Linear-quadratic optimal control under non-Markovian switching

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Abstract

We study a finite-dimensional continuous-time optimal control problem on finite horizon for a controlled diffusion driven by Brownian motion, in the linear-quadratic case. We admit stochastic coefficients, possibly depending on an underlying independent marked point process, so that our model is general enough to include controlled switching systems where the switching mechanism is not required to be Markovian. The problem is solved by means of a Riccati equation, which a backward stochastic differential equation driven by the Brownian motion and by the random measure associated to the marked point process.

Keywords: Linear-quadratic optimal control, optimal control with stochastic coefficients, Riccati backward stochastic differential equations (Riccati BSDE).

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1 Introduction

In order to present and motivate our results let us consider for a moment a classical linear-quadratic stochastic optimal control problem, with a controlled state equation driven by a d-dimensional Brownian motion $W = (W^1, \dots, W^d)$ of the form

$$\left\{ \begin{array}{lcl} dX_t &=& (A(t)X_t + B(t)u_t)\,dt + \sum_{j=1}^d C^j(t)X_t dW_t^j, \\ X_0 &=& x \in \mathbb{R}^n \end{array} \right.$$

and a cost functional

$$J(u) = \mathbb{E}\left[\int_0^T (< S(t)X_t, X_t > + |u_t|^2)\,dt + < GX_T, X_T >\right],$$

where $T > 0$ is a fixed finite time horizon, A, B, C^j, S are matrix-valued bounded functions and $S(t)$ and G are non-negative definite. The problem of minimizing $J(u)$ over all adapted, square-integrable, $\mathbb{R}^k$-valued processes can be solved via the classical Riccati equation which provides an optimal feedback control. More realistic models for many applications require the coefficients A, B, C^j, S, G to be stochastic. A simple instance is given by optimization problems for so called regime-switching diffusions, see [6], [21], [27], [26] among others, where the controlled process X is assumed to evolve under a number of regimes, represented as the elements of a finite set $K = \{1, \dots, m\}$, across which its behavior can be markedly

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different. The system is then described by another stochastic process $(I_t)_{t\geq 0}$, with values in K, which represents the running regime and which is often assumed to be piecewise constant, with random positions ξ_n on random time intervals $[T_n, T_{n+1})$, where T_n are an increasing sequence of switching times. The dynamic system of interest is now

$$\begin{cases} dX_t &= (A(t,I_t)X_t + B(t,I_t)u_t)\,dt + \sum_{j=1}^d C^j(t,I_t)X_t dW_t^j, \\ X_0 &= x, \end{cases}$$

where A, B and C^j are bounded functions defined on $[0,T]\times K$, and a similar modification is performed on the cost functional as well. For example, in mathematical finance, to model the price of a stock in a financial market, we can use an equation of the form

$$dS_t = \mu(t,I_t)S_t\,dt + \sigma(t,I_t)S_t\,dW_t,$$

where S represent the stock price, μ and σ the appreciation and volatility rates, which are modulated by the regime process I, which can be understood as representing the random environment, the market trends, an economic regime, a credit (reputation) state as well as other economic factors. These models are also called controlled hybrid diffusion systems or jump linear systems and are the object of intense study, since they are fairly general and appropriate for a wide variety of applications. For some recent applications in risk theory, financial engineering, and insurance modeling, we refer the reader to [12], [29], [34], [38], [39] and the references therein. Moreover these models have also been used in manufacturing, communication theory, signal processing, and wireless networks; see the many references cited in [25]. In the literature, a standard assumption is that the process I should be a continuous-time Markov chain with state space K, characterized by its transition rates, independent of the Wiener process W. In this case the pair (X, I) is a controlled Markov process with values in $\mathbb{R}^n \times K$, and extensions of the standard theory allow to solve the linear-quadratic optimization problem by means of a system of Riccati equations, indexed by $i \in K$, see for instance Chapter 4 in [11], in particular equation (4.17). As a general reference for linear quadratic problems for systems driven by multiplicative white noise perturbations and Markov switching we also refer the reader to [13]. It is the purpose of the present paper to generalize this framework and consider the case of a general piecewise-constant, non-Markovian process I, independent of W. In addition, we will consider more general regime sets K which can be possibly infinite (even uncountable). Thus, in the following, the sequence (T_n, ξ_n) (or equivalently the process I) will only be assumed to be a marked point process, satisfying a mild technical condition (Assumption (A) below). To allow for even greater generality we will consider a controlled state equation of the form

$$\begin{cases} dX_s = & (A_s X_s + B_s u_s)\,ds + \sum_{j=i}^d C_s^j X_s dW_s^j, \qquad s\in[t,T]\subset[0,T], \\ X_t = & x, \end{cases} \tag{1.1}$$

with a quadratic cost functional

$$J(t,x,u) = \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T (< S_s X_s, X_s > + |u_s|^2)ds + < G X_T, X_T > \right], \tag{1.2}$$

where now A, B, C^j, S (respectively, G) are matrix-valued bounded stochastic processes (resp. bounded random variable), which are assumed to be predictable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by I and W (resp. $\mathcal{F}_T$-measurable). S and G are non-negative, as before. Correspondingly, the control u will also be $(\mathcal{F}_t)$-predictable. The use of a random cost functional is customary when dealing with stochastic coefficients, but since $\mathbb{E}^{\mathcal{F}_0} = \mathbb{E}$ this models generalizes the previous ones when $t = 0$. Our main result states that the (stochastic) value function has the form

$$\inf_u J(t,x,u) =< P_t x, x >, \tag{1.3}$$

where P is the unique global solution to the following Riccati backward stochastic differential equation:

$$\begin{cases} -dP_t &= (A_t' P_t + P_t A_t + C_t' P_t C_t + C_t' Q_t + Q_t C_t - P_t' B_t B_t' P_t + S_t)\,dt \\ & \quad -Q_t dW_t - \displaystyle\int_K U_t\,\tilde{\mu}(dt,dx), \\ P_T &= G, \end{cases} \tag{1.4}$$

see Theorems 4.5 and 4.6 for more details. The unknown in (1.4) is a triple (P, Q, U), where P is a matrix-valued adapted process with cadlag paths and $Q = (Q^1, \dots, Q^d)$ is matrix-valued $(\mathcal{F}_t)$-predictable processes and U is a matrix-valued $(\mathcal{F}_t)$-predictable random field defined on $\Omega \times [0, T] \times K$. Finally, the optimal control is characterized by the optimal feedback control law $u_s = -B_s^{'} P_{s-} X_s$. Hence, we solve completely a linear quadratic stochastic optimal control problem under non-Markovian switching.

When only the Brownian motion is present the problem has been widely studied. It was introduced by Bismut in [5] as an open problem and firstly solved by Peng [31] without control dependent noise (as our case).In [5] the solution to the Riccati equation, although introduced, for the first time in literature, in the general brownian case, is proved to exist in the special case when the coefficients are adapted to a filtration independent from the noise. The case considered by Peng in [31] corresponds to ours and the techniques to solve the backward stochastic Riccati equations are similar once the linear Lyapunov equation is derived. In [31] the a-priori estimates for the Lyapunov equation is obtained directly from the equation and then a monotone convergence argument is performed, while we use a fixed point technique and a control theoretic argument to get the a-priori estimate. The two approaches are both classical, we decided for clearness to report our argument in full detail. On the other hand the Lyapunov equation has to be solved in different spaces due to the presence of the new martingale term arising from the switching. Only more recently, in a series of papers [22], [23], [24] and eventually [35], have the authors solved the more general case with control dependent case. The basic difficulty they had to face is the fact that the (multidimensional) stochastic Riccati backward equation becomes quadratic with respect to the martingale term; the technique used in [35], based on an inversion of stochastic flow, seems to be difficult to extend beyond the Brownian context. All these results treat the finite horizon case, in [16] and [17] there are some extensions to the infinite horizon and ergodic case.

The papers [18], [28], [33] contain results on linear quadratic optimal control when the driving noise consists of a Brownian motion and an independent Poisson random measure. In [33] the authors assume that the backward stochastic Riccati equation admits an appropriate solution and obtain an optimal state feedback control and a representation of the value function. In [18] the stochastic LQ control problem is studied under partial observation but only in the one-dimensional case; the associated Riccati BSDE is derived, but existence and uniqueness for this BSDE is not studied. In [28] more complete results were obtained: the author fully characterizes the value function by means of the so-called associated stochastic Hamiltonian system, a fully coupled forward-backward system related to the maximum principle; he proves well-posedness for this systems and derives the corresponding optimal feedback control law, even in the general case when the control parameter affects the integral with respect to the Brownian motion and with respect to the compensated Poisson random measure. However, the corresponding Riccati BSDE is solved only in a more restricted setting.

In the present paper our aim is not to obtain better results for the case of Wiener and Poisson noise (eventually reaching the generality of [35]), but rather to generalize the kind of noise occurring in the coefficients, allowing for a much more general random measure and having in mind mainly applications to non-Markovian switching systems, as explained above. This leads of course to additional technical difficulties, especially in the solution of the Riccati BSDE (for which we extend the method used in [31] rather than the one used in [28]).

We note that in recent times there is an increasing interest in addressing BSDEs driven by general random measures, often motivated by applications to stochastic control: see [1] [3], [9], [10] as a non-exhaustive list. To our knowledge, the present paper is the first attempt to apply these recent progresses to the topic of linear quadratic optimal control.

2 General framework and preliminaries.

This section sets out the notation and some assumptions that are supposed to hold in the sequel. We first describe the noise entering the system. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where a standard d-dimensional Brownian motion $W = (W^1, \dots, W^d)$ is defined as well as an independent multivariate point process (also called marked point process) on a space K. Next we recall some basic properties of such point processes for which we refer to [20] or [8] or [7]. We suppose that K a Borel space, i.e. a topological space homeomorphic to a Borel subset of a compact metric space (some authors call this a Lusin space); thus, K can be any complete separable metric space. The Borel σ-algebra of K is denoted by $\mathcal{B}(K)$ (a similar notation will also be used for other topological spaces as well). A marked point process is a double sequence $(T_n, \xi_n)_{n\geq 1}$ such that the random variables T_n take values in $(0,\infty]$ and satisfy $T_n < T_{n+1}$ whenever $T_n < \infty$, and the random variables ξ_n (called *marks*) take values in K and satisfy $\xi_n = \Delta$ whenever $T_n = \infty$, where Δ is a distinguished point in K. We will impose conditions implying that the process is non-explosive, that is $T_n \to \infty$. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the smallest complete right continuous filtration generated by I and W. Throughout the paper we only use this filtration. We denote the conditional expectation with respect to $\mathcal{F}_t$ by the symbol $\mathbb{E}^{\mathcal{F}_t}(\cdot)$ (rather than $\mathbb{E}[\cdot \mid \mathcal{F}_t]$). We let $\mathcal{P}$ denote the predictable σ-algebra corresponding to $(\mathcal{F}_t)_{t\geq 0}$. By abuse of notation, we use the same symbol to denote the trace of $\mathcal{P}$ on any subset $\Omega \times J$ for any interval $J \subset [0,\infty)$. For any auxiliary measurable space $(G, \mathcal{G})$, a function on the product $\Omega \times J \times G$ which is measurable with respect to $\mathcal{P} \otimes \mathcal{G}$ is also called *predictable*. To the marked point process we can associate a K-valued piecewise constant process I defined by $I_t = \xi_n$ for $t \in [T_n T_{n+1})$ (and $I_t = k_0$, some given point in K, for $t \in [0, T_1)$) and a random measure μ on $((0,\infty) \times K, \mathcal{B}((0,\infty)\times K))$ given by

$$\mu(dt, dx) = \sum_{T_n<\infty} \delta_{(T_n,\xi_n)}(dt, dx). \tag{2.1}$$

We need the concept of compensator (or dual predictable projection) of μ under $\mathbb{P}$, relative to the filtration $(\mathcal{F}_t)$. This is a predictable random measure $((0,\infty)\times K, \mathcal{B}((0,\infty)\times K))$, denoted $\nu(dt\, dx)$, satisfying

$$\mathbb{E}\int_0^\infty \int_K H_t(x)\, \mu(dt\,, dx) = \mathbb{E}\int_0^\infty \int_K H_t(x)\, \nu(dt\, dx), \tag{2.2}$$

for every nonnegative predictable process H. The measure ν admits the disintegration:

$$\nu(\omega, dt, dx) \;=\; da_t(\omega)\, \phi_{\omega,t}(dx), \tag{2.3}$$

where a is an increasing càdlàg predictable process starting at $a_0 = 0$ (which is also the compensator of the univariate point process $\mu((0,t]\times K)$, $t \geq 0$) and ϕ is a transition probability from $(\Omega \times (0,\infty), \mathcal{P})$ into $(K, \mathcal{K})$ We make the following

Assumption (A) $\mathbb{P}$-a.s., the process $(a_t)_{t>0}$ has continuous trajectories.

It can be proved that Assumption (A) implies that the process is non-explosive, and in fact it is equivalent to the requirement that the jump times T_n are non exposive and totally inaccessible. (A) holds if and only if, $\mathbb{P}$-a.s., $\nu(\{t\}\times K) = 0$ for every $t > 0$. We finally note that we will be interested in a control problem formulated for a fixed deterministic time horizon $T \in (0,\infty)$, so that we only need to have W defined on $[0,T]$ and μ a random measure defined on $(0,T]\times K$. For any Euclidean space E, we denote by $<\cdot,\cdot>$ the scalar product and by $\mathcal{B}(E)$ the Borel σ-algebra. We denote by $\mathcal{S}_n$ the space of symmetric matrices of dimension $n\times n$, and by $\mathcal{S}_n^+$ its subset of non-negative definite matrices. We denote by the same symbol $|\cdot|$ both the norm of a vector and the matrix operator norm. Let a, b be real numbers, $0 \leq a < b \leq T$. The following classes of processes will be used in the paper.

- $L^p_{\mathcal{P}}(\Omega\times[a,b]; E)$, for $p \in [1,\infty]$ denotes the standard L^p space constructed on the measurable space $(\Omega\times[a,b], \mathcal{P})$ endowed with the product measure $\mathbb{P}(d\omega)\, dt$. It is endowed with the natural norm

$$|Y|^p_{L^p_{\mathcal{P}}(\Omega\times[a,b];E)} = \mathbb{E}\int_a^b |Y_s|^p ds$$

  for $p < \infty$, replaced by the essential supremum of $|Y|$ for $p = \infty$. Elements of this space are identified up to almost sure equality with respect to $\mathbb{P}(d\omega)\, dt$.

- $L^p_{\mathcal{P}}(\Omega; D([a,b]; E))$, for $p \in [1,\infty]$, denotes the space of adapted processes Y with càdlàg paths in E (i.e., right-continuous on $[a,b)$ having finite left limits on $(a,b]$) such that the norm

$$\begin{array}{ll} |Y|^p_{L^p_{\mathcal{P}}(\Omega;D([a,b];E))} = \mathbb{E}\sup_{t\in[a,b]} |Y_t|^p & \text{if} \quad p<\infty, \\ |Y|_{L^\infty_{\mathcal{P}}(\Omega;D([a,b];E))} = ess\ sup_{\omega\in\Omega}\sup_{t\in[a,b]} |Y_t(\omega)| & \text{if} \quad p=\infty \end{array}$$

  is finite. Elements of this space are identified up to indistinguishability.

Remark 2.1 The previous notation is justified from the fact that a process $\tilde{Y} \in L^p_{\mathcal{P}}(\Omega; D([a,b]; E))$ is progressively measurable and it is well known that given such a process, it is possible to find $Y \in L^p_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^k)$ such that $Y = \tilde{Y}$ $\mathbb{P}(d\omega)\, dt$-a.s.,

- $L^p_{\mathcal{P}}(\Omega; C([a,b]; E))$, for $p \in [1,\infty)$, denotes the subspace of $L^p_{\mathcal{P}}(\Omega; D([a,b]; E))$ consisting of processes with continuous paths. It is endowed with the same norm and its elements are predictable processes.
- $L^p(a,b,\nu)$, for $p \in [1,\infty)$, denotes the set of equivalence classes, with respect to the measure $\phi_t(\omega, dx) da_t(\omega)\mathbb{P}(d\omega)$, of mappings $H : \Omega \times (a,b] \times K \to \mathcal{S}_n$ which are predictable (i.e. $\mathcal{P} \otimes \mathcal{B}(K)$-measurable) and such that

$$|H|^p_{L^p(a,b,\nu)} = \mathbb{E} \int_{(a,b]} \int_K |H_t(x)|^p \; \mu(dt, dx) = \mathbb{E} \int_{(a,b]} \int_K |H_t(x)|^p \, \nu(dt, dx) < \infty.$$

Moreover we denote with $L^p(\Omega, \mathcal{F}_T, \mathbb{P}; E)$ the subset of $\mathbb{P}$-equivalence classes of $L^p(\Omega, \mathcal{F}, \mathbb{P}; E)$ which have an $\mathcal{F}_T$-measurable representative, endowed with the same norm ($p \in [1,\infty]$). We recall that for any predictable real function satisfying $\int_0^T \int_K |H_t(y)| \; \phi_t(dy)\, da_t < \infty, \mathbb{P}$-a.s. one can define the stochastic integral with respect to $\tilde{\mu} = \mu - \nu$ as the difference of ordinary integrals with respect to μ and ν. Given an element H of $L^1(0,T,\nu)$, its stochastic integral with respect to $\tilde{\mu}$ turns out to be a finite variation martingale on $[0,T]$. Moreover if H is in $L^2(0,T,\nu)$ then its stochastic integral with respect to $\tilde{\mu}$ is a square integrable, purely discontinuous martingale with predictable quadratic variation $\int_0^{\cdot} \int_K |H_t(x)|^2 \; \phi_t(dx)\, da_t$. Finally we recall that the weak property of predictable representation holds with respect to $(\mathcal{F}_t)$ and $\mathbb{P}$ (see [3, Example 2.1 (2)]). This means that every square integrable martingale M has a representation

$$M_t = M_0 + \int_0^t Z_s \, dW_s + \int_0^t \int_K U(s,x) \tilde{\mu}(ds, dx)$$

where $Z \in L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^d)$ and $U \in L^2(0,T,\nu)$.

3 Assumptions and statement of the problem

Throughout the paper we assume that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a Brownian motion W and an independent multivariate point process $(T_n, \xi_n)_{n\geq 1}$ on a space K are given, satisfying the assumptions in the previous section, in particular Assumption (A) that will be recalled in the statements of the main results. We consider the following stochastic differential equation

$$\begin{cases} dX_t &= (A_t X_t dt + B_t u_t)\, dt + C_t X_t \, dW_t, \\ X_s &= x, \end{cases} \tag{3.1}$$

where the unknown process X is $\mathbb{R}^n$-valued and represents the state of a controlled system, u is the control process and the initial condition $x \in \mathbb{R}^n$ is deterministic. A precise notion of solution to the state equation (3.1) is given below. To stress its dependence on u, t, and x we will denote it by $X^{t,x,u}$ when needed. We introduce a cost functional of the form

$$J(t,x,u) = \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T (< S_s X_s, X_s > + |u_s|^2)\, ds + < G X_T, X_T > \right]$$

and we aim at finding an optimal control, relatively to the given data (t,x), that is $\bar{u} \in L^2_{\mathcal{P}}(\Omega \times [t,T]; \mathbb{R}^k)$ such that

$$J(t,x,\bar{u}) = \operatorname*{ess\,inf}_{u \in L^2_{\mathcal{P}}(\Omega \times [t,T]; \mathbb{R}^k)} J(t,x,u).$$

We also look for a characterization of the (random) value function, that is the essential infimum above. Elements of the space $L^2_{\mathcal{P}}(\Omega \times [t,T]; \mathbb{R}^k)$ are called admissible controls.

Remark 3.1 The minimization could also be equivalently performed over $L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^k)$ since the values of the process u over $[0,t]$ are irrelevant.

Another possible formulation consists in considering control processes $\tilde{u}$ satisfying $\mathbb{E}\int_0^T |\tilde{u}_s|^2 ds < \infty$ which are only progressively measurable (rather than predictable). However, given such a process $\tilde{u}$, it is possible to find $u \in L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^k)$ such that $u = \tilde{u}$ $\mathbb{P}(d\omega)\,dt$-a.s., so that the corresponding trajectories coincide and we clearly have $J(t,x,u) = J(t,x,\tilde{u})$. Therefore the two optimization problems are essentially the same. If one prefers to use progressively measurable control processes the optimal feedback law (4.21) simplifies to $\bar{u}_s = -B_s^{'} P_s \bar{X}_s$.

We will work under the following general assumptions on the coefficients.

Hypothesis 3.2

(A_1) We assume that the processes $A, B, C = (C^1, \dots, C^d)$ satisfy

$$A \in L^\infty_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^{n\times n}), \quad B \in L^\infty_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^{n\times k}), \quad C^j \in L^\infty_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^{n\times n}),$$

for $j = 1, \dots, d$.

(A_2) $G \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathcal{S}_n^+)$.

(A_3) $S \in L^\infty_{\mathcal{P}}(\Omega \times [0,T]; \mathcal{S}_n^+)$.

We denote M_A, M_B, M_C, M_G, M_S nonnegative constants such that $|G(\omega)| \leq M_G$ $\mathbb{P}(d\omega)$-a.s. and

$$|A_t(\omega)| \leq M_A, \quad |B_t(\omega)| \leq M_B, \quad |C^j_t(\omega)| \leq M_C, \quad |S_t(\omega)| \leq M_S,$$

$\mathbb{P}(d\omega)dt$-a.s. for $j = 1, \dots, d$.

Next we present precise statements that ensure that the formulation of the optimization problem makes sense.

Definition 3.1 *Given $x \in \mathbb{R}^n$, $t \in [0,T]$ and $u \in L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^k)$, a solution to (3.1) is a process $X \in L^2_{\mathcal{P}}(\Omega; C([0,T]; \mathbb{R}^n))$ such that, $\mathbb{P}$-a.s.,*

$$X_s = x + \int_t^s (A_r X_r + B_r u_r)\, dr + \sum_{j=1}^d \int_t^s C^j_r X_r \, dW^j_r, \qquad s \in [t,T]. \tag{3.2}$$

The following existence and uniqueness result is standard (see [15],[19] or [32]).

Theorem 3.3 *Let assumption (A_1) be satisfied. For any $p \geq 2$, given any $t \in [0,T]$, $x \in \mathbb{R}^n$ and predictable control u with*

$$\mathbb{E}\left(\int_t^T |u_s|^2 ds\right)^{p/2} < \infty,$$

the equation (3.1) has a unique solution $X \in L^p_{\mathcal{P}}(\Omega; C([t,T]; \mathbb{R}^n))$ and it satisfied the estimate

$$\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} |X_s|^p \leq C_p \left[|x|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T |u_s|^2 ds\right)^{p/2}\right] \tag{3.3}$$

for a suitable constant C_p depending on p, T, M_A, M_B and M_C. Notice that $C_p \geq 1$.

4 Solution of the optimal control problem

4.1 The Lyapunov equation.

We start from the linear part of the Riccati equation. Namely we consider the Lyapunov equation

$$\begin{cases} -dP_t &= (A_t'P_t + P_tA_t + C_t'P_tC_t + C_t'Q_t + Q_tC_t + L_t)\,dt \\ & \quad -Q_tdW_t - \displaystyle\int_K U_t\,\tilde{\mu}(dt,dx), \\ P_T &= H, \end{cases} \tag{4.1}$$

where $L \in L^2_{\mathcal{P}}(\Omega \times [0,T];\mathcal{S}_n)$ and $H \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathcal{S}_n)$. We use the shortened notation

$$C_t'P_tC_t + C_t'Q_t + Q_tC_t = \sum_{j=1}^{d}[(C_t^j)'P_tC_t^j + (C_t^j)'Q_t^j + Q_t^jC_t^j], \qquad Q_tdW_t = \sum_{j=1}^{d} Q_t^j dW_t^j. \tag{4.2}$$

Definition 4.1 *A solution to problem (4.1) is a process* $(P,Q,U) \in L^2_{\mathcal{P}}(\Omega; D([0,T];\mathcal{S}_n)) \times L^2_{\mathcal{P}}(\Omega \times [0,T];(\mathcal{S}_n)^d) \times L^2(0,T,\nu)$ *that verifies,* $\mathbb{P}$*-a.s.,*

$$P_t = H + \int_t^T (A_s'P_s + P_sA_s + C_s'P_sC_s + C_s'Q_s + Q_sC_s + L_s)\,ds - \int_t^T Q_sdW_s - \int_t^T\int_K U_t\,\tilde{\mu}(ds,dx),\ t \in [0,T]. \tag{4.3}$$

Proposition 4.1 ensures existence and uniqueness of the solution of the Lyapunov equation (4.1). We remark that Assumption (A) is used at this point, but it is not needed in the sequel.

Proposition 4.1 *Assume Hypotheses* (A_1)*. Then for any* $H \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathcal{S}_n)$ *and* $L \in L^2_{\mathcal{P}}(\Omega \times [0,T];\mathcal{S}_n)$ *problem (4.1) has a unique solution* (P,Q,U) *and we have moreover*

$$\mathbb{E}\sup_{s\in[t,T]}|P_s|^2 + \mathbb{E}\int_t^T |Q_s|^2ds + \mathbb{E}\int_t^T\int_K |U_s(x)|^2\nu(ds,dx) \le C_0\,\mathbb{E}\left[|H|^2 + \int_t^T |L_s|^2ds\right], \tag{4.4}$$

for every $t \in [0,T]$ *and for some constant* C_0 *depending only on* T, M_A, M_C *and the underlying marked point process.*

Proof. The proof of this and other similar results relies on the weak property of predictable representation mentioned above. In the case of a Poisson random measure (possibly however with infinite activity) the result was proved in Lemma 2.4 of [36], in Theorem 2.1 in [2] and in Theorem 53.1 in [30]. The result is also proved in [3] in the setting of a nonhomogeneous compensator ν assumed to be absolutely continuous with respect to Lebesgue measure. Under the Assumption (A), it is straightforward to generalize the established fixed point method of proof to the present setting, see for instance [9] and [10]. For this reason we omit the proof and leave the details to the reader. □

The following result is a key step towards the fundamental relation (see Proposition 4.4-1).

Theorem 4.2 *Assume Hypotheses* (A_1)*. Let* $H \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathcal{S}_n)$*,* $L \in L^\infty_{\mathcal{P}}(\Omega \times [0,T];\mathcal{S}_n)$ *and let* (P,Q,U) *be the unique solution to (4.1). Then for all* $t \in [0,T], x \in \mathbb{R}^n, u \in L^2_{\mathcal{P}}(\Omega \times [0,T];\mathbb{R}^k)$*, denoting by* $X^{t,x,u}$ *the corresponding solution to (3.1), it holds that,* $\mathbb{P}$*-a.s.,*

$$< P_tx, x >= \mathbb{E}^{\mathcal{F}_t} < HX_T^{t,x,u}, X_T^{t,x,u} > + \mathbb{E}^{\mathcal{F}_t}\int_t^T [< L_sX_s^{t,x,u}, X_s^{t,x,u} > -2 < P_sB_su_s, X_s^{t,x,u} >]ds \tag{4.5}$$

Moreover, for all $t \in [0,T]$*,*

$$|P_t| \le C_2\left[|H|_{L^\infty(\Omega,\mathcal{F}_T,\mathbb{P};\mathcal{S}_n)} + (T-t)|L|_{L^\infty_{\mathcal{P}}(\Omega\times[t,T];\mathcal{S}_n)}\right], \quad \mathbb{P}\text{-a.s.} \tag{4.6}$$

where $C_2 \ge 1$ *is the constant in* (3.3)*. In particular, we have* $P \in L^\infty_{\mathcal{P}}(\Omega \times [t,T];\mathcal{S}_n)$*.*

Proof. *First step.* We first prove (4.5) for $u \in L^8_{\mathcal{P}}(\Omega\times[0,T];\mathbb{R}^k)$. The corresponding process $X = X^{t,x,u}$ solution to (3.1) then belongs to $L^8_{\mathcal{P}}(\Omega\times[0,T];\mathbb{R}^n)$ by Theorem 3.3. Differentiating by the Itô rule (see e.g. [14], Theorem 9.35) we obtain

$$\begin{array}{rcl} d < P_sX_s, X_s > &=& \sum_{i=1}^d [< Q^i_s X_s, X_s > +2 < P_sX_s, C^i_sX_s >]\, dW^i_s \\ && + \int_K < U_s(y)X_s, X_s > \tilde{\mu}(ds,dy) - [< L_sX_s, X_s > -2 < P_sB_su_s, X_s >]ds. \end{array}$$

In order to prove that the local martingale terms have zero mean we introduce an approximating procedure. Let $\Psi \in C^2(\mathbb{R}^n)$ with $\Psi(y) = 1$ for $|y| \le 1$, $\Psi(y) = 0$ for $|y| \ge 2$ and $\Psi(y) \in [0,1]$, $\forall y \in \mathbb{R}^n$. Again by the Itô rule we obtain, for all integer $N \ge 1$,

$$\begin{array}{rcl} d[\Psi(X_s/N) < P_sX_s, X_s >] &=& N^{-1}F_N(s)ds + G_N(s)dW_s + \Psi(\frac{X_s}{N}) \int_K < U_s(y)X_s, X_s > \tilde{\mu}(ds,dy) \\ && -\Psi(X_s/N)[< L_sX_s, X_s > -2 < P_sB_su_s, X_s >]ds, \end{array} \tag{4.7}$$

where

$$\begin{array}{rcl} F_N(s) &=& < \Psi'(\frac{X_s}{N}), [A_sX_s + B_su_s] >< P_sX_s, X_s > \\ && +2 \sum_{i=1}^d < \Psi'(\frac{X_s}{N}), C^i_sX_s >< P_sC^i_sX_s, X_s > \\ && +\frac{1}{2N} \sum_{i=1}^d < \Psi''(\frac{X_s}{N})C^i_sX_s, C^i_sX_s >< P_sX_s, X_s > \\ && + \sum_{i=1}^d < \Psi'(\frac{X_s}{N}), C^i_sX_s >< Q^i_sX_s, X_s > \end{array}$$

and for $i = 1, ...d$

$$G^i_N(s) = \frac{1}{N} < \Psi'(\frac{X_s}{N}), C^i_sX_s >< P_sX_s, X_s > +\Psi(\frac{X_s}{N}) \left(2 < P_sC^i_sX_s, X_s > + < Q^i_sX_s, X_s >\right).$$

It can be easily verified that $\sup_N \mathbb{E}\int_t^T |F_N(s)|ds < \infty$.

Moreover, since $\Psi(N^{-1}y) = 0$ and $\Psi'(N^{-1}y) = 0$ if $|y| > 2N$ we have, for all fixed $N \ge 1$,

$$\sum_{i=1}^d \mathbb{E}\int_t^T |G^i_N(s)|^2 ds \le cN^4 \left(M_C^2 T\, \mathbb{E} \sup_{s\in[t,T]} |P_s|^2 + \mathbb{E}\int_t^T |Q_s|^2 ds \right) < \infty,$$

and

$$\mathbb{E}\int_t^T \int_K |\Psi(\frac{X_s}{N}) < U_s(y)X_s, X_s > |\, \nu(ds,dy) \le cN^2 \left(\mathbb{E}\int_t^T \int_K |U_s(y)|^2\, \nu(ds,dy) \right)^{1/2} < \infty,$$

for a suitable positive constant c.

Finally $< LX, X >$ and $< PBu, X >$ belong to $L^1_{\mathcal{P}}(\Omega\times[t,T];\mathbb{R})$, $< P_TX_T, X_T >$ belongs to $L^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, and $\Psi(X_s/N)$ boundedly converges to 1 $\mathbb{P}$-a.s. for all s.

Thus, first integrating in $[t,T]$ and then computing conditional expectation with respect to $\mathcal{F}_t$, and finally letting $N \to \infty$, from (4.7) we deduce:

$$< P_tx, x >= \mathbb{E}^{\mathcal{F}_t} < P_TX_T, X_T > +\mathbb{E}^{\mathcal{F}_t} \int_t^T [< L_sX_s, X_s > -2 < P_sB_su_s, X_s >]ds.$$

Second step. We prove estimate (4.6). From the first step we know that for all $x \in \mathbb{R}^n$, $\mathbb{P}$-a.s.

$$< P_tx, x >= \mathbb{E}^{\mathcal{F}_t} < HX^{t,x,0}_T, X^{t,x,0}_T > +\mathbb{E}^{\mathcal{F}_t} \int_t^T < L_sX^{t,x,0}_s, X^{t,x,0}_s > ds \tag{4.8}$$

and so

$$| < P_tx, x > | \le |H|_{L^\infty(\Omega,\mathcal{F}_T,\mathbb{P};\mathcal{S}_n)} \mathbb{E}^{\mathcal{F}_t} |X^{t,x,0}_T|^2 + |L|_{L^\infty_{\mathcal{P}}(\Omega\times[0,T];\mathcal{S}_n)} \int_t^T \mathbb{E}^{\mathcal{F}_t} |X^{t,x,0}_s|^2 ds \tag{4.9}$$

and by estimate (3.3) with $u = 0$ we have, for all $x \in \mathbb{R}^n$ with $|x| \le 1$,

$$| < P_tx, x > | \le C_2 |H|_{L^\infty(\Omega,\mathcal{F}_T,\mathbb{P};\mathcal{S}_n)} + C_2(T-t)|L|_{L^\infty_{\mathcal{P}}(\Omega\times[0,T];\mathcal{S}_n)}, \quad \mathbb{P}\text{-a.s.} \tag{4.10}$$

such bound, implies the estimate (4.6).

Third step. We extend (4.5) to all the admissible controls. For a general $u \in L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^k)$ we choose a sequence u_m such that $u_m \to u$ in $L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathbb{R}^k)$ and each u_m is bounded. By Theorem 3.3, $X^{t,x,u_m} \to X^{t,x,u}$ in $L^2_{\mathcal{P}}(\Omega; C([t,T]; \mathbb{R}^n))$ and, by the second step, $P \in L^\infty_{\mathcal{P}}(\Omega \times [0,T]; \mathcal{S}^n)$. Equality (4.5) holds for u_m and X^{t,x,u_m} and it is easy to verify that we obtain (4.5) for u and $X^{t,x,u}$ letting $m \to \infty$. For instance, we may verify that

$$\begin{aligned}
&\left|\int_t^T < L_s X_s^{t,x,u_m}, X_s^{t,x,u_m} > - < L_s X_s^{t,x,u}, X_s^{t,x,u} > ds\right| \\
&\leq \left[\left(\sup_{s\in[t,T]} |X_s^{t,x,u_m}|^2\right)^{1/2} + \left(\sup_{s\in[t,T]} |X_s^{t,x,u}|^2\right)^{1/2}\right] \cdot \\
&\cdot \left(\sup_{s\in[t,T]} |X_s^{t,x,u_m} - X_s^{t,x,u}|^2\right)^{1/2} T|L|_{L^\infty_{\mathcal{P}}(\Omega\times[0,T];\mathcal{S}_n)}
\end{aligned}$$

tends to 0 in L^1. The other terms are treated in a similar way. □

4.2 Existence and uniqueness for the Riccati equation

In this section we prove the existence of a unique solution for the Riccati equation

$$\left\{\begin{array}{ll} -dP_t & = (A'_t P_t + P_t A_t + C'_t P_t C_t + C'_t Q_t + Q_t C_t - P'_t B_t B'_t P_t + S_t)\, dt \\ & -Q_t dW_t - \displaystyle\int_K U_t\, \tilde{\mu}(ds, dx) \\ P_T & = H \end{array}\right. \tag{4.11}$$

where $H \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathcal{S}_n)$ is a general final datum while the other coefficients are the ones introduced in Assumption (A) and Hypothesis 3.2. We still use the shortened notation (4.2). The occurrence of a quadratic nonlinear term requires a specific approach to solve the problem, which is classical when dealing with the Riccati equation, see for instance [4] for the classical case and [31], Section 5, or [37], when the coefficients are random. First we will find a local solution and then we will prove some a priori estimate for the solution to guarantee the existence of a global solution. The method we use to prove the a priori bound is based on the so-called fundamental relation (see Proposition 4.4 below) and uses, in an essential way, the control-theoretic interpretation of the Riccati equation. We give the notion of solution for the equation (4.11), to be compared with Definition 4.1.

Definition 4.2 *Fix $T_0 \in [0,T]$. A solution for problem (4.11) on the interval $[T_0,T]$ is a triple (P,Q,U) with*

$$P \in L^\infty_{\mathcal{P}}(\Omega; D([T_0,T]; \mathcal{S}_n)), \quad Q \in L^2_{\mathcal{P}}(\Omega \times [T_0,T]; (\mathcal{S}_n)^d), \quad U \in L^2(T_0, T, \nu)$$

such that, $\mathbb{P}$-a.s.,

$$\begin{array}{rcl} P_t & = & H + \int_t^T [A'_s P_s + P_s A_s + C'_s P_s C_s + C'_s Q_s + Q_s C_s + S_s]\, ds \\ & & - \int_t^T Q_s dW_s - \int_t^T \int_K U_s(x) \tilde{\mu}(ds,dx) - \int_t^T P_s B_s B'_s P_s\, ds, \qquad t \in [T_0,T]. \end{array} \tag{4.12}$$

Proposition 4.3 *(local existence and uniqueness). Under Hypotheses 3.2, for every $R > 0$ there exists a $\delta = \delta(R) \in (0,T]$ such that problem (4.11), with $|H|_{L^\infty} \leq R$, has a unique solution on the interval $[T-\delta,T]$.*

Proof. Recall the notation M_B, M_S for the constants introduced in Hypothesis 3.2. Let C_p and C_0 be the constants in (3.3) and (4.4) respectively. We fix arbitrarily $r > C_2 R$ and choose $\delta \in]0,T]$ satisfying

$$C_2[R + \delta(r^2 M_B^2 + M_S)] \leq r, \qquad 4C_0 r^2 M_B^4 \delta \leq \frac{1}{2}. \tag{4.13}$$

We define

$$B(r) = \{P \in L^2_{\mathcal{P}}(\Omega; D([T-\delta,T]; \mathcal{S}_n)) : \sup_{t\in[T-\delta,T]} |P_t| \leq r \quad \mathbb{P}\text{-a.s.}\}$$

and note that $B(r)$ is a complete metric space when endowed with the distance of $L^2_{\mathcal{P}}(\Omega; D([T-\delta, T]; \mathcal{S}_n))$. We construct a contraction map $\Gamma : B(r) \to B(r)$, letting $\Gamma(P) = \widehat{P}$, where $(\widehat{P}, \widehat{Q}, \widehat{U})$ is the unique solution to the Lyapunov equation (4.1) on the time interval $[T-\delta, T]$ with $L = S - PBB'P$; that is,

$$\begin{array}{rcl} \widehat{P}_t = H & + & \int_t^T [A'_s \widehat{P}_s + \widehat{P}_s A_s + C'_s \widehat{P}_s C_s + C'_s \widehat{Q}_s + \widehat{Q}_s C_s + S_s]\, ds \\ & & - \int_t^T \widehat{Q}_s \, dW_s - \int_t^T \int_K \widehat{U}_s(x) \tilde{\mu}(ds, dx) - \int_t^T P_s B_s B'_s P_s \, ds. \end{array} \tag{4.14}$$

We first check that Γ maps $B(r)$ into itself. By Proposition 4.1 (applied on $[T-\delta, T]$) we know that $\Gamma(P) \in L^\infty_{\mathcal{P}}(\Omega; D([T-\delta, T]; \mathcal{S}_n))$, so it is enough to show that for all $t \in [T-\delta, T]$ it holds $|\Gamma(P)_t| \le r$ $\mathbb{P}$-a.s. Thanks to (4.6) we have, for all t,

$$\begin{array}{rcl} |\Gamma(P)_t| & \le & C_2 \left[|G|_{L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathcal{S}_n^+)} + \delta \, |S - PBB'P|_{L^\infty_{\mathcal{P}}(\Omega \times [T-\delta, T]; \mathcal{S}_n)} \right] \\ & \le & C_2 [R + \delta (r^2 M_B^2 + M_S)] \le r, \qquad \mathbb{P}\text{-a.s.} \end{array} \tag{4.15}$$

by (4.13). To check the contraction property, we take P^1 and P^2 in $B(r)$ and recall (4.4) obtaining

$$\begin{array}{rcl} \mathbb{E} \sup_{t \in [T-\delta, T]} |\Gamma(P^1)_t - \Gamma(P^2)_t|^2 & \le & C_0 \, \mathbb{E} \int_{T-\delta}^T |P_s^1 B_s B_s^{'} P_s^1 - P_s^2 B_s B_s^{'} P_s^2|^2 \, ds \\ & \le & 2C_0 \, \mathbb{E} \int_{T-\delta}^T [|(P_s^1 - P_s^2) B_s B_s^{'} P_s^1|^2 + |P_s^2 B_s B_s^{'} (P_s^1 - P_s^2)|^2] \, ds \\ & \le & 4 C_0 r^2 M_B^4 \delta \, \mathbb{E} \sup_{t \in [T-\delta, T]} |P_t^1 - P_t^2|^2 \end{array}$$

so that Γ is indeed a contraction in $B(r)$ by (4.13).

If P is its unique fixed point, the solution (P, Q, U) of (4.1) with $L = S - PBB'P$ is a solution to (5.11). Notice that $P \in L^\infty_{\mathcal{P}}(\Omega; D([T-\delta, T]; \mathcal{S}_n))$ thus (Q, U) are well defined by Proposition 4.1.

Conversely, given two solution (P^i, Q^i, U^i) in $[T-\delta_0, T]$, $i = 1, 2$ let $R' = |P^1|_{L^\infty(\Omega; D([T-\delta_0, T]; S_n)} + |P^2|_{L^\infty(\Omega; D([T-\delta_0, T]; S_n)}$ and fix r' and $\delta' \le \delta_0$ such that $4C_0 r^2 M_B^4 \delta' < 1/2$ and $C_2[R' + \delta'(r' M_B)^2 + \delta' M_S] \le r'$ (therefore $r' \ge R'$ since $C_2 \ge 1$).

Both P^i lie in the ball of radius r' in $L^\infty(\Omega; D([T-\delta', T]; S_n)$ and are fixed points of the above defined mapping Γ which is a contraction on such a ball. Therefore they must coincide. Proceeding iteratively we get that P^1 and P^2 coincide on the whole $[T-\delta_0, T]$. This implies that the other components Q^i, U^i must coincide as well by the uniqueness result in Proposition 4.1. □

We prove the following a priori bound for any solution with nonnegative final point.

Proposition 4.4 *Assume Hypothesis 3.2 and let (P, Q, U) be any solution to (4.11) in the sense of Definition 4.2 on an interval $[T_0, T]$. Moreover suppose that $H \ge 0$. Then the following holds.*

1. *(The fundamental relation) For all $t \in [T_0, T]$, $x \in \mathbb{R}^n$, $u \in L^2_{\mathcal{P}}(\Omega \times [t, T]; \mathbb{R}^k)$ it holds*

$$< P_t x, x >= J(t, x, u) - \mathbb{E}^{\mathcal{F}_t} \int_t^T |u_s + B'_s P_s X_s^{t,x,u}|^2 ds, \qquad \mathbb{P}\text{-a.s.} \tag{4.16}$$

2. *(Positivity) For every $t \in [T_0, T]$ and $x \in \mathbb{R}^n$ we have $< P_t x, x > \ge 0$ $\mathbb{P}$-a.s. In particular, $P \in L^\infty_{\mathcal{P}}(\Omega; D([T_0, T]; \mathcal{S}_n^+))$.*

3. *(A priori estimate) For every $t \in [T_0, T]$ we have $|P_t| \le C_2(|H|_{L^\infty} + T M_S)$ $\mathbb{P}$-a.s., where C_2 is the constant in* (3.3).

Proof. We note that (P, Q, U) is the solution to the Lyapunov equation (4.1) with $L = S - PBB'P$. Hence by (4.5)

$$\begin{array}{rcl} < P_t x, x > & = & \mathbb{E}^{\mathcal{F}_t} < G X_T^{t,x,u}, X_T^{t,x,u} > + E^{\mathcal{F}_t} \int_t^T < S_s X_s^{t,x,u}, X_s^{t,x,u} > ds \\ & & - \mathbb{E}^{\mathcal{F}_t} \int_t^T < P_s B_s B'_s P_s X_s^{t,x,u}, X_s^{t,x,u} > - \mathbb{E}^{\mathcal{F}_t} \int_t^T < P_s B_s u_s, X_s^{t,x,u} > ds. \end{array} \tag{4.17}$$

The fundamental relation then follows adding and subtracting $\mathbb{E}^{\mathcal{F}_t} \int_t^T |u_s|^2 \, ds$ to the right-hand side. To prove positivity, consider the following closed loop equation, starting at any time $t \in [T_0, T]$ with an arbitrary initial data $x \in \mathbb{R}^n$:

$$\left\{ \begin{array}{rcl} d\bar{X}_s & = & [A \bar{X}_s - B_s B_s^{'} P_s \bar{X}_s] \, ds + C_s \bar{X}_s dW_s \\ \bar{X}_t & = & x. \end{array} \right. \tag{4.18}$$

Such equation fulfills the hypotheses of proposition 3.3. Then applying the fundamental relation (4.16) to the control $\bar{u} = -B^{'}P\bar{X}$ and to $\bar{X}^{t,x,\bar{u}} = \bar{X}$ we get $< P_t x, x >= J(t,x,\bar{u}) \geq 0$, $\mathbb{P}$-a.s., which proves the claim. Equality (4.16), with $u = 0$, gives for all $x \in \mathbb{R}^n$ and all $t \in [T_0, T]$,

$$\begin{array}{rcl} < P_t x, x > & \leq & J(t,x,0) \\ & = & \mathbb{E}^{\mathcal{F}_t} < G X_T^{t,x,0}, X_T^{t,x,0} > + \mathbb{E}^{\mathcal{F}_t} \int_t^T < S_s X_s^{t,x,0}, X_s^{t,x,0} > ds \\ & \leq & M_G \, \mathbb{E}^{\mathcal{F}_t} |X_T^{t,x,0}|^2 + M_S \int_t^T \mathbb{E}^{\mathcal{F}_t} |X_s^{t,x,0}|^2 \, ds \end{array} \tag{4.19}$$

and from (3.3) it follows that $< P_t x, x > \leq C_2 \, [M_G + T M_S] \; |x|^2$, which proves the required estimate. □

Now using the a priori bound in Proposition 4.4 we are in a position to extend the local existence and uniqueness shown in 4.1 to the whole $[0,T]$.

Theorem 4.5 *Suppose that Assumption (A) and Hypothesis 3.2 hold true. Then the Riccati equation (4.11) with $H = G$ has a unique solution (P,Q,U) such that $P \in L^\infty_{\mathcal{P}}(\Omega; D([0,T]; \mathcal{S}_n^+))$, $Q \in L^2_{\mathcal{P}}(\Omega \times [0,T]; (\mathcal{S}_n)^d)$ and $U \in L^2(0,T,\nu)$.*

Proof. Let $R = C_2(M_G + T M_S)$- By Proposition 4.3 there exists a unique solution (P,Q,U) of equation (4.11) in $[T - \delta(R), T]$. Moreover by Proposition 4.4 we know that $|P_{T-\delta(R)}| \leq R$ and $P_{T-\delta(R)} \geq 0$. Then we can again can apply the local existence in $[T - 2\delta(R), T - \delta(R)]$ with final datum $P_{T-\delta(R)}$. We can then argue iteratively and cover the whole interval $[0,T]$ by a finite number of intervals of length $\delta(R)$ and obtain the required global solution. Uniqueness is proved in a similar way: we already know that any two solutions (P^i, Q^i, U^i), $i = 1,2$ must satisfy $0 \leq P^i_t \leq RI$ for all $t \in [0,T]$. Using iteratively the uniqueness result in Proposition (4.3), fist on $[T - \delta(R), T]$ then on $[T - 2\delta(R), T - \delta(R)]$ and so on we get that they must coincide.

Finally, the fact that P takes values in $\mathcal{S}_n^+$ follows from the positivity property in Proposition 4.4. □

4.3 Synthesis of the optimal control

The following theorem provides a solution to the control problem.

Theorem 4.6 *Suppose that Assumption (A) and Hypothesis 3.2 hold true. Fix $t \in [0,T]$ and $x \in \mathbb{R}^n$. Then the following holds.*

1. *There exists a unique optimal control $\bar{u} \in L^2_{\mathcal{P}}(\Omega \times [t,T]; \mathbb{R}^k)$.*

2. *If $\bar{X} = X^{t,x,\bar{u}}$ denotes the corresponding solution (that is the optimal state), then $\bar{X}$ is the unique solution to the closed loop equation on $[t,T]$:*

$$\left\{ \begin{array}{rcl} d\bar{X}_s & = & [A_s \bar{X} - B_s B_s^{'} P_s \bar{X}_s] \, ds + C_s \, \bar{X}_s dW_s, \\ \bar{X}_t & = & x. \end{array} \right. \tag{4.20}$$

3. *The following feedback law holds $\mathbb{P}$-a.s. for almost every $s \in [t,T]$:*

$$\bar{u}_s = -B_s^{'} P_{s-} \bar{X}_s. \tag{4.21}$$

4. *The value function, i.e. the optimal cost, is given by $J(t,x,\bar{u}) =< P_t x, x >$, $\mathbb{P}$-a.s.*

Proof. The optimal control, if it exists, is unique by the strict convexity of the map $u \mapsto J(t,x,u)$ on $L^2_{\mathcal{P}}(\Omega \times [t,T]; \mathbb{R}^k)$. Let (P,Q,U) be the unique solution to the Riccati equation (4.11). From the fundamental relation (4.16) we have

$$J(t,x,u) =< P_t x, x > + \mathbb{E} \int_0^T |u_s + B_s^{'} P_s \bar{X}_s^{t,x,u}|^2 ds =< P_t x, x > + \mathbb{E} \int_0^T |u_s + B_s^{'} P_{s-} \bar{X}_s^{t,x,u}|^2 ds,$$

where the last inequality follows from the fact that $P_t = P_{t-}$, $\mathbb{P}(d\omega)\, dt$-a.s., since P_t has càdlàg paths. Then $J(t,x,u) \geq < P_t x, x >$ for all $u \in L^2_{\mathcal{P}}(\Omega \times [t,T]; \mathbb{R}^k)$ and the equality holds if and only if (4.21) holds, that is, if and only if $X = \bar{X}$ solves (4.20) and $u = \bar{u}$. □

5 Conclusions

As already mentioned in the introduction this paper, to authors knowledge, is a first attempt to solve linear quadratic controlled switching systems where the switching mechanism is not required to be Markovian. The problem as been solved by means of a Riccati equation, which turned out to be a backward stochastic differential equation driven by the Brownian motion and by the random measure associated to the marked point process. To separate difficulties we have not considered the control dependent noise case which, it is well knonwn, also in the Brownian setting has been an open problem for decades because of the presence of a quadratic martingale term. This choice allows to follow a classical scheme of resolution passing through the study of the Lyapunov equation.

We believe that our results in the non-Markovian case can be generalized in several directions, for instance to the case of control on infinite horizon, both for a discounted or an ergodic cost functional, and to the more difficult situation when the control affects the diffusion coefficient (along the lines of [35], where however the Wiener process is the only source of randomness) or even when the controlled equation is driven by some discontinuous integrator in addition to the Brownian motion. These extensions are left for future work.

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