Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory

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1. Introduction

Quantum Markov semigroups, i.e. semigroups of contractive, completely positive maps, are mathematical models for open quantum systems. In this paper we study a class of such semigroups on the C*-algebras of compact quantum groups (also called Woronowicz C*-algebras).

A compact quantum group $\mathbb{G}$ is a unital C*-algebra $C(\mathbb{G})$ equipped with additional structure, that generalizes the C*-algebra of continuous functions on a compact group (see Section 2 for the precise definition). In particular, any commutative compact quantum group is isomorphic to the C*-algebra of continuous functions on a compact group. The quantum group structure thus plays two roles: on one hand, positive functionals on $C(\mathbb{G})$ replace the states of a classical Markov process and, on the other hand, the actions of $C(\mathbb{G})$ on itself, allow us to formulate important symmetry properties for quantum Markov semigroups.

We show that quantum Markov semigroups on the (reduced or universal) C*-algebra of a compact quantum group that are translation invariant w.r.t. the coproduct are in one-to-one correspondence with Lévy processes on its *-Hopf algebra $\mathcal{A}$, see Theorems 3.2 and 3.4. This shows that the characterization of Lévy processes in topological groups as the Markov processes which are invariant under time and space translations extends to compact quantum groups. In particular, if the compact quantum group is commutative, the associated stochastic processes reduce to Lévy processes with values in a compact group, i.e., stochastic processes with stationary and independent increments.

In general, a KMS-symmetry property of a quantum Markov semigroup on a C*-algebra with respect to given KMS state on it, allows to study the semigroup on the scale of associated noncommutative L_p-spaces. On compact quantum groups

the natural state to refer to is the unique translation invariant state: it is called the Haar state because it reduces to the Haar measure of compact group when C($\mathbb{G}$) is commutative. In Section 4 we show that the quantum Markov semigroup is KMS-symmetric (with respect to the Haar state) if and only if the generating functional of its associated Lévy process is invariant under the unitary antipode, and that the quantum Markov semigroup satisfies the stronger condition of GNS-symmetry if and only if the generating functional of its associated Lévy process is invariant under the antipode.

In Section 5 we characterize the Schürmann triples of KMS-symmetric Lévy processes.

In the classical literature on Brownian motion or Lévy processes on (simple) Lie groups, the analysis of their invariance under the adjoint action of the group on itself has been particularly intense. We formulate this invariance property for compact quantum groups and show that it imposes a very strong restriction. In Section 6 we develop a method that allows to determine ad-invariant generating functionals on compact quantum groups of Kac type, i.e. when the Haar state is a trace. Using this method we find a complete classification of the ad-invariant Lévy processes on the free orthogonal quantum groups in Section 10.

In Section 7 we give a complete description of the Dirichlet form associated to KMS symmetric Lévy processes and, in the GNS symmetric case, a characterization of the associated quadratic forms on the Hopf algebra $\mathcal{A}$, arising in this way, in terms of their translation invariance.

In the framework of Alain Connes' noncommutative geometry [16], efforts have been directed towards the construction and investigation of Dirac operators and spectral triples on the Woronowicz quantum groups $SU_q(2)$ and related homogeneous noncommutative spaces (see for example [7,8,17–19]). The relevance of this point of view relies in the fact that a spectral triple allows to construct topological invariants as cyclic co-cycles in cyclic cohomology and local couplings with the K-theory of the C*-algebra (Connes–Chern character).

In Section 8 we construct a Hilbert bimodule derivation, giving rise to differential calculus on the compact quantum group C*-algebra C($\mathbb{G}$), which, in the GNS symmetric case, allows to represent the Dirichlet form as a generalized Dirichlet integral

$$\mathcal{E}[a] = \frac{1}{2}\|da\|^2.$$

Using the derivation, we then construct a Dirac operator D whose spectrum is explicitly determined by the spectrum of the Dirichlet form on the GNS Hilbert space of the Haar state. Later we show that the Dirac operator D is part of spectral triple with respect to which the elements of the Hopf algebra $\mathcal{A}$ are Lipschitz.

In the last three Sections 9, 10, 11, we discuss in detail examples of the above constructions on compact Lie groups, group C*-algebras of countable discrete groups, the free orthogonal quantum groups O_N^+ and the Woronowicz quantum groups $SU_q(2)$.

2. Preliminaries

Sesquilinear forms will be linear in the right entry. For an algebra $\mathcal{A}$, $\mathcal{A}'$ will denote the algebraic dual of $\mathcal{A}$, i.e. the space of all linear functionals from $\mathcal{A}$ to $\mathbb{C}$. For a C*-algebra A, by A' we will mean the dual space of all linear continuous functionals on A. The symbol $\otimes$ will denote the spatial tensor product of C*-algebras and $\odot$ the algebraic tensor product, see, e.g., [31] for tensor products and other facts about C*-algebras.

2.1. Compact quantum groups

The notion of compact quantum groups has been introduced in [46]. Here we adopt the definition from [48] (Definition 1.1 of that paper).

Definition 2.1. A C*-*bialgebra* (a compact quantum semigroup) is a pair (A, Δ), where A is a unital C*-algebra, $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital, $*$-homomorphic map which is coassociative, i.e.

$$(\Delta \otimes \mathrm{id}_\mathsf{A}) \circ \Delta = (\mathrm{id}_\mathsf{A} \otimes \Delta) \circ \Delta.$$

If the quantum cancellation properties

$$\overline{\mathrm{Lin}}\big((1 \otimes \mathsf{A})\Delta(\mathsf{A})\big) = \overline{\mathrm{Lin}}\big((\mathsf{A} \otimes 1)\Delta(\mathsf{A})\big) = \mathsf{A} \otimes \mathsf{A}$$

are satisfied, then the pair (A, Δ) is called a *compact quantum group* (CQG).

If the algebra A of a compact quantum group is commutative, then A is isomorphic to the algebra $C(G)$ of continuous functions on a compact group G. To emphasis that for an arbitrary (i.e. not necessarily non-commutative) compact quantum group (A, Δ) the algebra A replaces the algebra of continuous functions on an (abstract) quantum analog of a group, the notation $\mathbb{G} = (\mathsf{A}, \Delta)$ and $\mathsf{A} = C(\mathbb{G})$ is also frequently used.

The map Δ is called the *coproduct* of A and it induces the convolution product of functionals

$$\lambda \star \mu := (\lambda \otimes \mu) \circ \Delta, \quad \lambda, \mu \in \mathsf{A}'.$$

The following fact is of fundamental importance, cf. [48, Theorem 2.3].

Proposition 2.2. *Let A be a compact quantum group. There exists a unique state $h \in \mathsf{A}'$ (called the* Haar state *of A) such that for all $a \in \mathsf{A}$*

$$(h \otimes \mathrm{id}_\mathsf{A}) \circ \Delta(a) = h(a)1 = (\mathrm{id}_\mathsf{A} \otimes h) \circ \Delta(a).$$

The left (resp. right) part of the equation above is usually referred to as left- (resp. right-) invariance property of the Haar state. In general, the Haar state of a compact quantum group need not be faithful or tracial.

2.2. Corepresentations

An element $u = (u_{jk})_{1 \leqslant j,k \leqslant n} \in M_n(\mathsf{A})$ is called an *n-dimensional corepresentation of* $\mathbb{G} = (\mathsf{A}, \Delta)$ if for all $j, k = 1, \ldots, n$ we have $\Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk}$. All corepresentations considered in this paper are supposed to be finite-dimensional. A corepresentation u is said to be *non-degenerate*, if u is invertible, *unitary*, if u is unitary, and *irreducible*, if the only matrices $T \in M_n(\mathbb{C})$ with $Tu = uT$ are multiples of the identity matrix. Two corepresentations $u, v \in M_n(\mathsf{A})$ are called *equivalent*, if there exists an invertible matrix $U \in M_n(\mathbb{C})$ such that $Uu = vU$.

An important feature of compact quantum groups is the existence of the dense $*$-subalgebra $\mathcal{A}$ (the algebra of the *polynomials* of A), which is in fact a Hopf $*$-algebra – so for example $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$. With the notation $\mathbb{G} = (\mathsf{A}, \Delta)$, the $*$-algebra $\mathcal{A}$ is often denoted in the literature as $\mathrm{Pol}(\mathbb{G})$.

Fix a complete family $(u^{(s)})_{s \in \mathcal{I}}$ of mutually inequivalent irreducible unitary corepresentations of A, then $\{u_{k\ell}^{(s)}; \ s \in \mathcal{I}, \ 1 \leqslant k, \ell \leqslant n_s\}$ (where n_s denotes the dimension of $u^{(s)}$) is a linear basis of $\mathcal{A}$, cf. [48, Proposition 5.1]. We shall reserve the index $s = 0$ for the trivial corepresentation $u^{(0)} = \mathbf{1}$. The Hopf algebra structure on $\mathcal{A}$ is defined by

$$\varepsilon\big(u_{jk}^{(s)}\big) = \delta_{jk}, \qquad S\big(u_{jk}^{(s)}\big) = \big(u_{kj}^{(s)}\big)^* \quad \text{for } s \in \mathcal{I}, \ j, k = 1, \ldots, n_s,$$

where $\varepsilon : \mathcal{A} \to \mathbb{C}$ is the counit and $S : \mathcal{A} \to \mathcal{A}$ is the antipode. They satisfy

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta, \tag{2.1}$$

$$m_{\mathcal{A}} \circ (\mathrm{id} \otimes S) \circ \Delta = \varepsilon(a)\mathbf{1} = m_{\mathcal{A}} \circ (S \otimes \mathrm{id}) \circ \Delta, \tag{2.2}$$

$$\big(S\big(a^*\big)^*\big) = a \tag{2.3}$$

for all $a \in \mathcal{A}$. Let us also remind that the Haar state is always faithful on $\mathcal{A}$.

Set $V_s = \mathrm{span}\{u_{jk}^{(s)}; \ 1 \leqslant j, k \leqslant n_s\}$ for $s \in \mathcal{I}$. By [48, Proposition 5.2], there exists an irreducible unitary corepresentation $u^{(s^c)}$, called the *contragredient* representation of $u^{(s)}$, such that $V_s^* = V_{s^c}$. Clearly $(s^c)^c = s$.

We shall frequently use *Sweedler notation* for the coproduct of an element $a \in \mathcal{A}$, i.e. omit the summation and the index in the formula $\Delta(a) = \sum_i a_{(1),i} \otimes a_{(2),i}$ and write simply $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

2.3. The dual discrete quantum group

To every compact quantum group $\mathbb{G} = (\mathsf{A}, \Delta)$ there exists a dual discrete quantum group $\hat{\mathbb{G}}$, cf. [32]. For our purposes it will be most convenient to introduce $\hat{\mathbb{G}}$ in the

setting of Van Daele's algebraic quantum groups, cf. [41–43]. However, the reader should be aware that we adopt a slightly different convention for the Fourier transform.

A pair (A, Δ), consisting of a $*$-algebra A (with or without identity) and a coassociative comultiplication $\Delta : A \to M(A \odot A)$, is called an *algebraic quantum group* if the product is non-degenerate (i.e. $ab = 0$ for all a implies $b = 0$), if the two operators $T_1 : A \odot A \ni a \otimes b \mapsto \Delta(a)(b \otimes 1) \in A \odot A$ and $T_2 : A \odot A \ni a \otimes b \mapsto \Delta(a)(1 \otimes b) \in A \odot A$ are well-defined bijections and if there exists a nonzero left-invariant positive functional on A. Here, $M(B)$ denotes the set of multipliers on B. We refer the reader to [41] for proofs and further details and will just recall a few facts here that we shall need later.

If (A, Δ) is a compact quantum group then $(\mathcal{A}, \Delta|_\mathcal{A})$ is an algebraic quantum group ("of compact type") and the Haar state is a faithful left- and right-invariant functional.

For $a \in \mathcal{A}$ we can define $h_a \in \mathcal{A}'$ by the formula

$$h_a(b) = h(ab) \quad \text{for } b \in \mathcal{A},$$

where h is the Haar state, and we denote by $\hat{\mathcal{A}}$ the space of linear functionals on $\mathcal{A}$ of the form h_a for $a \in \mathcal{A}$.

The set $\hat{\mathcal{A}}$ becomes an associative $*$-algebra with the convolution of functionals as the multiplication: $\lambda \star \mu = (\lambda \otimes \mu) \circ \Delta$, and the involution $\lambda^*(x) = \overline{\lambda(S(x)^*)}$ ($\lambda, \mu \in \hat{\mathcal{A}}$). Note that $\hat{\mathcal{A}}$ is closed under the convolution by [41, Proposition 4.2]. The Hopf structure is given as follows: the coproduct $\hat{\Delta}$ is the dual of the product on $\mathcal{A}$, the antipode $\hat{S}$ is the dual to S and the counit $\hat{\varepsilon}$ is the evaluation in $\mathbf{1}$. In particular, we have $\hat{S}(\lambda)(x) = \lambda(Sx)$ for $\lambda \in \hat{\mathcal{A}}$, $x \in \mathcal{A}$ and if $\hat{\Delta}(\lambda) \in \hat{\mathcal{A}} \odot \hat{\mathcal{A}}$ then

$$\hat{\Delta}(\lambda)(x \otimes y) = \lambda_{(1)}(x) \otimes \lambda_{(2)}(y) = \lambda(xy), \quad x, y \in \mathcal{A}.$$

The pair $\hat{\mathbb{G}} = (\hat{\mathcal{A}}, \hat{\Delta})$ is an algebraic quantum group, called the *dual* of $\mathbb{G}$.

The linear map which associates to $a \in \mathcal{A}$ the functional $h_a \in \hat{\mathcal{A}}$ is called the *Fourier transform*. Let us note that, due to the faithfulness of the Haar state h, $\hat{\mathcal{A}}$ separates the points of $\mathcal{A}$.

2.4. Woronowicz characters and modular automorphism group

A nice introduction to this part can be found in [46,48,25] or [39].

For $a \in \mathsf{A}$, $\lambda \in \mathsf{A}'$ we define

$$\lambda \star a = (\mathrm{id} \otimes \lambda)\Delta(a),$$
$$a \star \lambda = (\lambda \otimes \mathrm{id})\Delta(a).$$

If $a \in \mathcal{A}$ and $\lambda \in \mathcal{A}'$, then $\lambda \star a, a \star \lambda \in \mathcal{A}$.

For a compact quantum group A with dense $*$-Hopf algebra $\mathcal{A}$, there exists a unique family $(f_z)_{z \in \mathbb{C}}$ of linear multiplicative functionals on $\mathcal{A}$, called *Woronowicz characters* (cf. [48, Theorem 1.4]), such that

(1) $f_z(\mathbf{1}) = 1$ for $z \in \mathbb{C}$,

(2) the mapping $\mathbb{C} \ni z \mapsto f_z(a) \in \mathbb{C}$ is an entire holomorphic function for all $a \in \mathcal{A}$,

(3) $f_0 = \varepsilon$ and $f_{z_1} \star f_{z_2} = f_{z_1+z_2}$ for any $z_1, z_2 \in \mathbb{C}$,

(4) $f_z(S(a)) = f_{-z}(a)$ and $f_{\bar{z}}(a^*) = \overline{f_{-z}(a)}$ for any $z \in \mathbb{C}$, $a \in \mathcal{A}$,

(5) $S^2(a) = f_{-1} \star a \star f_1$ for $a \in \mathcal{A}$,

(6) the Haar state h satisfies:

$$h(ab) = h\big(b(f_1 \star a \star f_1)\big), \quad a, b \in \mathcal{A}.$$

The formulas

$$\rho_{z,z'}(a) = f_z \star a \star f_{z'}, \qquad \sigma_z = \rho_{iz,iz} \quad \text{and} \quad \tau_z = \rho_{iz,-iz} \tag{2.4}$$

define automorphisms of $\mathcal{A}$, in terms of which $\sigma_t = \rho_{it,it}$ and $\tau_t = \rho_{it,-it}$, $t \in \mathbb{R}$, define one parameter groups of automorphisms of $\mathcal{A}$. The former is known as *modular automorphism group*. Moreover, h is the $(\sigma, -1)$-KMS state, which means that it satisfies

$$h(ab) = h\big(b\sigma_{-i}(a)\big), \quad a, b \in \mathcal{A}, \tag{2.5}$$

cf. [6, Definition 5.3.1] or [31, Section 8.12]. For $z, z' \in \mathbb{C}$, $h(\rho_{z,z'}(a)) = h(a)$, so

$$h\big(\sigma_z(a)\big) = h\big(\tau_z(a)\big) = h(a), \quad a \in \mathcal{A}. \tag{2.6}$$

The matrix elements of the irreducible unitary corepresentations satisfy the famous generalized Peter–Weyl orthogonality relations

$$h\big((u_{ij}^{(s)})^* u_{k\ell}^{(t)}\big) = \frac{\delta_{st}\delta_{j\ell}f_{-1}(u_{ki}^{(s)})}{D_s}, \qquad h\big(u_{ij}^{(s)}(u_{k\ell}^{(t)})^*\big) = \frac{\delta_{st}\delta_{ik}f_1(u_{\ell j}^{(s)})}{D_s}, \tag{2.7}$$

where $f_1 : \mathcal{A} \to \mathbb{C}$ is the Woronowicz character and

$$D_s = \sum_{\ell=1}^{n_s} f_1\big(u_{\ell\ell}^{(s)}\big)$$

is the quantum dimension of $u^{(s)}$, cf. [46, Theorem 5.7.4]. Note that unitarity implies that the matrix

$$\big(f_1\big((u_{jk}^{(s)})^*\big)\big)_{1 \leqslant j,k \leqslant n_s} \in M_{n_s}(\mathbb{C})$$

is invertible, with inverse $(f_1(u_{jk}^{(s)}))_{jk} \in M_{n_s}(\mathbb{C})$, cf. [46, Eq. (5.24)].

Remark 2.3. The Haar state on a compact quantum group is a trace if and only if the antipode is involutive, i.e. we have $S^2(a) = a$ for all $a \in \mathcal{A}$. In this case we say that

(A, Δ) is of Kac type. This is also equivalent to the following equivalent conditions, cf. [48, Theorem 1.5],

(1) $f_z = \varepsilon$ for all $z \in \mathbb{C}$,
(2) $\sigma_t = \mathrm{id}$ for all $t \in \mathbb{R}$.

The antipode S and the automorphism $\tau_{\frac{i}{2}}$ are closable operators on A, and the closure $\overline{S}$ admits the polar decomposition

$$\overline{S} = R \circ T, \qquad (2.8)$$

where T is the closure of $\tau_{\frac{i}{2}}$ and $R : A \to A$ is a linear antimultiplicative norm preserving involution that commutes with hermitian conjugation and with the semigroup $(\tau_t)_{t \in \mathbb{R}}$, i.e. $\tau_t \circ R = R \circ \tau_t$ for all $t \in \mathbb{R}$, see [48, Theorem 1.6]. The operator R is called the *unitary antipode* and is related to Woronowicz characters through the formula

$$R(a) = S(f_{\frac{1}{2}} \star a \star f_{-\frac{1}{2}}) \quad \text{for } a \in \mathcal{A}. \qquad (2.9)$$

2.5. Lévy processes on involutive bialgebras

We recall the definition of Lévy processes on $*$-bialgebras, cf. [36]. An introduction to this topic can also be found in [20]. Lindsay and Skalski have developed an analytic theory of Lévy processes on C*-bialgebras, see [29] and the references therein.

Definition 2.4. A family of unital $*$-homomorphisms $(j_{st})_{0 \leqslant s \leqslant t}$ defined on a $*$-bialgebra $\mathcal{A}$ with values in a unital $*$-algebra $\mathcal{B}$ with some fixed state $\Phi : \mathcal{B} \to \mathbb{C}$ is called a *Lévy process on $\mathcal{A}$* (w.r.t. Φ), if the following conditions are satisfied:

(i) the images corresponding to disjoint time intervals commute, i.e. $[j_{st}(\mathcal{A}), j_{s't'}(\mathcal{A})] = \{0\}$ for $0 \leqslant s \leqslant t \leqslant s' \leqslant t'$, and expectations corresponding to disjoint time intervals factorize, i.e.

$$\Phi\big(j_{s_1 t_1}(a_1) \cdots j_{s_n t_n}(a_n)\big) = \Phi\big(j_{s_1 t_1}(a_1)\big) \cdots \Phi\big(j_{s_n t_n}(a_n)\big),$$

for all $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathcal{A}$ and $0 \leqslant s_1 \leqslant t_1 \leqslant \cdots \leqslant t_n$;
(ii) $m_{\mathcal{B}} \circ (j_{st} \otimes j_{tu}) \circ \Delta = j_{su}$ for all $0 \leqslant s \leqslant t \leqslant u$, where $m_{\mathcal{B}}$ denotes the multiplication of $\mathcal{B}$ and Δ is the comultiplication on $\mathcal{A}$;
(iii) the functionals $\varphi_{st} = \Phi \circ j_{st} : \mathcal{A} \to \mathbb{C}$ depend only on $t - s$;
(iv) $\lim_{t \searrow s} j_{st}(a) = j_{ss}(a) = \varepsilon(a) 1_{\mathcal{B}}$ for all $a \in \mathcal{A}$, where $1_{\mathcal{B}}$ denotes the unit of $\mathcal{B}$.

We do not distinguish two Lévy processes on the same $*$-bialgebra $\mathcal{A}$ which are *equivalent*. By this we mean that two processes $(j_{st})_{0 \leqslant s \leqslant t}$ and $(k_{st})_{0 \leqslant s \leqslant t}$ with values in unital $*$-algebras $(\mathcal{B}, \Phi)$ and $(\mathcal{B}', \Phi')$, respectively, agree on all their finite joint moments, i.e.

$$\Phi\big(j_{s_1 t_1}(b_1) \cdots j_{s_n t_n}(b_n)\big) = \Phi'\big(k_{s_1 t_1}(b_1) \cdots k_{s_n t_n}(b_n)\big),$$

for all $n \in \mathbb{N}$, $s_1 \leqslant t_1, \ldots, s_n \leqslant t_n$ and $b_1, \ldots, b_n \in \mathcal{A}$.

If $(j_{st})_{0 \leqslant s \leqslant t}$ is a Lévy process, then the functionals $\varphi_t := \varphi_{0,t} = \varphi_{s,t+s}$ $(t \geqslant 0)$ form a convolution semigroup of states, i.e.

- $\varphi_0 = \varepsilon$, $\varphi_s \star \varphi_t = \varphi_{s+t}$, $\lim_{t \to 0} \varphi_t(a) = \varepsilon(a)$ for all $a \in \mathcal{A}$,
- $\varphi_t(\mathbf{1}) = 1$, $\varphi_t(a^* a) \geqslant 0$ for all $a \in \mathcal{A}$ and $t \geqslant 0$.

For such a semigroup there exists a linear functional ϕ which is hermitian (i.e. $\phi(a^*) = \overline{\phi(a)}$ for $a \in \mathcal{A}$), conditionally positive ($\phi(a^* a) \geqslant 0$ when $a \in \ker \varepsilon$), vanishes on $\mathbf{1}$, and is such that

$$\varphi_t = \exp_\star t\phi = \varepsilon + t\phi + \frac{t^2}{2} \phi \star \phi + \cdots + \frac{t^n}{n!} \phi^{\star n} + \cdots. \tag{2.10}$$

Conversely, by the Schoenberg correspondence (cf. [20]), for every hermitian conditionally positive linear functional $\phi : \mathcal{A} \to \mathbb{C}$ with $\phi(\mathbf{1}) = 0$ there exists a unique convolution semigroup of states $(\varphi_t)_{t \geqslant 0}$ which satisfies (2.10) and a unique (up to equivalence) Lévy process $(j_{st})_{0 \leqslant s \leqslant t}$. The functional ϕ will be called the *generating functional* of the Lévy process $(j_{st})_{0 \leqslant s \leqslant t}$.

Given the convolution semigroup of states $(\varphi_t)_{t \geqslant 0}$, we can also define the semigroup of operators on $\mathcal{A}$ (called the *Markov semigroup on $\mathcal{A}$* associated to $(j_{st})_{0 \leqslant s \leqslant t}$)

$$T_t = (\mathrm{id} \otimes \varphi_t) \circ \Delta, \quad t \geqslant 0.$$

The infinitesimal generator of this semigroup is an operator $L : \mathcal{A} \to \mathcal{A}$, which is related to ϕ by the relations

$$L(a) = (\mathrm{id} \otimes \phi) \circ \Delta(a) = \phi \star a \quad \text{and} \quad \phi(a) = \varepsilon \circ L(a).$$

In this case we write $L = L_\phi$. As usual the formula to recover the semigroup from the generator is $T_t = \exp(tL)$ for $t \geqslant 0$. The fundamental theorem of coalgebra ensures that all this makes sense in the bialgebra $\mathcal{A}$.

Let $\mathcal{L}(\mathcal{A})$ denote the algebra of linear operators from $\mathcal{A}$ to $\mathcal{A}$. Operators $L \in \mathcal{L}(\mathcal{A})$ of the form $L = L_\phi = (\mathrm{id} \otimes \phi) \circ \Delta$ for some linear functional $\phi \in \mathcal{A}'$ will play an important role in the paper and we will refer to them as *convolution operators.*

An operator $L \in \mathcal{L}(\mathcal{A})$ is a convolution operator if and only if it is *translation invariant* on $\mathcal{A}$, i.e.

$$\Delta \circ L = (\mathrm{id} \otimes L) \circ \Delta,$$

and, if this is the case, the linear functional ϕ can be recovered from L using the formula

$$\phi = \varepsilon \circ L.$$

The map $\mathcal{A}' \ni \phi \to L_\phi \in \mathcal{L}(\mathcal{A})$ is also called *the dual right representation*. It is a unital algebra homomorphism for the convolution product, i.e. we have

$$L_\varepsilon = \mathrm{id},$$

$$L_\phi \circ L_\psi = L_{\phi \star \psi},$$

for $\phi, \psi \in \mathcal{A}'$. Moreover, L_ϕ is hermitian, i.e.

$$L_\phi\big(a^*\big) = (L_\phi a)^* \quad \text{for } a \in \mathcal{A}$$

iff ϕ is hermitian, i.e. $\phi(a^*) = \overline{\phi(a)}$, $a \in \mathcal{A}$.

3. Translation invariant Markov semigroups

Our goal is to construct Markov semigroups on compact quantum groups that reflect the structure of the quantum group. In this section we show that it is exactly the *translation invariant* Markovian semigroups that can be obtained from Lévy processes on the algebra of smooth functions $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$ of the quantum group $\mathbb{G} = (\mathsf{A}, \Delta)$.

For this purpose we first prove that the Markov semigroup $(T_t)_{t \geqslant 0}$ of a Lévy process on $\mathcal{A}$ has a unique extension to a strongly continuous Markov semigroup on both its reduced and its universal C*-algebra. We then show that the characterization of Lévy processes in topological groups as the Markov processes which are invariant under time and space translations extends to compact quantum groups.

Definition 3.1. A strongly continuous semigroup of operators $(T_t)_{t \geqslant 0}$ on a C*-algebra A is called a *quantum Markov semigroup on* A if every T_t is a unital, completely positive contraction.

If $(j_{st})_{0 \leqslant s \leqslant t}$ is a Lévy process on a $*$-bialgebra $\mathcal{A}$ with the convolution semigroup of states $(\varphi_t)_{t \geqslant 0}$ on $\mathcal{A}$ and the Markov semigroup $(T_t)_{t \geqslant 0}$ on $\mathcal{A}$, then, by a result of Bédos, Murphy and Tuset [3, Theorem 3.3], each φ_t extends to a continuous functional on A_u, the universal C*-algebra generated by $\mathcal{A}$. Then the formula $T_t = (\mathrm{id} \otimes \varphi_t) \circ \Delta$ makes sense on A_u (where $\Delta : \mathsf{A}_u \to \mathsf{A}_u \otimes \mathsf{A}_u$ denotes the unique unital $*$-homomorphism that extends $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$) and one easily shows (in the same way as in proposition below) that $(T_t)_t$ becomes a Markov semigroup on A_u (in the sense of Definition 3.1).

For us, however, it will be more natural to consider the reduced C*-algebra generated by $\mathcal{A}$. This is the C*-algebra A_r obtained by taking the norm closure of the GNS representation of $\mathcal{A}$ with respect to the Haar state h. The Haar state h is by construction faithful on A_r. The coproduct on $\mathcal{A}$ extends to a unique unital $*$-homomorphism $\Delta : \mathsf{A}_r \to \mathsf{A}_r \otimes \mathsf{A}_r$ which makes the pair (A_r, Δ) a compact quantum group. The following result shows that, even though $\varphi_t : \mathcal{A} \to \mathbb{C}$ can be unbounded with respect to the reduced C*-norm and therefore may not extend to A_r, $(T_t)_{t \geqslant 0}$ always extends to a Markov semigroup on A_r.

Michael Brannan showed that states on any C*-algebraic version $C(\mathbb{G})$ of $\mathbb{G}$ define a continuous convolution operator on the reduced version $C_r(\mathbb{G})$, cf. [5, Lemma 3.4]. We will need a similar result for convolution semigroups of states on $\mathrm{Pol}(\mathbb{G})$.

Theorem 3.2. *Each Lévy process $(j_{st})_{0 \leqslant s \leqslant t}$ on the Hopf $*$-algebra $\mathcal{A}$ gives rise to a unique strongly continuous Markov semigroup $(T_t)_{t \geqslant 0}$ on A_r, the reduced C*-algebra generated by $\mathcal{A}$.*

Proof. Let $(\lambda, \mathcal{H}, \xi)$ be the GNS representation of $\mathcal{A}$ for the Haar state h, thus $h(a) = \langle \xi, \lambda(a)\xi \rangle$ for $a \in \mathcal{A}$. We denote by $\|.\|_r$ the norm in A_r, that is $\|a\|_r = \|\lambda(a)\|$, where $\|.\|$ denotes the operator norm.

Similarly, let $(\rho_t, \mathcal{H}_t, \xi_t)$ be the GNS representation of $\mathcal{A}$ for the state $\varphi_t = \Phi \circ j_{0t}$, so that $\varphi_t(a) = \langle \xi_t, \rho_t(a)\xi_t \rangle$ for $a \in \mathcal{A}$.

We define the operators

$$
i_t : \mathcal{H} \ni v \to v \otimes \xi_t \in \mathcal{H} \otimes \mathcal{H}_t,
$$
$$
\pi_t : \mathcal{H} \otimes \mathcal{H}_t \ni v \otimes w \to \langle \xi_t, w \rangle_{\mathcal{H}_t} v \in \mathcal{H},
$$
$$
E_t : B(\mathcal{H} \otimes \mathcal{H}_t) \ni X \to \pi_t \circ X \circ i_t \in B(\mathcal{H}).
$$

Since for each t, i_t is an isometry and π_t is contractive, E_t is contractive too: $\|E_t(X)\| = \|\pi_t \circ X \circ i_t\| \leqslant \|X\|$.

Next we define

$$
U : \lambda(\mathcal{A})\xi \otimes \rho_t(\mathcal{A})\xi_t \ni \lambda(a)\xi \otimes \rho_t(b)\xi_t \mapsto \lambda(a_{(1)})\xi \otimes \rho_t(a_{(2)}b)\xi_t \in \mathcal{H} \otimes \mathcal{H}_t
$$

and we check that it is an isometry with adjoint given by

$$
U^*\big(\lambda(a)\xi \otimes \rho_t(b)\xi_t\big) = \lambda(a_{(1)})\xi \otimes \rho_t\big(S(a_{(2)})b\big)\xi_t.
$$

Indeed, using the invariance of the Haar measure, we show that U is isometric

$$
\begin{aligned}
&\big\langle U\big(\lambda(a)\xi \otimes \rho_t(b)\xi_t\big), U\big(\lambda(c)\xi \otimes \rho_t(d)\xi_t\big) \big\rangle \\
&\quad = \big\langle \lambda(a_{(1)})\xi \otimes \rho_t(a_{(2)}b)\xi_t, \lambda(c_{(1)})\xi \otimes \rho_t(c_{(2)}d)\xi_t \big\rangle \\
&\quad = h\big(a_{(1)}^* c_{(1)}\big)\varphi_t\big(b^* a_{(2)}^* c_{(2)} d\big) = \big(h \otimes \varphi_t^{b,d}\big)\big(a_{(1)}^* c_{(1)} \otimes a_{(2)}^* c_{(2)}\big) = \big(h \star \varphi_t^{b,d}\big)\big(a^* c\big) \\
&\quad = h\big(a^* c\big)\varphi_t^{b,d}(\mathbf{1}) = h\big(a^* c\big)\varphi_t\big(b^* d\big) = \big\langle \lambda(a)\xi \otimes \rho_t(b)\xi_t, \lambda(c)\xi \otimes \rho_t(d)\xi_t \big\rangle,
\end{aligned}
$$

where $\varphi_t^{b,d}(x) := \varphi_t(b^* x d)$. Moreover, by the antipode property (2.2) we have

$$
\begin{aligned}
UU^*\big(\lambda(a)\xi \otimes \rho_t(b)\xi_t\big) &= U\big(\lambda(a_{(1)})\xi \otimes \rho_t\big(S(a_{(2)})b\big)\xi_t\big) = \lambda(a_{(1)})\xi \otimes \rho_t\big(a_{(2)}S(a_{(3)})b\big)\xi_t \\
&= \lambda\big(a_{(1)}\varepsilon(a_{(2)})\big)\xi \otimes \rho_t(b)\xi_t = \lambda(a)\xi \otimes \rho_t(b)\xi_t,
\end{aligned}
$$

which implies that U is an isometry with dense image and therefore extends to a unique unitary operator denoted again by U.

Now the fact that the Markov semigroup $(T_t)_t$ is bounded on A_r, i.e.

$$\big\|T_t(a)\big\|_r = \big\|\lambda\big(T_t(a)\big)\big\|_{B(\mathcal{H})} \leqslant \big\|\lambda(a)\big\|_{B(\mathcal{H})} = \|a\|_r,$$

follows immediately from the relation

$$\lambda\big(T_t(a)\big) = E_t\big(U\big(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}\big)U^*\big), \tag{3.1}$$

since

$$\big\|\lambda\big(T_t(a)\big)\big\| = \big\|E_t\big(U\big(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}\big)U^*\big)\big\| \leqslant \big\|U\big(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}\big)U^*\big\|$$
$$= \big\|\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}\big\| = \big\|\lambda(a)\big\|.$$

To see that (3.1) holds, let us fix $v \in \mathcal{H}$ and $b \in \mathcal{A}$ such that $v = \lambda(b)\xi$. Then

$$\begin{aligned}
E_t\big(U\big(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}\big)U^*\big)v &= \big(\pi_t \circ U \circ \big(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}\big) \circ U^* \circ i_t\big)\big(\lambda(b)\xi\big) \\
&= \big(\pi_t \circ U \circ \big(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}\big) \circ U^*\big)\big(\lambda(b)\xi \otimes \xi_t\big) \\
&= \pi_t \circ U \circ \big(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}\big)\big(\lambda(b_{(1)})\xi \otimes \rho_t\big(S(b_{(2)})\big)\xi_t\big) \\
&= \pi_t \circ U \big(\lambda(ab_{(1)})\xi \otimes \rho_t\big(S(b_{(2)})\big)\xi_t\big) \\
&= \pi_t\big(\lambda(a_{(1)}b_{(1)})\xi \otimes \rho_t\big(a_{(2)}b_{(2)}S(b_{(3)})\big)\xi_t\big) \\
&= \pi_t\big(\lambda(a_{(1)}b)\xi \otimes \rho_t(a_{(2)})\xi_t\big) \\
&= \big\langle \xi_t, \rho_t(a_{(2)})\xi_t\big\rangle \lambda(a_{(1)}b)\xi \\
&= \lambda\big(a_{(1)}\varphi_t(a_{(2)})\big)\lambda(b)\xi = \lambda\big(T_t(a)\big)v.
\end{aligned}$$

This way we showed that each T_t extends to a contraction on A_r. The extensions again form a semigroup and since both Δ and φ_t are completely positive and unital, T_t is too. Let us now check that $(T_t)_t$ forms a strongly continuous semigroup on A_r.

For a given $a \in \mathsf{A}_r$ we choose, by density, an element $b \in \mathcal{A}$ such that $\|a - b\|_r < \epsilon$. By definition for $b \in \mathcal{A}$, $T_t(b) = \varphi_t \star b = (\mathrm{id} \otimes \varphi_t) \circ \Delta(b)$, where $(\varphi_t)_t$ is the convolution semigroup of states on $\mathcal{A}$ (cf. Section 2.5). Thus

$$\begin{aligned}
\big\|T_t(a) - a\big\|_r &\leqslant \big\|T_t(a) - T_t(b)\big\|_r + \big\|T_t(b) - b\big\|_r + \|b - a\|_r \\
&\leqslant 2\|a - b\|_r + \big\|(\varphi_t \star b) - b\big\|_r \leqslant 2\epsilon + \sum\big\|b_{(1)}\varphi_t(b_{(2)}) - b_{(1)}\varepsilon(b_{(2)})\big\|_r \\
&= 2\epsilon + \sum\big|\varphi_t(b_{(2)}) - \varepsilon(b_{(2)})\big|\|b_{(1)}\|_r.
\end{aligned}$$

Since $\lim_{t\to 0+} \varphi_t(b) = \varepsilon(b)$ for any $b \in \mathcal{A}$ and the sum is finite, we conclude that

$$\lim_{t\to 0+}\big\|T_t(a) - a\big\|_r = 0 \quad \text{for each } a \in \mathsf{A}_r. \qquad \square$$

The next results give a characterization of Markov semigroups which are related to Lévy processes on compact quantum groups.

Lemma 3.3. *Let* (A, Δ) *be a compact quantum group and let* $T : \mathsf{A} \to \mathsf{A}$ *be a completely bounded linear map.*

If T *is translation invariant, i.e. satisfies*

$$\Delta \circ T = (\mathrm{id} \otimes T) \circ \Delta$$

then $T(V_s) \subseteq V_s$ *for all* $s \in \mathcal{I}$ *and therefore* T *also leaves the* $*$-*Hopf algebra* $\mathcal{A}$ *invariant.*

Proof. Let $s, s' \in \mathcal{I}$, $s \neq s'$, and $1 \leqslant j, k \leqslant n_s$, $1 \leqslant p, q \leqslant n_{s'}$. Since the Haar state is idempotent, we have

$$h\big(\big(u_{pq}^{(s')}\big)^* T\big(u_{jk}^{(s)}\big)\big) = (h \star h)\big(\big(u_{pq}^{(s')}\big)^* T\big(u_{jk}^{(s)}\big)\big)$$

$$= \sum_{r=1}^{n_{s'}} (h \otimes h)\big(\big(\big(u_{pr}^{(s')}\big)^* \otimes \big(u_{rq}^{(s')}\big)^*\big) \Delta\big(T\big(u_{jk}^{(s)}\big)\big)\big)$$

$$= \sum_{r=1}^{n_{s'}} \sum_{\ell=1}^{n_s} (h \otimes h)\big(\big(u_{pr}^{(s')}\big)^* \otimes \big(u_{rq}^{(s')}\big)^* \big(u_{j\ell}^{(s)} \otimes T\big(u_{\ell k}^{(s)}\big)\big)\big)$$

$$= \sum_{\ell=1}^{n_s} \delta_{ss'} \frac{\overline{f_1\big(\big(u_{jp}^{(s)}\big)^*\big)}}{D_s} h\big(\big(u_{\ell q}^{(s')}\big)^* T\big(u_{\ell k}^{(s)}\big)\big),$$

i.e. $h\big(\big(u_{pq}^{(s')}\big)^* T\big(u_{jk}^{(s)}\big)\big) = 0$ for all $s, s' \in \mathcal{I}$, with $s \neq s'$, and all $1 \leqslant j, k \leqslant n_s$, $1 \leqslant p, q \leqslant n_{s'}$. Therefore $T\big(u_{jk}^{(s)}\big) \in V_s$. $\square$

Theorem 3.4. *Let* (A, Δ) *be a compact quantum group and* $(T_t)_{t \geqslant 0}$ *a quantum Markov semigroup on* A.

Then $(T_t)_{t \geqslant 0}$ *is the quantum Markov semigroup of a* (*uniquely determined*) *Lévy process on* $\mathcal{A}$ *if and only if* T_t *is translation invariant for all* $t \geqslant 0$.

Proof. If $(T_t)_t$ comes from a Lévy process on $\mathcal{A}$, then, on $\mathcal{A}$, $T_t = (\mathrm{id} \otimes \varphi_t) \circ \Delta$ and so

$$\Delta \circ T_t = (\mathrm{id} \otimes \mathrm{id} \otimes \varphi_t) \circ (\Delta \otimes \mathrm{id}) \circ \Delta = \big(\mathrm{id} \otimes \big((\mathrm{id} \otimes \varphi_t) \circ \Delta\big)\big) \circ \Delta = (\mathrm{id} \otimes T_t) \circ \Delta.$$

Hence T_t is translation invariant on $\mathcal{A}$, and therefore also on A by continuity.

Conversely, if every T_t is translation invariant, then Lemma 3.3 implies that, for all $a \in V_s$, $T_t a \in V_s$ and so, since V_s is finite dimensional, $\varepsilon(T_t a) \to \varepsilon(a)$ as $t \to 0$. It now follows easily that $\varphi_t := \varepsilon \circ T_t|_{\mathcal{A}}$ defines a convolution semigroups of states whose generating functional defines a Lévy process whose Markov semigroup is $(T_t)_t$. $\square$

The corresponding result, for counital multiplier C*-bialgebras satisfying a residual vanishing at infinity condition, was proved by Lindsay and Skalski [28, Proposition 3.2]. Their result covers coamenable compact quantum groups (where the counit extends continuously to the C*-algebra). The above proof, for all compact quantum groups, is simpler.

4. GNS-symmetry and KMS-symmetry of convolution operators

In this section we study symmetry properties of convolution operators $L_\phi(a) = \phi \star a$ on $\mathcal{A}$ and we show that they can be translated into invariance properties of the corresponding generating functional ϕ.

We will use two antilinear involutions $\#$ and $\star$ on $\mathcal{A}'$, defined by

$$\phi^{\#}(a) = \overline{\phi(a^*)},$$
$$\phi^{\star}(a) = \phi^{\#}(S(a)),$$

for $a \in \mathcal{A}$. A functional $\phi \in \mathcal{A}'$ is hermitian if and only if $\phi^{\#} = \phi$. Furthermore, we have $\varepsilon^{\#} = \varepsilon^{\star} = \varepsilon$ and $h^{\#} = h^{\star} = h$. Note that $\#$ is multiplicative whereas $\star$ is anti-multiplicative with respect to the convolution of functionals:

$$(\phi \star \psi)^{\#} = \phi^{\#} \star \psi^{\#}, \qquad (\phi \star \psi)^{\star} = \psi^{\star} \star \phi^{\star}$$

for $\phi, \psi \in \mathcal{A}'$.

Let us denote by $L^2(\mathsf{A}, h)$ the GNS Hilbert space of (A, h), by $\xi_h = 1_{\mathsf{A}} \in L^2(\mathsf{A}, h)$ the cyclic vector representing the Haar state: $h(a) = \langle \xi_h, a\xi_h \rangle$ and let us assume that we are given an embedding, i.e. an injective linear map $i : \mathcal{A} \to L^2(\mathsf{A}, h)$ with a dense range. We say that a linear operator $L : \mathcal{A} \to \mathcal{A}$ *admits an i-adjoint* if there exists $L^{\dagger} : \mathcal{A} \to \mathcal{A}$ such that

$$\langle i(a), i(Lb) \rangle = \langle i(L^{\dagger}a), i(b) \rangle$$

for any $a, b \in \mathcal{A}$. Since h is faithful on $\mathcal{A}$ and since i has a dense range, the adjoint is unique if it exists. Then, an operator $L \in \mathcal{L}(\mathcal{A})$ is called *i-symmetric* if L equals to its i-adjoint.

In this paper we shall consider two embeddings. The first one is the natural inclusion coming from the GNS construction

$$i_h : \mathcal{A} \ni a \to a\xi_h \in L^2(\mathsf{A}, h).$$

Definition 4.1. A map $L^{\star} \in \mathcal{L}(\mathcal{A})$ such that

$$h(a^* L(b)) = h(L^{\star}(a)^* b) \tag{4.1}$$

for all $a, b, \in \mathcal{A}$ will be called a *GNS-adjoint*, or simply *adjoint* of L w.r.t. h. A map L will be called *GNS-symmetric* if $L = L^\star$.

Let us observe that a convolution operator always admits a GNS-adjoint.

Proposition 4.2. *Let $\phi \in \mathcal{A}'$. Then there exists a unique convolution operator $L_\phi^\star$ that is adjoint to L_ϕ w.r.t. the Haar state, i.e. that satisfies*

$$h\big(a^* L_\phi(b)\big) = h\big(L_\phi^\star(a)^* b\big)$$

for all $a, b \in \mathcal{A}$. The adjoint of L_ϕ is given by

$$L_\phi^\star = L_{\phi^\star}.$$

Therefore L_ϕ is GNS-symmetric if and only if $\phi^\star = \phi$.

Proof. This is simply the fact that the dual right representation is a $*$-representation w.r.t. to the involution $\star$ and the inner product defined by the Haar state as $\mathcal{A} \times \mathcal{A} \ni (a, b) \mapsto \langle a, b \rangle = h(a^* b) \in \mathbb{C}$. The proof is the same as in the finite-dimensional case, see [40, Proposition 2.3]. See also [21, Proposition 3.4]. $\square$

The second embedding we can consider is the *symmetric embedding*

$$i_s : \mathcal{A} \ni a \mapsto i_s(a) = \sigma_{-\frac{i}{4}}(a) \xi_h \in L^2(A, h)$$

and the related notion of symmetry is the following.

Definition 4.3. We shall call a map $L^\flat \in \mathcal{L}(\mathcal{A})$ the *KMS-adjoint* of $L \in \mathcal{L}(\mathcal{A})$, if we have

$$h\big(\sigma_{-\frac{i}{2}}(a)^* L(b)\big) = h\big(L^\flat(a)^* \sigma_{-\frac{i}{2}}(b)\big) \tag{4.2}$$

for all $a, b, \in \mathcal{A}$. An operator $L \in \mathcal{L}(\mathcal{A})$ is called *KMS-symmetric* if $L^\flat = L$.

Let us note here that a definition of KMS-symmetric operator on a von Neumann algebra was introduced by Goldstein and Lindsay [22] in the framework of (noncommutative) Haagerup L^p-spaces and by Cipriani in his PhD thesis (cf. [10]) in the context of the standard form of von Neumann algebras. Later, in [12, Definition 2.31], a definition of a KMS-symmetric operator on a C*-algebra was provided.

In the sequel, we shall also need the definition of KMS-symmetric operators on the whole C*-algebra. It is stated as follows.

Definition 4.4. A linear map $L : \mathsf{A} \to \mathsf{A}$ is called *KMS-symmetric* w.r.t. h with modular automorphism group $(\sigma_t)_{t \in \mathbb{R}}$, if

$$h\big(aL(b)\big) = h\big(\sigma_{\frac{i}{2}}(b)L\big(\sigma_{-\frac{i}{2}}(a)\big)\big) \tag{4.3}$$

for all a, b in a dense σ-invariant $*$-subalgebra B of the C*-algebra A.

Note that a continuous map $L : \mathsf{A} \to \mathsf{A}$ is KMS-symmetric in the sense of Definition 4.4 if it is $\mathcal{A}$-invariant (i.e. $L(\mathcal{A}) \subset \mathcal{A}$), hermitian and $(\sigma, -1)$-KMS-symmetric in the sense of [12, Definition 2.31]. The temperature $\beta = -1$ is chosen according to the KMS-property of the Haar state $h(ab) = h(b\sigma_{-i}(a))$, see Eq. (2.5).

The analogue of Proposition 4.2 for KMS-symmetric operators on $\mathcal{A}$ is now the following.

Theorem 4.5. *Let $\phi \in \mathcal{A}'$. Then there exists a unique convolution operator $L_\phi^\flat$ that is the KMS-adjoint to L_ϕ w.r.t. the Haar state, i.e. that satisfies*

$$h\big(\sigma_{-\frac{i}{2}}(a)^* L_\phi(b)\big) = h\big(L_\phi^\flat(a)^* \sigma_{-\frac{i}{2}}(b)\big)$$

for all $a, b \in \mathcal{A}$. The KMS-adjoint of L_ϕ is given by $L_\phi^\flat = L_{\phi^\# \circ R}$, where R denotes the unitary antipode.

Proof. Let us observe first that a linear map $L \in \mathcal{L}(\mathcal{A})$ admits a KMS-adjoint if and only if it admits a GNS-adjoint, and that the two adjoints are related by

$$L^\flat = \sigma_{\frac{i}{2}} \circ L^\star \circ \sigma_{-\frac{i}{2}}. \tag{4.4}$$

Indeed, if the GNS-adjoint exists then, by (4.1) and the σ-invariance of h, we have

$$h\big(\sigma_{-\frac{i}{2}}(a)^* L(b)\big) = h\big(L^\star\big(\sigma_{-\frac{i}{2}}(a)\big)^* b\big) = h\big(\big(\sigma_{\frac{i}{2}} \circ L^\star \circ \sigma_{-\frac{i}{2}}\big)(a)^* \sigma_{-\frac{i}{2}}(b)\big).$$

Comparing with Eq. (4.2) and using the faithfulness of the Haar state, we deduce that $L^\flat$ exists and satisfies (4.4). Conversely, if the KMS-adjoint exists then, using similar arguments, we show that the GNS-adjoint exists and $L^\star = \sigma_{-\frac{i}{2}} \circ L^\flat \circ \sigma_{\frac{i}{2}}$.

Now, it follows from Proposition 4.2 that $L_\phi^\flat$ exists and for all $a \in \mathcal{A}$ we have

$$\begin{aligned}
L_\phi^\flat(a) &= f_{-\frac{1}{2}} \star \big(L_{\phi^\# \circ S}(f_{\frac{1}{2}} \star a \star f_{\frac{1}{2}})\big) \star f_{-\frac{1}{2}} \\
&= f_{-\frac{1}{2}} \star \big(\phi^\# \circ S\big) \star f_{\frac{1}{2}} \star a \\
&= a_{(1)}\big(f_{-\frac{1}{2}} \star \big(\phi^\# \circ S\big) \star f_{\frac{1}{2}}\big)(a_{(2)}) \\
&= a_{(1)} f_{-\frac{1}{2}}(a_{(2)})\big(\phi^\# \circ S\big)(a_{(3)}) f_{\frac{1}{2}}(a_{(4)}) \\
&= a_{(1)}\big(\big(\phi^\# \circ S\big)(f_{\frac{1}{2}} \star a_{(2)} \star f_{-\frac{1}{2}})\big) \\
&= a_{(1)}\big(\phi^\# \circ R\big)(a_{(2)}) = L_{\phi^\# \circ R}(a),
\end{aligned}$$

since $R(a) = S(f_{\frac{1}{2}} \star a \star f_{-\frac{1}{2}})$, see Eq. (2.9). $\square$

Corollary 4.6. *Suppose that $\phi \in \mathcal{A}'$. Then*

(1) *L_ϕ is GNS-symmetric if and only if ϕ satisfies $\phi^\# \circ S = \phi$.*
(2) *L_ϕ is KMS-symmetric if and only if ϕ satisfies $\phi^\# \circ R = \phi$.*

The generating functionals ϕ of Lévy processes are necessarily hermitian ($\phi^\# = \phi$). We call a hermitian functional ϕ on $\mathcal{A}$ ϕ *GNS-symmetric* if it is invariant under the antipode: $\phi \circ S = \phi$, and *KMS-symmetric* if it is invariant under the unitary antipode: $\phi \circ R = \phi$.

Remark 4.7. A hermitian ϕ is GNS-symmetric if and only if each matrix $\phi^{(s)} = [\phi(u_{jk}^{(s)})]_{j,k}$ is hermitian:

$$\phi(u_{jk}^{(s)}) = (\phi \circ S)(u_{jk}^{(s)}) = \phi((u_{kj}^{(s)})^*) = \overline{\phi(u_{kj}^{(s)})}.$$

We shall show now that invariance under the phase in the polar decomposition of the antipode has also an influence on the properties of L_ϕ.

Proposition 4.8. *Let $\phi \in \mathcal{A}'$. Then the following conditions are equivalent*:

(1) *L_ϕ commutes with the modular automorphism group σ,*
(2) *ϕ commutes with the Woronowicz characters: $\phi \star f_z = f_z \star \phi$ for $z \in \mathbb{C}$,*
(3) *$\phi \circ \tau_{\frac{i}{2}} = \phi$.*

Proof. By Eq. (2.4), we have $L_\phi \circ \sigma_t = \sigma_t \circ L_\phi$ if and only if

$$\phi \star f_{it} \star a \star f_{it} = f_{it} \star \phi \star a \star f_{it}$$

for all $a \in \mathcal{A}$. Convolving by f_{-it} from the right and applying the counit, we see that L_ϕ commutes with the modular automorphism group, if and only if

$$\phi \star f_{it} = f_{it} \star \phi$$

for all $t \in \mathbb{R}$, which is equivalent to

$$\phi \star f_z = f_z \star \phi \tag{4.5}$$

for all $z \in \mathbb{C}$ by uniqueness of analytic continuation. We have shown this way that (1) $\Leftrightarrow$ (2).

From Eq. (4.5) we deduce immediately that

$$\phi \circ \tau_z(a) = \phi(f_{iz} \star a \star f_{-iz}) = (f_{-iz} \star \phi \star f_{iz})(a) = \phi(a),$$

so (2) implies (3).

Finally, let us see that (3) implies (2). For that we adopt the matrix notation from [46]:

$$F^{(s)} = \left[f_{-1}\big(u_{jk}^{(s)}\big) \right]_{j,k=-s}^{s} \quad \text{and} \quad \phi^{(s)} = \left[\phi\big(u_{jk}^{(s)}\big) \right]_{j,k=-s}^{s}.$$

From therein we know that $F^{(s)}$ is invertible and positive and that $f_z(u^{(s)}) = (F^{(s)})^{-z}$. If $\phi \circ \tau_{\frac{i}{2}} = \phi$, then by the definition of τ_z we have $\phi \star f_{-\frac{1}{2}} = f_{-\frac{1}{2}} \star \phi$ and also $\phi \star f_{-1} = f_{-1} \star \phi$. This means that

$$\phi^{(s)} F^{(s)} = F^{(s)} \phi^{(s)}$$

and by the functional calculus $\phi^{(s)}$ must commute with all $(F^{(s)})^z$ for $z \in \mathbb{C}$. This translates into $\phi \star f_z = f_z \star \phi$ for all $z \in \mathbb{C}$. $\quad\square$

It is known that on von Neumann algebras GNS-symmetry is a stronger condition than the KMS-one (cf. [12, Remarks after Definition 2.31]). The previous observation allows to provide a simple proof of this fact in our setting.

Corollary 4.9. *If ϕ is GNS-symmetric, then ϕ commutes with all Woronowicz characters and is KMS-symmetric.*

Proof. For GNS-symmetric ϕ we have $\phi = \phi \circ S^2 = \phi \circ \tau_i$, which translates into $\phi \star f_{-1} = f_{-1} \star \phi$. From the proof of Proposition 4.8 we see that this implies that ϕ is invariant under all τ_z ($z \in \mathbb{C}$) or, equivalently, commutes with all Woronowicz characters. In particular $\phi = \phi \circ \tau_{\frac{i}{2}}$ and

$$\phi = \phi \circ \tau_{\frac{i}{2}} = (\phi \circ S) \circ \tau_{\frac{i}{2}} = \phi \circ R. \quad\square$$

Remark 4.10. If the algebra $\mathcal{A}$ is of Kac type ($S^2 = \mathrm{id}$), then $R = S$ and the notions of GNS-symmetry and KMS-symmetry coincide. However, Example 11.5 shows that in general KMS-symmetry is a weaker condition than GNS-symmetry.

We end this section with an observation linking the symmetries of the generators and the related Markov semigroups.

Theorem 4.11. *Let $(T_t)_{t \geqslant 0}$ be the Markov semigroup of a Lévy process on $\mathcal{A}$ with generating functional ϕ.*

(a) *The following three conditions are equivalent*:
   (a1) *ϕ is KMS-symmetric.*
   (a2) *L_ϕ is KMS-symmetric.*
   (a3) *For each $t \geqslant 0$, T_t is KMS-symmetric on A (see Definition 4.4).*
(b) *The following four conditions are equivalent*:
   (b1) *ϕ is GNS-symmetric.*

(b2) L_ϕ *is GNS-symmetric.*

(b2′) L_ϕ *satisfies the* quantum detailed balance *condition, i.e. we have*

$$h\big(aL_\phi(b)\big) = h\big(L_\phi(a)b\big) \quad for\ a, b \in \mathcal{A}. \tag{4.6}$$

(b3) $(T_t)_{t \geqslant 0}$ *satisfies the* quantum detailed balance *condition, i.e.* (4.6) *holds for all* T_t, $t \geqslant 0$.

Proof. The equivalences (x1) $\Leftrightarrow$ (x2) follow from Corollary 4.6.

The KMS-symmetry as well as the GNS-symmetry of ϕ is preserved under the convolution powers (for example, if $\phi(Sa) = \phi(a)$, then $(\phi \star \phi)(Sa) = (\phi \otimes \phi)(S(a_{(2)}) \otimes S(a_{(1)})) = \phi(a_{(1)})\phi(a_{(2)}) = (\phi \star \phi)(a)$.) Since $L_\phi^n(a) = \phi^{\star n} \star a$, we see that both kinds of symmetry are also preserved for the powers of L_ϕ. This implies that for $(T_t)_{t \geqslant 0}$, being of the form $T_t = \exp t L_\phi$, the KMS-symmetry or condition (4.6) of $(T_t)_{t \geqslant 0}$ is equivalent to KMS-symmetry or (4.6) of L_ϕ.

Finally we need to check that (b2′) $\Leftrightarrow$ (b1). Assume that L_ϕ satisfies (4.6). Then, by Proposition 4.2, L_ϕ satisfies

$$L_\phi(a) = L^\star\big(a^*\big)^* = \big(\phi^\star \star a^*\big)^* = a_{(1)}\phi\big(S\big(a_{(2)}^*\big)^*\big) = L_{\phi \circ S^{-1}}(a),$$

which implies $\phi \circ S = \phi$.

Conversely, if $\phi \circ S = \phi$, then by the same calculation we see that

$$h\big(L_\phi(a)b\big) = h\big(L_{\phi \circ S^{-1}}(a)b\big) = h\big(L_\phi^\star\big(a^*\big)^*b\big) = h\big(aL_\phi(b)\big). \qquad \square$$

5. Schürmann triples corresponding to KMS-symmetric generators

In this section we give a method to produce KMS-symmetric generating functionals. To this aim, we recall the notion of a Schürmann triple and describe its behavior under the composition of an arbitrary generator with the unitary antipode.

Our steps are motivated by the following easy observation.

Proposition 5.1. *Let ϕ be a generating functional of a Lévy process. Then $\phi + \phi \circ R$ is a KMS-symmetric generating functional of a Lévy process.*

Proof. Since $R(a^*) = R(a)^*$ and $\varepsilon(R(a)) = \varepsilon(a)$, we easily check that the Schoenberg criteria for a generating functional are satisfied for $\phi + \phi \circ R$. Moreover, $R^2 = \mathrm{id}$ implies that $\phi + \phi \circ R$ is invariant under the unitary antipode. $\square$

Note that the same procedure cannot be applied to the GNS-symmetric case, since S does not preserve the positivity and is not involutive.

For a pre-Hilbert space D we denote by $\mathcal{L}^\#(D)$ the set of all operators from D to D which admit an adjoint.

Definition 5.2. A *Schürmann triple* on a $*$-bialgebra $\mathcal{A}$ with counit ε is a triple $((\pi, D), \eta, \phi)$ consisting of:

(1) a unital $*$-representation $\pi : \mathcal{A} \to \mathcal{L}^{\#}(D)$ of $\mathcal{A}$ on some pre-Hilbert space D,
(2) a linear map $\eta : \mathcal{A} \to D$, called *cocyle*, such that

$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b) \quad \text{for all } a, b \in \mathcal{A},$$

(3) a hermitian linear functional $\phi : \mathcal{A} \to \mathbb{C}$ satisfying

$$\phi(ab) = \langle \eta(a^*), \eta(b) \rangle \quad \text{for } a, b \in \ker \varepsilon.$$

Schürmann proved (cf. [36]) that for any generating functional ϕ of a Lévy process there exists a Schürmann triple $((\pi, D), \eta, \phi)$ (such that the generating functional is the last ingredient of the triple). Moreover, the Schürmann triple is uniquely determined (modulo unitary equivalence) provided that η is surjective.

Definition 5.3. Given a pre-Hilbert space D, the *opposite space* D^{op} is defined as $D^{\mathrm{op}} = \{\bar{v} : v \in D\}$ (the set of the same elements as D) with the same addition $\bar{v} + \bar{w} = \overline{v + w}$, but with the scalar multiplication given by $\lambda \cdot \bar{v} = \overline{\bar{\lambda}v}$ and with the scalar product $\langle \bar{v}, \bar{w} \rangle_{\mathrm{op}} = \langle w, v \rangle$.

Given a unital $*$-representation $\pi : \mathcal{A} \to \mathcal{L}^{\#}(D)$ we define $\pi^{\mathrm{op}} : \mathcal{A} \to \mathcal{L}^{\#}(D^{\mathrm{op}})$ by the formula

$$\pi^{\mathrm{op}}(a)\bar{v} = \overline{(\pi \circ R)(a^*)v}, \quad \bar{v} \in D^{\mathrm{op}}.$$

We check directly that π^{op} is unital, multiplicative, and $*$-preserving, so it is a $*$-representation of $\mathcal{A}$ on D^{op}. We shall call it the *opposite representation*.

Theorem 5.4. *If ϕ is a generating functional of a Lévy process with the Schürmann triple $((\pi, D), \eta, \phi)$ on $\mathcal{A}$, then $\phi \circ R$ is a generating functional of a Lévy process with the Schürmann triple $((\pi^{\mathrm{op}}, D^{\mathrm{op}}), \eta^{\mathrm{op}}, \phi \circ R)$ on $\mathcal{A}$ where π^{op} is the opposite representation with the representation space D^{op} and $\eta^{\mathrm{op}} : \mathcal{A} \to D^{\mathrm{op}}$ is defined by $\eta^{\mathrm{op}}(a) = \overline{\eta(R(a^*))}$.*

Proof. Let ϕ be a generating functional of a Lévy process. Then it follows from the properties of R, mentioned after formula (2.8), that $\phi \circ R$ is hermitian, conditionally positive and vanishes at $\mathbf{1}$. By the Schoenberg correspondence, $\phi \circ R$ is a generating functional of a Lévy process.

Now we want to check that $((\pi^{\mathrm{op}}, D^{\mathrm{op}}), \eta^{\mathrm{op}}, \phi \circ R)$ is a Schürmann triple. For that, note that η^{op} is linear and by the cocycle property of η we have

$$\eta^{\mathrm{op}}(ab) = \overline{\eta\big(R((ab)^*)\big)} = \overline{\eta\big(R(a^*)R(b^*)\big)}$$

$$= \overline{\pi\big(R(a^*)\big)\eta\big(R(b^*)\big)} + \overline{\eta\big(R(a^*)\big)\varepsilon\big(R(b^*)\big)}$$

$$= \pi^{\mathrm{op}}(a)\overline{\eta\big(R(b^*)\big)} + \overline{\eta\big(R(a^*)\big)}\varepsilon(b)$$

$$= \pi^{\mathrm{op}}(a)\eta^{\mathrm{op}}(b) + \eta^{\mathrm{op}}(a)\varepsilon(b).$$

Moreover, $\phi \circ R$ is linear and hermitian, and for $a, b \in \ker \varepsilon$ we have

$$\big\langle \eta^{\mathrm{op}}\big(a^*\big), \eta^{\mathrm{op}}(b)\big\rangle_{\mathrm{op}} = \big\langle \overline{\eta\big(R(a)\big)}, \overline{\eta\big(R(b^*)\big)}\big\rangle_{\mathrm{op}} = \big\langle \eta\big(R(b^*)\big), \eta\big(R(a)\big)\big\rangle$$

$$= \phi\big(R(b)R(a)\big) = (\phi \circ R)(ab). \qquad \square$$

Corollary 5.5. *If ϕ is invariant under R and $((\pi, D), \eta, \phi)$ is the related surjective Schür-mann triple, then π is equivalent to its opposite representation π^{op}.*

Corollary 5.6. *If ϕ is a generating functional of a Lévy process with surjective Schürmann triple $((\pi, D), \eta, \phi)$ on $\mathcal{A}$, then $((\pi \oplus \pi^{\mathrm{op}}, D \oplus D^{\mathrm{op}}), \eta \oplus \eta^{\mathrm{op}}, \phi + \phi \circ R)$ is a Schürmann triple of a KMS symmetric generator $\phi + \phi \circ R$.*

Note that the Schürmann triple $((\pi \oplus \pi^{\mathrm{op}}, D \oplus D^{\mathrm{op}}), \eta \oplus \eta^{\mathrm{op}}, \phi + \phi \circ R)$ in Corollary 5.6 is not necessarily surjective, even if the triple $((\pi, D), \eta, \phi)$ is surjective. This is for example the case if ϕ is already KMS symmetric – then the range of $\eta \oplus \eta^{\mathrm{op}}$ is the diagonal of $D \oplus D^{\mathrm{op}}$.

Remark 5.7. Let ϕ be a generating functional of a Lévy process on $\mathcal{A}$ with the associated Schürmann triple $((\pi, D), \eta, \phi)$, i.e. $((\pi, D), \eta, \phi)$ is the unique Schürmann triple for ϕ with a surjective cocycle. If $\mathcal{A}$ is an algebraic quantum group "of compact type", i.e. it is the $*$-subalgebra of polynomials of a compact quantum group $\mathbb{G} = (\mathsf{A}, \Delta)$, then $\mathcal{A}$ is linearly spanned by the coefficients of unitary corepresentations and thus for every $a \in \mathcal{A}$, $\pi(a)$ is a bounded operator in D. In this case the space D can be completed to a Hilbert space H, $\eta : \mathcal{A} \to D$ turns into a cocycle $\eta : \mathcal{A} \to H$ with dense image, and π maps $\mathcal{A}$ to $\mathcal{B}(H)$.

6. Generating functionals invariant under adjoint action

On classical Lie groups, central measures play an important role in harmonic analysis and the study of Lévy processes. A measure μ on a topological group G is called *central*, if it commutes with all other measures (w.r.t. to the convolution). This is the case if

$$\int\limits_G f\big(gxg^{-1}\big)\,\mathrm{d}\mu(x) = \int\limits_G f(x)\,\mathrm{d}\mu(x)$$

for all $g \in G$ and $f \in C(G)$, or, equivalently, if $\delta_g \star \mu \star \delta_{g^{-1}} = \mu$ for all $g \in G$. On compact quantum groups we don't have Dirac measures, but we can translate this condition to

$$\psi_{(1)} \star \mu \star \hat{S}(\psi_{(2)}) = \psi(1)\mu$$

for all functionals $\psi : \mathcal{A} \to \mathbb{C}$, for which $(\mathrm{id} \otimes \hat{S}) \circ \hat{\Delta}(\psi) = \psi_{(1)} \otimes \hat{S}(\psi_{(2)})$ can be defined, i.e. for functionals which belong to the algebra of smooth functions $\hat{\mathcal{A}}$ on the dual discrete quantum group. This condition is equivalent to invariance of the functional μ under the adjoint action, see below.

In this section we will study ad-invariance for functionals on compact quantum groups. On cocommutative compact quantum groups (i.e. such that $\tau \circ \Delta = \Delta$, where τ is the flip operator $\tau(x \otimes y) = y \otimes x$) all functionals are ad-invariant, but on non-cocommutative compact quantum groups, ad-invariance characterizes an interesting class of functionals that share many similar properties with central measures. After reviewing several characterizations and showing that the ad-invariant functionals are exactly those that belong to the center of $\mathcal{A}'$, we show that it is possible to construct from a given functional an ad-invariant one. But this construction does not preserve positivity.

Recall that the *adjoint action* of a Hopf algebra is defined by $\mathrm{ad} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$,

$$\mathrm{ad}(a) = a_{(1)}S(a_{(3)}) \otimes a_{(2)}$$

for $a \in \mathcal{A}$, see, e.g., [30], [25, Section 1.3.4].

The adjoint action is a left coaction, i.e. we have

$$(\mathrm{id} \otimes \mathrm{ad}) \circ \mathrm{ad} = (\Delta \otimes \mathrm{id}) \circ \mathrm{ad},$$
$$(\varepsilon \otimes \mathrm{id}) \circ \mathrm{ad} = \mathrm{id}.$$

But note that ad is not an algebra homomorphism.

Definition 6.1. We call a linear functional $\phi \in \mathcal{A}'$ ad-*invariant*, if it satisfies

$$(\mathrm{id} \otimes \phi) \circ \mathrm{ad} = \phi \mathbf{1}_{\mathcal{A}}.$$

Similarly, a linear map $L \in \mathcal{L}(\mathcal{A})$ is called ad-*invariant*, if it satisfies

$$(\mathrm{id} \otimes L) \circ \mathrm{ad} = \mathrm{ad} \circ L.$$

If the quantum group is cocommutative, then the adjoint action is the trivial coaction $\mathrm{ad}(a) = \mathbf{1} \otimes a$. Therefore in this case all functionals are ad-invariant.

It is straightforward to verify that the counit ε and the Haar state h are ad-invariant.

The following characterizations show that the ad-invariant functionals are a natural generalization of central measures.

Proposition 6.2. *Let $\phi \in \mathcal{A}'$. The following conditions are equivalent.*

(a) ϕ *is* ad-*invariant.*
(b) *We have*

$$\psi_{(1)} \star \phi \star \hat{S}(\psi_{(2)}) = \psi(1)\phi$$

for all $\psi \in \hat{\mathcal{A}}$.
(c) ϕ *commutes with all elements of $\hat{\mathcal{A}}$: $\phi \star \psi = \psi \star \phi$ for all $\psi \in \hat{\mathcal{A}}$.*
(d) ϕ *belongs to the center of $\mathcal{A}'$: $\phi \star \psi = \psi \star \phi$ for all $\psi \in \mathcal{A}'$.*

Proof. (a) $\Leftrightarrow$ (b): If $\phi : \mathcal{A} \to \mathbb{C}$ is ad-invariant, then we have

$$a_{(1)} S(a_{(3)}) \phi(a_{(2)}) = \phi(a)1.$$

Applying the functional $\psi = h_b \in \hat{\mathcal{A}}$ with $b \in \mathcal{A}$ to this, we get

$$\psi(1)\phi(a) = \psi\big(a_{(1)} S(a_{(3)})\big)\phi(a_{(2)}) = \psi_{(1)}(a_{(1)})\psi_{(2)}\big(S(a_{(3)})\big)\phi(a_{(2)})$$
$$= \psi_{(1)}(a_{(1)})\phi(a_{(2)})\hat{S}(\psi_{(2)})(a_{(3)}) = \big(\psi_{(1)} \star \phi \star \hat{S}(\psi_{(2)})\big)(a)$$

for all $a \in \mathcal{A}$ and all $\psi \in \hat{\mathcal{A}}$. The converse follows, because by the faithfulness of the Haar state on $\mathcal{A}$ we have

$$\forall b \in \mathcal{A}, \quad \psi(a) = h(ba) = 0 \quad \Rightarrow \quad a = 0.$$

(c) $\Rightarrow$ (b): This follows directly from the antipode axiom,

$$\psi_{(1)} \star \phi \star \hat{S}(\psi_{(2)}) = \psi_{(1)} \star \hat{S}(\psi_{(2)}) \star \phi = \hat{\varepsilon}(\psi)\hat{1} \star \phi = \psi(1)\phi,$$

where $\hat{1} = \varepsilon$ is the unit of $\mathcal{A}'$.

(a) $\Rightarrow$ (c): Suppose that ϕ is ad-invariant and apply $\psi \circ m \circ (\mathrm{id} \otimes \phi \otimes \mathrm{id}) \circ (\mathrm{ad} \otimes \mathrm{id})$ to $\Delta(a)$, then this gives

$$\psi\big(a_{(1)} S(a_{(3)})a_{(4)}\big)\phi(a_{(2)})$$

which is equal to $\psi(a_{(1)})\phi(a_{(2)}) = (\psi \star \phi)(a)$ by the antipode axiom. On the other hand, using the ad-invariance of ϕ, the same expression becomes

$$\psi(1a_{(2)})\phi(a_{(1)}) = (\phi \star \psi)(a).$$

(c) $\Leftrightarrow$ (d): This follows, because $\mathcal{A}'$ embeds into the multiplier algebra $\mathcal{M}(\hat{\mathcal{A}})$ of $\hat{\mathcal{A}}$, since

$$\psi \star h_a = h_c, \qquad h_a \star \psi = h_d$$

with $c = \psi(S(a_{(1)}))a_{(2)}$, $d = \psi(S^{-1}(a_{(2)}))a_{(1)}$. $\quad\square$

Corollary 6.3. *The* ad-*invariant functionals form a unital subalgebra of* $\mathcal{A}'$ *with respect to the convolution.*

The following formula shows that the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is ad-invariant, if we define the adjoint action of $\mathcal{A} \otimes \mathcal{A}$ by $\mathrm{ad}^{\otimes} = (m \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{ad} \otimes \mathrm{ad})$.

Lemma 6.4. *The adjoint action satisfies the relation*

$$(m \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{ad} \otimes \mathrm{ad}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \mathrm{ad}.$$

Proof. Using Sweedler notation, we get

$$
\begin{aligned}
(m \otimes \mathrm{id} &\otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{ad} \otimes \mathrm{ad}) \circ \Delta(a) \\
&= a_{(1)}S(a_{(3)})a_{(4)}S(a_{(6)}) \otimes a_{(2)} \otimes a_{(5)} \\
&= a_{(1)}\varepsilon(a_{(3)})\mathbf{1}_{\mathcal{A}}S(a_{(5)}) \otimes a_{(2)} \otimes a_{(4)}
\end{aligned}
$$

for $a \in \mathcal{A}$, where we used the antipode property (2.2). After further simplification, using the counit property (2.1), we get

$$= a_{(1)}S(a_{(4)}) \otimes a_{(2)} \otimes a_{(3)} = (\mathrm{id} \otimes \Delta) \circ \mathrm{ad}(a). \qquad \square$$

Lemma 6.5. *Let* $\phi \in \mathcal{A}'$. *Then* ϕ *is* ad-*invariant if and only if* L_ϕ *is* ad-*invariant.*

Proof. Let us observe that

$$
\begin{aligned}
(\mathrm{id} \otimes L_\phi) \circ \mathrm{ad}(a) &= (\mathrm{id} \otimes \mathrm{id} \otimes \phi)\big(a_{(1)}S(a_{(4)}) \otimes a_{(2)} \otimes a_{(3)}\big) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \phi)\big(a_{(1)}S(a_{(3)})a_{(4)}S(a_{(6)}) \otimes a_{(2)} \otimes a_{(5)}\big) \quad \text{(cf. Lemma 6.4)} \\
&= a_{(1)}S(a_{(3)})a_{(4)}S(a_{(6)})\phi(a_{(5)}) \otimes a_{(2)}.
\end{aligned}
$$

If we assume that ϕ is ad-invariant, then

$$(\mathrm{id} \otimes L_\phi) \circ \mathrm{ad}(a) = a_{(1)}S(a_{(3)})\phi(a_{(4)}) \otimes a_{(2)} = \phi(a_{(2)})\,\mathrm{ad}(a_{(1)}) = \mathrm{ad} \circ L_\phi(a).$$

On the other hand, if we suppose that L_ϕ is ad-invariant, then the application of $(\mathrm{id} \otimes \varepsilon)$ to both sides of the equation

$$\phi(a_{(4)})a_{(1)}S(a_{(3)}) \otimes a_{(2)} = a_{(1)}S(a_{(3)})a_{(4)}S(a_{(6)})\phi(a_{(5)}) \otimes a_{(2)}$$

gives the ad-invariance of ϕ. $\quad\square$

We can use the Haar state to produce ad-invariant functionals.

Proposition 6.6. *Denote by* $\mathrm{ad}_h \in \mathcal{L}(\mathcal{A})$ *the linear map given by*

$$\mathrm{ad}_h = (h \otimes \mathrm{id}) \circ \mathrm{ad}\,.$$

Then $\phi_{\mathrm{ad}} := \phi \circ \mathrm{ad}_h$ *is* ad-*invariant for all* $\phi \in \mathcal{A}'$.

Proof. Observe that by definition we have $\phi_{\mathrm{ad}} = \phi \circ \mathrm{ad}_h = (h \otimes \phi) \circ \mathrm{ad}$. Using the invariance of the Haar measure (Proposition 2.2) we check that

$$\phi_{\mathrm{ad}}(a)\mathbf{1} = h\big(a_{(1)}S(a_{(3)})\big)\phi(a_{(2)})\mathbf{1} = a_{(1)}S(a_{(5)})h\big(a_{(2)}S(a_{(4)})\big)\phi(a_{(3)})$$

$$= a_{(1)}S(a_{(3)})\phi_{\mathrm{ad}}(a_{(2)}) = (\mathrm{id} \otimes \phi_{\mathrm{ad}}) \circ \mathrm{ad}(a). \qquad \square$$

Let us collect the basic properties of ad_h.

Proposition 6.7.

(a) $\mathrm{ad}_h \circ \mathrm{ad}_h = \mathrm{ad}_h$.
(b) $(\phi \circ \mathrm{ad}_h)^\star = \phi^\star \circ \mathrm{ad}_h$ *for all* $\phi \in \mathcal{A}'$.
(c) *A linear functional* $\phi \in \mathcal{A}'$ *is* ad-*invariant if and only if* $\phi = \phi \circ \mathrm{ad}_h$.

Proof. Ad (a). Explicit calculations give

$$\mathrm{ad}_h \circ \mathrm{ad}_h(a) = h\big(h\big[a_{(1)}S(a_{(5)})\big]a_{(2)}S(a_{(4)})\big)a_{(3)}.$$

Apply the invariance of the Haar measure (Proposition 2.2) to the element under the Haar state, and after the appropriate renumbering, we get

$$\mathrm{ad}_h \circ \mathrm{ad}_h(a) = h\big(h\big(a_{(1)}S(a_{(3)})\big)\mathbf{1}\big)a_{(2)} = \mathrm{ad}_h(a).$$

Ad (b). Recall that $\phi^\star = \phi^\# \circ S$, where S is the antipode and $\phi^\#(a) = \overline{\phi(a^*)}$. Then the assertion will follow if we show that

$$\big[\mathrm{ad}_h \circ S(a)^*\big]^* = S \circ \mathrm{ad}_h(a).$$

Using the properties that $S \circ * \circ S \circ * = \mathrm{id}$ and $\Delta(S(a)) = \tau \circ (S \otimes S) \circ \Delta(a)$ we check that $\mathrm{ad}(S(a)^*) = S(a_{(3)})^* a_{(1)}^* \otimes S(a_{(2)})^*$. Then since h is hermitian, we have

$$\big[\mathrm{ad}_h \circ S(a)^*\big]^* = \big[(h \otimes \mathrm{id}) \circ (\mathrm{ad} \circ S)(a)^*\big]^* = \overline{h\big(S(a_{(3)})^* a_{(1)}^*\big)}S(a_{(2)})$$

$$= h\big(a_{(1)}S(a_{(3)})\big)S(a_{(2)}) = S\big(\mathrm{ad}_h(a)\big).$$

Ad (c). First we check that for an ad-invariant functional ϕ we have $\phi = \phi \circ \mathrm{ad}_h$:

$$\phi \circ \mathrm{ad}_h(a) = \phi \circ (h \otimes \mathrm{id}) \circ \mathrm{ad}(a) = h \circ (\mathrm{id} \otimes \phi) \circ \mathrm{ad}(a) = h\big(\phi(a)\mathbf{1}\big) = \phi(a).$$

The converse follows immediately from Proposition 6.6. $\square$

Applying Lemma 6.5 and Corollary 6.3, we get an analogue of Theorem 4.11 for ad-invariance.

Corollary 6.8. *Let $(T_t)_{t \geqslant 0}$ be the Markov semigroup of a Lévy process on $\mathcal{A}$ with generating functional ϕ. The following three conditions are equivalent:*

(a1) ϕ *is* ad-*invariant.*
(a2) L_ϕ *is* ad-*invariant.*
(a3) *For each $t \geqslant 0$, T_t is* ad-*invariant.*

In the next proposition we show that ad-invariance of functionals can be characterized by the form of their characteristic matrices.

Proposition 6.9. *A functional ϕ is* ad-*invariant if and only if its characteristic matrices $(\phi(u_{jk}^{(s)}))_{1 \leqslant j,k \leqslant n_s}$ are multiples of the identity matrix for all $s \in \mathcal{I}$, i.e. if there exist complex numbers c_s, $s \in \mathcal{I}$, such that $\phi(u_{jk}^{(s)}) = c_s \delta_{jk}$ for all $s \in \mathcal{I}$ and all $1 \leqslant j, k \leqslant n_s$.*

Proof. We use the orthogonality relation for the Haar measure (2.7) to show that for the ad-invariant functional ϕ we have

$$\phi\big(u_{jk}^{(s)}\big) = \phi_{\mathrm{ad}}\big(u_{jk}^{(s)}\big) = \sum_{p,r=1}^{n} h\big(u_{jp}^{(s)}\big(u_{kr}^{(s)}\big)^*\big) \phi\big(u_{pr}^{(s)}\big) = \frac{1}{D_s} \sum_{p,r=1}^{n} f_1\big(u_{rp}^{(s)}\big) \phi\big(u_{pr}^{(s)}\big) \cdot \delta_{jk},$$

and we observe that the constant $\frac{1}{D_s} \sum_{p,r=1}^{n} f_1(u_{rp}^{(s)}) \phi(u_{pr}^{(s)})$ does not depend on j or k. Reciprocally, if ϕ is of this form, then we check that $\phi = \phi_{\mathrm{ad}}$ and, by (c) in Proposition 6.7, ϕ is ad-invariant. $\square$

In general, the mapping $\mathrm{ad}_h^* : \phi \mapsto \phi_{\mathrm{ad}}$ in Proposition 6.6 preserves neither hermiticity nor positivity, see Example 11.7. But [45, Lemma 4.1] and [4, Theorem 4.5] suggest that some properties of ad_h can be improved if we replace the antipode by the twisted antipode defined by $\widetilde{S}(a) = f_1 \star S(a)$ for $a \in \mathcal{A}$.

Theorem 6.10. *Let $\mathbb{G}$ be a compact quantum group with dense $*$-Hopf algebra $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$. Denote by $\widetilde{S}$ the twisted antipode defined by $\widetilde{S}(a) = f_1 \star S(a) = f_{-1}(a_{(1)}) S(a_{(2)})$ and denote by $\widetilde{\mathrm{ad}}$ the twisted adjoint action $\widetilde{\mathrm{ad}}(a) = a_{(1)} \widetilde{S}(a_{(3)}) \otimes a_{(2)}$, $a \in \mathcal{A}$.*

(a) *The map* $\widetilde{\mathrm{ad}}_h : \mathcal{A} \to \mathcal{A}$ *defined by*

$$\widetilde{\mathrm{ad}}_h(a) = (h \otimes \mathrm{id}) \circ \widetilde{\mathrm{ad}}(a) = h\big(a_{(1)}\widetilde{S}(a_{(3)})\big)a_{(2)}$$

satisfies

$$\widetilde{\mathrm{ad}}_h\big(a^*a\big) = (h \otimes \mathrm{id})\big(\big(\widetilde{\mathrm{ad}}(a)\big)^*\widetilde{\mathrm{ad}}(a)\big)$$

for $a \in \mathcal{A}$ *and therefore preserves positivity.*

(b) *If* $\mathbb{G}$ *is of Kac-type, then we have*

$$\widetilde{\mathrm{ad}}_h \circ \widetilde{\mathrm{ad}}_h = \widetilde{\mathrm{ad}}_h.$$

Proof. (b) was already shown in Proposition 6.7, since in the Kac case we have $\widetilde{\mathrm{ad}}_h = \mathrm{ad}_h$.

Let us now prove (a). Since the twisted antipode is an algebra anti-homomorphism, we have

$$\widetilde{\mathrm{ad}}\big(a^*a\big) = a^*_{(1),j}a_{(1),k}\widetilde{S}(a_{(3),k})\widetilde{S}\big(a^*_{(3),j}\big) \otimes a^*_{(2),j}a_{(2),k},$$

where we put back summation indices to distinguish the sums coming from the first and the second factor.

Therefore

$$(h \otimes \mathrm{id})\big(\widetilde{\mathrm{ad}}\big(a^*a\big)\big) = h\big(a^*_{(1),j}a_{(1),k}\widetilde{S}(a_{(3),k})\widetilde{S}\big(a^*_{(3),j}\big)\big)a^*_{(2),j}a_{(2),k}$$
$$= h\big(\sigma_i\big(\widetilde{S}\big(a^*_{(3),j}\big)\big)a^*_{(1),j}a_{(1),k}\widetilde{S}(a_{(3),k})\big)a^*_{(2),j}a_{(2),k}.$$

Now for any $b \in \mathcal{A}$,

$$\sigma_i\big(\widetilde{S}(b^*)\big) = f_{-1} \star \widetilde{S}(b^*) \star f_{-1} = S(b^*) \star f_{-1} = S^{-1}(b)^* \star f_{-1}$$
$$= \big(S^{-1}(b) \star f_1\big)^* = \big(f_1 \star S(b)\big)^* = \widetilde{S}(b)^*,$$

and so we get

$$(h \otimes \mathrm{id})\big(\widetilde{\mathrm{ad}}\big(a^*a\big)\big) = h\big(\big(a^*_{(1),j}\widetilde{S}(a_{(3),j})\big)^* a_{(1),k}\widetilde{S}(a_{(3),k})\big)a^*_{(2),j}a_{(2),k}$$
$$= (h \otimes \mathrm{id})\big(\big(\widetilde{\mathrm{ad}}(a)\big)^*\widetilde{\mathrm{ad}}(a)\big). \qquad \square$$

Recall that the linear span of the characters of the irreducible unitary corepresentations of a compact quantum group is an algebra

$$\mathcal{A}_0 = \mathrm{span}\bigg\{\chi_s = \sum_{j=1}^{n_s} u_{jj}^{(s)} : s \in \mathcal{I}\bigg\}, \tag{6.1}$$

called the *algebra of central functions* on $\mathbb{G}$.

Note that $\widetilde{\mathrm{ad}}_h(\mathcal{A}) \subseteq \mathcal{A}_0$. Indeed, $\widetilde{\mathrm{ad}}_h$ acts on coefficients of irreducible unitary corepresentations as

$$
\begin{aligned}
\widetilde{\mathrm{ad}}_h\big(u_{jk}^{(s)}\big) &= \sum_{p,q=1}^{n_s} h\big(u_{jp}^{(s)}\widetilde{S}(u_{qk}^{(s)})\big)u_{pq}^{(s)} \\
&= \sum_{p,q,\ell=1}^{n_s} h\big(u_{jp}^{(s)}S(u_{\ell k}^{(s)})\big)f_{-1}\big(u_{q\ell}^{(s)}\big)u_{pq}^{(s)} \\
&= \sum_{p,q,\ell=1}^{n_s} h\big(u_{jp}^{(s)}\big(u_{k\ell}^{(s)}\big)^*\big)f_{-1}\big(u_{q\ell}^{(s)}\big)u_{pq}^{(s)} \\
&= \frac{1}{D_s}\sum_{p,q,\ell=1}^{n_s} \delta_{jk}f_1\big(u_{\ell p}^{(s)}\big)f_{-1}\big(u_{q\ell}^{(s)}\big)u_{pq}^{(s)} \\
&= \frac{\delta_{jk}}{D_s}\sum_{p=1}^{n_s} u_{pp}^{(s)}.
\end{aligned}
\tag{6.2}
$$

We see that we can use $\widetilde{\mathrm{ad}}_h^{\,*} : \phi \mapsto \phi \circ \widetilde{\mathrm{ad}}_h$ to produce ad-invariant functionals.

Corollary 6.11. *The linear map* $\widetilde{\mathrm{ad}}_h^{\,*} : (\mathcal{A}_0)' \to \mathcal{A}'$, $\widetilde{\mathrm{ad}}_h^{\,*}(\phi) = \phi \circ \widetilde{\mathrm{ad}}_h$, *maps functionals on* $\mathcal{A}_0$ *to ad-invariant functionals on* $\mathcal{A}$. *It maps states on* $\mathcal{A}_0$ *to states on* $\mathcal{A}$.

If $\mathbb{G}$ *is of Kac type, then* $\widetilde{\mathrm{ad}}_h^{\,*}$ *defines bijections between states on* $\mathcal{A}_0$ *and ad-invariant states on* $\mathcal{A}$, *and between generating functionals on* $\mathcal{A}_0$ *and ad-invariant generating functionals on* $\mathcal{A}$.

Proof. It follows immediately from Eq. (6.2) and Proposition 6.9 that for any $\phi \in \mathcal{A}'$ the functional $\widetilde{\mathrm{ad}}_h^{\,*}(\phi) = \phi \circ \widetilde{\mathrm{ad}}_h$ is ad-invariant, since

$$
\phi \circ \widetilde{\mathrm{ad}}_h\big(u_{jk}^{(s)}\big) = \frac{\delta_{jk}}{D_s}\phi\bigg(\sum_{\ell=1}^{n_s} u_{\ell\ell}^{(s)}\bigg).
$$

By Theorem 6.10, $\widetilde{\mathrm{ad}}_h^{\,*}$ maps positive functionals to positive functionals. Since we have also $\widetilde{\mathrm{ad}}(\mathbf{1}) = \mathbf{1}$ and

$$
\phi \circ \widetilde{\mathrm{ad}}_h(\mathbf{1}) = \phi(\mathbf{1}),
$$

it follows that $\widetilde{\mathrm{ad}}_h^{\,*}$ maps states on $\mathcal{A}$ onto ad-invariant states on $\mathcal{A}$.

In the Kac case we have furthermore $\varepsilon \circ \widetilde{\mathrm{ad}}_h = \varepsilon \circ \mathrm{ad}_h = \varepsilon$, so in this case $\widetilde{\mathrm{ad}}_h$ maps the kernel of the counit onto itself and therefore $\widetilde{\mathrm{ad}}_h^{\,*}$ maps generating functionals on $\mathcal{A}$ onto ad-invariant generating functionals on $\mathcal{A}$.

Conversely, any state or generating functional ψ on $\mathcal{A}_0$ can be extended to a state or generating functional $\hat{\psi} = \psi \circ \widetilde{\mathrm{ad}}_h$ on $\mathcal{A}$. By Proposition 6.9 it is clear that $\hat{\psi}$ is the unique ad-invariant extension of ψ. $\square$

In Section 10, we will show that this corollary allows to completely characterize the ad-invariant generating functionals on the free orthogonal quantum group O_n^+.

7. Dirichlet forms

In this section we determine explicitly the structure of the Dirichlet forms associated to KMS-symmetric generating functionals on compact quantum groups. In case of GNS symmetry, we also characterize the invariance under translation of generators on the algebra $\mathcal{A}$, in terms of an associated quadratic form on $\mathcal{A}$.

Recall that $L^2(\mathsf{A}, h)$ denotes the GNS Hilbert space of (A, h) and that the cyclic vector $\xi_h = 1_\mathsf{A} \in L^2(\mathsf{A}, h)$ represents the Haar state as $h(a) = \langle \xi_h, a\xi_h \rangle$. From now on and until the end of Section 8, we assume that the Haar state is faithful on the C*-algebra A so that we can identify A with an involutive subalgebra of the von Neumann algebra $L^\infty(\mathsf{A}, h)$ of bounded operators on $L^2(\mathsf{A}, h)$, generated by A by the GNS representation. As a consequence, the vector ξ is cyclic for the von Neumann algebra $L^\infty(\mathsf{A}, h)$ too.

Notice that, as the Haar state h is a $(\sigma, -1)$-KMS state for the modular automorphism group σ (see Section 2.4), it follows, by the KMS theory (in particular [6, Corollary 5.3.9]), that the vector ξ is also separating for the von Neumann algebra $L^\infty(\mathsf{A}, h)$. This fact allows to apply the Tomita–Takesaki modular theory to the Haar state h on the C*-algebra A of the compact quantum group.

Recall also that the symmetric embedding is defined by

$$i_s : \mathsf{A} \to L^2(\mathsf{A}, h), \qquad i_s(a) = \Delta^{\frac{1}{4}} a \xi_h,$$

where Δ denotes (exceptionally) the Tomita–Takesaki modular operator. This definition agrees with the one from Section 4 (page 2803), since for $a \in \mathcal{A}$ we have

$$i_s(a) = \Delta^{\frac{1}{4}} a \Delta^{-\frac{1}{4}} \xi_h = \sigma_{-\frac{i}{4}}(a)\xi_h.$$

For a given KMS-symmetric generating functional ϕ of a Lévy process on $\mathcal{A}$, and the related convolution operator $L_\phi(a) = \phi \star a$, we define the sesquilinear and the quadratic forms

$$\mathcal{E}_\phi\big(i_s(a), i_s(b)\big) = \big\langle i_s(a), i_s\big(-L_\phi(b)\big)\big\rangle = -h\big(\sigma_{-\frac{i}{4}}(a)^*(\sigma_{-\frac{i}{4}} \circ L_\phi)(b)\big),$$
$$\mathcal{E}_\phi\big[i_s(a)\big] = \mathcal{E}_\phi\big(i_s(a), i_s(a)\big),$$

on the domain

$$D(\mathcal{E}_\phi) = \big\{i_s(a) \in L^2(\mathsf{A}, h) \colon a \in D(L_\phi) \text{ and } \mathcal{E}_\phi\big[i_s(a)\big] < \infty\big\}.$$

The explicit values of the sesquilinear form $\mathcal{E}_\phi$ on the basis of the coefficients of the unitary corepresentations are the following

$$\mathcal{E}_\phi\big(i_s\big(u_{jk}^{(s)}\big), i_s\big(u_{lm}^{(t)}\big)\big) = \big\langle i_s\big(u_{jk}^{(s)}\big), i_s\big(-L_\phi\big(u_{lm}^{(t)}\big)\big)\big\rangle = \sum_r \big\langle i_s\big(u_{jk}^{(s)}\big), i_s\big(u_{lr}^{(t)}\big)\big\rangle \phi\big(u_{rm}^{(t)}\big)$$

$$= \sum_r h\big(\sigma_{-\frac{i}{4}}\big(u_{jk}^{(s)}\big)^* \sigma_{-\frac{i}{4}}\big(u_{lr}^{(t)}\big)\big)\phi\big(u_{rm}^{(t)}\big)$$

$$= \sum_{r,p,p'} h\big(\big(u_{jk}^{(s)}\big)^* u_{pp'}^{(t)}\big) f_{\frac{1}{2}}\big(u_{lp}^{(t)}\big) f_{\frac{1}{2}}\big(u_{p'r}^{(t)}\big)\phi\big(u_{rm}^{(t)}\big)$$

$$= \frac{\delta_{st}}{D_s} \sum_p f_{-1}\big(u_{pj}^{(t)}\big) f_{\frac{1}{2}}\big(u_{lp}^{(t)}\big) \sum_r f_{\frac{1}{2}}\big(u_{kr}^{(t)}\big)\phi\big(u_{rm}^{(t)}\big)$$

$$= \frac{\delta_{st}}{D_s} f_{-\frac{1}{2}}\big(u_{lj}^{(t)}\big)\big(f_{\frac{1}{2}} \star \phi\big)\big(u_{km}^{(t)}\big).$$

Since σ_z leaves the subspaces V_s invariant, hence

$$i_s(V_s) = \sigma_{-\frac{i}{4}}(V_s)\xi_h = V_s\xi_h = i_h(V_s)$$

and the operator defined by

$$H_\phi i_s(a) := i_s(-L_\phi a), \qquad a \in D(H_\phi) := i_s(\mathcal{A}) \subset L^2(\mathsf{A}, h)$$

leaves invariant the subspaces

$$E_s = V_s\xi_h = \mathrm{Span}\big\{u_{jk}^{(s)}\xi_h\colon j, k = 1, \dots, n_s\big\} \subset L^2(\mathsf{A}, h), \quad s \in \mathcal{I}.$$

Therefore, since $L^2(A, h) = \bigoplus_{s \in \mathcal{I}} E_s$, the operator H_ϕ decomposes as

$$H_\phi = \bigoplus_{s \in \mathcal{I}} H_\phi^s$$

a direct sum of its restrictions H_ϕ^s on each finite dimensional subspace E_s.

Theorem 7.1. *Let ϕ be a KMS-symmetric generating functional of a Lévy process on $\mathcal{A}$. Then the operator H_ϕ is essentially self-adjoint, the quadratic form $\mathcal{E}_\phi$ is closable and its closure is a Dirichlet form.*

Proof. The operator H_ϕ is a direct sum of bounded operators and is symmetric as L_ϕ is KMS symmetric. It follows that H_ϕ is essentially self-adjoint and its closure is given by

$$D(\overline{H_\phi}) = \Big\{ \xi = \bigoplus_{s \in \mathcal{I}} \xi_s \in L^2(A, h)\colon \sum_{s \in \mathcal{I}} \|H_\phi\xi_s\|^2 < +\infty \Big\},$$

$$\overline{H_\phi}\Big(\bigoplus_{s \in \mathcal{I}} \xi_s\Big) = \bigoplus_{s \in \mathcal{I}} H_\phi\xi_s, \qquad \bigoplus_{s \in \mathcal{I}} \xi_s \in D(\overline{H_\phi}).$$

As, by definition, $\mathcal{E}_\phi[\xi] = \langle \xi, H_\phi \xi \rangle$ for $\xi \in D(H_\phi)$, we have that $\mathcal{E}_\phi$ is closable and its closure is given by

$$D(\overline{\mathcal{E}_\phi}) = \left\{ \xi = \bigoplus_{s \in \mathcal{I}} \xi_s \in L^2(A, h) \colon \sum_{s \in \mathcal{I}} \langle \xi_s, H_\phi \xi_s \rangle < +\infty \right\},$$

$$\overline{\mathcal{E}_\phi}\left[\bigoplus_{s \in \mathcal{I}} \xi_s \right] = \sum_{s \in \mathcal{I}} \langle \xi_s, H_\phi \xi_s \rangle, \qquad \bigoplus_{s \in \mathcal{I}} \xi_s \in D(\overline{\mathcal{E}_\phi}).$$

Now, the quantum Markov semigroup T_t on the C*-algebra A, generated by L_ϕ, is KMS symmetric, i.e. it is $(\sigma, -1)$-KMS symmetric in the sense of Definition 2.1 in [11] (see also Definition 2.31 in [12]). By Theorem 2.3 and Theorem 2.4 in [11] (see also Theorem 2.39 and Theorem 2.44 in [12]) the semigroup $e^{-t\overline{H}_\phi}$ on $L^2(\mathsf{A}, h)$ is Markovian so that the quadratic form $\overline{\mathcal{E}_\phi}$ is a Dirichlet form by Theorem 4.11 in [10] (see also Theorem 2.52 in [12]). □

Remark 7.2. Using the embedding $i_h \colon \mathsf{A} \to L^2(\mathsf{A}, h)$, we can identify the Dirichlet form on $L^2(\mathsf{A}, h)$, associated to a KMS-symmetric generating functional ϕ, with the following quadratic form on the C*-algebra A

$$\mathcal{Q}_\phi[a] = \mathcal{E}_\phi\big[i_h(a)\big] = -h\big(a^*(\sigma_{-\frac{i}{4}} \circ L_\phi \circ \sigma_{\frac{i}{4}})(b)\big)$$

defined on $dom(\mathcal{Q}_\phi) := \{a \in A \colon i_h(a) \in dom(\mathcal{E}_\phi)\}$. If furthermore, ϕ is GNS-symmetric, then L_ϕ commutes with the modular group $(\sigma_z)_z$ (Proposition 4.8 and Corollary 4.9) and one has

$$\mathcal{Q}_\phi[a] = -h\big(a^* L_\phi(a)\big).$$

The next theorem shows that the Dirichlet forms associated to GNS-symmetric Lévy processes admit an additional invariance.

Theorem 7.3. *Let L be a GNS symmetric operator on $\mathcal{A} \subset L^2(\mathsf{A}, h)$. Then the following conditions are equivalent:*

(1) *There exists a functional $\phi \in \mathcal{A}'$ such that $L = L_\phi$, where $L_\phi = (\mathrm{id} \otimes \phi) \circ \Delta$;*
(2) *L is translation invariant on $\mathcal{A}$;*
(3) *The semigroup $(T_t)_{t \geqslant 0}$ on $\mathcal{A}$ (or A_r or A_u) associated to L by the formula $T_t|_{\mathcal{A}} = \exp_\star tL$ is translation invariant;*
(4) *The sesquilinear form $\mathcal{Q}$ defined by $\mathcal{Q}(a, b) = -h(a^* L(b))$ on $\mathcal{A}$ satisfies*

$$\mathcal{Q}(a, b)\mathbf{1} = (m_* \otimes \mathcal{Q})\big(\Delta(a), \Delta(b)\big), \quad a, b \in \mathcal{A}, \tag{7.1}$$

where m_* denotes the sesquilinear map obtained from the multiplication, namely, $m_*(a, b) = a^* b$.

Proof. We already observed in Section 2.5 the equivalence (1) ⇔ (2) whereas the equivalence (2) ⇔ (3) follows from Theorem 3.4, so that we need to prove only (1) ⇔ (4). Let us assume that L satisfies

$$(\mathrm{id} \otimes L) \circ \Delta = \Delta \circ L.$$

Then, using Sweedler notation and the invariance of the Haar state,

$$
\begin{aligned}
(m_* \otimes \mathcal{Q})\big(\Delta(a), \Delta(b)\big) &= a^*_{(1)} b_{(1)} \mathcal{Q}(a_{(2)}, b_{(2)}) \\
&= -a^*_{(1)} b_{(1)} h\big(a^*_{(2)} L(b_{(2)})\big) = -(\mathrm{id} \otimes h)\big(\big(a^*_{(1)} \otimes a^*_{(2)}\big)(\mathrm{id} \otimes L)\Delta(b)\big) \\
&= -(\mathrm{id} \otimes h)\big(\big(a^*_{(1)} \otimes a^*_{(2)}\big)\big(L(b)_{(1)} \otimes L(b)_{(2)}\big)\big) \\
&= -(\mathrm{id} \otimes h)\Delta\big(a^* L(b)\big) = -h\big(a^* L(b)\big)\mathbf{1} = \mathcal{Q}(a,b)\mathbf{1}.
\end{aligned}
$$

On the other hand, if we assume that Eq. (7.1) holds, then

$$
\begin{aligned}
(h \otimes h)\big(a^* \otimes \mathbf{1}\big)\Delta\big(b^*\big)\big((\mathrm{id} \otimes L)\Delta(c)\big) \\
= (h \otimes h)\big(a^* b^*_{(1)} c_{(1)} \otimes b^*_{(2)} L(c_{(2)})\big) = -h\big(a^* b^*_{(1)} c_{(1)}\big)\mathcal{Q}(b_{(2)}, c_{(2)}) \\
= -h\big(a^*(m_* \otimes \mathcal{Q})\big(\Delta(b), \Delta(c)\big)\big) = -h\big(a^*\big)\mathcal{Q}(b,c)
\end{aligned}
$$

and

$$
\begin{aligned}
(h \otimes h)\big(a^* \otimes \mathbf{1}\big)\Delta\big(b^*\big)(\Delta \circ L)(c) \\
= (h \otimes h)\big(a^* b^*_{(1)} (Lc)_{(1)} \otimes b^*_{(2)} (Lc)_{(2)}\big) = h\big(a^*(\mathrm{id} \otimes h)\Delta\big(b^* L(c)\big)\big) \\
= h\big(a^*\big)h\big(b^* L(c)\big) = -h\big(a^*\big)\mathcal{Q}(b,c).
\end{aligned}
$$

Since $\mathcal{A} \odot \mathcal{A}$ is the linear span of $(\mathcal{A} \otimes 1)\Delta(\mathcal{A})$ and $h \otimes h$ is faithful on $\mathcal{A} \odot \mathcal{A}$, we conclude that L is translation invariant. □

Corollary 7.4. *Let $\phi \in \mathcal{A}'$ be the generating functional of a GNS-symmetric Lévy process and $\mathcal{E}_\phi$ the associated Dirichlet form. Then the sesquilinear form $\mathcal{Q}$ on $\mathcal{A}$ defined by*

$$\mathcal{Q}(a,b) := \mathcal{E}_\phi\big(i_h(a), i_h(b)\big), \quad a, b \in \mathcal{A},$$

satisfies Eq. (7.1). Conversely, let L be a GNS-symmetric operator on $\mathcal{A}$ such that $L(\mathbf{1}) = 0$, L is hermitian and positive on $\ker \varepsilon$, and the sesquilinear form on $\mathcal{A}$ defined by

$$\mathcal{Q}(a,b) := -h\big(a^* Lb\big), \quad a, b \in \mathcal{A},$$

satisfies Eq. (7.1). Then $L = L_\phi$ for a generating functional ϕ of a GNS-symmetric Lévy process.

8. Derivations, cocycles and spectral triples

In this section we associate to any Lévy process on a CQG $\mathbb{G} = (\mathsf{A}, \Delta)$, a natural derivation on its Hopf $*$-subalgebra $\mathcal{A}$, with values in a Hilbert bimodule over the C*-algebra $\mathsf{A} = C(\mathbb{G})$. This gives rise, on the same bimodule, to a self-adjoint operator D, with respect to which we prove that the elements of $\mathcal{A}$ are "Lipschitz" in a natural, suitable sense. The construction makes essential use of the Schürmann triple associated to the generator of the process.

In case the GNS symmetry holds true, we will show that the derivation is, essentially, a differential square root of the generator H_ϕ. Moreover, if the spectrum of H_ϕ on $L^2(A, h)$ is discrete, then the Hilbert bimodule and the operator D form a spectral triple in the sense of the noncommutative geometry of A. Connes [16]. This fact suggests to refer to D as the *Dirac operator* associated to the process.

We remark that the role of GNS symmetry of the process is to provide a suitable closability property of the derivation, needed to prove that the Dirac operator D is self-adjoint and that the spectrum of the *Dirac Laplacian* D^2 coincides with that of the generator H_ϕ, away from zero.

We will show in Section 9 that in case the CQG is a compact Lie group and the Lévy process is the Brownian motion associated to a given Riemannian metric, the differential calculus illustrated above reduces to the familiar one: the derivation coincides with the gradient operator and the Lipschitz property has the usual meaning.

Consider on the Hopf $*$-subalgebra $\mathcal{A}$ of a compact quantum group $\mathbb{G} = (\mathsf{A}, \Delta)$, the generating functional $\phi \in \mathcal{A}'$ of a Lévy process and its associated Schürmann triple $((\pi, H_\pi), \eta, \phi)$ on a Hilbert space H_π (see Remark 5.7).

Denote by $\lambda_L, \lambda_R : \mathsf{A} \to B(L^2(\mathsf{A}, h))$ the left and right actions of A on the Hilbert space $L^2(A, h)$

$$\lambda_L(a)(b\xi_h) := ab\xi_h,$$
$$\lambda_R(a)(b\xi_h) := ba\xi_h, \quad a, b \in \mathsf{A},$$

where $\xi_h \in L^2(\mathsf{A}, h)$ denotes the cyclic vector representing the Haar state. Recall now that $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a morphism of C*-algebras and λ_L, π are representations of the C*-algebra A, so that $\lambda_L \otimes \pi$ is a representation of the C*-algebra $\mathsf{A} \otimes \mathsf{A}$. Correspondingly, consider the left and right actions $\rho_L, \rho_R : \mathsf{A} \to B(L^2(\mathsf{A}, h) \otimes H_\pi)$ of A on the Hilbert space $L^2(\mathsf{A}, h) \otimes H_\pi$ defined by

$$\rho_L := (\lambda_L \otimes \pi) \circ \Delta,$$
$$\rho_R := \lambda_R \otimes \mathrm{id}_{H_\pi},$$

or, more explicitly, by

$$\rho_L(a)(b\xi_h \otimes v) = \big((\lambda_L \otimes \pi) \circ \Delta(a)\big)(b\xi_h \otimes v) = \sum a_{(1)}b\xi_h \otimes \pi(a_{(2)})v,$$
$$\rho_R(a)(b\xi_h \otimes v) = \lambda_R(a)(b\xi_h) \otimes v = ba\xi_h \otimes v,$$

for $a, b \in \mathsf{A}$ and $v \in H_\pi$. The actions λ_L, ρ_L are continuous and form representations of the C*-algebra A. Likewise, also the actions λ_R, ρ_R are continuous and form antirepresentations of the C*-algebra A or representations of the opposite C*-algebra A^{op}. Moreover, as λ_L, λ_R (resp. ρ_L, ρ_R) commute, they provide a A-bimodule structure on the Hilbert space $L^2(A, h)$ (resp. $L^2(\mathsf{A}, h) \otimes H_\pi$).

In the following we shall adopt the simplified notations: for $a \in \mathsf{A}$ and $\xi \in L^2(\mathsf{A}, h) \otimes H_\pi$ we write

$$a \cdot \xi := \rho_L(a)\xi,$$

$$\xi \cdot a := \rho_R(a)\xi.$$

Recall that we denote by $i_h : \mathsf{A} \to L^2(\mathsf{A}, h)$ the GNS embedding (cf. page 2802)

$$i_h(a) := a\xi_h, \quad a \in \mathsf{A}.$$

Proposition 8.1. *Consider on the Hopf $*$-subalgebra $\mathcal{A}$ of a compact quantum group $\mathbb{G} = (\mathsf{A}, \Delta)$, the generating functional $\phi \in \mathcal{A}'$ of a Lévy process, its associated Schürmann triple $((\pi, H_\pi), \eta, \phi)$ and the induced A-bimodule structure on $L^2(\mathsf{A}, h) \otimes H_\pi$. Then the linear map defined by*

$$\partial : \mathcal{A} \to L^2(\mathsf{A}, h) \otimes H_\pi \qquad \partial := (i_h \otimes \eta) \circ \Delta$$

or, more explicitly, by

$$\partial a = (i_h \otimes \eta)(\Delta a) = \sum a_{(1)}\xi_h \otimes \eta(a_{(2)}), \quad a \in \mathcal{A},$$

is a derivation in the sense that it satisfies the Leibniz rule

$$\partial(ab) = (\partial a) \cdot b + a \cdot (\partial b), \quad a, b \in \mathcal{A}.$$

Proof. The map is well defined because $\Delta(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{A}$ (where, forcing notation a little bit, we denoted by $\mathcal{A} \otimes \mathcal{A}$ the image in $\mathsf{A} \otimes \mathsf{A}$ of the subspace $\mathcal{A} \odot \mathcal{A} \subseteq \mathsf{A} \odot \mathsf{A}$ under the canonical quotient map from $\mathsf{A} \odot \mathsf{A}$ to $\mathsf{A} \otimes \mathsf{A}$).

As the map $\eta : \mathcal{A} \to H_\pi$ is a 1-cocycle and the counit $\varepsilon : \mathcal{A} \to \mathbb{C}$ satisfies the identity $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$, i.e. $\sum b_{(1)}\varepsilon(b_{(2)}) = b$, we have

$$\partial(ab) = (i_h \otimes \eta)\big(\Delta(ab)\big) = (i_h \otimes \eta)\big(\Delta(a)\Delta(b)\big)$$

$$= \sum_{j,k} a_{(1),j} b_{(1),k}\xi_h \otimes \eta(a_{(2),j} b_{(2),k})$$

$$= \sum_{j,k} a_{(1),j} b_{(1),k}\xi_h \otimes \big[\pi(a_{(2),j})\eta(b_{(2),k}) + \eta(a_{(2),j})\varepsilon(b_{(2),k})\big]$$

$$= \left(\sum_j \lambda_L(a_{(1),j}) \otimes \pi(a_{(2),j}) \right) \left(\sum_k b_{(1),k}\xi_h \otimes \eta(b_{(2),k}) \right)$$

$$+ \left(\sum_k \lambda_R(b_{(1),k}) \otimes \varepsilon(b_{(2),k})\mathrm{id}_{H_\pi} \right) \left(\sum_j a_{(1),j}\xi_h \otimes \eta(a_{(2),j}) \right)$$

$$= \rho_L(a)\big(\partial(b)\big) + \left(\lambda_R\left(\sum_k b_{(1),k}\varepsilon(b_{(2),k}) \right) \otimes \mathrm{id}_{H_\pi} \right)\big(\partial(a)\big)$$

$$= \rho_L(a)\big(\partial(b)\big) + \big(\lambda_R(b) \otimes \mathrm{id}_{H_\pi}\big)\big(\partial(a)\big)$$

$$= \rho_L(a)\big(\partial(b)\big) + \rho_R(b)\big(\partial(a)\big)a \cdot \partial(b) + \partial(a) \cdot b. \qquad \square$$

Proposition 8.2. *Let $\phi \in \mathcal{A}'$ be a GNS-symmetric generating functional and consider the hermitian convolution generator $L_\phi : \mathcal{A} \to \mathcal{A}$, its Hilbert-space extension $(H_\phi, D(H_\phi))$ as well as the Dirichlet form $(\mathcal{E}_\phi, D(\mathcal{E}_\phi))$ (see Section 7).*

Then the operator $d : D(d) \to L^2(A,h) \otimes H_\pi$ defined as

$$D(d) := i_h(\mathcal{A}) = \mathcal{A}\xi_h \subset L^2(A,h), \qquad d\big(i_h(a)\big) := \partial a, \quad a \in \mathcal{A},$$

is closable and

$$\big\| d\big(i_h(a)\big) \big\|^2_{L^2(A,h)\otimes H_\pi} = 2\langle i_h(a), H_\phi i_h(a) \rangle_{L^2(A,h)} = 2\mathcal{E}_\phi\big[i_h(a)\big], \quad a \in \mathcal{A}.$$

Proof. We have

$$\big\| d\big(i_h(a)\big) \big\|^2 = \left\langle \sum_j a_{(1),j} \otimes \eta(a_{(2),j}), \sum_k a_{(1),k} \otimes \eta(a_{(2),k}) \right\rangle$$

$$= \sum_{j,k} h\big(a^*_{(1),j}a_{(1),k}\big)\big[\phi\big(a^*_{(2),j}a_{(2),k}\big) - \varepsilon\big(a^*_{(2),j}\big)\phi(a_{(2),k}) - \phi\big(a^*_{(2),j}\big)\varepsilon(a_{(2),k})\big]$$

$$= \sum_{j,k} h\big(a^*_{(1),j}a_{(1),k}\big)\phi\big(a^*_{(2),j}a_{(2),k}\big) - \sum_{j,k} h\big(a^*_{(1),j}a_{(1),k}\big)\varepsilon\big(a^*_{(2),j}\big)\phi(a_{(2),k})$$

$$- \sum_{j,k} h\big(a^*_{(1),j}a_{(1),k}\big)\phi\big(a^*_{(2),j}\big)\varepsilon(a_{(2),k}).$$

The first term vanishes because, by the GNS symmetry of L_ϕ, we have

$$\sum_{j,k} h\big(a^*_{(1),j}a_{(1),k}\big)\phi\big(a^*_{(2),j}a_{(2),k}\big) = h\big(\mathbf{1} \cdot L_\phi\big(a^*a\big)\big) = h\big(L_\phi(\mathbf{1})a^*a\big) = 0.$$

The second term becomes

$$\sum_{j,k} h\big(a^*_{(1),j}a_{(1),k}\big)\varepsilon\big(a^*_{(2),j}\big)\phi(a_{(2),k})$$

$$= h\left(\left(\sum_j a_{(1),j}\varepsilon(a_{(2),j}) \right)^* \left(\sum_k a_{(1),k}\phi(a_{(2),k}) \right) \right) = h\big(a^* L_\phi(a)\big).$$

The third term is the complex conjugate of the second term and, since it is real, they are the same. Hence the identity $\|d\xi\|^2 = 2\langle \xi, H_\phi\xi\rangle_{L^2(A,h)} = 2\mathcal{E}_\phi[\xi]$ holds for $\xi \in D(d)$. Since ϕ is also KMS symmetric, Theorem 7.1 implies that the quadratic form is closable. It follows that the operator d is closable, too. $\quad\square$

From now on we will denote by the same symbol $(d, D(d))$ the closure of the closable operator considered in the previous result.

One of the conclusion of the above result reads $\overline{H}_\phi = \frac{1}{2}d^* \circ d$, i.e. the L^2-generator has the aspect of a "generalized Laplacian" composed of a "generalized divergence" operator d^* and a "generalized gradient" operator d. In other words, the operator d (essentially the derivation ∂) is a differential square root of the L^2-generator.

The next result shows that in the noncommutative space $C(\mathbb{G})$, the elements of the dense subalgebra $\mathcal{A}$ have a *noncommutative Lipschitz property*. Below we denote by $\mathsf{A} \widehat{\otimes} H_\pi$ the projective tensor product of the C*-algebra A and the Hilbert space H_π which is the completion of the algebraic tensor product of A and H_π with respect to the norm

$$\|x\|_{A \widehat{\otimes} H_\pi} := \inf\Big\{ \sum \|a_i\|_A \|\xi_i\|_{H_\pi}, \text{ where } x = \sum a_i \otimes \xi_i \Big\},$$

see [23] or [34].

Proposition 8.3. *Let $\phi \in \mathcal{A}'$ be a GNS-symmetric generating functional with associated Schürmann triple $((\pi, H_\pi), \eta, \phi)$. Let us consider the Hilbert space $\mathcal{H}_\phi := (L^2(\mathsf{A}, h) \otimes H_\pi) \oplus L^2(\mathsf{A}, h)$ as a A-bimodule under the commuting left and right actions*

$$\pi_L := \rho_L \oplus \lambda_L, \qquad \pi_R := \rho_R \oplus \lambda_R.$$

Consider also on $\mathcal{H}_\phi$ the self-adjoint operator

$$D := \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}.$$

Then the commutator $[D, \pi_L(a)]$ is bounded for all $a \in \mathcal{A}$ with norm bounded by

$$\big\| [D, \pi_L(a)] \big\| \leqslant \|\partial a\|_{\mathsf{A} \widehat{\otimes} H_\pi}.$$

Proof. It follows from

$$[D, \pi_L(a)] = D \circ \pi_L(a) - \pi_L(a) \circ D$$

$$= \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}\begin{pmatrix} \rho_L(a) & 0 \\ 0 & \lambda_L(a) \end{pmatrix} - \begin{pmatrix} \rho_L(a) & 0 \\ 0 & \lambda_L(a) \end{pmatrix}\begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & d \circ \lambda_L(a) - \rho_L(a) \circ d \\ d^* \circ \rho_L(a) - \lambda_L(a) \circ d^* & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & d \circ \lambda_L(a) - \rho_L(a) \circ d \\ -(d \circ \lambda_L(a^*) - \rho_L(a^*) \circ d)^* & 0 \end{pmatrix}$$

that $[D, \pi_L(a)]$ is bounded for all $a \in \mathcal{A}$ if and only if $d \circ \lambda_L(a) - \rho_L(a) \circ d$ is bounded for all $a \in \mathcal{A}$. To check that the latter is actually the case, let us observe that, for $b \in \mathcal{A}$, we have

$$\big(d \circ \lambda_L(a) - \rho_L(a) \circ d\big) i_h(b) = d\big(i_h(ab)\big) - \rho_L(a)(\partial b)$$

$$= \partial(ab) - \rho_L(a)(\partial b) = \rho_R(b)(\partial a).$$

For any presentation $\partial a = \sum_{k=1}^{n} a_k \otimes \xi_k \in \mathsf{A} \otimes H_\pi$ we then have

$$\big\|\rho_R(b)(\partial a)\big\|_{L^2(\mathsf{A},h) \otimes H_\pi} = \big\|(\lambda_R \otimes \mathrm{id}_{H_\pi})(\partial a)\big\|_{L^2(\mathsf{A},h) \otimes H_\pi}$$

$$= \left\|\sum_{k=1}^{n} a_k b \xi_h \otimes \xi_k\right\|_{L^2(\mathsf{A},h) \otimes H_\pi}$$

$$\leqslant \sum_{k=1}^{n} \|a_k b \xi_h\|_{L^2(\mathsf{A},h)} \|\xi_k\|_{H_\pi}$$

$$\leqslant \big\|i_h(b)\big\|_{L^2(A,h)} \sum_{k=1}^{n} \|a_k\|_A \cdot \|\xi_k\|_{H_\pi}.$$

Optimizing among all presentations $\partial a = \sum_{k=1}^{n} a_k \otimes \xi_k \in \mathsf{A} \otimes H_\pi$ we get

$$\big\|\rho_R(b)(\partial a)\big\|_{L^2(\mathsf{A},h) \otimes H_\pi} \leqslant \big\|i_h(b)\big\|_{L^2(A,h)} \cdot \|\partial a\|_{A \widehat{\otimes} H_\pi}, \quad a, b \in \mathcal{A},$$

so that

$$\big\|d \circ \lambda_L(a) - \rho_L(a) \circ d\big\| \leqslant \|\partial a\|_{A \widehat{\otimes} H_\pi}, \quad a \in \mathcal{A}.$$

Finally notice that, setting $T_a := d \circ \lambda_L(a) - \rho_L(a) \circ d$, we have $\|T_a\| \leqslant \|\partial a\|_{A \widehat{\otimes} H_\pi}$,

$$[D, \pi_L(a)] = \begin{pmatrix} 0 & T_a \\ -T_a^* & 0 \end{pmatrix}$$

and

$$\big|[D, \pi_L(a)]\big|^2 = \begin{pmatrix} T_a T_a^* & 0 \\ 0 & T_a^* T_a \end{pmatrix},$$

so that

$$\big\|\,[D, \pi_L(a)]\,\big\| = \|T_a\|, \quad a \in \mathcal{A}. \qquad \square$$

Theorem 8.4. *Consider, on the Hopf $*$-subalgebra $\mathcal{A}$ of a compact quantum group $\mathbb{G} = (\mathsf{A}, \Delta)$, the GNS-symmetric generating functional $\phi \in \mathcal{A}'$ with Schürmann triple $((\pi, H_\pi), \eta, \phi)$.*

Consider also the GNS-symmetric, hermitian convolution generator $L_\phi : \mathcal{A} \to \mathcal{A}$ and its closed extension $(H_\phi, D(H_\phi))$ on the space $L^2(A, h)$, characterized by

$$H_\phi\big(i_h(a)\big) := -i_h(L_\phi a)$$

on its core $i_h(\mathcal{A}) = \mathcal{A}\xi_h \subset D(H_\phi)$.

If the spectrum of $(H_\phi, D(H_\phi))$ is discrete and considering the representation of $\mathsf{A} = C(\mathbb{G})$ constructed above

$$\pi_L := \rho_L \oplus \lambda_L : \mathsf{A} \to B(\mathcal{H}_\phi) \qquad \mathcal{H}_\phi := \big(L^2(\mathsf{A}, h) \otimes H_\pi\big) \oplus L^2(\mathsf{A}, h),$$

we have that $(\mathcal{A}, D, (\pi_L, \mathcal{H}_\phi))$ is a (possibly kernel-degenerate) spectral triple in the sense that

- *$[D, \pi_L(a)]$ is a bounded operator for all $a \in \mathcal{A}$,*
- *D has discrete spectrum on the orthogonal complement of its kernel.*

Proof. By construction

$$D^2 = \begin{pmatrix} dd^* & 0 \\ 0 & d^*d \end{pmatrix},$$

so that the spectrum of D^2 is the union of the spectra of dd^* and d^*d. Since these two operators are unitarily equivalent on the orthogonal complement of their kernels and zero belongs to the spectrum of d^*d, the spectrum of D^2 coincides with the spectrum of $2H_\phi$, by Proposition 8.2. Since, by assumption, the spectrum of H_ϕ is discrete we have that the spectrum of D^2, hence the one of D, are discrete too on the orthogonal complement of their kernels. This result, together with Theorem 8.3 allows us to conclude the proof. $\square$

The fact that the kernel of the Dirac operator D may be infinite dimensional is a variation with respect to the original definition of spectral triple given in [16], due to the definition of D as an anti-diagonal matrix. To construct the associated K-homology invariants this fact has to be taken into account, for example using the methods developed in Section 3 of [13]. This degeneracy raises the problem if, in the hypothesis of the above theorem, the spectral triple may be used to construct a trace on the C*-algebra as in the Connes Trace Theorem (proved in [14]) for the usual spectral triple.

9. Two classical examples: commutative and cocommutative CQGs

9.1. Algebras of functions on compact groups

Let G be a compact Lie group and let $C(G)$ denote the commutative C*-algebra of all continuous functions on G. Then $C(G)$ is a compact quantum group with the comultiplication

$$\Delta : C(G) \to C(G) \otimes C(G) \cong C(G \times G),$$

defined by

$$\Delta(f)(s,t) = f(st), \quad f \in C(G), \ s,t \in G.$$

The counit and the antipode are defined on the dense $*$-subalgebra $C_c(G)$ generated by the coefficients of arbitrary continuous finite-dimensional representation π, i.e. functions $\pi_{ij} : G \to \mathbb{C}$, and they are given by

$$\varepsilon(f) = f(e), \qquad S(f)(x) = f(x^{-1}), \quad f \in C_c(G).$$

This is a general example of a commutative compact quantum group, in the sense that if A is the algebra of continuous functions on a compact quantum group which is commutative as a C*-algebra, then there exists a unique compact group G such that A is isomorphic to $C(G)$ with coproduct corresponding to the classical one given above (cf. [46, Theorem 1.5]).

The quantum group $C(G)$ is cocommutative if and only if the group G is commutative. It is always of Kac type, i.e. $S^2 = \mathrm{id}$. This implies that the modular automorphism group is trivial and that the Haar state is tracial (see Remark 2.3), and so the notions of GNS- and KMS-symmetry coincide (see Remark 4.10).

The generating functionals of Lévy processes in G are classified by Hunt's formula as follows (cf. [27]). Let $\{X_1, X_2, \ldots, X_d\}$ be a fixed basis of the Lie algebra $\mathfrak{g}$ associated to the Lie group G and let $x_1, x_2, \ldots, x_d \in C_c^\infty(G)$ be the local coordinates associated to this basis, i.e. $X_i = \frac{\partial}{\partial x_i}$ at the neutral element e. Then an arbitrary generating functional ϕ is of the form

$$\phi(f) = \sum_{i=1}^d c_i X_i f(e) + \frac{1}{2} \sum_{j,k=1}^d a_{jk} X_j X_k f(e)$$

$$+ \int_{G \backslash \{e\}} \left(f(g) - f(e) - \sum_{i=1}^d x_i(g) X_i f(e) \right) \nu(dg)$$

for twice differentiable f. Here c_i, a_{jk} are real constants, $(a_{jk})_{j,k=1}^d$ is a positive definite symmetric matrix and the measure ν on G satisfies

$$\nu(\{e\}) = 0, \qquad \int\limits_{U} \sum_{i=1}^{d} x_i^2 \, \mathrm{d}\nu < \infty, \qquad \nu(G \setminus U) < \infty,$$

for any neighborhood U of e in G. The first term in the decomposition above is called the *drift*, whereas the second one is called the *diffusion*. The measure ν is called *Lévy measure*.

The GNS-symmetric processes correspond to functionals with no drift part and symmetric Lévy measures, i.e. $\nu(E) = \nu(E^{-1})$ for measurable E (see [27, Proposition 4.3], where such processes are called *invariant under the inverse map*).

The characterization of ad-invariant processes (called *conjugate invariant* in [27]) depends on the particular group structure. The two extreme cases are abelian Lie groups and simple Lie groups. In the first case, as observed in Section 6, the adjoint action is trivial and all functionals are ad-invariant. If the Lie group is simple and connected, then the adjoint action $\mathrm{ad}(f)(x, y) = f(xyx^{-1})$ has trivial kernel. Then the Lévy measure of an ad-invariant process must be *conjugate-invariant* (or *central*, cf. [1]), that is $\nu(gEg^{-1}) = \nu(E)$ for all measurable E. Moreover, the drift part vanishes and the diffusion part is (up to a constant) the Beltrami–Laplace operator on G (see [27, Propositions 4.4, 4.5]). In the case the Dirichlet form reduces to the Dirichlet integral on G

$$\mathcal{E}[a] = \int\limits_{G} \big|\nabla a(g)\big|^2 \, \mathrm{d}g$$

defined on the Sobolev space $H^{1,2}(G)$ of functions having square integrable gradient and the derivation is just the gradient operator. We refer to [27] for details on this topic.

9.2. C*-algebra of a countable discrete group

Let Γ be a countable discrete group and let $\ell^2(\Gamma)$ denote the Hilbert space of all square-summable functions on Γ. The space $\ell^2(\Gamma)$ is spanned by the orthonormal basis $\{\delta_g \colon g \in \Gamma\}$, where as usual $\delta_g(h) = 1$ if $g = h$ and $\delta_g(h) = 0$ otherwise. Then each element $g \in \Gamma$ defines the linear operator $\lambda_g \colon \ell^2(\Gamma) \to \ell^2(\Gamma)$ by the formula

$$\lambda_g(\delta_h) = \delta_{gh}, \quad h \in \Gamma.$$

Each λ_g is a unitary operator and the mapping $g \to \lambda_g$ is called the left regular unitary representation of the Hilbert space Γ on $\ell^2(\Gamma)$.

The closure of the $*$-algebra generated by $\{\lambda_g \colon g \in \Gamma\}$ in $B(\ell^2(\Gamma))$ is denoted by $C_r^*(\Gamma)$ and called the *reduced* C*-*algebra* or the *group algebra* of Γ. One can also define the *universal* C*-algebra of the group, denoted by $C_u^*(\Gamma)$, by taking the direct sum of all cyclic representations of Γ (universal representation) instead of the left regular one. The two algebras are isomorphic if and only if Γ is amenable, cf. [31].

The mapping Δ defined by $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ extends (in a unique way) to a $*$-homomorphism from $C_r^*(\Gamma)$ to $C_r^*(\Gamma) \otimes C_r^*(\Gamma)$ which preserves the unit. The pair

$(C_r^*(\Gamma), \Delta)$ is a compact quantum group. The linear span $\mathcal{A}$ of $\{\lambda_g \colon g \in \Gamma\}$ in $B(\ell^2(\Gamma))$ is a $*$-Hopf algebra on which counit and antipode are defined by $\varepsilon(\lambda_g) = 1$ and $S(\lambda_g) = \lambda_{g^{-1}}$ respectively, for $g \in \Gamma$.

The quantum group $C_r^*(\Gamma)$ is always cocommutative (i.e. the comultiplication is invariant under the flip). Moreover, each algebra of continuous functions on a compact quantum group which is cocommutative is essentially of this form (there exists a unique discrete group Γ and $*$-homomorphisms $C_u^*(\Gamma) \to \mathsf{A} \to C_r^*(\Gamma))$, see [46, Theorem 1.7]. Cocommutativity implies that the adjoint action is trivial: $\mathrm{ad}(a) = \mathbf{1} \otimes a$, $\mathrm{ad}_h(a) = a$ and all functionals are ad-invariant $\phi \circ \mathrm{ad}_h = \phi$.

The algebra $C_r^*(\Gamma)$ is of Kac type so that the modular automorphism group is trivial. The Haar state is a trace and, on generators, it is explicitly given by $h(\delta_g) = 0$ for $g \neq e$ and $h(\delta_e) = 1$. The GNS Hilbert space $L^2(C_r^*(\Gamma), h)$ can then be identified with $l^2(\Gamma)$.

In this case the notions of GNS and KMS symmetry coincide, and ϕ is symmetric iff $\phi(\lambda_g) = \phi(\lambda_{g^{-1}})$ for any $g \in \Gamma$. Moreover, symmetric generating functionals of Lévy processes are in one-to-one correspondence with (obviously continuous) positive, conditionally negative-type functions

$$d : \Gamma \to [0, \infty), \qquad d(g) = -\phi(\lambda_g), \quad g \in \Gamma$$

(cf. [15, Example 10.2]). The associated Dirichlet form is given by

$$\mathcal{E}[a] = \sum_{g \in \Gamma} d(g) |a(g)|^2, \quad a \in l^2(\Gamma),$$

and the generator of the Markovian semigroup on $l^2(\Gamma)$ is just the multiplication operator

$$(H_\phi a)(g) = d(g) a(g),$$

defined for those $a \in l^2(\Gamma)$ such that the right hand side in square integrable.

The derivation associated to the KMS symmetric generating functional ϕ (recall Section 8) is given by $\partial(\lambda_g) = \lambda_g \otimes \eta(\lambda_g)$ for $g \in \Gamma$, where η is the 1-cocycle corresponding to ϕ in the Schürmann triple $((\pi, D), \eta, \phi)$. Composing the 1-cocycle η on the C*-algebra $C_r^*(\Gamma)$ with the left regular representation, one obtains the 1-cocycle

$$c : \Gamma \to D, \qquad c(g) = \eta(\lambda_g)$$

on the group Γ. In terms of this, the negative type function is given by

$$d(g) = \big\| c(g) \big\|_D^2.$$

Identifying $l^2(\Gamma) \otimes D$ with $l^2(\Gamma, D)$, one obtains that the derivation above reduces to the multiplication operator

$$(\partial a)(g) = c(g)a(g), \quad g \in \Gamma,$$

defined for all a in the domain of the Dirichlet form.

The spectrum of the generator H_ϕ is discrete if and only if the negative-type function d is proper on Γ (a condition which is met, for example, for some length functions of finitely generated groups, see for example [9]). In these situations the construction of a spectral triple shown in Theorem 8.4 applies.

10. Example: free orthogonal quantum groups O_N^+

Let $N \geqslant 2$. The compact quantum group $(C_u(O_N^+), \Delta)$ is the universal unital C*-algebra generated by N^2 self-adjoint elements v_{jk}, $1 \leqslant j, k \leqslant N$, subject to the condition that the matrix $V = (v_{jk}) \in M_N \otimes C_u(O_N^+)$ is a unitary corepresentation, i.e. that

$$\sum_{\ell=1}^{N} v_{\ell j} v_{\ell k} = \delta_{jk} = \sum_{\ell=1}^{N} v_{j\ell} v_{k\ell}$$

and

$$\Delta(v_{jk}) = \sum_{\ell=1}^{N} v_{j\ell} \otimes v_{\ell k}$$

for all $1 \leqslant j, k \leqslant N$, see [44,2]. The equivalence classes of the irreducible unitary corepresentations of this compact quantum group can be indexed by $\mathbb{N}$, with $u^{(0)} = \mathbf{1}$ the trivial corepresentation and $u^{(1)} = (v_{jk})_{1 \leqslant j,k \leqslant N}$ the corepresentation whose coefficients are exactly the N^2 generators of $C_u(O_N^+)$ (this is also called the *fundamental* corepresentation of O_N^+). The dense *-Hopf algebra $\text{Pol}(O_N^+)$ associated to O_N^+, also called the *-algebra of polynomial on O_N^+, is the *-algebra generated by v_{jk}, $1 \leqslant j, k \leqslant N$. The compact quantum group O_N^+ is called the *free orthogonal compact quantum group*. For $N > 2$ it is not co-amenable, i.e. the Haar state of O_N^+ is not faithful on $C_u(O_N^+)$, therefore we will study the Markov semigroups of Lévy processes on $\text{Pol}(O_N^+)$ on the reduced C*-algebraic version $C_r(O_N^+)$ of O_N^+.

The compact quantum group O_N^+ is of Kac type, and therefore a generating functional ϕ is KMS-symmetric if and only if it is GNS-symmetric, which is the case if the characteristic matrices are symmetric, i.e. if $\phi(u_{jk}^{(s)}) = \phi(u_{kj}^{(s)})$ for all $s \in \mathbb{N}$ and j, k running from 1 up to the dimension of the sth corepresentation.

Corollary 6.11 reduced the problem of classifying ad-invariant generating functionals on a compact quantum group to the classification of generating functionals on the subalgebra of central functions. For the free orthogonal quantum group O_N^+ the algebra of central functions is isomorphic to the C*-algebra of continuous functions on the interval $[-N, N]$, cf. [5, Corollary 4.3]. Furthermore, the restriction of the counit to this subalgebra is the evaluation of a function in a boundary point.

Let us begin by describing linear functionals which are positive on a given interval and vanish in a given point.

Proposition 10.1. *Denote by $\tau_x : C([0,1]) \to \mathbb{C}$ the evaluation of a function in $x \in [0,1]$.*

(a) *Suppose $0 < x < 1$. A linear functional $\varphi : \mathbb{C}[x] \to \mathbb{C}$ with $\varphi(1) = 0$ is positive on the cone*

$$K_x([0,1]) = \mathbb{C}[x] \cap C([0,1])_+ \cap \ker(\tau_x)$$

if and only if there exist real numbers a, b with $a \geqslant 0$ and a finite measure ν on $[0,1]$ with $\nu(\{x\}) = 0$ such that

$$\varphi(f) = bf'(x) + af''(x) + \int_0^1 \left(f(y) - f(x) - yf'(x)\right) \frac{\nu(dy)}{(y-x)^2}$$

for all polynomials $f \in C([0,1])$.
The triple (a, b, ν) is uniquely determined by φ. We will call (a, b, ν) the characteristic triple of the linear functional φ.

(b) *Suppose $x \in \{0, 1\}$. Then a linear functional $\varphi : \mathbb{C}[x] \to \mathbb{C}$ with $\varphi(1) = 0$ is positive on the cone*

$$K_x([0,1]) = \mathbb{C}[x] \cap C([0,1])_+ \cap \ker(\tau_x)$$

if and only if there exist a real number d with $d \geqslant 0$ if $x = 0$, and $d \leqslant 0$ if $x = 1$, and a finite measure μ on $[0,1]$ with $\mu(\{0\}) = 0$ such that

$$\varphi(f) = df'(x) + \int_0^1 \left(f(y) - f(x)\right) \frac{\mu(dy)}{y}$$

for all polynomials $f \in C([0,1])$.
The pair (d, μ) is uniquely determined by φ. We will call (b, ν) the characteristic pair of the linear functional φ.

Proof. (a) This is actually the classical Lévy–Khinchin formula for Lévy processes on $\mathbb{R}$, see, e.g., [35, Theorem 8.1], which can be viewed as a special case of Hunt's formula [24]. Skeide [38] has given a C*-algebraic proof which doesn't use the group structure, but works for the C*-algebra of continuous functions on a compact set, with a character given by evaluation in a fixed point which has neighborhood with Euclidean coordinates (i.e. smooth functions admit a Taylor expansion around the fixed point).

(b) This is actually the classical Lévy–Khinchin formula for subordinators, cf. [35, Theorem 21.5]. We prove the formula for $x = 0$, the case $x = 1$ follows easily by a change of variable $t \mapsto 1 - t$.

By (a), since φ has to be positive also on the smaller cone given by polynomials that vanish in $x = 0$ and which are positive on $[-\varepsilon, 1]$ for any $\varepsilon > 0$, there exists a unique triple (a, b, ν) with $a, b \in \mathbb{R}$ with $a \geqslant 0$ and ν a finite measure on $[0, 1]$, such that

$$\varphi(f) = bf'(0) + af''(0) + \int_0^1 \big(f(y) - f(0) - yf'(0) \big) \frac{\nu(\mathrm{d}y)}{y^2}.$$

For $n \in \mathbb{N}$ we set $g_n(y) = \frac{1}{n+1} y \sum_{k=0}^n (1 - y)^k$. We have $g_n \in K_0([0, 1])$, $g_n'(0) = 1$, and $g_n''(0) = -n$, therefore

$$0 \leqslant \varphi(g_n) = b - na + \int_0^1 \big(g_n(y) - y \big) \frac{\nu(\mathrm{d}y)}{y^2}$$

for all $n \in \mathbb{N}$. The sequence $(g_n)_{n \in \mathbb{N}}$ is decreasing and therefore we must have $a = 0$. By monotone convergence we get

$$b \geqslant \int_0^1 \big(y - g_n(y) \big) \frac{\nu(\mathrm{d}y)}{y^2} \xrightarrow{n \to \infty} \int_0^1 \frac{\nu(\mathrm{d}y)}{y}$$

which proves that the measure $\frac{1}{y}\nu$ is finite. Putting $\mu = \frac{1}{y}\nu$ and $d = b - \int_0^1 \frac{\nu(\mathrm{d}y)}{y} \geqslant 0$, we get the desired formula

$$\varphi(f) = df'(0) + \int_0^1 \big(f(y) - f(0) \big) \frac{\mu(\mathrm{d}y)}{y}.$$

Conversely, since a polynomial f which vanishes in $x = 0$ and is positive on $[0, 1]$ has a positive derivative at $x = 0$, it is clear that any such functional is positive on $K_0([0, 1])$. Uniqueness follows from (a). $\square$

This result allows us to describe all ad-invariant generating functionals on $\mathrm{Pol}(O_N^+)$. This result can be considered as Hunt's formula for ad-invariant Lévy processes on the free orthogonal quantum group O_N^+.

Let us denote by $\mathrm{Pol}_0(O_N^+)$ the algebra of central polynomial functions on O_N^+, see Eq. (6.1). We will use the same isomorphism between $\mathrm{Pol}_0(O_N^+)$ and polynomials $\mathrm{Pol}([-N, N])$ as Brannan [5]. Recall that Banica [2] showed that the equivalence classes of irreducible unitary corepresentations of O_N^+ can be labelled by non-negative integers and that they satisfy the "fusion rules"

$$u^{(s)} \otimes u^{(t)} \cong u^{(|s-t|)} \oplus u^{(|s-t|+2)} \oplus \cdots \oplus u^{(s+t)}$$

for $s, t \in \mathbb{N}$. Since the trivial corepresentation $u^{(0)} = \mathbf{1}$ has dimension 1 and the fundamental corepresentation $u^{(1)} = (v_{jk})_{1 \leqslant j,k \leqslant N}$ has dimension N, one can show by induction that the dimensions of the irreducible unitary corepresentations are given by Chebyshev polynomials of the second kind, $D_s = U_s(N)$. The conditional expectation $\widetilde{\mathrm{ad}}_h : \mathrm{Pol}(O_N^+) \to \mathrm{Pol}_0(O_N^+)$ onto the algebra of central functions is therefore given by

$$\widetilde{\mathrm{ad}}_h \left(u_{jk}^{(s)} \right) = \frac{1}{U_s(N)} \delta_{jk} \chi_s,$$

where $\chi_s = \sum_{j=1}^{D_s} u_{jj}^{(s)}$ denotes the trace of $u^{(s)}$.

The fusion rules imply that the characters satisfy the three-term recurrence relation

$$\chi_1 \chi_s = \chi_{s+1} + \chi_{s-1}$$

for $s \geqslant 1$, then we get the desired isomorphism $\mathrm{Pol}(O_N^+)_0 \cong \mathrm{Pol}([-N, N])$ by setting $\chi_s \mapsto U_s$ for $s \in \mathbb{N}$, where U_s denotes the sth Chebyshev polynomial of the second kind, defined by $U_0(x) = 1$, $U_1(x) = x$, and $U_{s+1}(x) = xU_s(x) - U_{s-1}(x)$.

Theorem 10.2. *The* ad-*invariant generating functional on* $\mathrm{Pol}(O_N^+)$ *are of the form*

$$\hat{L} = L \circ \widetilde{\mathrm{ad}}_h$$

with L *defined on* $\mathrm{Pol}(O_N^+)_0 \cong \mathrm{Pol}([-N, N])$ *by*

$$Lf = -bf'(N) + \int\limits_{-N}^{N} \frac{f(x) - f(N)}{N - x} \, \mathrm{d}\nu(x)$$

where $b \geqslant 0$ *is a real number and* ν *is a finite measure on* $[-N, N]$ *with* $\nu(\{N\}) = 0$.

Proof. This follows from Theorem 6.10 and Proposition 10.1. $\quad\square$

Using the discussion above, we can give a formula for the values of ad-invariant generating functionals on the coefficients of the irreducible unitary corepresentations of O_N^+.

Corollary 10.3. *The* ad-*invariant generating functional on* $\mathrm{Pol}(O_N^+)$ *given in Theorem* 10.2 *with characteristic pair* (b, ν) *acts on the coefficients of unitary irreducible corepresentations of* O_N^+ *as*

$$L\left(u_{jk}^{(s)} \right) = \frac{\delta_{jk}}{U_s(N)} \left(-bU_s'(N) + \int\limits_{-N}^{N} \frac{U_s(x) - U_s(N)}{N - x} \nu(\mathrm{d}x) \right)$$

for $s \in \mathbb{N}$, *where* U_s *denotes the* sth *Chebyshev polynomial of the second kind.*

Remark 10.4. Since the characteristic matrices of L are diagonal, we can read off the eigenvalues of T_L from Corollary 10.3. Assume for simplicity $b = 1$, $\nu = 0$. Then the eigenvalues of T_L are given by

$$\lambda_s = -\frac{U'_s(N)}{U_s(N)}, \quad s \in \mathbb{N},$$

with multiplicities given by the square of the dimension $m_s = D_s^2 = (U_s(N))^2$ of $u^{(s)}$.

Recall that the "spectral dimension" d_D of the associated spectral triple is, by definition (see [17,18]), the abscissa of convergence of the zeta function $z \mapsto \mathcal{Z}_D(z) := \mathrm{Tr}(|D|^{-z})$, initially defined for $z \in \mathbb{C}$ with $\mathrm{Re}\, z > 0$. It coincides with the infimum of all $d > 0$ such that the sum $\sum_s m_s(-\lambda_s)^{-d/2}$ is finite. In the present situation of Corollary 10.3 and assuming $N = 2$, $b = 1$, $\nu = 0$, we have $U_s(2) = s + 1$, $U'_s(2) = \frac{s(s+1)(s+2)}{6}$,

$$\lambda_s = -\frac{s(s+2)}{6}$$

and finally $d_D = 3$. This value of the spectral dimension agrees nicely with the known fact that O_2^+ is isomorphic to $SU_{-1}(2)$, see [2], and that $C(SU_{-1}(2))$ can be realized by matrix-valued functions on the three-dimensional Lie group $SU(2)$, cf. [49]. On the other hand, for $N > 2$, we have

$$U_s(N) = \frac{q(N)^{s+1} - q(N)^{-s-1}}{q(N) - q(N)^{-1}},$$

$$U'_s(N) = \frac{q'(N)}{q(N)} \frac{s(q(N)^{s+2} - q(N)^{-s-2}) - (s+2)(q(N)^s - q(N)^{-s})}{(q(N) - q(N)^{-1})^2}$$

$$= \frac{q'(N)}{q(N)} \left(s\frac{q(N)^{s+1} - q(N)^{-s-1}}{q(N) - q(N)^{-1}} - 2\frac{q(N)^s - q(N)^{-s}}{(q(N) - q(N)^{-1})^2} \right),$$

with $q(N) = \frac{1}{2}(N + \sqrt{N^2 - 4}) > 1$, $q'(N) = \frac{1}{2}(1 + \frac{N}{\sqrt{N^2-4}}) > 0$, and

$$\lambda_s = -\frac{q'(N)}{q(N)} \left(s - 2\frac{q(N)^s - q(N)^{-s}}{(q(N)^{s+1} - q(N)^{-s-1})(q(N) - q(N)^{-1})} \right).$$

Since $q(N)$ is bigger then 1 (and fixed), the term

$$\frac{q(N)^s - q(N)^{-s}}{(q(N)^{s+1} - q(N)^{-s-1})(q(N) - q(N)^{-1})} \to 0 \quad \text{as } s \to \infty.$$

This implies that the growth of the eigenvalues λ_s (as a function of s) is asymptotically linear $\lambda_s \cong -\frac{q'(N)}{q(N)}s$, while the multiplicities $m_s = U_s(N)^2 \cong \frac{q(N)^{2s}}{(1-q(N)^{-2})^2}$ grow exponentially, therefore the sum

$$\sum_s m_s(-\lambda_s)^{-d/2} \cong \sum_s \frac{q(N)^{2s}}{s^{d/2}}$$

can never converge, which means that $d_D = +\infty$.

11. Example: Woronowicz quantum group $SU_q(2)$

Let us fix $q \in (0, 1)$. The compact quantum group $C(SU_q(2))$ (see [47]) is the universal unital C*-algebra generated by α and γ subject to the following relations

$$\alpha^* \alpha + \gamma^* \gamma = 1, \qquad \alpha \alpha^* + q^2 \gamma \gamma^* = 1,$$
$$\gamma^* \gamma = \gamma \gamma^*, \qquad \alpha \gamma = q \gamma \alpha, \qquad \alpha \gamma^* = q \gamma^* \alpha$$

with the comultiplication extended uniquely to a unit-preserving $*$-homomorphism from the formulas

$$\Delta(\alpha) = \alpha \otimes \alpha - q \gamma^* \otimes \gamma, \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

For $C(SU_q(2))$ the equivalence classes of irreducible unitary corepresentations are indexed by non-negative half-integers $s \in \frac{1}{2}\mathbb{N}$ and are of dimension $n_s = 2s + 1$. For each $u^{(s)} = (u_{jk}^{(s)})_{j,k}$ the indices j, k run over the set $\{-s, -s+1, \ldots, s-1, s\}$ (see e.g. [33] for the detailed description of $u^{(s)}$). Moreover, every corepresentation is equivalent to its contragredient one and we have

$$S(u_{kj}) = \left(u_{jk}^{(s)}\right)^* = (-q)^{k-j} u_{-j,-k}^{(s)}. \tag{11.1}$$

The quantum group is neither commutative nor cocommutative. The Woronowicz characters, the modular automorphism group, the unitary antipode and the quantum dimension are the following (cf. [46, Appendix A1]):

$$f_z\left(u_{jk}^{(s)}\right) = q^{2jz} \delta_{jk}, \tag{11.2}$$

$$\sigma_z\left(u_{jk}^{(s)}\right) = f_{iz} \star u_{jk}^{(s)} \star f_{iz} = q^{2iz(j+k)} u_{jk}^{(s)}, \tag{11.3}$$

$$R\left(u_{jk}^{(s)}\right) = S\left(f_{\frac{1}{2}} \star u_{jk}^{(s)} \star f_{-\frac{1}{2}}\right) = q^{k-j} \left(u_{kj}^{(s)}\right)^*, \tag{11.4}$$

$$D_s = \sum_{k=-s}^{s} f_1\left(u_{kk}^{(s)}\right) = \sum_{k=-s}^{s} q^{2k} = q^{-2s}[2s+1]_{q^2}. \tag{11.5}$$

The following example describes the irreducible representations of $C(SU_q(2))$ and the related opposite representations (cf. Section 5).

Example 11.1. On $\mathsf{A} = C(SU_q(2))$ we have two families of irreducible $*$-representation indexed by $\theta \in [0, 2\pi)$:

(1) the 1-dimensional representations $\delta_\theta : \mathsf{A} \to \mathbb{C}$:

$$\delta_\theta(\alpha) = e^{i\theta}, \qquad \delta_\theta(\gamma) = 0;$$

(2) the infinitely-dimensional representations on a Hilbert space $\rho_\theta : \mathsf{A} \to B(\ell^2)$:

$$\rho_\theta(\alpha)e_n = We_n, \qquad \rho_\theta(\gamma)e_n = e^{i\theta}q^n e_n,$$

where $(e_n)_{n\in\mathbb{N}}$ is the standard orthonormal basis of ℓ^2 and W is the weighted shift defined by $We_0 = 0$ and $We_n = \sqrt{1-q^{2n}}e_{n-1}$ for $n \geqslant 1$.

We check directly that

$$\delta_\theta^{\mathrm{op}} = \delta_{-\theta} \quad \text{and} \quad \rho_\theta^{\mathrm{op}} = \rho_{\pi+\theta}.$$

Indeed, we note first that $R(\alpha) = \alpha^*$ and $R(\gamma) = -\gamma$. So for δ_θ we have

$$\delta_\theta^{\mathrm{op}}(\alpha)\bar{1} = \overline{\delta_\theta\big(R(\alpha^*)\big)1} = \overline{\delta_\theta(\alpha)1} = \overline{e^{i\theta}1} = e^{-i\theta}\bar{1} = \delta_{-\theta}(\alpha)\bar{1}$$

and

$$\delta_\theta^{\mathrm{op}}(\gamma)\bar{1} = \overline{\delta_\theta\big(R(\gamma^*)\big)1} = \overline{-\delta_\theta\big(\gamma^*\big)1} = 0 = \delta_{-\theta}(\gamma)\bar{1}.$$

Similarly, for ρ_θ we compute

$$\rho_\theta^{\mathrm{op}}(\gamma)\bar{e}_n = \overline{-\rho_\theta\big(\gamma^*\big)e_n} = \overline{-e^{-i\theta}q^n e_n} = e^{i(\pi+\theta)}q^n\bar{e}_n = \rho_{\pi+\theta}(\gamma)\bar{e}_n,$$
$$\rho_\varphi^{\mathrm{op}}(\alpha)\bar{e}_n = \overline{\rho_\varphi(\alpha)e_n} = \overline{We_n} = \sqrt{1-q^{2n}}\bar{e}_n = \rho_{\pi+\theta}(\alpha)\bar{e}_n.$$

11.1. GNS-symmetric generators

We first describe a generic GNS-symmetric functional on $SU_q(2)$ and provide an example of an unbounded generating functional.

Proposition 11.2. *A hermitian functional ϕ on $SU_q(2)$ defined by*

$$\phi\big(u_{jk}^{(s)}\big) = c_{s,j}\delta_{jk} \tag{11.6}$$

with real constants $(c_{s,j})_{s\in\frac{1}{2}\mathbb{N},-s\leqslant j\leqslant s}$ is GNS-symmetric.

Reciprocally, any GNS-symmetric generator ϕ on $SU_q(2)$ must be of the form (11.6) *and it is hermitian if and only if the constants satisfy the supplementary symmetry condition: $c_{s,j} = c_{s,-j}$ for all s and j.*

Proof. We calculate explicitly that ϕ of the form (11.6) is invariant under the antipode:

$$\phi \circ S\big(u_{jk}^{(s)}\big) = \phi\big((u_{kj}^{(s)})^*\big) = \overline{\phi\big(u_{kj}^{(s)}\big)} = \overline{c_{s,j}}\delta_{jk} = c_{s,j}\delta_{jk} = \phi\big(u_{jk}^{(s)}\big).$$

Conversely, suppose that $\phi \circ S = \phi$, then also $\phi \circ S^2 = \phi$. On $SU_q(2)$ we have $S^2(u_{jk}^{(s)}) = q^{2(j-k)}u_{jk}^{(s)}$, thus $\phi(u_{jk}^{(s)}) = \phi \circ S^2(u_{jk}^{(s)}) = q^{2(j-k)}\phi(u_{jk}^{(s)})$ and ϕ can have non-zero values only on the diagonal. By Remark 4.7, all $c_{s,j}$ must be real.

Finally, $c_{s,j} = \bar{c}_{s,j} = \phi(u_{jj}^{(s)})$ and $c_{s,-j} = \phi(u_{-j,-j}^{(s)}) = \phi((u_{jj}^{(s)})^*)$ from which the last part follows. $\square$

11.2. Unbounded GNS-symmetric generator

Let π be the $*$-representation of $SU_q(2)$ on $\ell^2(\mathbb{N} \times \mathbb{Z})$ given by

$$\pi(\alpha)e_{k,n} = \sqrt{1-q^{2k}}e_{k-1,n} \quad (k \geqslant 1), \qquad \pi(\alpha)e_{0,n} = 0,$$

$$\pi(\alpha^*)e_{k,n} = \sqrt{1-q^{2k+2}}e_{k+1,n} \quad (k \geqslant 0),$$

$$\pi(\gamma)e_{k,n} = q^k e_{k,n-1}, \qquad \pi(\gamma^*)e_{k,n} = q^k e_{k,n+1},$$

where $\{e_{k,n};\ k \geqslant 0,\ n \in \mathbb{Z}\}$ is the standard orthonormal basis of $\ell^2(\mathbb{N} \times \mathbb{Z})$. For a fixed $0 < \lambda < 1$ let us consider a Poisson type generator

$$\phi_\lambda(a) = \langle v_\lambda, (\pi - \varepsilon)(a)v_\lambda \rangle \quad \text{with } v_\lambda = \sum_{k=0}^{\infty} \lambda^k e_{k,0}.$$

The related cocycle $\eta_\lambda(a) = (\pi - \varepsilon)(a)v_\lambda$ is uniquely determined by the value on α^* (see [37]), where it equals

$$\eta_\lambda(\alpha^*) = \sum_{k=0}^{\infty} \lambda^k (\pi - \varepsilon)(\alpha^*)e_{k,0} = \sum_{k=0}^{\infty} \lambda^k \left(\sqrt{1-q^{2k+2}}e_{k+1,0} - e_{k,0}\right)$$

$$= \sum_{k=1}^{\infty} \lambda^{k-1}\sqrt{1-q^{2k}}e_{k,0} - \sum_{k=0}^{\infty} \lambda^k e_{k,0}$$

$$= -e_{0,0} + \sum_{k=1}^{\infty} \left(\lambda^{k-1}\sqrt{1-q^{2k}} - \lambda^k\right)e_{k,0}.$$

Note that $\eta_\lambda(\alpha^*) \in H$ since

$$\left\|\eta_\lambda(\alpha^*)\right\|^2 = 1 + \sum_{k=1}^{\infty}\left|\lambda^{k-1}\sqrt{1-q^{2k}} - \lambda^k\right|^2 \leqslant 1 + 4\sum_{k=1}^{\infty}\lambda^{2(k-1)} < +\infty.$$

Define also a cocyle η_∞ by its value on α^*:

$$\eta_\infty(\alpha^*) = -e_{0,0} + \sum_{k=1}^{\infty}\left(\sqrt{1-q^{2k}} - 1\right)e_{k,0} = -\sum_{k=0}^{\infty}\left(1 - \sqrt{1-q^{2k}}\right)e_{k,0}.$$

We shall show that $\eta_\infty(\alpha^*) \in H$ and that $\eta_\lambda(\alpha^*) \to \eta_\infty(\alpha^*)$ in H when $\lambda \to 1^-$. For the first part, we check directly that

$$\left\|\eta_\infty(\alpha^*)\right\|^2 = \sum_{k=0}^\infty \left(1 - \sqrt{1 - q^{2k}}\right)^2 = \sum_{k=0}^\infty \frac{q^{4k}}{(1 + \sqrt{1 - q^{2k}})^2} < \frac{1}{1 - q^4} < +\infty.$$

Next, we show the convergence:

$$\left\|\eta_\lambda(\alpha^*) - \eta_\infty(\alpha^*)\right\|^2 = \left\|\sum_{k=1}^\infty \left(\lambda^{k-1}\sqrt{1 - q^{2k}} - \lambda^k - \sqrt{1 - q^{2k}} + 1\right)e_{k,0}\right\|^2$$

$$= \sum_{k=1}^\infty \left|\left(1 - \sqrt{1 - q^{2k}}\right)\left(1 - \lambda^{k-1}\right) + \lambda^{k-1}(1 - \lambda)\right|^2$$

$$= \sum_{k=1}^\infty \left(1 - \sqrt{1 - q^{2k}}\right)^2\left(1 - \lambda^{k-1}\right)^2 + (1 - \lambda)^2 \sum_{k=1}^\infty \lambda^{2(k-1)}$$

$$+ 2(1 - \lambda)\sum_{k=1}^\infty \left(1 - \sqrt{1 - q^{2k}}\right)\left(1 - \lambda^{k-1}\right)\lambda^{k-1}.$$

Note that $1 - \lambda^{k-1} = (1 - \lambda)(\lambda^{k-2} + \cdots + 1) \leqslant (k-1)(1 - \lambda)$. This implies

$$\left\|\eta_\lambda(\alpha^*) - \eta_\infty(\alpha^*)\right\|^2 \leqslant (1 - \lambda)^2 \sum_{k=1}^\infty (k-1)^2 \frac{q^{4k}}{(1 + \sqrt{1 - q^{2k}})^2} + \frac{(1-\lambda)^2}{1 - \lambda^2}$$

$$+ 2(1 - \lambda)^2 \sum_{k=1}^\infty \frac{q^{2k}}{1 + \sqrt{1 - q^{2k}}}(k-1)\lambda^{k-1}$$

$$\leqslant (1 - \lambda)^2 \sum_{k=1}^\infty (k-1)^2 q^{4k} + \frac{1 - \lambda}{1 + \lambda} + 2(1 - \lambda)^2 \sum_{k=1}^\infty q^{2k}(k-1)$$

and we see that each term tends to 0 when $\lambda \to 1^-$.

Let us now define by ϕ_∞ the functional related to the cocycle η_∞ by the formula

$$\phi_\infty(ab) = \langle \eta_\infty(a^*), \eta_\infty(b) \rangle, \quad a, b \in \ker \varepsilon,$$

with the additional conditions that $\phi_\infty(\mathbf{1}) = 0$ and that the 'drift' part is zero (which remains to say that $\phi_\infty(\alpha) = \phi_\infty(\alpha^*) \in \mathbb{R}$). This way ϕ_∞ is uniquely determined on the whole of $\mathcal{A} = \mathrm{Lin}\{\mathbf{1}, \alpha - \alpha^*, K_2\}$, where K_2 is the linear span of products of two elements from $\ker \varepsilon$ (cf. [37]). By the Schoenberg correspondence, if well-defined, ϕ_∞ is a generating functional of a Lévy process.

To see that ϕ_∞ is well-defined, we can check that on K_2 the functional is just a limit of functionals related to η_λ. Indeed, if $a, b \in \ker \varepsilon$ then

$$\phi_\infty(ab) = \big\langle \eta_\infty\big(a^*\big), \eta_\infty(b)\big\rangle = \Big\langle \lim_{\lambda \to 1^-} \eta_\lambda\big(a^*\big), \lim_{\mu \to 1^-} \eta_\mu(b)\Big\rangle$$

and since both limits exist we have

$$\phi_\infty(ab) = \lim_{\lambda \to 1^-} \big\langle \eta_\lambda\big(a^*\big), \eta_\lambda(b)\big\rangle = \lim_{\lambda \to 1^-} \phi_\lambda(ab) = \lim_{\lambda \to 1^-} \big\langle v_\lambda, (\pi - \varepsilon)(ab)v_\lambda\big\rangle.$$

We conclude that

$$\phi_\infty(a) = \lim_{\lambda \to 1^-} \big\langle v_\lambda, (\pi - \varepsilon)(a)v_\lambda\big\rangle \quad \text{for } a \in K_2. \tag{11.7}$$

Our aim now is to show that ϕ_∞ is GNS-symmetric and unbounded.

Proposition 11.3. *The functional ϕ_∞ is GNS-symmetric.*

Proof. By Proposition 11.2, it is enough to show that ϕ_∞ vanishes on the non-diagonal coefficients of the corepresentations $u^{(s)}$, $s \in \frac{1}{2}\mathbb{N}$. These coefficients are of the form $b_{m,n} := \alpha^m p(\gamma^*\gamma)\gamma^n$ for $m, n \in \mathbb{Z}$ (with the notation $a^{-n} = (a^*)^n$ for $n > 0$ and p denoting a polynomial, see [33]) and they are off diagonal iff $n \neq 0$. So it is enough to check that ϕ_∞ vanishes on $\alpha^m(\gamma^*\gamma)^k\gamma^n$ ($n \neq 0$).

We first observe that $\gamma, \gamma^* \in K_2$. Indeed, the relation $\alpha\gamma = q\gamma\alpha$ together with $\gamma, \alpha - \mathbf{1} \in \ker \varepsilon$ imply that

$$\gamma = \frac{q}{1-q}\gamma(\alpha - \mathbf{1}) - \frac{1}{1-q}(\alpha - \mathbf{1})\gamma \in K_2.$$

Therefore an element $\alpha^m(\gamma^*\gamma)^k\gamma^n$ belongs to K_2 provided $k \neq 0$ or $n \neq 0$. So the formula (11.7) can be applied

$$\phi_\infty\big(\alpha^m\big(\gamma^*\gamma\big)^k\gamma^n\big) = \lim_{\lambda \to 1^-} \sum_{p,r=0}^\infty \lambda^{p+r}\big\langle e_{p,0}, \pi(\alpha)^m\pi\big(\gamma^*\gamma\big)^k\pi(\gamma)^n e_{r,0}\big\rangle.$$

Since $\pi(\gamma)$ and $\pi(\gamma^*)$ move (down and up, respectively) the second index of the basis vectors $e_{k,n}$ and since none of $\pi(\alpha)$, $\pi(\alpha^*)$ and $\pi(\gamma^*\gamma)$ move the second index, we immediately see that if $n \neq 0$ then $\pi(\alpha)^m\pi(\gamma^*\gamma)^k\pi(\gamma)^n e_{r,0} \in \mathbb{C} \cdot e_{r+m,n}$, which is orthogonal to $e_{p,0}$ for any m, k and p, q. So the sum under the limit equals to 0 and thus ϕ_∞ is of the form (11.6). $\square$

Proposition 11.4. *The functional ϕ_∞ is unbounded.*

Proof. We shall show that $|\phi_\infty(\alpha^{*m}\alpha^m)| \to +\infty$ when $m \to +\infty$. Since

$$\big\|\alpha^{*m}\alpha^m\big\|_\mathsf{A} \leqslant \|\alpha\|_\mathsf{A}^{2m} \leqslant 1,$$

this will imply that ϕ_∞ is unbounded.

Observe first that

$$\alpha^{*m}\alpha^m = \alpha^{*(m-1)}(\alpha^*\alpha)\alpha^{m-1} = \alpha^{*(m-1)}(1 - \gamma^*\gamma)\alpha^{m-1}$$
$$= \alpha^{*(m-1)}\alpha^{m-1}(1 - q^{-2(m-1)}\gamma^*\gamma)$$

and by induction

$$\alpha^{*m}\alpha^m = (1 - \gamma^*\gamma)(1 - q^{-2}\gamma^*\gamma)\cdots(1 - q^{-2(m-1)}\gamma^*\gamma), \quad m \geqslant 1.$$

Applying the standard formula from the q-calculus (cf. [26, Eq. (0.3.5)]):

$$(a; q)_n = \sum_{k=0}^{n} \frac{(q^2; q^2)_n}{(q^2; q^2)_k (q^2; q^2)_{n-k}} q^{k(k-1)}(-a)^k$$

we arrive at

$$\alpha^{*m}\alpha^m - 1 = (1 - \gamma^*\gamma)\cdots(1 - q^{-2(m-1)}\gamma^*\gamma) - 1$$
$$= \sum_{k=1}^{m}(-1)^k \frac{(q^2; q^2)_m}{(q^2; q^2)_k (q^2; q^2)_{m-k}} q^{k(k-1)}(\gamma^*\gamma)^k.$$

We see that each term under the sum contains $\gamma^*\gamma$ and thus belongs to K_2.

Now that we have proved that $\alpha^{*m}\alpha^m - 1 \in K_2$, we can apply the formula (11.7) to calculate the value of ϕ_∞ on $\alpha^{*m}\alpha^m$. Namely,

$$\phi_\infty(\alpha^{*m}\alpha^m)$$
$$= \phi_\infty(\alpha^{*m}\alpha^m - 1)$$
$$= \lim_{\lambda \to 1^-} \sum_{j,k=0}^{\infty} \lambda^{j+k}\langle e_{j,0}, (\pi - \varepsilon)(\alpha^{*m}\alpha^m - 1)e_{k,0}\rangle$$
$$= \lim_{\lambda \to 1^-} \sum_{j,k=0}^{\infty} \lambda^{j+k}\langle e_{j,0}, [(I_H - \pi(\gamma^*\gamma))\cdots(I_H - q^{-2(m-1)}\pi(\gamma^*\gamma)) - I_H]e_{k,0}\rangle$$
$$= \sum_{k=0}^{\infty}((1 - q^{2k})\cdots(1 - q^{2k-2m+2}) - 1)$$
$$= -\sum_{k=0}^{m-1} 1 - \sum_{k=m}^{\infty}(1 - (1 - q^{2k})\cdots(1 - q^{2k-2m+2})).$$

We finally note that the infinite sum is non-negative, and so

$$|\phi_\infty(\alpha^{*m}\alpha^m)| = m - 1 + \sum_{k=m}^{\infty}(1 - (1 - q^{2k})\cdots(1 - q^{2k-2m+2})) \geqslant m - 1. \qquad \square$$

11.3. KMS-symmetry

In case of $C(SU_q(2))$ it is easy to check that a hermitian ϕ is KMS-symmetric iff for each $s \in \frac{1}{2}\mathbb{N}$ the matrix $\phi_q^{(s)} = [q^j \phi(u_{jk}^s)]$ is hermitian. Moreover, if a functional ϕ is hermitian and KMS-symmetric, then the values of ϕ on the corepresentation matrix $u^{(s)}$ are determined by the values $\phi(u_{jk}^{(s)})$ for $|k| \leqslant j$ and the conditions

$$\phi\big(u_{-j,-k}^{(s)}\big) = (-q)^{j-k}\overline{\phi\big(u_{jk}^{(s)}\big)} \quad \text{and} \quad \phi\big(u_{kj}^{(s)}\big) = q^{j-k}\overline{\phi\big(u_{j,k}^{(s)}\big)}.$$

Below we provide an example of a KMS-symmetric generator which is not GNS-symmetric.

Example 11.5 *(KMS-symmetric generator which is not GNS-symmetric).* Let us consider the Poisson type generating functional on $SU_q(2)$

$$\phi(a) = \big\langle e_k, (\rho_\theta - \varepsilon)(a)e_k\big\rangle,$$

where ρ_θ is the infinite-dimensional representation of $C(SU_q(2))$ on ℓ^2, described in Example 11.1, $\theta \in [0, 2\pi)$, and e_k is the kth standard orthonormal basis vector of ℓ^2.

If (and only if) $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, then ϕ is KMS-symmetric. Indeed, by Theorem 5.4 and Example 11.1 the condition for the generating functional to be KMS-symmetric, $\phi(a) = \phi \circ R(a)$, reduces to

$$\big\langle e_k, \big(\rho_\theta(a) - \rho_{\theta+\pi}\big(a^*\big)\big)e_k\big\rangle = \varepsilon(a) - \varepsilon\big(a^*\big) \tag{11.8}$$

for any $a \in \mathcal{A}$. For $a = \gamma$ the left hand side of (11.8) is

$$\big\langle e_k, \big(\rho_\theta(\gamma) - \rho_{\theta+\pi}\big(\gamma^*\big)\big)e_k\big\rangle = \big\langle e_k, \big(e^{i\theta} - e^{-i(\theta+\pi)}\big)q^k e_k\big\rangle = \big(e^{i\theta} + e^{-i\theta}\big)q^k,$$

which equals $|\varepsilon(\gamma)|^2 = 0$ only when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. For such θ we check by a direct calculation that Eq. (11.8) holds true for each element of the form $a = (\alpha^*)^l \gamma^m (\gamma^*)^n$ (such elements form a linear basis of $\mathcal{A}$).

The Poisson generator related to ρ_θ with $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ is not GNS-symmetric since $S(\gamma) = -q\gamma$ and $q \neq 1$ imply $\phi(\gamma) \neq \phi \circ S(\gamma)$.

For $k = 0$ and $\theta = \frac{\pi}{2}$ we can calculate explicitly the values of the generating functional (11.8). Namely, using the explicit formula for the coefficient of the corepresentations (cf. [33, (B.19)]), the Vandermonde summation formula (cf. (0.5.9) in [26]) and the standard q-transformation (cf. (0.2.14) therein), we get

$$\phi\big(u_{jk}^{(s)}\big) = \begin{cases} -1, & j = k, \\ i^{2s} q^{(s-j)(s+j+1)} - \delta_{0,j}, & j \geqslant 0, \ k = -j, \\ i^{-2s} q^{(s-j)(s+j+1)-2j}, & j < 0, \ k = -j, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, it has non-zero entries only on the diagonal and the anti-diagonal.

The eigenvalues of the matrix $\phi^{(s)} = [\phi(u_{jk}^{(s)})]$, $\phi(a) = \langle e_0, (\rho_{\frac{\pi}{2}} - \varepsilon)(a)e_0 \rangle$ are:

$$\lambda_j^+ = -\big(1 + q^{(s-j)(s+j+1)+j}\big), \qquad \lambda_j^- = -\big(1 - q^{(s-j)(s+j+1)+j}\big)$$

with $j = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{k}{2}$ when $s = \frac{k}{2}$ $(k \in 2\mathbb{N}+1)$ or

$$\lambda_0 = -1 + (-1)^s q^{s(s+1)}, \qquad \lambda_j^+ = -\big(1 + q^{(s-j)(s+j+1)+j}\big),$$
$$\lambda_j^- = -\big(1 - q^{(s-j)(s+j+1)+j}\big)$$

for $j = 1, 3, \ldots, k$ when $s = k$ $(k \in \mathbb{N})$.

11.4. ad-*invariance*

We already noted that ad-invariance is a strong constraint on the functional. Namely, it is necessarily a multiple of the identity on each of the corepresentation matrices, in particular of diagonal form. A comparison of this notion with that of GNS-symmetry (suggested by Proposition 11.2), shows that ad-invariant generating functionals of a Lévy process on $C(SU_q(2))$ are necessarily GNS-symmetric.

Corollary 11.6. *Let ϕ be a functional on $SU_q(2)$. If ϕ is* ad-*invariant and hermitian, then ϕ is GNS-symmetric.*

Proof. By Proposition 6.9, ϕ is of the form $\phi(u_{jk}^{(s)}) = c_s \delta_{jk}$. By hermiticity, $c_s = \phi(u_{jj}^{(s)}) = \phi((u_{-j,-j}^{(s)})^*) = \overline{\phi(u_{-j,-j}^{(s)})} = \bar{c}_s$ and we conclude by Proposition 11.2 that ϕ is GNS-symmetric. $\square$

The following example shows that the map $\phi \to \phi_{\mathrm{ad}}$ preserves neither hermiticity nor positivity.

Example 11.7. Let ϕ be the functional on $SU_q(2)$ defined by $\phi(\alpha) = e^{it}$, $\phi(\alpha^*) = e^{-it}$ and zero otherwise, where $t \notin 2\pi\mathbb{Z}$. Then ϕ_{ad} is ad-invariant and $\phi_{\mathrm{ad}}(\alpha) = \phi_{\mathrm{ad}}(\alpha^*) = (1 - q^2)^{-1}(e^{it} + q^2 e^{-it})$, so it is not hermitian.

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