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estratto da **AEROTECNICA - MISSILI E SPAZIO**
Volume 67, numero 1-4, marzo-dicembre 1988

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ABSTRACT

A semi-variational approach is followed in order to derive a set of ordinary differential linearized equations for static analysis of axial structures under end loads. Both a finite element model of the cross-section and a roto-translation model fit for slender beams are drawn. An eigengroblem is set up and faced looking at the load intensity as a parametr: by increasing load, both central and extremity solutions evolve and a wavy undecaying mode is reached at a lower bound for buckling loads. Simple examples relative to both global and local instability are discussed, while the parametric analysis of a shell of revolution shows some unforeseen aspects correlating to experimental test results.

1. INTRODUCTION

If an elastic, shallow shell of revolution is acted on by self-balanced static forces at a given cross-section, it will respond with a deformation that decays moving apart from the loaded section (according to De Saint Venant's principle) and that is a combination of natural modes of the shell itself, often referred to as extremity solutions: circumferential waves on the shell cross-section characterize the shape of such modes, see for example Gol'denveizer (1961). Instead if the same shell is loaded in axial compression, it will buckle at a given load with an undecaying deformation once again with circumferential waves. Therefore it is expected that extremity solutions evolve naturally towards undecaying modes while increasing the axial compression, or possibly towards modes easy to snap-through into undecaying solutions.

This way of facing an instability phenomenon can be generalized to some different buckling problems of a wide class of structural elements, such as those depicted in Fig. 1: beams, struts, tubes, stringers, flat and curved sheets, longitudinally stiffened panels and shells, all showing a definite direction and a constant cross-section; that's why is this paper they are referred to as axial structures. An elastic axial structure is fit for a semi-variational analysis approach: following the displacement method, if the displacement is approximated by a discrete set of unknowns pertaining to the section by means of suitable functions (either interpolating functions as in the finite element method or appropriate shapes as in Ritz's method), applying a variational principle just to the section itself leads to an ordinary differential equation set, the unknowns depending only on the axial coordinate. A linearized analysis yields homogeneous equations with constant coefficients in several cases of practical interest, when elastic properties and pre-stress do not vary along the structure axis, as in axial compression, panel shear and shell torsion: an eigenproblem is therefore set up and static modes are sought, looking at the pre-stress intensity as a parameter. When this is null, the same elastic problem is found as pointed out in linear beam analysis by Giavotto *et al.* (1983, 1986), allowing for both central and extremity solutions. For increasing load, wavy-decaying solutions are in general found, with the wavy content continuosly growing and the decaying one lessening, until an undecaying wavy solution is achieved: thus a lower bound for linear buckling loads is found, while at the actual critical load a solution will agree in wave length with extremity boundary conditions. The same considerations apply as well to overall beam buckling, the instability modes being the natural evolution of central solutions.

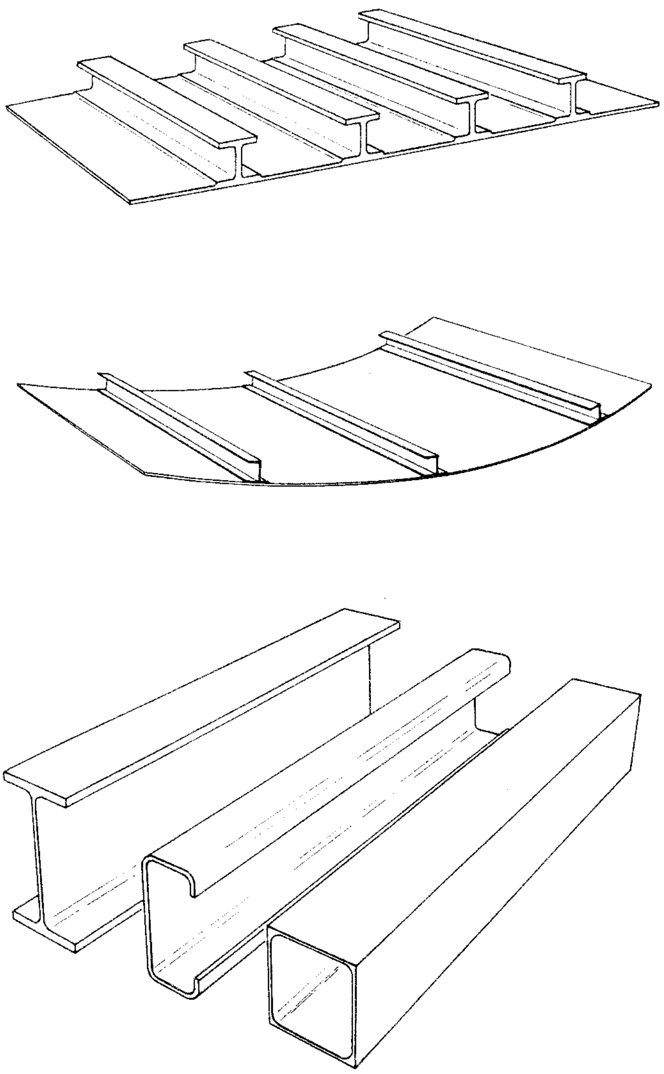

Fig. 1 - Some typical axial structures

The main purpose of this paper is to set up a right formulation of the problem to be of use in subsequent search, and to show the behaviour of static modes in some very simple and typical cases. The semi-variational approach is stated in section 2 going over the topics of classical mechanics with an unusual and yet straightforward vector notation. Then two displacement models are examined: the general finite element model (section 3) seems effective for application to cross-sections of any shape and material properties, allowing for anisotropy and unhomogeneity; the roto-translation model (section 4) makes it possible to study the evolution of modes with end load transfer (central solutions), therefore giving the whole picture of static modes. Whatever the displacement model, the derivation is fully nonlinear and so it is ready for future work although linearized equations are discussed in this paper. The examples enclosed in section 5 are quite simple and still contribute to explain the discussion.

Two advantages appear from such an approach to stability problems. First, an investigative interest in the development of buckling mechanisms. The eigensolution trend can give some insight even into the buckling of a cylinder, that is a nonlinear phenomenon: perhaps undecaying modes could be more effectively used in interactive buckling analysis, a problem being now faced with finite strips (see Benito, Sridharan (1985) and Ali, Sridharan (1988)). Second, the relatively small size of the problem in a computing process: the unknowns of the ordinary differential equations are the discrete degrees-of-freedom of one section; thus local buckling of columns and shell buckling can be dealt with using detailed models of the cross-section redardless to the structure length. Such a possibility becomes valuable when sections are not homogeneous as for composite laminates: indeed the effectiveness of a semi-variational approach to the analysis of composite axial structures has been shown in several application fields by Giavotto *et al.* (1979, 1983, 1986), Borri, Mantegazza (1985), Borri, Merlini (1986) and Ghiringhelli, Sala (1986).

2. SEMI-VARIATIONAL APPROACH

The structural element we are concerned with is an elastic prismatic solid generated by a mere translation of its cross-section along its axis: material properties are constant along the axial direction, although generally anisotropic and unhomogeneous on the section; external constraints, if any, are constant along one or more generatrices. Such a body can be referred to as an elastic axial structure. A coordinate system appropriate to describe geometric and material properties is assumed: it will in general be curvilinear on the section plane (coordinates x^1 and x^2, see Fig. 2) and rectangular with respect to the structure axis (coordinate x^3). A frame of covariant base vectors $\mathbf{g}_j = \mathbf{x}_{,j}$ and a reciprocal frame of contravariant base vectors $\mathbf{g}^j$ such that $\mathbf{g}^j \cdot \mathbf{g}_k = \delta^j_k$ (the Kronecker symbol) are associated with each point with position vector $\mathbf{x}$: each frame is not generally orthonormal, nevertheless the conditions $\mathbf{g}_j \cdot \mathbf{g}_3 = \delta_{j3}$ and $\mathbf{g}^j \cdot \mathbf{g}^3 = \delta^{j3}$ hold true anywhere, $\mathbf{g}^3 = \mathbf{g}_3$ being a constant unit vector (1).

(1) Throughout the paper latin indexes are intended to vary from 1 to 3, while greek ones will be used on narrower ranges. Standard convention of implicit summation for repeated indexes and the comma for partial derivatives with respect to the coordinate directions are adopted.

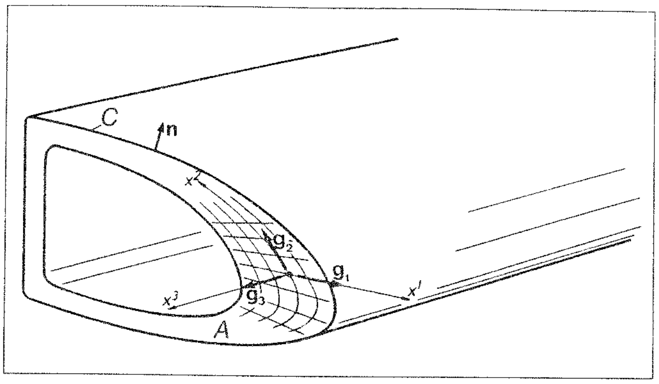

Fig. 2 - General reference frame for an axial structure

As the body deforms each point undergoes a displacement $\mathbf{s}$ and its position becomes defined by a new vector $\mathbf{x}' = \mathbf{x} + \mathbf{s}$. The differential of the position vector $d\mathbf{x} = \mathbf{g}_j\, dx^j$ (with contravariant components $dx^j = \mathbf{g}^j \cdot d\mathbf{x}$) changes in the deformed configuration to vector $d\mathbf{x}' = J d\mathbf{x}$: the linear operator J, *i. e.* the Jacobian of the coordinate transformation $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, is the so-called deformation gradient or Jacobian of the deformation (Haber, 1984), and in a fully Lagrangian description it is given by

$$J = (\mathbf{g}_j + \mathbf{s}_{,j})(\mathbf{g}^j)\cdot \, . \qquad (1)$$

The difference between the squared differentials, $d\mathbf{x}'\cdot d\mathbf{x}' - d\mathbf{x}\cdot d\mathbf{x} = d\mathbf{x}\cdot 2e d\mathbf{x}$, defines the strain symmetric linear operator

$$e = \frac{1}{2}\,(J^{\mathrm{T}} J - 1)\,, \qquad (2)$$

where J^{T} is the adjoint of J (²). Operator e transforms the base vectors $\mathbf{g}_k$ into vectors $e\mathbf{g}_k$ of covariant components $e_{jk} = \mathbf{g}_j \cdot (e\mathbf{g}_k)$: scalars $e_{jk} = e_{kj}$ are called the matrix covariant elements of the operator e and are just the covariant components of the Green strain tensor (³).

The local equilibrium equations on the lateral surface and within the body can be written

$$\mathbf{f} - J\sigma\mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathbf{F} + (J\sigma)_{,k}\, \mathbf{g}^k = \mathbf{0}\,, \qquad (3)$$

where $\mathbf{f}$ and $\mathbf{F}$ are the surface and volume densities (with respect to the undeformed body) of external forces, $\mathbf{n}$ is the unit outwards normal to the undeformed surface, and σ is the stress symmetric operator. The matrix contravariant elements $\sigma^{jk} = \mathbf{g}^j \cdot (\sigma\mathbf{g}^k)$ of operator σ are just the contravariant components of the 2nd Piola-Kirchhoff stress tensor: it is seen that $\mathbf{g}_j \cdot ((J\sigma)_{,n}\, \mathbf{g}^k)$ are the tensorial derivatives along coordinate x^n of the components of the so-called 1st Piola-Kirchhoff or Piola-Lagrange stress tensor.

Scalar product of balance equations by an arbitrary vector field, for example the virtual displacement $\delta\mathbf{s}$, and integration over the cross-section contour line C and surface area A, yields a scalar condition equivalent to Eqs. (3) for the whole section,

$$\int_C \delta\mathbf{s}\cdot(\mathbf{f} - J\sigma\mathbf{n})\,dC + \int_A \delta\mathbf{s}\cdot(\mathbf{F} + (J\sigma)_{,k}\mathbf{g}^k)\,dA = 0.$$

Function $\delta\mathbf{s}\cdot((J\sigma)_{,k}\, \mathbf{g}^k) = ((\sigma J^{\mathrm{T}})_{,k}\, \delta\mathbf{s})\cdot \mathbf{g}^k$ is now integrated by parts by means of the theorem of the divergence for vector $\sigma J^{\mathrm{T}}\delta\mathbf{s}$; as scalar product of vectors $\delta\mathbf{e}_k = \delta e \mathbf{g}_k$ by $\boldsymbol{\sigma}^k = \sigma\mathbf{g}^k$, because of Eqs. (1), (2) and symmetry of σ, leads to

$$\delta\mathbf{e}_k \cdot \boldsymbol{\sigma}^k = (J^{\mathrm{T}}\delta\mathbf{s}_{,k}) \cdot \boldsymbol{\sigma}^k\,, \qquad (4)$$

it follows that:

$$\int_C \delta\mathbf{s}\cdot\mathbf{f}\,dC + \int_A \delta\mathbf{s}\cdot\mathbf{F}\,dA + \frac{d}{dx^3}\int_A \delta\mathbf{s}\cdot(J\boldsymbol{\sigma}^3)\,dA - \int_A \delta\mathbf{e}_k\cdot\boldsymbol{\sigma}^k\,dA = 0\,. \qquad (5)$$

The left-hand side of Eq. (5) is nothing but the total virtual work per unit length, with work of stress $J\boldsymbol{\sigma}^3$ acting on the section surface correctly appearing: thus Eq. (5) is the equation of principle of virtual work for finite displacements stated for an infinitesimal length of structure. Within the context of a direct method, displacement $\mathbf{s}$ is expressed as a function of a number of free unknowns, depending on the specific structure modelling: it is noted that this function can be nonlinear with respect to the unknowns of the analysis, as it will be shown later while dealing with a particular model.

To study possible configurations in the neighborhood of an equilibrium configuration, a linearized approach is established. A necessary condition for the equilibrium of a configuration $\mathbf{s} + d\mathbf{s}$ is obtained by differentiating Eq. (5): recalling Eq. (4), it follows

$$\int_C (\delta\mathbf{s}\cdot d\mathbf{f} + d\delta\mathbf{s}\cdot\mathbf{f})\,dC + \int_A (\delta\mathbf{s}\cdot d\mathbf{F} + d\delta\mathbf{s}\cdot\mathbf{F})\,dA$$
$$+ \frac{d}{dx^3}\int_A (J^{\mathrm{T}}\,\delta\mathbf{s}\cdot d\boldsymbol{\sigma}^3 + d\,(J^{\mathrm{T}}\,\delta\mathbf{s})\cdot\boldsymbol{\sigma}^3)\,dA$$
$$- \int_A (\delta\mathbf{e}_k\cdot d\boldsymbol{\sigma}^k + d\,(J^{\mathrm{T}}\delta\mathbf{s}_{,k})\cdot\boldsymbol{\sigma}^k)\,dA = 0\,. \qquad (6)$$

(²) The adjoint or transpose operator f^{T} of f is defined (Hestenes, 1986) by the condition that, for any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a}\cdot(f\mathbf{b}) = (f^{\mathrm{T}}\mathbf{a})\cdot\mathbf{b}$: it is seen that, for any two reciprocal frames $\mathbf{g}_k$ and $\mathbf{g}^k$, $f^{\mathrm{T}} = \mathbf{g}^k(f\mathbf{g}_k)\cdot = \mathbf{g}_k(f\mathbf{g}^k)\cdot$. An operator f is said to be symmetric or self-adjoint if $f^{\mathrm{T}} = f$.

(³) Other useful operators are the identity associated with two reciprocal frames, $g = \mathbf{g}_j\,\mathbf{g}^j\cdot = \mathbf{g}^j\mathbf{g}_j\cdot = 1$ (its matrix elements are the components of the metric tensor), and the displacement gradient $s_{/} = \mathbf{s}_{,j}\,\mathbf{g}^j\cdot$. Hence the deformation gradient reads $J = 1 + s_{/}$ and the strain becomes defined as $e = 1/2\,(s_{/} + s^{\mathrm{T}}_{/} + s^{\mathrm{T}}_{/}\, s_{/})$.

Variation ds (4) represents a possible infinitesimal displacement or a *mode* within the body of the axial structure, that can be called a static mode: the actual solution must of course match the boundary conditions at the extremities. This approach includes both linear solutions from the natural state (extremity modes and central solutions under end loads) and linear buckling modes arising at the onset of instability. With regard to variations of external actions $d\mathbf{f}$, $d\mathbf{F}$ and stress $d\sigma$, they must of course be related to unknown variations: elastic constitutive laws provide a stress-strain differential relation that can be written

$$d\boldsymbol{\sigma}^k = L^{k\mathrm{T}} D L^j \, d\mathbf{e}_j \ , \qquad (7)$$

where D is a symmetric array operator and L^j array operators are used to select six independent strain components (5). A choice for L^j could be

$$L^1 = \begin{Bmatrix} \mathbf{g}^1\mathbf{g}_1\cdot + \mathbf{g}^3\mathbf{g}_2\cdot \\ \mathbf{g}^1\mathbf{g}_3\cdot \end{Bmatrix} ,$$
$$L^2 = \begin{Bmatrix} \mathbf{g}^2\mathbf{g}_2\cdot + \mathbf{g}^3\mathbf{g}_1\cdot \\ \mathbf{g}^2\mathbf{g}_3\cdot \end{Bmatrix} , \qquad (8)$$
$$L^3 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} ,$$

that separates transverse strain and stress components from those facing on the section surface. Finally, Eq. (6) depends on the actual external actions $\mathbf{f}$ and $\mathbf{F}$ and stress σ, and accounts for the finite displacement $\mathbf{s}$ reached during the loading process, that can be computed via an incremental analysis: in some problems however, as in initial buckling, the reference configuration is characterized by an infinitesimal displacement that can be dropped from Eq. (6), giving a linear prebuckling state.

A semi-variational approach is suitably applied to condition (6), with a direct method based on a discrete model of the section surface, as for example the Kantorovich (1964) method: once displacement is expanded using properly chosen functions of coordinates x^1 and x^2, a set of simultaneous ordinary differential equations can be drawn, giving the static modes as a function of coordinate x^3. In the next sections the linear differential equations fit for initial instability analysis will be derived in two extreme and significant cases of displacement expression; another model will be referred to in the examples.

(4) The second variation $d\delta\mathbf{s}$ obviously cancels if (and only if) displacement is a linear function of the analysis unknowns.

(5) Because of symmetry of operators e and σ it must be $L^j\mathbf{g}^k = L^k\mathbf{g}^j$.

3. FINITE ELEMENT MODEL

In many discrete structural models the displacement vector field is assumed to be linearly dependent on the analysis unknowns: this is typical of the most classical finite element representations, the displacement being interpolated among a number of significant displacements. A development of such a discrete model for linear elastic analysis of anisotropic beam cross-sections was given by Giavotto *et al.* (1983), including the extremity mode analysis; this section is devoted to the application of condition (6) to models of this kind: in such case the second variation $d\delta\mathbf{s}$ is of course null.

Without loss of generality rectangular Cartesian coordinates are referred to: therefore vector components are easily worked on, and matrix notation will be adopted. An approximate displacement expansion can be formally written as

$$\{s\} = [N(x^1, x^2)]\{u(x^3)\} \ .$$

Independent components of stress and strain tensors are conveniently grouped in columns

$$\{\sigma\} = [\sigma^{11}\ \sigma^{22}\ \sigma^{12}\ \sigma^{13}\ \sigma^{23}\ \sigma^{33}]^{\mathrm{T}} ,$$
$$\{e\} = [e_{11}\ e_{22}\ 2e_{12}\ 2e_{13}\ 2e_{23}\ e_{33}]^{\mathrm{T}},$$

and the elastic stress-strain relation (7) with (8) becomes

$$\{d\sigma\} = [D]\{de\} \ .$$

The linearized strain-displacement differential relationship takes the form,

$$\{de\} = [L]\{ds_{,3}\} + [\mathcal{D}]\{ds\} \ ,$$

where $[L]$ and $[\mathcal{D}]$ are operators of order 6×3 defined by

$$[L] = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad \text{and} \quad [\mathcal{D}] = \begin{bmatrix} \partial/\partial x^1 & 0 & 0 \\ 0 & \partial/\partial x^2 & 0 \\ \partial/\partial x^2 & \partial/\partial x^1 & 0 \\ 0 & 0 & \partial/\partial x^1 \\ 0 & 0 & \partial/\partial x^2 \\ 0 & 0 & 0 \end{bmatrix} .$$

By using the above formulae, condition (6), written for a reference configuration with infinitesimal displacement, becomes

$$\{\delta u\}^{\mathrm{T}}\{dP\} + \frac{d}{dx^3}\left(\{\delta u\}^{\mathrm{T}}\left[M_e + M_\sigma \quad C_e^{\mathrm{T}} + C_\sigma^{\mathrm{T}}\right]\begin{Bmatrix} du' \\ du \end{Bmatrix}\right)$$

$$-\begin{Bmatrix} \delta u' \\ \delta u \end{Bmatrix}^{\mathrm{T}}\begin{bmatrix} M_e + M_\sigma & C_e^{\mathrm{T}} + C_\sigma^{\mathrm{T}} \\ C_e + C_\sigma & K_e + K_\sigma \end{bmatrix}\begin{Bmatrix} du' \\ du \end{Bmatrix} = 0 \ , \qquad (9)$$

where the prime indicates the derivative with respect to axis x^3, and the following integrals have been introduced ([6]):

$$\{dP\} = \int_C [N]^T \{df\}\, dC + \int_A [N]^T \{dF\}\, dA \ ,$$

$$\begin{bmatrix} M_e & C_e^T \\ C_e & K_e \end{bmatrix} = \int_A \begin{bmatrix} (\mathcal{L}N)^T D\, (\mathcal{L}N) & (\mathcal{L}N)^T D\, (\mathcal{D}N) \\ (\mathcal{D}N)^T D\, (\mathcal{L}N) & (\mathcal{D}N)^T D\, (\mathcal{D}N) \end{bmatrix} dA$$

$$\begin{bmatrix} M_\sigma & C_\sigma^T \\ C_\sigma & K_\sigma \end{bmatrix} = \int_A \begin{bmatrix} N^T \sigma^{33} N & N^T \sigma^{3\beta} N_{,\beta} \\ N_{,\alpha}^T \sigma^{\alpha 3} N & N_{,\alpha}^T \sigma^{\alpha\beta} N_{,\beta} \end{bmatrix} dA$$

Column $\{dP\}$ refers to the loaded or constrained regions, if any, of the section. Matrices $[M_e]$, $[C_e]$ and $[K_e]$ represent the elastic behaviour of the whole section with respect to unknowns $\{du\}$ and their derivatives $\{du'\}$ (elastic matrices), while $[M_\sigma]$, $[C_\sigma]$ and $[K_\sigma]$ are the counterpart accounting for the actual stress (pre-stress matrices). It is noted that the same Eq. (9) could be obtained if general curvilinear coordinates were used on the section plane, with only a different definition of the matrices involved.

An equivalent form for condition (9) is,

$$\{\delta u\}^T \Big(\{dP\} + [M_e + M_\sigma]\{du''\}$$
$$- [(C_e + C_\sigma) - (C_e + C_\sigma)^T - M'_\sigma]\{du'\}$$
$$- [K_e + K_\sigma - C_\sigma']\{du\} \Big) = 0 \ . \qquad (10)$$

It is seen that coefficient matrices in Eqs. (9) and (10) are not in general constant, depending on the state of stress along the structure: a beam loaded in shear is an example of variable stress along x^3. If stress is constant along the axis (as for example in axially loaded structures) and in absence of surface and volume forces, the following ordinary differential linear equations with constant coefficients are obtained from Eq. (10),

$$[M_e + \mu M_\sigma]\{du''\} - [(C_e - C^T_e) + \mu(C_\sigma - C^T_\sigma)]\{du'\}$$
$$- [K_e + \mu K_\sigma]\{du\} = \{0\}, \qquad (11)$$

which are valid either without lateral constraints, either with fixed lateral constraints provided the related degrees-of-freedom have been removed. Pre-stress matrices in Eqs. (11) are now intended to represent the stress distribution on the section, while their magnitude is controlled by a load parameter μ.

Mostly, in initial instability analyses, functions $\{u(x^3)\}$ are given an expression satisfying end conditions and a first order eigenvalue problem with respect to μ is set up, giving the critical buckling loads and modes. Another way of looking at homogeneous Eqs. (11) is to seek possible eigensolutions of exponential type $\{du\} = \{dU\}\exp(px^3)$, thus drawing a second order eigenvalue problem with respect to p (this being in general complex), while μ is treated as a parameter. Indeed when μ vanishes, Eqs. (11) become those governing the linear elastic problem of axial structures, already worked out in beam analysis by Giavotto *et al.* (1983, 1986), and will undergo either polynomial or exponential solutions: the first ones group both rigid motions and central solutions, viz. deformation modes excited by end loads; the last ones, referred to as extremity solutions, are modes excited by locally self-balanced loads and always decay along axis x^3, *i. e. Re(p)* is never null. As μ increases these modes evolve and exponential solutions can be generally expected (evolution in such way of central solutions will be discussed in the next section). An instability criterion is now of course needed in order to ascertain when an eigensolution of Eqs. (11) can be a buckling mode: in any case for a critical load to be caught, a solution must match the actual end boundary conditions.

This way of facing Eqs. (11) has some affinity with the aeroelastic analysis of flutter, where the unknowns are function of time and relevant matrices are in part constant and in part dependent on aerodynamic forces, controlled by the airflow speed as a parameter. The present conservative problem with symmetric and skew-symmetric matrices is simpler, but the same instability criterion is likely to hold: when motion ceases to be damped in time (flutter) or when displacement shape ceases to decay in space (present case) then instability can occur. Yet it is the real part of eigenvalue p that controls stability and, similarly to flutter analysis, the knowledge of how eigensolutions vary with the load parameter μ seems to be worthwhile for a better understanding of the instability mechanisms themselves. Obviously the actual end conditions must be taken into account, but in several cases their influence on the valuation of critical loads is negligible: this happens when the mode wave lengths are many times smaller than the structure length, as for local buckling of columns with slender sections.

The analogy with the flutter analysis allows to take advantage of the rich experience gained in numerical aeroelasticity. Among several methods, two techniques are here quoted as promising with very complex cross-sections: the modal condensation and truncation, possibly based on the modes at μ=0, and the sweep of a single static mode versus the load parameter, see for instance Cardani, Mantegazza (1978, 1979). Indeed such powerful techniques could be very useful when studying modal interaction in a subsequent nonlinear analysis.

([6]) Greek indexes are intended to range from 1 to 2.

4. ROTO-TRANSLATION MODEL

In the analysis of axial structures it is somewhat useful to resolve displacement **s** into a part which does not deform the cross-section and a residual part. The first term is a roto-translation of the section and so depends on six parameters, if no lateral constraints exist: it is not in general a rigid displacement as these parameters vary along the structure axis, but anyway it includes and can completely describe any rigid motion; consequently the residual part, that in a broad sense can be referred to as section warping, always represents a deformation. Such a displacement resolution proves very suitable in the analysis of beam central solutions (Giavotto *et al.*, 1983, 1986), but its usefulness seems poor in stability problems where warping is important and mostly comparable in magnitude with roto-translation, as for local buckling of thin-walled columns or shell buckling.

Nevertheless an interest exists in mere roto-translation displacement in some problems with insignificant section warping, such as overall buckling of slender beams with gross cross-section: when radii of gyration are small in comparison with section dimensions, warping is generally so small with respect to roto-translation that it can be neglected in a linear stability analysis. More precisely we can say that at the onset of instability two infinitely close equilibrium configurations **s** and **s**+d**s** differ only by the roto-translational part, warping remaining unchanged: the number of unknowns lowers to a few parameters (six in general) and a reduced stability analysis is established as already pointed out by Borri, Merlini (1986). Neglecting warping is of course an approximation that must be sounded out on the particular problem at hand, and yet a general formulation based on roto-translational displacement variations will be derived here and profitably compared with the broader one stated above.

A nonlinear setting out enters into the field of finite rotations to which great attention has been paid during the past two decades: among several and significant works, papers by Green, Naghdi (1970), Bathe, Bolourchi (1979), Tang, Yeung and Chon (1980), Argyris (1982), Maewal (1983), Shkutin (1985) and Hodges (1987A, B) are here quoted. Following a suggestion by Shkutin (1985), the nonlinear model in 1-D field of a straight slender beam is drawn here from a 3-D macro-polar model consistently with the displacement method: derivation runs plain mainly due to the identification of two finite strain vector 1-D fields via a well defined rotation-derivative operator, and the definition of two work-conjugate stress integral vector 1-D fields. This way lengthy and troublesome component expressions for strains are avoided, and concise vectorial equations are written which keep valid for rotations however large.

A reference axis parallel to coordinate x^3 is chosen in order to set up a Cosserat-type kinematic description: the position vector of a point is resolved into the pole position $\mathbf{x}(x^3)$ on the axis plus the point position $\mathbf{y}(x^1, x^2)$ on the section; the covariant base vectors will be $\mathbf{g}_\alpha = \mathbf{y}_{,\alpha}$ $(\alpha = 1,2)$ and the unit vector $\mathbf{g}_3 = \mathbf{x}_{,3}$. After displacement, vectors $\mathbf{x}$ and $\mathbf{y}$ become $\mathbf{x} + \mathbf{v}$ and $\Phi\mathbf{y}$ respectively, where $\mathbf{v}(x^3)$ is the pole translation and operator $\Phi(x^3)$ is the section rotation. As Φ represents a proper orthogonal transformation ($\Phi^T\Phi = 1$), it is defined by three independent parameters: among the several parametrization formulae proposed in literature (see Hughes (1986), Hestenes (1986), and Hodges (1987A)), the following one is chosen as the most appropriate for present purposes, as already suggested by Borri, Merlini (1986),

$$\Phi = \exp(\boldsymbol{\varphi}\times) = 1 + \sum_{n=1}^{\infty} \frac{(\boldsymbol{\varphi}\times)^n}{n!} \quad , \qquad (12)$$

$\boldsymbol{\varphi}(x^3)$ being the finite rotation vector (7). Displacement of a point is therefore given the form

$$\mathbf{s} = \mathbf{v} + (\Phi - 1)\mathbf{y} \; ,$$

that is a nonlinear expression of the section free unknowns $\mathbf{v}$ and $\boldsymbol{\varphi}$: thus non-null terms depending on the second variation $d\delta\mathbf{s}$ must be expected.

In order to characterize section strain, the derivative operator Φ' is solved for the derivative rotation vector $\boldsymbol{\varphi}'$. Derivation follows from the orthogonality property of Φ as shown by Hughes (1986) and yields

$$\Phi' = (\Psi\boldsymbol{\varphi}')\times\Phi \quad , \qquad \Psi = 1 + \sum_{n=1}^{\infty} \frac{(\boldsymbol{\varphi}\times)^n}{(n+1)!} \quad , \qquad (13)$$

where Ψ can be called the rotation-derivative operator (8). A multiplicative macro-polar decomposition is now easily introduced for the Jacobian operator (1) on the basis of section rotation Φ, leading to the expression

$$J = \Phi(1 + \varepsilon) \; , \qquad \varepsilon = \boldsymbol{\varepsilon}_3\, \mathbf{g}^{3\cdot} = (\boldsymbol{\chi} - \mathbf{y}\times\boldsymbol{\omega})\mathbf{g}^{3\cdot} \quad ,$$

where,

$$\boldsymbol{\chi} = \Phi^T(\mathbf{v}' - (\Phi - 1)\mathbf{g}_3), \qquad \boldsymbol{\omega} = \Phi^T\Psi\boldsymbol{\varphi}' . \qquad (14)$$

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(7) Rotation Φ is more correctly expressed, with the language of geometric algebra (Hestenes, 1986), as a function of a bivector, the rotation angle $\boldsymbol{\phi}$, dual of the rotation vector $\boldsymbol{\varphi}$ ($\boldsymbol{\phi} = i\,\boldsymbol{\varphi}$, i being the dextral unit pseudoscalar of the 3-D Euclidean space).

(8) Operators Φ and Ψ are usually known in a more appropriate form for computation, namely

$$\Phi = 1 + \varphi^{-1}(\sin\varphi)\boldsymbol{\varphi}\times + \varphi^{-2}(1 - \cos\varphi)\,\boldsymbol{\varphi}\times\boldsymbol{\varphi}\times$$

$$\Psi = 1 + \varphi^{-2}(1 - \cos\varphi)\boldsymbol{\varphi}\times + \varphi^{-2}(1 - \varphi^{-1}\sin\varphi)\,\boldsymbol{\varphi}\times\boldsymbol{\varphi}\times$$

where $\varphi = \sqrt{\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}}$. Series expansion of trigonometric functions and property $(\boldsymbol{\varphi}\times)^n = -\varphi^2(\boldsymbol{\varphi}\times)^{n-2}$ for $n>2$, give formulae (12), (13). The following properties are quoted:

$$\Phi = 1 + \Psi\boldsymbol{\varphi}\times = 1 + \boldsymbol{\varphi}\times\Psi \quad , \qquad \Psi = \Phi\Psi^T = \Psi^T\Phi \quad .$$

The unsymmetric operator ε will be called (local) stretch as it is fully responsible for strain, since from Eq. (2) it follows that ([9])

$$e = \frac{1}{2}\,(\varepsilon + \varepsilon^T + \varepsilon^T \varepsilon)\ ;$$

local stretch ε again depends directly on two vectors $\boldsymbol{\chi}$ and $\boldsymbol{\omega}$ unique for the whole section, namely the section strain vectors (whose components are shear, axial, flexural and torsional deformations), and Eqs. (14) provide a finite nonlinear strain-displacement vectorial relationship ([10]).

Although a linearized principle, written for a reference configuration with infinitesimal displacement, could now be worked out from Eq. (6) by proper truncation of operators Φ and Ψ treading in Borri, Merlini (1986) steps, a much more straightforward way will be followed here. For simplicity sake no external forces nor lateral constraints are considered; thereby the principle of virtual work Eq. (5) can be stated as

$$\frac{d}{dx^3}\int_A \delta s \cdot \Phi\,(1+\varepsilon)\boldsymbol{\sigma}^3 dA - \int_A \delta\boldsymbol{\varepsilon}_3 \cdot (1+\varepsilon)\boldsymbol{\sigma}^3 dA = 0\ ,$$

and, if Eq. (13) is used to differentiate operator Φ, it can be written

$$\frac{d}{dx^3}\left(\begin{Bmatrix}\delta \mathbf{v}\\ \Psi\delta\boldsymbol{\varphi}\end{Bmatrix}^T \cdot \begin{Bmatrix}\Phi\mathbf{t}\\ \Phi\mathbf{m}\end{Bmatrix}\right) - \begin{Bmatrix}\delta\boldsymbol{\chi}\\ \delta\boldsymbol{\omega}\end{Bmatrix}^T \cdot \begin{Bmatrix}\mathbf{t}\\ \mathbf{m}\end{Bmatrix} = 0 \qquad (15)$$

having introduced the stress integrals defined as

$$\mathbf{t} = \int_A (1+\varepsilon)\boldsymbol{\sigma}^3 dA\ ,$$
$$\mathbf{m} = \int_A \mathbf{y}\times(1+\varepsilon)\boldsymbol{\sigma}^3 dA\ . \qquad (16)$$

Vectors **t** and **m** work for virtual strain vectors $\delta\boldsymbol{\chi}$ and $\delta\boldsymbol{\omega}$ and so they are properly assumed as the stress resultant and moment on the section ([11]). It must be noted that the "true" actions on the section balancing external loads are the rotated vectors Φ**t** and Φ**m**, namely the integrals of the Piola-Lagrange stress vector $J\boldsymbol{\sigma}^3$ over the rotated section, while in analogy with micro-polar continua (Atluri, 1984) vectors **t** and **m** can be said to be the integrals of the Biot-Lure-type stress vector $(1+\varepsilon)\boldsymbol{\sigma}^3$. As strain vector variations can be easily brought to the form ([12])

$$d\boldsymbol{\chi} = \Phi^T(d\mathbf{v}' + (\mathbf{g}_3 + \mathbf{v}')\times\Psi d\boldsymbol{\varphi})\ ,$$
$$d\boldsymbol{\omega} = \Phi^T(\Psi d\boldsymbol{\varphi})',$$

Eq. (15) can also be written as

$$\begin{Bmatrix}\delta \mathbf{v}\\ \Psi\delta\boldsymbol{\varphi}\end{Bmatrix}^T \cdot \begin{Bmatrix}(\Phi\mathbf{t})'\\ (\Phi\mathbf{m})' + (\mathbf{g}_3+\mathbf{v}')\times\Phi\mathbf{t}\end{Bmatrix} = 0\,. \qquad (17)$$

Due to arbitrariness of virtual variation of unknowns **v** and $\boldsymbol{\varphi}$(operator Ψ is non-singular), Eq. (17) immediately gives the intrinsic balance equations for finite roto-translation in terms of section force and moment (in absence of external forces).

Linearization of Eq. (17) leads to

$$\begin{Bmatrix}\delta \mathbf{v}\\ \Psi\delta\boldsymbol{\varphi}\end{Bmatrix}^T \cdot$$

$$\begin{Bmatrix}(\Phi d\mathbf{t})' - \Phi\mathbf{t}\times d\boldsymbol{\omega}\\ (\Phi d\mathbf{m})' + (\mathbf{g}_3+\mathbf{v}')\times\Phi d\mathbf{t} - \Phi\mathbf{t}\times d\boldsymbol{\chi} - \Phi\mathbf{m}\times d\boldsymbol{\omega}\end{Bmatrix} =$$

$$= 0\ . \qquad (18)$$

As in the present case the elastic stress-strain relation (7) with (8) reads

$$d\boldsymbol{\sigma}^3 = L^3 D L^3 (1+\varepsilon^T) d\boldsymbol{\varepsilon}_3\ ,$$

definitions (16) give the following nonlinear symmetric differential stress-strain integral relation:

$$\begin{Bmatrix}d\mathbf{t}\\ d\mathbf{m}\end{Bmatrix} =$$

$$\int_A \begin{Bmatrix}1\\ \mathbf{y}\times\end{Bmatrix}\left((1+\varepsilon)L^{3^T} D L^3 (1+\varepsilon^T) + \sigma^{33}\right)\begin{Bmatrix}1\\ \mathbf{y}\times\end{Bmatrix}^T dA\ \begin{Bmatrix}d\boldsymbol{\chi}\\ d\boldsymbol{\omega}\end{Bmatrix}.$$

When displacement of the reference configuration is infinitesimal, Eq. (18) becomes

$$\{\delta r\}^T \cdot \left(([E+\Pi]\{d\eta\})' + (G[E+\Pi] - \Sigma)\{d\eta\}\right) = 0\ , \qquad (19)$$

where columns

$$\{dr\} = \begin{Bmatrix}d\mathbf{v}\\ d\boldsymbol{\varphi}\end{Bmatrix}$$

and

$$\{d\eta\} = \begin{Bmatrix}d\mathbf{v}' + \mathbf{g}_3\times d\boldsymbol{\varphi}\\ d\boldsymbol{\varphi}'\end{Bmatrix} = \{dr'\} - G^T\{dr\}$$

group the linearized variations of the unknown and strain vectors, and the array operators are defined by

$$E = \int_A \begin{Bmatrix}1\\ \mathbf{y}\times\end{Bmatrix} L^{3^T} D L^3 \begin{Bmatrix}1\\ \mathbf{y}\times\end{Bmatrix}^T dA\ , \qquad G = \begin{bmatrix}0 & 0\\ \mathbf{g}_3\times & 0\end{bmatrix},$$

$$\Pi = \int_A \begin{Bmatrix}1\\ \mathbf{y}\times\end{Bmatrix}\sigma^{33}\begin{Bmatrix}1\\ \mathbf{y}\times\end{Bmatrix}^T dA\ , \qquad \Sigma = \begin{bmatrix}0 & \mathbf{t}\times\\ \mathbf{t}\times & \mathbf{m}\times\end{bmatrix},$$

where now pre-stress vectors (16) are evaluated with operator 1+ε replaced by the unity.

In condition (19) the most important of the two pre-stress operators Π and Σ is by far the second one,

---

([9]) Transverse strain components $\varepsilon_{\alpha\beta}$ (α, β = 1, 2) are obviously null, consistently with a section roto-translation.

([10]) It can be seen that formulae (14) also derive from intrinsic definitions of generalized strains of the beam axis, *i.e.* relative change in tangent vector and curvature, as stated by Borri, Mantegazza (1985) for curved beams.

([11]) The moment too is more precisely a bivector, dual of vector **m**.

([12]) Note that $\boldsymbol{\omega}\times = \Phi^T\Phi'$.

which depends on integral vectors **t** and **m**. Operator Π depends on the distribution of axial stress σ^{33}, which is mostly smaller than material elastic moduli enclosed in D by several orders of magnitude, and therefore it vanishes when adding to elastic operator E ([13]). Indeed operator Π is found to be responsible for finite strains (as it derives from operator $1+\varepsilon$ in integrals (16)) and it can be neglected in a reduced stability analysis. Pre-stress operator Σ is skew-symmetric, but it corresponds to a symmetric geometric stiffness operator, as it is seen by differentiating Eq. (15). As regards to elastic operator E, which allows for anisotropy and unhomogeneity, it can be seen that it differs from the section stiffness found in linear beam theory, as proposed by Giavotto *et al.* (1983, 1986), because of the different displacement expansion assumed: note that if the correct stiffness is used for E in Eq. (19) a better approximation of the elastic behaviour may be expected, as warping variations are implicitly solved into section strain variations.

If operator Π is neglected, the ordinary differential equations

$$E\{d\eta'\} - (\mu\Sigma E^{-1}-G)E\{d\eta\} = \{0\} \qquad (20)$$

are derived from Eq. (19), having introduced the load parameter μ to quantify pre-stress. In the following, vectors **t** and **m**, and then operator Σ, are assumed to be constant ([14]). Obviously Eqs. (20) are satisfied by any rigid displacement $dv_0 + d\varphi_0 \times (x^3 g_3 + y)$, *i. e.* with parameters

$$\{dr\} = (1+x^3\, G^T)\{dr_0\} \;;$$

indeed this formula is a general solution of the second order differential set with $\{dr\}$ as unknowns deriving from Eqs. (20), and it allows lowering the derivative rank to the first one with $\{d\eta\}$ as unknowns. Besides, Eqs. (20) are satisfied if

$$\{d\eta\} = E^{-1}\,\exp((\mu\Sigma E^{-1}- G)x^3)\,E\;\{d\eta_0\}\;.$$

When pre-stress is null (μ=0) the general solution becomes

$$\{d\eta\} = (1 - x^3\,E^{-1}GE)\{d\eta_0\}\;,$$

and corresponds to the beam cubic central solutions (Giavotto, 1986)

$$\{dr\} = \Big(x^3 + \frac{1}{2}\,(x^3)^2 E^{-1}\,(EG^T - GE) - \frac{1}{6}\,(x^3)^3\,G^T E^{-1} GE\Big)\,\{d\eta_0\}\;.$$

([13]) Exceptions are offered by operators E defective of some principal stiffness, as for example in wires or in thin-walled beams with open sections.

([14]) Hence balance equations entail $\mathbf{t} = t\,^3\mathbf{g}_3$, viz. null shear forces.

As μ increases the general solution $\{d\eta\}$, and consequently the displacement $\{dr\}$, will be of exponential type whenever operator $\mu\Sigma E^{-1}$-G is non-singular, as it can generally be expected (eventually non-pure exponential if the above operator has multiple eigenvalues and non-independent eigenvectors). Thus roto-translational exponential solutions are the natural evolution of central solutions for varying load parameter, and the same instability criterion can be applied as discussed when dealing with Eqs. (11).

Now, recalling the above displacement resolution, it can be seen that solutions here derived are the roto-translational modes (warping has been neglected) among the whole eigensolution set allowed by the more general formulation Eqs. (11): thus in a sense they play here an analogous rôle as rigid motions do in flutter analysis. Therefore when seeking exponential solutions of the general problem Eqs. (11), a number of modes (six for laterally unconstrained structures) have to be expected being the natural evolution of central solutions. This distinction of possible instability modes according to their derivation is perhaps the right way to classify beam instabilities as being global or local, despite the fact that global modes can often show a significant generalized warping of the section. In fact the behaviour of eigenvalues p with respect to the load parameter μ is in general quite different for global and local modes, as shown by classical investigations and data available about buckling loads of thin-walled struts: a significant example is offered in a paper by Williams (1974).

5. EXAMPLES

The following examples deal with some of the most classical initial buckling problems: the aim is here to quickly obtain analytical solutions and discuss their trend with increasing load. Therefore isotropic material is considered and the simplest models are adopted to handle the actually significant unknowns.

The first example refers to the primary overall buckling problem and shows an application of the roto-translation model giving an insight into the evolution of central solutions. The other two examples refer to panel and shell instabilities and show some typical trends in local buckling. In order to be concise, the linear model adopted in these two examples is borrowed from the well-known shallow shell theory ([15]). For an isotropic

([15]) A nonlinear model based on a polar kinematic description of the shell and involving finite rotations has been developed in a similar way as the slender beam model: it gives the same linearized equations as the classical shallow shell theory and will be used in planned nonlinear investigations.

and homogeneous circular cylinder of radius r, thickness h, Young modulus E, Poisson ratio ν, referred to intrinsic rectangular Cartesian coordinates with a dextral base frame and vector $\mathbf{g}_1$ pointing outwards, the middle surface differential equations (see for instance Dikmen, 1982) give rise to the following linearized semi-variational hybrid principle for a prebuckling state in axial compression,

$$-\int_0^{2\pi r}\begin{Bmatrix}\delta w\\ \delta f\end{Bmatrix}^{T'} \begin{Bmatrix} B(dw_{,2222}+2dw_{,2233}+dw_{,3333})+\frac{1}{r}df_{,33}-t^{33}dw_{,33} \\ \frac{1}{r}dw_{,33}-\frac{1}{Eh}(df_{,2222}+2df_{,2233}+df_{,3333})\end{Bmatrix} dx^2 = 0 \ , \qquad (21)$$

where w is the displacement normal to the surface, f the Airy membrane-stress function, $t^{22}=f_{,33}$, $t^{33}=f_{,22}$ and $t^{23}=t^{32}=f_{,23}$ the specific membrane forces, and $B = Eh^3/12(1-\nu^2)$ the shell flexural rigidity.

5.1 The Euler rod

For a slender beam referred to a centroidal orthogonal frame and symmetric with respect to x^1x^3-plane, the following equations pertaining to in-plane stability under axial load are derived from Eq. (19),

$$\begin{bmatrix}GA_1 & 0\\ 0 & EJ_2\end{bmatrix}\begin{Bmatrix}d\chi_1'\\ d\omega_2'\end{Bmatrix} - \begin{bmatrix}0 & t^3\\ GA_1 - t^3 & 0\end{bmatrix}\begin{Bmatrix}d\chi_1\\ d\omega_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \qquad (22)$$

with strain components

$$\begin{Bmatrix}d\chi_1\\ d\omega_2\end{Bmatrix} = \begin{Bmatrix}dv_1' - d\varphi_2\\ d\varphi_2'\end{Bmatrix}$$

as unknowns. GA_1 and EJ_2 are the shear and bending rigidities of the section; terms due to operator Π have been neglected and consistently the axial force t^3 will be neglected in front of rigidity GA_1.

Eqs. (22) allow for a rigid displacement variation

$$\begin{Bmatrix}dv_1\\ d\varphi_2\end{Bmatrix} = \begin{Bmatrix}dv_{10} + x^3 d\varphi_{20}\\ d\varphi_{20}\end{Bmatrix} \ ,$$

and when $t^3=0$ they are satisfied by the linear solution

$$\begin{Bmatrix}d\chi_1\\ d\omega_2\end{Bmatrix} = \begin{Bmatrix}d\chi_{10}\\ d\omega_{20} - x^3(GA_1/EJ_2)d\chi_{10}\end{Bmatrix} .$$

Under axial load t^3, searching for a solution of type

$$\begin{Bmatrix}d\chi_1\\ d\omega_2\end{Bmatrix} = \begin{Bmatrix}d\chi_{10}\\ d\omega_{20}\end{Bmatrix} \exp(px^3)$$

leads to an eigenproblem which gives ([16])

$$p = \pm\sqrt{t^3/EJ_2} \quad , \qquad d\chi_{10}/d\omega_{20} = 0 \ .$$

Real and imaginary parts of eigenvalues p are plotted versus compressive load $-t^3$ in Fig. 3; while p is always real when in traction, for any compressive load eigenvalues are merely imaginary: this means that $t^3=0$ is a lower bound for the buckling load of any compressed rod.

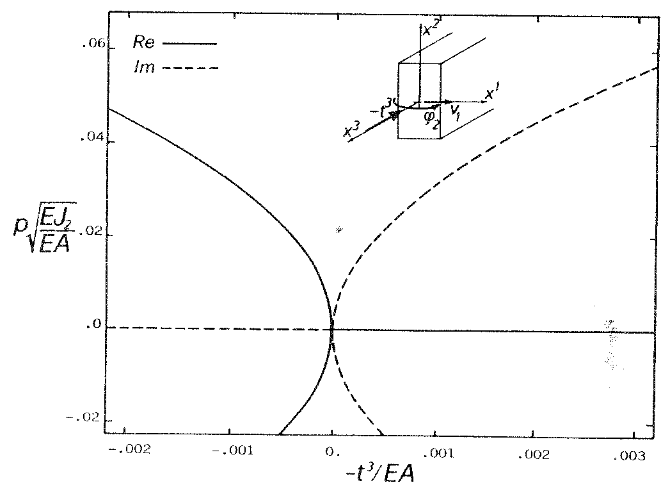


Fig. 3 - Eigenvalue-load plot for an axially compressed rod

The actual eigenvalue (hence the critical buckling load) depends on the boundary conditions, as undecaying modes for $t^3<0$ must develop along the beam according to end constraints. If the half-wave length λ is used instead of p ([17]),

$$p = \pm i\,\pi/\lambda \quad , \qquad \lambda = \sqrt{\pi^2 EJ_2/-t^3} \ ,$$

the general solution of Eqs. (22) can be written as

$$dv_1 = dv_{10} + x^3 d\varphi_{20} + dv_{1C}\cos(\pi x^3/\lambda) + dv_{1S}\sin(\pi x^3/\lambda) \ ,$$

$$d\varphi_2 = dv_1' \quad , \qquad d\chi_1 = 0 \quad , \qquad d\omega_2 = dv_1'' \ .$$

Boundary compatibility determines λ: for example for a simply supported beam at $x^3=0$ and $x^3=\ell$ it results

$$\lambda = \ell/n \quad , \qquad dv_1 = dv_{1S}\sin(n\pi x^3/\ell) \ ,$$

n being any positive integer value. Hence for such a compressed beam a series of instability modes is possible with different wave lengths and eigenvalues

([16]) Eigenvalues and eigenvectors have been approximated for $t^3/GA_1 \ll 1$ and $t^3/pGA_1 \ll 1$. The same result would be obtained if the shear deformation was neglected.

([17]) In the following, some caution is required to distinguish between component indexes and power exponents.

$p=\pm\, i n\pi/\ell$: the corresponding loads will be $-t^3 = EJ_2(n\pi/\ell)^2$, and the actual critical load is found for n=1. It is seen that plot of λ (*i. e.* ℓ for n=1) versus $-t^3$ results in the classical Euler buckling curve.

5.2 Simply-supported flat panel in axial compression

For a simply-supported thin plate of width b in uniform axial compression, the shell flexural behaviour becomes uncoupled from the inplane one and principle (21) can be written

$$-\int_0^b \delta w\,(B(dw_{,2222}+2dw_{,2233}+dw_{,3333})-t^{33}dw_{,33})dx^2=0\ .$$

The Fourier expansion

$$dw=\sum_{m=1}^{\infty} dw_m \sin(m\pi x^2/b)$$

satisfies the side constraints and yields the following uncoupled equations

$$(b/m\pi)^4 dw_m'''' + (\mu/m^2-2)(b/m\pi)^2 dw_m'' + dw_m = 0$$
$$(m=1,2,\dots\infty)\ ,$$

the load parameter μ being defined as for

$$t^{33}=-\mu\,\frac{\pi^2 Eh}{12(1-\nu^2)}\left(\frac{h}{b}\right)^2$$

Seeking exponential solutions

$$dw_m = dW_m \exp(m\pi p_m x^3/b) \qquad (m=1,2,\dots\infty)$$

gives the characteristic equations

$$p_m^4-2(1-\mu/2m^2)p_m^2+1=0 \qquad (m=1,2,\dots\infty)\ ,$$

showing that p_m is real for $\mu\leq 0$, complex for $0<\mu<4m^2$ and imaginary for $\mu\geq 4m^2$. As the interest is for the lowest critical load, the sole first eigensolution with half-wave shape on the cross-section will be studied and hereinafter m=1 will be understood. Plot of eigenvalues

$$p=\pm\sqrt{1-\mu/2\pm\sqrt{(1-\mu/2)^2-1}}$$

is given in Fig.4 versus the load parameter μ: when in traction eigensolutions decay and so do the extremity solutions (μ=0); under compressive stress eigenvalues become complex showing a wave-shaped decaying solution; $Re(p)$ decreases for increasing μ and vanishes for μ=4 (the lower bound of any buckling load) therefrom leaving undecaying eigensolutions.

By resorting to the half-wave length λ,

$$p=\pm ib/\lambda \quad , \quad \lambda=b\sqrt{\mu/2-1\pm\sqrt{(\mu/2-1)^2-1}}\ ,$$

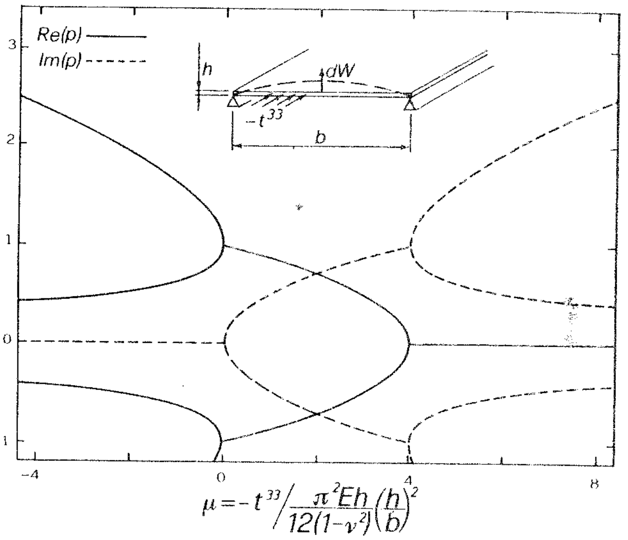


Fig. 4 - Eigenvalue-load plot for a simply-supported axially compressed panel.

it is noted that two different wave lengths are associated with any value of $\mu>4$, namely λ and b^2/λ, and conversely it results $\mu=(\lambda/b+b/\lambda)^2$. Of course the actually possible eigenvalue depends on the end constraints: for instance a simply supported panel of length ℓ will require $\lambda=\ell/n$, with n any positive integer, and the actual buckling load will be found for n such that $\mu=(nb/\ell+\ell/nb)^2$ is minimum. It is seen that plotting of $n\lambda/b$ (*i.e.* ℓ/b) versus μ for different values of n gives the well-known buckling curves of the flat panel.

Results obtained about the static modes of panel and Euler rod can be compared in Fig. 5, relative to a slender square tube analyzed with a rough uncoupled model. For column lengths $\ell>16b$ overall buckling occurs at $\mu=(32b/\ell)^2<4$, while the wall mode keeps decaying. For loads $\mu\geq 4$ the column will buckle locally as the steep eigenvalue slope can show: indeed within a narrow range of μ (4÷4.5), half-wave lengths between $b/\sqrt{2}$ and $b\sqrt{2}$ can freely develop in the tube walls and therefore local buckling will occur for any column length $\ell\geq b/\sqrt{2}$.

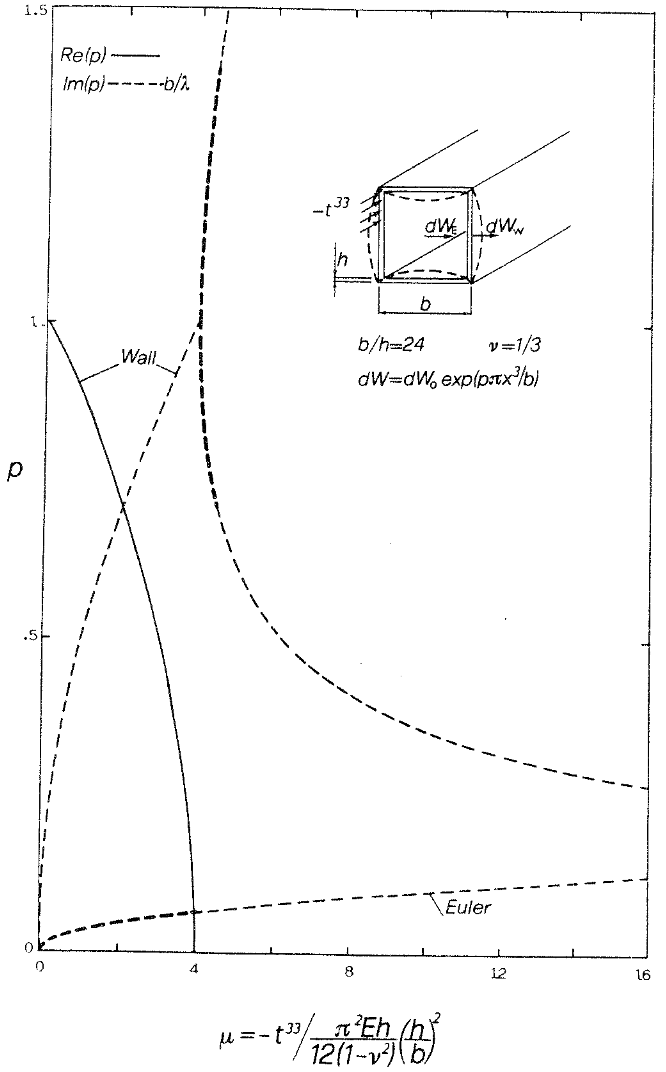


Fig. 5 - Eigenvalue-load plot for an axially compressed square tube.

5.3 Circular cylindrical shell in axial compression

Although the critical buckling load predicted via a linearized theory is not of practical interest for shallow shells of revolution, as real cylinders buckle well below that load (see for instance Bushnell, 1981), nevertheless the evolution of static modes while increasing compression can show some interesting features related to the actual shell behaviour.

Fourier expansion over the circumference of the unknowns in Eq. (21),

$$\begin{Bmatrix} dw \\ df \end{Bmatrix} = b^2 r \sum_{m=0}^{\infty} \left(\begin{Bmatrix} dw_{cm} \\ Ehb^2 r \cdot df_{cm} \end{Bmatrix} \cos(mx^2/r) + \begin{Bmatrix} dw_{sm} \\ Ehb^2 r \cdot df_{sm} \end{Bmatrix} \sin(mx^2/r) \right),$$

leads to a series of uncoupled problems governed by the ordinary differential equations

$$\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} (br)^4 \begin{Bmatrix} dw_{tm} \\ df_{tm} \end{Bmatrix}'''' + \begin{bmatrix} 2\mu-2n^2 & 1 \\ 1 & 2n^2 \end{bmatrix} (br)^2 \begin{Bmatrix} dw_{tm} \\ df_{tm} \end{Bmatrix}'' \tag{23}$$

$$+ \begin{bmatrix} n^4 & \\ & -n^4 \end{bmatrix} \begin{Bmatrix} dw_{tm} \\ df_{tm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (t=c,s\ ;\ m=0,1,2,\dots\infty)$$

Here b is the square root of the shell slenderness parameter, a number <<1 defined by

$$b^2 = \frac{h/r}{\sqrt{12(1-\nu^2)}} ,$$

$n=bm$ is a normalized circumferential frequency, and the load parameter μ is defined as for

$$t^{33} = -\mu \frac{Eh}{\sqrt{3(1-\nu^2)}} \left(\frac{h}{r}\right) = -\mu 2Ehb^2 .$$

Each equation set (23) allows exponential solutions like

$$\begin{Bmatrix} dw_{tm} \\ df_{tm} \end{Bmatrix} = \begin{Bmatrix} dW \\ dF \end{Bmatrix} \exp(px^3/br)$$

and yields the eigenproblem

$$\begin{bmatrix} \left(\dfrac{p^2-n^2}{p}\right)^2 + 2\mu & 1 \\ 1 & -\left(\dfrac{p^2-n^2}{p}\right)^2 \end{bmatrix} \begin{Bmatrix} dW \\ dF \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} ,$$

with characteristic equation

$$\left(\frac{p^2-n^2}{p}\right)^4 + 2\mu\left(\frac{p^2-n^2}{p}\right)^2 + 1 = 0 . \tag{24}$$

Roots of Eq. (24) give the eigenvalue-load plot shown in Fig. 6 for a few different circumferential frequencies n. The plot pattern evolves remarkably while increasing the wave number m: it is seen that the axisymmetric mode (m=0) and the chessboard modes with $n \leq .5$ become undecaying ($Re(p)$=0) for μ=1 (the exact linear multimode buckling load), while more wavy modes show a higher critical load. At μ=1, eigenvalues of modes with n=0 and n=.5 are respectively

$$p_{(0)} = \pm i \quad \text{and} \quad p_{(.5)} = \pm i/2 ,$$

and entail a chessboard shape with square buckles of size $\lambda=2\pi rb$ and an axisymmetric one with half-wave length $\pi rb=\lambda/2$. Combination of two modes gives the classical diamond pattern with equation

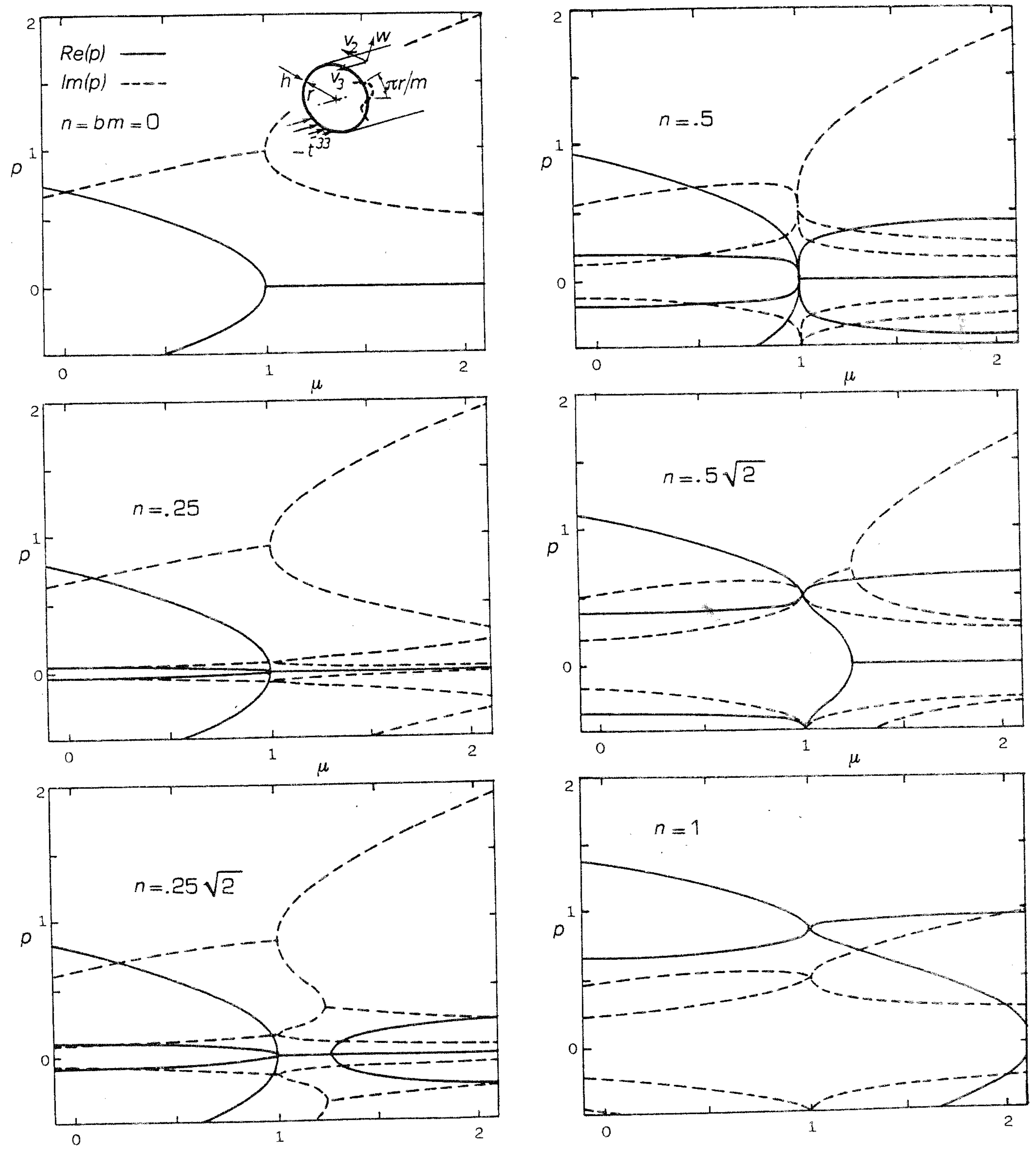
$$dw = b^2 r \left(dW_{(0)} \sin(2\pi x^3/\lambda) + dW_{(.5)} \sin(\pi x^2/\lambda) \sin(\pi x^3/\lambda)\right).$$

It is noted that two distinct modes exist with the same wave number. Although their eigenvectors

$$\begin{Bmatrix} dW \\ dF \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\mu \pm i\sqrt{1-\mu^2} \end{Bmatrix}$$

are conjugate for $\mu<1$, they are quite different in diffusion characteristics and therefore in membrane-like behaviour: in an expansion of specific stresses,

$$\begin{Bmatrix} dt^{22} \\ dt^{33} \\ dt^{23} \end{Bmatrix} = Ehb^2 \sum_{m=0}^{\infty} \left(\begin{Bmatrix} dt^{22}_{cm} \\ dt^{33}_{cm} \\ -dt^{23}_{sm} \end{Bmatrix} \cos(mx^2/r) + \begin{Bmatrix} dt^{22}_{sm} \\ dt^{33}_{sm} \\ dt^{23}_{cm} \end{Bmatrix} \sin(mx^2/r) \right)$$



$$\mu = -t^{33} \Big/ \frac{Eh}{\sqrt{3(1-\nu^2)}} \frac{h}{r}$$

Fig. 6 - Eigenvalue-load plot for an axially compressed cylindrical shell, for different circumferential frequencies *n*

the coefficient functions are of type

$$\begin{Bmatrix} dt^{22}_{tm} \\ dt^{33}_{tm} \\ dt^{23}_{tm} \end{Bmatrix} = \begin{Bmatrix} p^2 \\ -n^2 \\ np \end{Bmatrix} dF \exp(px^3/br) \; ,$$

showing that the amplitudes of the alternating stretch and shear along the circumference are different for the two modes. The relevant components of the tangent displacement are found to be ([18])

$$\begin{Bmatrix} dv_2 \\ dv_3 \end{Bmatrix} = b^3 r \sum_{m=0}^{\infty} \Big(\begin{Bmatrix} -dv_{2sm} \\ dv_{3cm} \end{Bmatrix} \cos(mx^2/r) +$$

$$+ \begin{Bmatrix} dv_{2cm} \\ dv_{3sm} \end{Bmatrix} \sin(mx^2/r) \; ,$$

with coefficients of type

$$\begin{Bmatrix} dv_{2tm} \\ dv_{3tm} \end{Bmatrix} = \begin{Bmatrix} n[(p^2-n^2)/p^2+(1+\nu)] \\ p[(p^2-n^2)/p^2-(1+\nu)] \end{Bmatrix} dF \exp(px^3/br) \; .$$

The eigenvalue analysis based on the instability criteria stated above just confirms the results of the classical linear theory (linear bifurcation). However, some interesting insight comes out from studying the eigenvalue moduli for $\mu<1$. As shown in Fig. 7, any wavy shape with $n<1$ has an eigenvalue modulus $|p|$ that decreases for increasing μ, crossing the unit value, while the axisymmetric shape has a constant eigenvalue modulus $|p|=1$ (the flat panel behaves the same way). By solving Eq. (24) for μ (a real value), the load parameter can be expressed as a function of the eigenvalue modulus as follows,

$$\mu = 1 - \frac{1}{2n^2} \Big(\frac{|p|^2-n^2}{|p|}\Big)^2 \Big[\Big(\frac{|p|^2+n^2}{|p|}\Big)^2 - 1\Big] \, . \qquad (25)$$

Plot of Eq. (25) versus $1/n^2$ is given in Fig. 8 for a narrow range of $|p|$ close to the unity. While for low values of $1/n^2$ the same modulus as the axisymmetric mode is attained at rather definite and relatively high loads, for higher $1/n^2$, moduli close to the unity occur on a wider and lower load range. A new instability criterion can perhaps be now suggested: indeed it can be thought that a decaying chessboard mode with an eigenvalue modulus lower than the axisymmetric mode is in a favourable position for an energy exchange between the modes themselves, leading to the onset of an undecaying compound mode (diamond shape with not necessarily square buckles, as experimental evidence shows). Moreover, the well known imperfection-sensitivity of shells in load-carrying capability (growing with the slenderness $1/b^2$) is in a sense accounted for by the loss of definiteness of the curve of Fig. 8 for increasing $1/n^2$. As a matter of fact, these considerations seem to agree encouragingly with experimental buckling results: a comparison with the data gathered by Kollár, Dulácska (1984) is given in Fig. 9, where Eq. (25) for $|p|$ close to the unity is plotted versus the slenderness $1/b^2$ for some different wave numbers m.

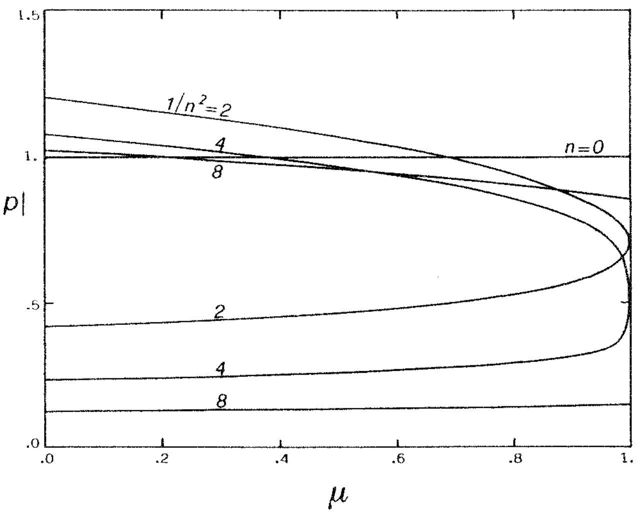


Fig. 7 - Eigenvalue modulus vs load prameter for an axially compressed cylindrical shell

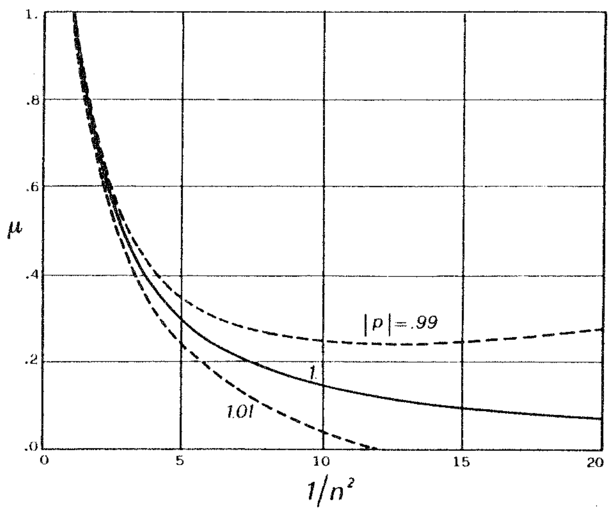


Fig. 8 - Load causing an eigenvalue modulus close to the unity vs the circumferential frequency

([18]) These formulae have been derived within a parallel formulation fully based on the displacement method, that matches the analysis of Gol'denveizer (1961) for $\mu = 0$.

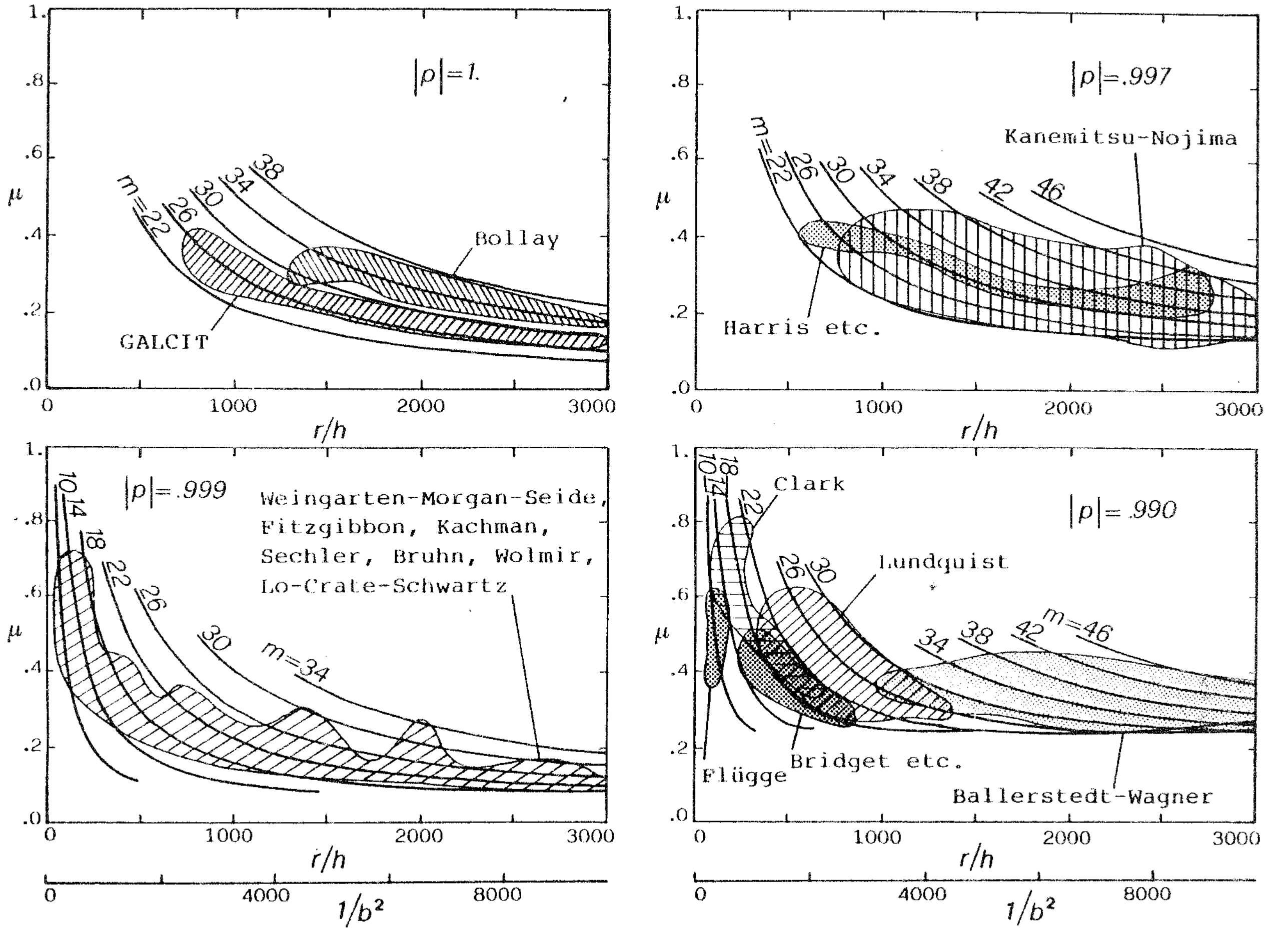


Fig. 9 - Load-slenderness plot for eigenvalue moduli close to the unity and comparison with experimental buckling results by several authors (ref. Kollár, Dulácska, 1984). Ratio *r/h* computed for v=1/3

6. CONCLUSIONS

To summarize, the subject of this paper is the study of the possible adjacent configurations (linear static modes) of the body of an axial structure, while varying a pre-stress parameter that controls the end load intensity: either the modes related to end load transfer (central solutions) or the decaying extremity solutions evolve and in general become the undecaying modes of linear buckling. The actual configuration will depend on the end conditions in hand. Therefore, in a sense the present analysis links two separate and well-known fields, and offers a new approach to the stability problem.

A possible application is a parametric analysis for computing local buckling critical loads of stringers and stiffened panels made of composite laminates. The finite element model is ready for implementation within a computing code able to give the extremity solutions of anisotropic cross-sections, like the one developed by Giavotto *et al.* (1983). The effectiveness of such a model relies on the possibility of a detailed representation of complex sections, no further unknowns being involved.

Moreover, the trend of eigenvalues and eigenvectors can give an insight even into the behaviour of imperfection-sensitive structures, as the last example shows. This subject is worthy of further investigation, including experimental work: buckling tests on circular cylinders with an imposed extremity mode are planned. At the same time an analytical search is starting on nonlinear modal interaction, possibly based on the decaying solutions. In this field appropriate structural models are needed, and it is believed that models based on a macro-polar kinematic description are the right ones for thin-walled axial structures and shallow shells, leading to quite simple nonlinear equations as it is shown in this paper for the slender beam. Indeed, when one or two dimensions are very shallow, the parameters of an average roto-translation across the structure shallowness can suitably be assumed as primary displacement unknowns: rotations must of course be carefully handled in order to correctly describe the geometric stiffening effect of pre-stress in wave-shaped modes.

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SOMMARIO

Sulla base di un approccio semivariazionale, si derivano le equazioni differenziali ordinarie linearizzate per l'analisi statica delle strutture assiali caricate agli estremi, impiegando sia un modello ad elementi finiti della sezione trasversale, sia un modello polare per travi snelle. Ne deriva un problema agli autovalori che viene affrontato considerando l'intensità del carico come un parametro: all'aumentare di questo, sia le soluzioni centrali che quelle di estremità evolvono verso modi che si propagano con legge sinusoidale lungo l'asse (modi di instabilità). Vengono discussi due semplici esempi di instabilità globale e locale, mentre dall'analisi parametrica di un cilindro sottile si rilevano alcuni aspetti significativi correlati ai risultati sperimentali.