C, M, P, Q, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log \phi}{\partial X_i}\right|_{\mu_X} \sigma_{\Delta K_{th,0}, X_i}\right] \\
&+ \left.\frac{\partial \log \phi}{\partial K_{IC}}\right|_{\mu_X}\Bigg) \left[\sum_{X_i = [\log C, M, P, Q, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log \phi}{\partial X_i}\right|_{\mu_X} \sigma_{K_{IC}, X_i}\right]
\end{aligned} \tag{A.16}$$

The characterisation of all the material properties and parameters requires the analysis of large amount of FCG data and the practical application of Eq. (A.16) is unfeasible looking at the complexity of the derivatives of the model with respect to the employed parameters. For the best author's knowledge, there are no statistical descriptions of the parameters p and q of NASGRO law in literature. Moreover, the variability of threshold SIF range and fracture toughness affect the first and third regions of the FCG rate largely more than the uncertainty of p and q. According to the above considerations, the statistical model considers deterministic values of p and q. The new formulation of NASGRO variance is expressed in (A.17).

$$\sigma^2_{\log dY/dN} \approx \left.\frac{\partial \log\phi}{\partial \log C}\right|_{\mu_X}\Bigg) \left[\sum_{X_i=[\log C, M, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log\phi}{\partial X_i}\right|_{\mu_X} \sigma_{\log C, X_i}\right] + \left.\frac{\partial \log\phi}{\partial M}\right|_{\mu_X}\Bigg) \left[\sum_{X_i=[\log C, M, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log\phi}{\partial X_i}\right|_{\mu_X} \sigma_{M, X_i}\right] + \left.\frac{\partial \log\phi}{\partial \Delta K_{th,0}}\right|_{\mu_X}\Bigg) \left[\sum_{X_i=[\log C, M, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log\phi}{\partial X_i}\right|_{\mu_X} \sigma_{\Delta K_{th,0}, X_i}\right] + \left.\frac{\partial \log\phi}{\partial K_{IC}}\right|_{\mu_X}\Bigg) \left[\sum_{X_i=[\log C, M, \Delta K_{th0}, K_{IC}]} \left.\frac{\partial \log\phi}{\partial X_i}\right|_{\mu_X} \sigma_{K_{IC}, X_i}\right] \quad (A.17)$$

The different terms of the equation are reported hereafter (A.18), (A.19), (A.20), (A.21). The complete formulation of the variance according to the last hypothesis is shown in (A.22).

$$\frac{\partial \log\phi}{\partial \log C} = 1 \quad (A.18)$$

$$\frac{\partial \log\phi}{\partial M} = \log\left(\frac{1-f}{1-R}\Delta K\right) \quad (A.19)$$

$$\frac{\partial \log\phi}{\partial \Delta K_{th,0}} = \frac{p\omega_0}{\Delta K\left(\frac{\mu_{\Delta K_{th,0}}\omega_0}{\Delta K \omega_1} - 1\right)\omega_1} \quad (A.20)$$

$$\frac{\partial \log\phi}{\partial K_{IC}} = \frac{q\left[\frac{K_{\max}}{\mu^2_{K_{IC}}\alpha_0} + \frac{\alpha_2}{\mu^6_{K_{IC}}\alpha_1\alpha_0^2}\right]}{\frac{K_{\max}}{\mu_{K_{IC}}\alpha_0} - 1} \quad (A.21)$$

$$\begin{aligned}\sigma^2_{\log dY/dN} \approx{} & \sigma^2_{\log C} + \log\left(\frac{1-f}{1-R}\Delta K\right)^2 \sigma^2_M \\ & + p^2\omega_0^2\left[\Delta K\left(\frac{\mu_{\Delta K_{th,0}}\omega_0}{\Delta K\omega_1} - 1\right)\omega_1\right]^{-2} \sigma^2_{\Delta K_{th,0}} \\ & + q^2\left[\frac{K_{\max}}{\mu^2_{K_{IC}}\alpha_0} + \frac{\alpha_2}{\mu^6_{K_{IC}}\alpha_1\alpha_0^2}\right]^2 \left(\frac{K_{\max}}{\mu_{K_{IC}}\alpha_0} - 1\right)^{-2} \sigma^2_{K_{IC}} \\ & + 2\log\left(\frac{1-f}{1-R}\Delta K\right)\rho_{\log C, M}\sigma_{\log C}\sigma_M \\ & + 2p\omega_0\left[\Delta K\left(\frac{\mu_{\Delta K_{th,0}}\omega_0}{\Delta K\omega_1} - 1\right)\omega_1\right]^{-1} \rho_{\log C, \Delta K_{th,0}}\sigma_{\log C}\sigma_{\Delta K_{th,0}} \\ & + 2q\left[\frac{K_{\max}}{\mu^2_{K_{IC}}\alpha_0} + \frac{\alpha_2}{\mu^6_{K_{IC}}\alpha_1\alpha_0^2}\right]\left(\frac{K_{\max}}{\mu_{K_{IC}}\alpha_0} - 1\right)^{-1} \rho_{\log C, K_{IC}}\sigma_{\log C}\sigma_{K_{IC}} \\ & + 2\log\left(\frac{1-f}{1-R}\Delta K\right)p\omega_0\left[\Delta K\left(\frac{\mu_{\Delta K_{th,0}}\omega_0}{\Delta K\omega_1} - 1\right)\omega_1\right]^{-1} \rho_{M, \Delta K_{th,0}}\sigma_M\sigma_{\Delta K_{th,0}} \\ & + 2\log\left(\frac{1-f}{1-R}\Delta K\right)q\left[\frac{K_{\max}}{\mu^2_{K_{IC}}\alpha_0} + \frac{\alpha_2}{\mu^6_{K_{IC}}\alpha_1\alpha_0^2}\right]\left(\frac{K_{\max}}{\mu_{K_{IC}}\alpha_0} - 1\right)^{-1} \rho_{M, K_{IC}}\sigma_M\sigma_{K_{IC}} \\ & + 2p\omega_0\left[\Delta K\omega_1\left(\frac{\mu_{\Delta K_{th,0}}\omega_0}{\Delta K\omega_1} - 1\right)\left(\frac{K_{\max}}{\mu_{K_{IC}}\alpha_0} - 1\right)\right]^{-1} q\left[\frac{K_{\max}}{\mu^2_{K_{IC}}\alpha_0} \right. \\ & \left. + \frac{\alpha_2}{\mu^6_{K_{IC}}\alpha_1\alpha_0^2}\right]\rho_{\Delta K_{th,0}, K_{IC}}\sigma_{\Delta K_{th,0}}\sigma_{K_{IC}} \end{aligned} \quad (A.22)$$

where $\omega_0 = \sqrt{\frac{y}{y+a_0}}$, $\omega_1 = \left(\frac{f-1}{A_0-1}\right)^{c^+_{th}R+1}$, $\alpha_0 = \left(\frac{B_K}{\alpha_1} + 1\right)$, $\alpha_1 = \exp\left(\frac{0.16A_K^2S_{ys}^4t^2}{\mu^4_{K_{IC}}}\right)$, $\alpha_2 = 0.64A_K^2B_KK_{\max}S_{ys}^4t^2$. The term ω_1 is made explicit under the hypothesis of positive load ratio $R \geqslant 0$ according to NASGRO manual [61]. However, the same consideration can be made for $R < 0$. S_{ys} is the yielding stress of material, t is the thickness of the structure at issue, A_0, A_K, B_K, c^+_{th} are NASGRO constants, a_0 is the El-haddad parameter and f is the crack closure function. Please refer to [61] for additional information about the NASGRO formulation.

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